UNITED ARAB EMIRATES
MINISTRY OF EDUCATION

YEAR OF
ZAYED

TEACHER EDITION

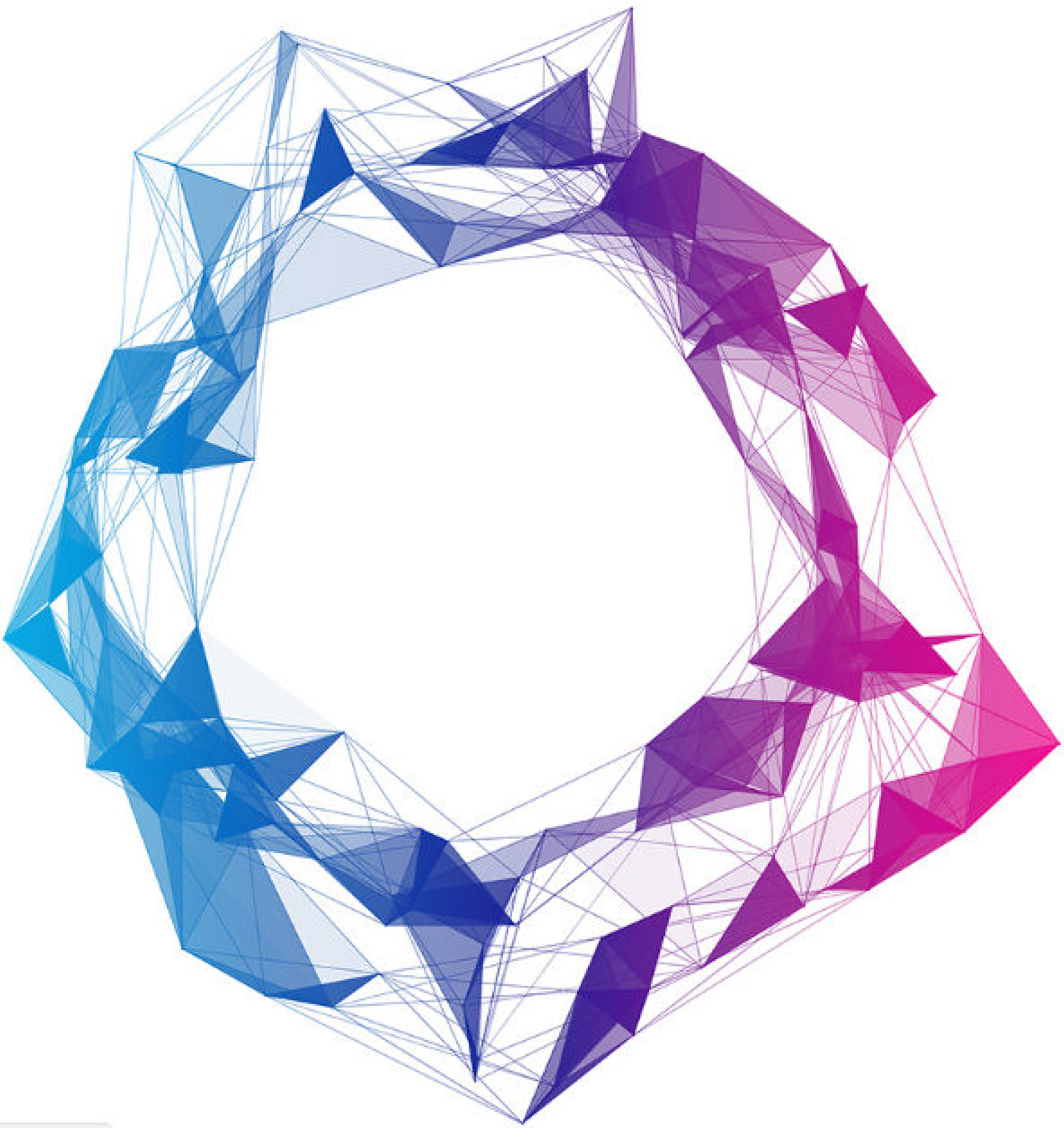
MATH

9

McGraw-Hill Education

Integrated Math

United Arab Emirates Edition



Mc Graw Hill Education

United Arab Emirates
Ministry of Education

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VISION
2021

Teacher Edition

McGraw-Hill Education

Integrated Math

United Arab Emirates Edition

GRADE **9** • VOLUME 2

McGraw Hill Education

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"Extensive knowledge and modern science must be acquired. The educational process we see today is in an ongoing and escalating challenge which requires hard work.
We succeeded in entering the third millennium, while we are more confident in ourselves."

H.H. Sheikh Khalifa Bin Zayed Al Nahyan
President of the United Arab Emirates

Contents in Brief

Authors

Our lead authors ensure that the McGraw-Hill Education mathematics programs are truly vertically aligned by beginning with the end in mind—success in Integrated Math Grade 9 and beyond. By "backmapping" the content from the high school programs, all of our mathematics programs are well articulated in their scope and sequence.

Lead Authors

John A. Carter, Ph.D.
Principal
Adlai E. Stevenson High School
Lincolnshire, Illinois

Areas of Expertise: Using technology and manipulatives to visualize concepts; mathematics achievement of English-language learners

Gilbert J. Cuevas, Ph.D.
Professor of Mathematics Education
Texas State University—San Marcos
San Marcos, Texas

Areas of Expertise: Applying concepts and skills in mathematically rich contexts; mathematical representations

Roger Day, Ph.D., NBCT
Mathematics Department Chairperson
Pontiac Township High School
Pontiac, Illinois

Areas of Expertise: Understanding and applying probability and statistics; mathematics teacher education

Carol Malloy, Ph.D.
Associate Professor
University of North Carolina at Chapel Hill
Chapel Hill, North Carolina

Areas of Expertise: Representations and critical thinking; student success in Algebra 1

Program Authors

Ruth Casey
Mathematics Consultant
Regional Teacher Partner
University of Kentucky
Lexington, Kentucky

Areas of Expertise: Graphing technology and mathematics

Jerry Cummins
Mathematics Consultant
Former President, National Council of Supervisors of Mathematics
Western Springs, Illinois

Areas of Expertise: Graphing technology and mathematics

Dr. Berchie Holliday, Ed.D.
National Mathematics Consultant
Silver Spring, Maryland

Areas of Expertise: Using mathematics to model and understand real-world data; the effect of graphics on mathematical understanding

Beatrice Moore Luchin
Mathematics Consultant
Houston, Texas

Areas of Expertise: Mathematical literacy; working with English language learners

Contributing Author

Dinah Zike FOLDABLES
Educational Consultant
Dinah-Might Activities, Inc.
San Antonio, Texas

These professionals were instrumental in providing valuable input and suggestions for improving the effectiveness of the mathematics instruction.

Lead Consultants

Viken Hovsepian
Professor of Mathematics
Rio Hondo College
Whittier, California

Jay McTiche
Educational Author and Consultant
Columbia, Maryland

Consultants

Mathematical Content

Grant A. Fraser, Ph.D.
Professor of Mathematics
California State University, Los Angeles
Los Angeles, California

Arthur K. Wayman, Ph.D.
Professor of Mathematics Emeritus
California State University, Long Beach
Long Beach, California

Gifted and Talented

Shelbi K. Cole
Research Assistant
University of Connecticut
Storrs, Connecticut

College Readiness

Robert Lee Kimball, Jr.
Department Head, Math and Physics
Wake Technical Community College
Raleigh, North Carolina

Graphing Calculator

Ruth M. Casey
T^3 National Instructor
Frankfort, Kentucky

Jerry Cummins
Former President
National Council of Supervisors of Mathematics
Western Springs, Illinois

Mathematical Fluency

Robert M. Capraro
Associate Professor
Texas A&M University
College Station, Texas

Pre-AP

Dixie Ross
Lead Teacher for Advanced Placement Mathematics
Pflugerville High School
Pflugerville, Texas

Reading and Writing

ReLeah Cossett Lent
Author and Educational Consultant
Morgantown, GA

Lynn T. Havens
Director of Project CRISS
Kalispell, Montana

Multiple Paths to Learning

Students will engage with tasks they find interesting, challenging, and important. In the classroom, key factors are teacher-student relationships, pedagogy, and classroom climate.

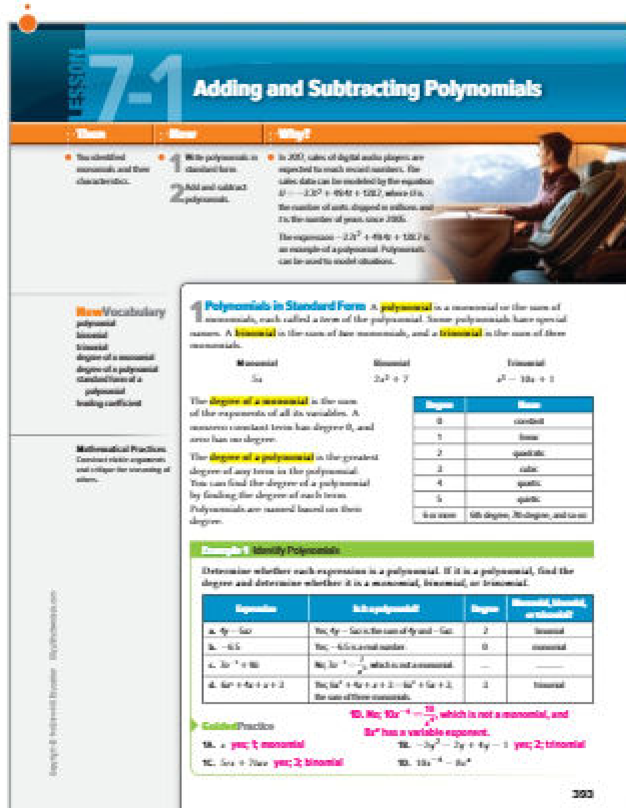
Integrated Math Grade 9 is designed to provide your students with a balanced approach to mathematics. Your students become successful through a variety of teaching modalities.

Keeping It Interesting

Students who are engaged are more likely to pay attention in class. ***Integrated Math Grade 9*** discusses topics that relate to today's students.

Content that Connects

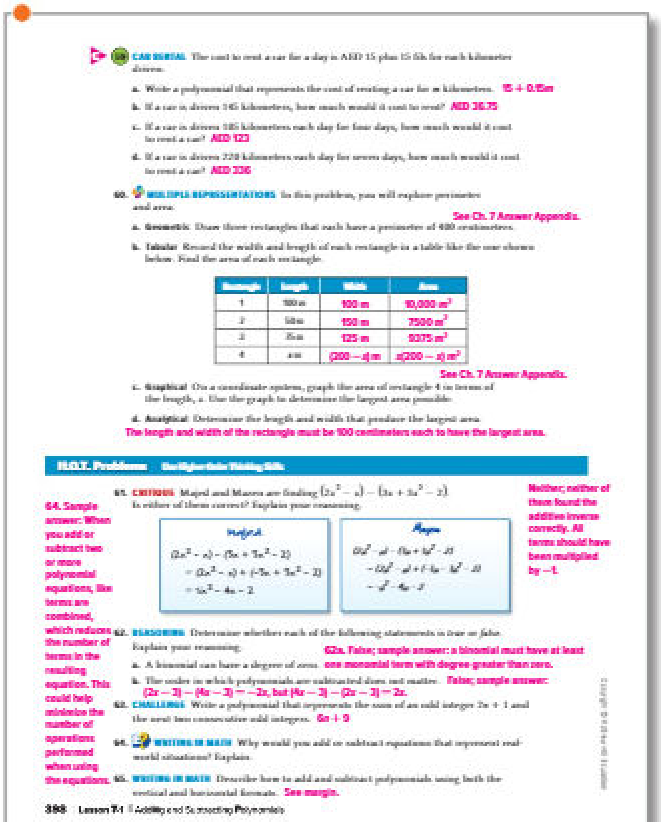
Students use a Then, Now, Why instructional plan that connects what they know with what they are learning. New Vocabulary and Review Vocabulary help students learn to "talk math."



Problem-Based Learning

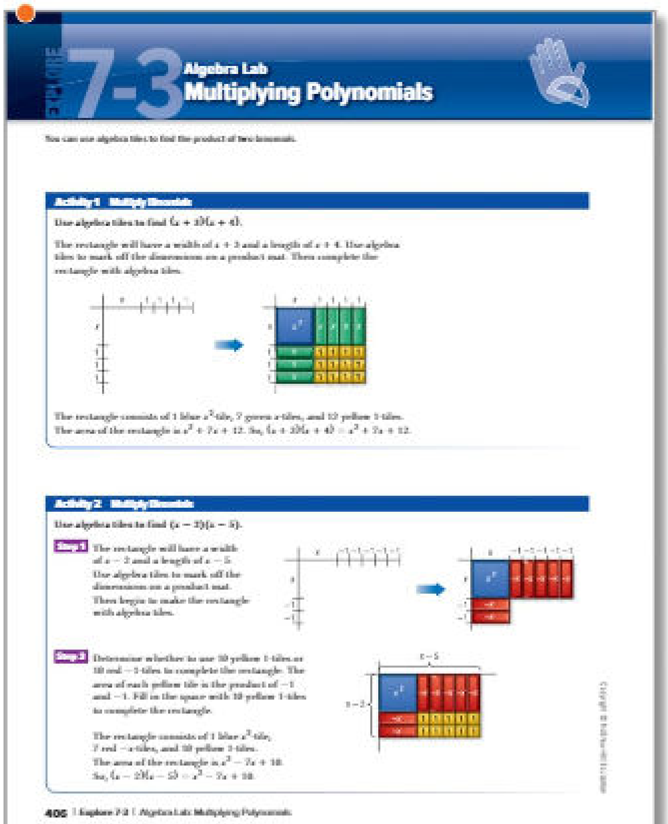
A wealth of problem-solving opportunities include:

- **Multiple Representations** in every chapter
- **H**igher **O**rder **T**hinking **problems** in every lesson
- Worked-out **Examples** that follow a four-step plan
- **Problem-Solving Strategy Tips** throughout
- **Test-Taking Strategies** in every chapter
- **Word-Problem Practice** masters for every lesson



Learning by Doing

Labs help to maintain learner motivation. Algebra Labs introduce and reinforce concepts using manipulatives like algebra tiles. Graphing Technology Labs allow students to explore concepts using graphing calculators.

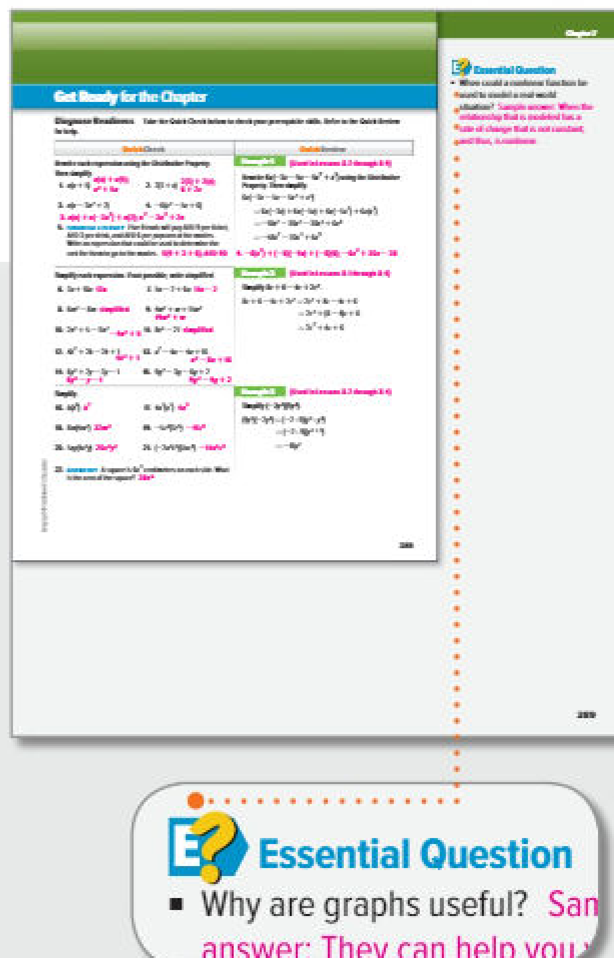


Understanding by Design

What should students know and be able to do? Understanding by Design can be used to help teachers identify learning goals, develop revealing assessments of student understanding, and plan effective and engaging learning activities.

Backward Design

Understanding by Design (UbD) is a framework that uses backward design to create a coherent curriculum by considering the desired results first and then planning instruction.



Identifying Desired Results

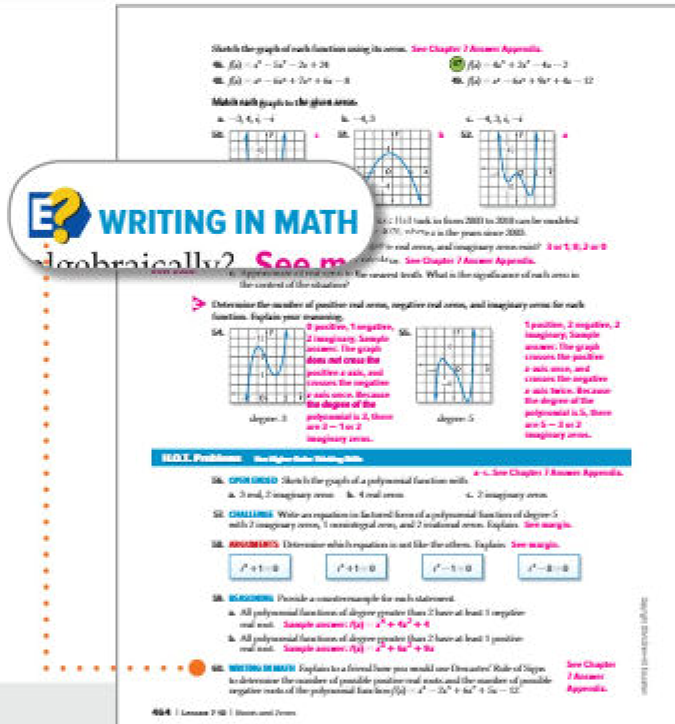
The first step in developing an effective curriculum using the UbD framework is to consider the goals. What should students know and be able to do?

Integrated Math Grade 9 focuses student attention on **Essential Questions**, which are located within each chapter of the Student and Teacher Editions.

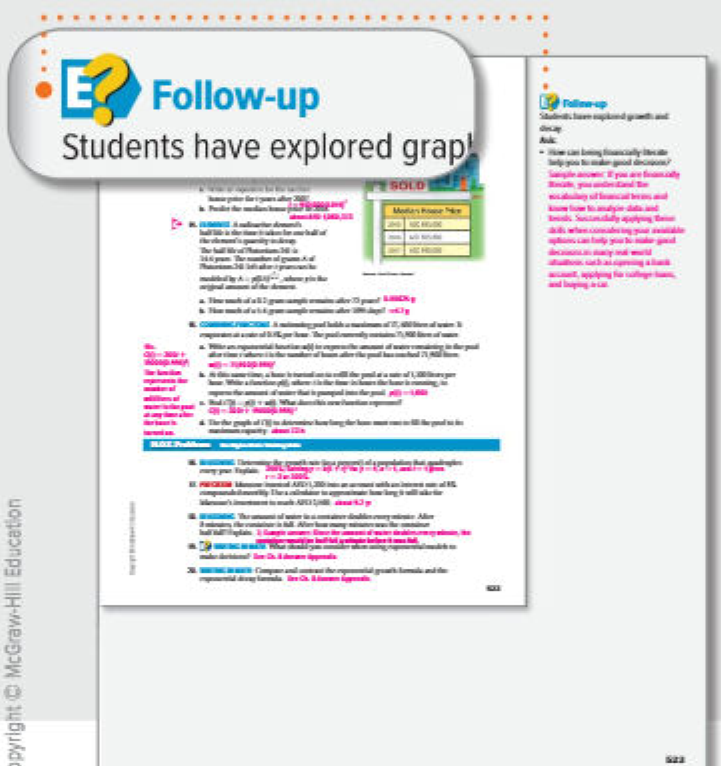
An Essential Question is provided at the beginning of each chapter. These thought provoking questions can be used as:

- a discussion starter for your class; throughout the discussion identify what the students already know and what they would like to know about the topic. Revisit these notes throughout the chapter.
- a benchmark of understanding; post these questions in a prominent place and have students expand upon their initial response as their understanding of the subject material grows.

> **"** *In mathematics, essential questions are used to develop students' understanding of key concepts as well as core processes.* **"**
>
> *— Jay McTighe,*
> *co-author of Understanding by Design*



Follow-up Essential Questions can be found throughout each chapter. These questions challenge students to apply specific knowledge to a broader context, thus deepening their understanding.



Determine Acceptable Evidence

A variety of assessment opportunities are available that enable students to show evidence of their understanding.

- **Practice and Problem Solving** and **H.O.T. Problems** allow students to explain, interpret, and apply mathematical concepts.
- **Mid-Chapter Quizzes** and **Chapter Tests** offer more traditional methods of assessment.

Plan Learning Experiences and Instruction

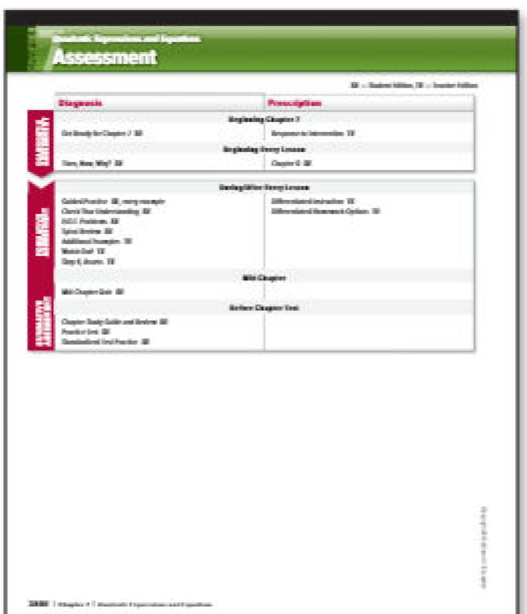
There are numerous performance activities available throughout the program, including:

- **Algebra and Geometry Labs** that offer students hands-on learning experiences, and
- **Graphing Technology Labs** that use graphing calculators to aid student understanding.

Assessment that Validates Learning

Only 48% of teachers feel that standardized tests are effective in helping them track student performance.

Integrated Math Grade 9 offers a variety of frequent and meaningful assessments built right into the curriculum structure and teacher support materials. The program includes both traditional and nontraditional methods of assessment, including quizzes and tests, performance tasks, and open-ended assessments.



1 Diagnostic

Initial Assessment

Assess students' knowledge at the beginning of the year with *Diagnostic and Placement Tests.* The results and scoring guides identify students who may need additional resources to meet grade-level standards.

Entry-Level Assessment

Assess students' prior knowledge at the beginning of a chapter or lesson.

Student Edition

- Get Ready for the Chapter

2 Formative

Progress Monitoring

Determine if students are progressing adequately as you teach each lesson. Use the assessments to differentiate lesson instruction and practice.

Student Edition

- Guided Practice
- Check Your Understanding
- H.O.T. Problems
- Mid-Chapter Quiz
- Study Guide and Review

3 Summative

Summative Assessment

Assess student success in learning the concepts in each chapter. Use remediation suggestions to address problem areas.

Student Edition

- Practice Test
- Standardized Test Practice

Built-In Differentiated Instruction

Approximately 43% of teachers feel their classes are so mixed in terms of students' learning abilities that they can't teach them effectively.

AL Approaching Grade Level
OL On Grade Level
BL Beyond Grade Level

Integrated Math Grade 9 provides resources to diagnose students, identify areas of need, and conduct short, frequent assessments for accurate data-driven decision making. Every lesson considers the needs of all students.

STEM Careers

Student intentions to go to college have increased over the past 20 years. In 1988, 80% said they were likely to go to college, compared to 90% today.

Developing STEM Careers

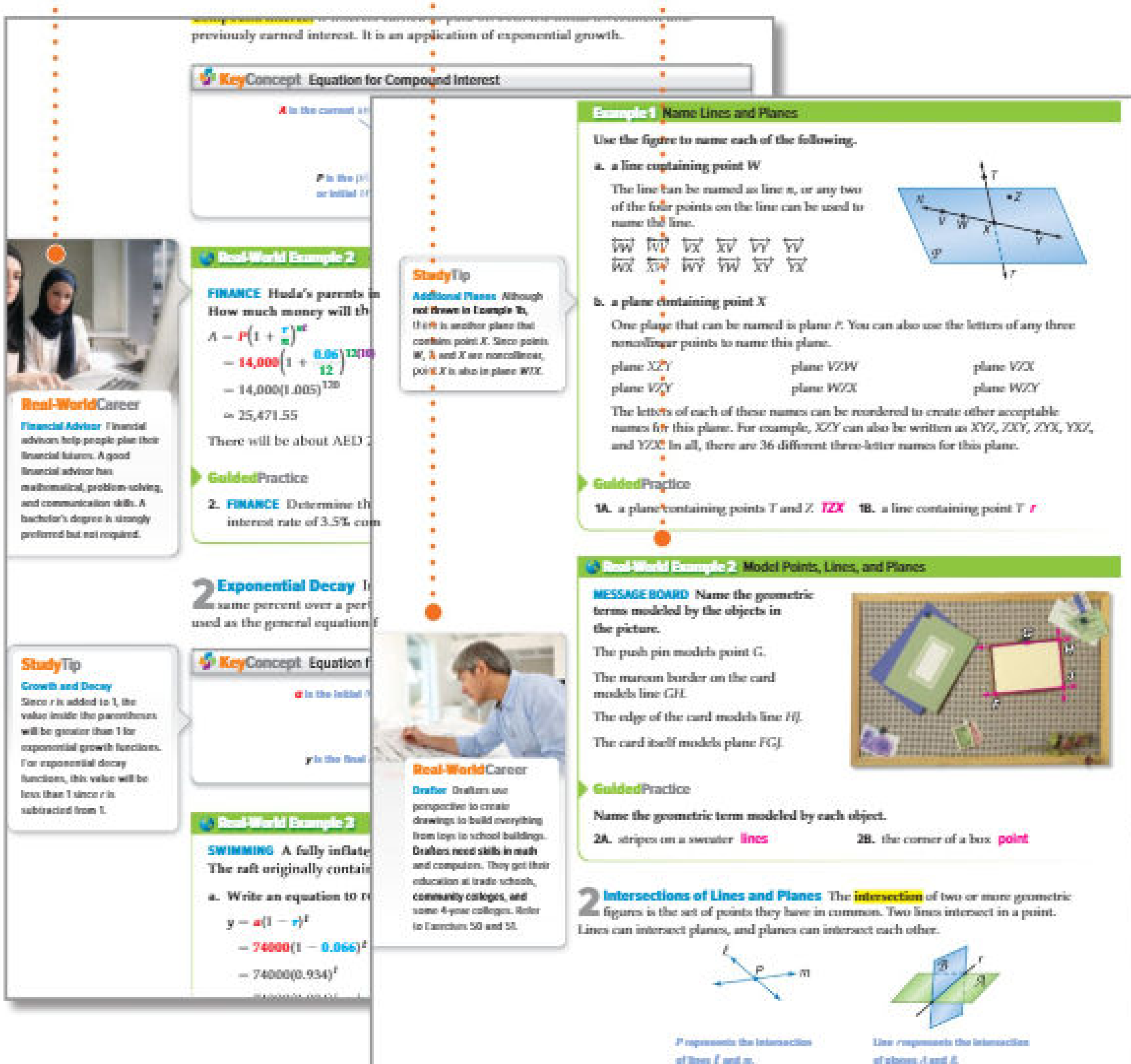
With ***Integrated Math***, you can unleash your students' curiosity about the world around them and prepare them for exciting **STEM** (**S**cience, **T**echnology, **E**ngineering, and **M**ath) careers.

Real-World Careers

Real-World Careers are engaging, providing information on exciting careers.

Examples

Examples are relevant, connecting in-class experiences to the world beyond the classroom.



21st Century Skills

The current and future health of America's 21st Century Economy depends directly on how broadly and deeply Americans reach a new level of literacy—'21st Century Literacy'—that includes strong academic skills, thinking, reasoning, teamwork skills, and proficiency in using technology.

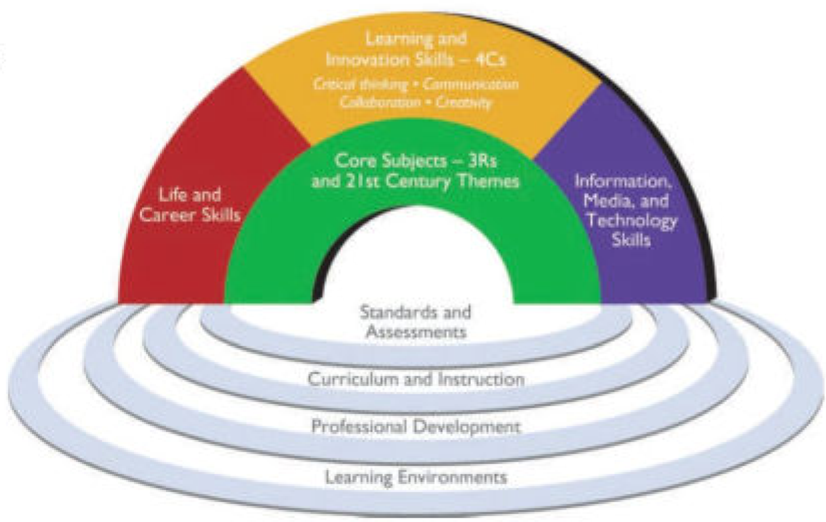
–– 21st Century Workforce Commission National Alliance of Business

Developing 21st Century Skills

The Partnership for 21st Century Skills identifies the following key student elements of a 21st century education.

Throughout the Integrated Math series, students solve problems that incorporate 21st century themes, such as financial literacy. In addition, there is also a project within each chapter that incorporates 21st Century Skills.

21st Century Student Outcomes and Support Systems



Learning and Innovation Skills

Students who are prepared for increasingly complex life and work environments are creative and innovative critical thinkers, problem solvers, effective communicators, and know how to work collaboratively. Throughout the Integrated Math series, students are required to write, explain, justify, prove, and analyze. Students can hone critical thinking skills through the use of **H.O.T. (Higher Order Thinking) Problems** and are encouraged to work collaboratively in labs.

Life and Career Skills

McGraw-Hill is committed to helping educators ensure that all students graduate from high school equipped to succeed in college or in their chosen careers. McGraw-Hill programs bridge the gap between secondary and postsecondary curricula by including pre-college content as well as the study skills and transitional life skills that are necessary for both secondary and college academic success.

Information, Media, and Technology Skills

Throughout the Integrated Math series, students use technology, including graphing calculators to develop 21st century mathematics knowledge and skills.

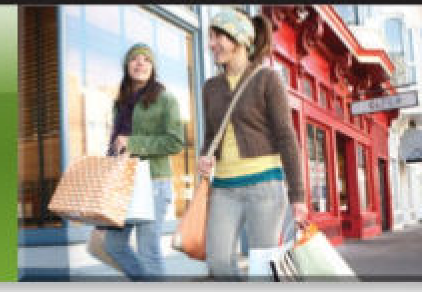
21st Century Assessments

The Integrated Math series offer a variety of frequent and meaningful assessments built right into the curriculum structure and teacher support materials. These programs include both traditional and nontraditional methods of assessment, including quizzes and tests, performance tasks, and open-ended assessments.

CHAPTER 1

Expressions, Equations, and Functions

CHAPTER 2 Linear Equations



CHAPTER 3

Linear Functions

CHAPTER 4

Equations of Linear Functions

CHAPTER 5

Linear Inequalities

CHAPTER 6

Systems of Linear Equations and Inequalities

CHAPTER 7 Quadratic Expressions and Equations

CHAPTER 8

Exponents and Exponential Functions

CHAPTER 9

Radical and Rational Functions, and Equations

CHAPTER 10

Tools of Geometry

CHAPTER 11

Parallel and Perpendicular Lines

CHAPTER 12

Congruent Triangles

CHAPTER 13

Quadrilaterals

CHAPTER 14

Similarity, Transformations and Symmetry

CHAPTER 15 Circles

Student Handbook

Reference

Chapter Project

Greens Going Green

Students use what they have learned about factoring quadratic equations to complete a project.

This chapter project addresses environmental literacy, as well as several specific skills identified as being essential to student success by the Framework for 21st Century Learning.

KeyVocabulary Introduce the key vocabulary in the chapter using the routine below.

Define: Writing an integer or polynomial in factored form is writing the integer or a polynomial as a product of its prime factors.

Example: The factored form of 12 is $2^2 \cdot 3$. The factored form of $x^2 + 5x + 6$ is $(x + 2)(x + 3)$.

Ask: What is the factored form of 48? $2^4 \cdot 3$

CHAPTER 7

Quadratic Expressions and Equations

Then

You applied the laws of exponents and explored exponential functions.

Now

In this chapter, you will:

- Add, subtract, and multiply polynomials.
- Factor trinomials.
- Factor differences of squares.
- Graph quadratic functions.
- Solve quadratic equations.

Why?

ARCHITECTURE Quadratic equations can be used to model the shape of architectural structures such as the tallest memorial in the United States, the Gateway Arch in Missouri.

Essential Question

- When could a nonlinear function be used to model a real-world situation? Sample answer: When the relationship that is modeled has a rate of change that is not constant, and thus, is nonlinear.

Get Ready for the Chapter

Diagnose Readiness Take the Quick Check below to check your prerequisite skills. Refer to the Quick Review for help.

QuickCheck

Rewrite each expression using the Distributive Property. Then simplify.

1. $a(a+5)$ $a(a)+a(5)$; a^2+5a
2. $2(3+x)$ $2(3)+2(x)$; $6+2x$
3. $n(n-3n^2+2)$
4. $-6(x^2-5x+6)$

3. $n(n)+n(-3n^2)+n(2)$; n^2-3n^3+2n

5. **FINANCIAL LITERACY** Five friends will pay AED 9 per ticket, AED 3 per drink, and AED 6 per popcorn at the movies. Write an expression that could be used to determine the cost for them to go to the movies. $5(9+3+6)$; AED 90

4. $-6(x^2)+(-6)(-5x)+(-6)(6)$; $-6x^2+30x-36$

Simplify each expression. If not possible, write *simplified*.

6. $3u+10u$ $13u$
7. $5a-2+6a$ $11a-2$
8. $6m^2-8m$ simplified
9. $4w^2+w+15w^2$ $19w^2+w$
10. $2x^2+5-11x^2$ $-9x^2+5$
11. $8v^3-27$ simplified
12. $4k^2+2k-2k+1$ $4k^2+1$
13. $a^2-4a-4a+16$ $a^2-8a+16$
14. $6y^2+2y-3y-1$ $6y^2-y-1$
15. $9g^2-3g-6g+2$ $9g^2-9g+2$

Simplify.

16. $b(b^6)$ b^7
17. $4n^3(n^2)$ $4n^5$
18. $8m(4m^2)$ $32m^3$
19. $-5z^4(3z^5)$ $-15z^9$
20. $5xy(4x^3y)$ $20x^4y^2$
21. $(-2a^4c^5)(7ac^4)$ $-14a^5c^9$
22. **GEOMETRY** A square is $6x^3$ centimeters on each side. What is the area of the square? $36x^6$

QuickReview

Example 1 (Used in Lessons 8-2 through 8-9)

Rewrite $6x(-3x-5x-5x^2+x^3)$ using the Distributive Property. Then simplify.

$$6x(-3x-5x-5x^2+x^3)$$
$$=6x(-3x)+6x(-5x)+6x(-5x^2)+6x(x^3)$$
$$=-18x^2-30x^2-30x^3+6x^4$$
$$=-48x^2-30x^3+6x^4$$

Example 2 (Used in Lessons 8-1 through 8-4)

Simplify $8c+6-4c+2c^2$.

$$8c+6-4c+2c^2=2c^2+8c-4c+6$$
$$=2c^2+(8-4)c+6$$
$$=2c^2+4c+6$$

Example 3 (Used in Lessons 8-2 through 8-4)

Simplify $(-2y^3)(9y^4)$.

$$(9y^3)(-2y^4)=(-2\cdot 9)(y^3\cdot y^4)$$
$$=(-2\cdot 9)(y^{3+4})$$
$$=-18y^7$$

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FOLDABLES StudyOrganizer

Dinah Zike's Foldables®

Focus Students write about factoring and quadratic equations as these concepts are presented in the lessons of this chapter.

Teach Have students make and label their Foldables as illustrated. Suggest that students use their Foldables to take notes, record concepts, and define terms. They can also use them to record the direction and progress of learning, to describe positive and negative experiences during learning, to write about personal associations and experiences, and to list examples of ways in which this new knowledge has been or will be used in their daily lives.

When to Use It Encourage students to add to their Foldables as they work through the chapter and to use them to review for the chapter test.

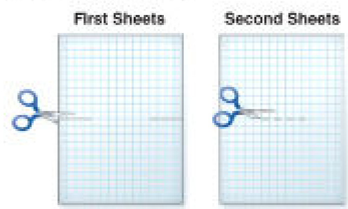
Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study this chapter. To get ready, identify important terms and organize your resources. You may wish to refer to previous chapters to review prerequisite skills.

FOLDABLES StudyOrganizer

Quadratic Expressions and Equations Make this Foldable to help you organize your notes about quadratic expressions and equations. Begin with five sheets of grid paper.

1 **Fold** in half along the width. On the first three sheets, cut 5 centimeters along the fold at the ends. On the second two sheets cut in the center, stopping 5 centimeters from the ends.



2 **Insert** the first sheets through the second sheets and align the folds. Label the front Chapter 8, Quadratic Expressions and Equations. Label the pages with lesson numbers and the last page with vocabulary.

NewVocabulary

polynomial
binomial
trinomial
degree of a monomial
degree of a polynomial
standard form of a polynomial
leading coefficient
FOIL method
quadratic expression
factoring
factoring by grouping
Zero Product Property
quadratic equation
prime polynomial
difference of two squares
perfect square trinomial
Square Root Property

ReviewVocabulary

absolute value the absolute value of any number n is the distance the number is from zero on a number line and is written $|n|$

The absolute value of −2 is 2 because it is 2 units from 0.

perfect square a number with a square root that is a rational number

390 | Chapter 7 | Quadratic Expressions and Equations

CHAPTER 7

Quadratic Expressions and Equations

Chapter Planner

	Diagnostic Assessment Quick Check EXPLORE 7-1 45 min: 0.5 day 90 min: 0.5 day	LESSON 7-1 45 min: 1.5 days 90 min: 0.5 day	LESSON 7-2 45 min: 1 day 90 min: 0.5 day
Title	**Algebra Lab: Adding and Subtracting Polynomials**	**Adding and Subtracting Polynomials**	**Multiplying a Polynomial by a Monomial**
Objectives	▪ Use algebra tiles to add and subtract polynomials.	▪ Write polynomials in standard form. ▪ Add and subtract polynomials.	▪ Multiply a polynomial by a monomial. ▪ Solve equations involving the products of monomials and polynomials.
Key Vocabulary		polynomial, binomial, trinomial, degree of a monomial, degree of a polynomial, leading coefficient	

EXPLORE 7-3 (45 min: 0.5 day; 90 min: 0.5 day)	LESSON 7-3 (45 min: 1.5 days; 90 min: 1 day)	LESSON 7-4 (45 min: 1 day; 90 min: 1 day)
Algebra Lab: Multiplying Polynomials	**Multiplying Polynomials**	**Special Products**
▪ Use algebra tiles to multiply polynomials.	▪ Multiply binomials by using the FOIL method. ▪ Multiply polynomials by using the Distributive Property.	▪ Find squares of sums and differences. ▪ Find the product of a sum and a difference.
	FOIL method quadratic expression	
		Formative Assessment Mid-Chapter Quiz

CHAPTER 7

Quadratic Expressions and Equations

Chapter Planner

	EXPLORE 7-5 (45 min: 0.5 day; 90 min: 0.25 day)	LESSON 7-5 (45 min: 1.5 days; 90 min: 0.75 day)	EXPLORE 7-6 (45 min: 0.5 day; 90 min: 0.25 day)	LESSON 7-6 (45 min: 1.5 days; 90 min: 0.75 day)
Title	**Algebra Lab: Factoring Using the Distributive Property**	**Using the Distributive Property**	**Algebra Lab: Factoring Trinomials**	**Solving $x^2 + bx + c = 0$**
Objectives	▪ Use algebra tiles to model using the Distributive Property to factor binomials.	▪ Use the Distributive Property to factor polynomials. ▪ Solve quadratic equations of the form $ax^2 + bx = 0$.	▪ Use algebra tiles to model factoring trinomials.	▪ Factor trinomials of the form $x^2 + bx + c$. ▪ Solve equations of the form $x^2 + bx + c = 0$.
Key Vocabulary		factoring factoring by grouping Zero Product Property		quadratic equation

LESSON 7-7 (45 min: 2 days; 90 min: 1 day)	LESSON 7-8 (45 min: 1 day; 90 min: 0.5 day)	LESSON 7-9 (45 min: 1 day; 90 min: 0.5 day)	LESSON 7-10 (45 min: 1 day; 90 min: 0.5 day)
Solving $ax^2 + bx + c = 0$	**Differences of Squares**	**Perfect Squares**	**Roots and Zeros**
▪ Factor trinomials of the form $ax^2 + bx + c$. ▪ Solve equations of the form $ax^2 + bx + c = 0$.	▪ Factor binomials that are the difference of squares. ▪ Use the difference of squares to solve equations.	▪ Factor perfect square trinomials. ▪ Solve equations involving perfect squares.	▪ Determine the number and type of roots for a polynomial equation. ▪ Find the zeros of a polynomial function.
prime polynomial	difference of two squares	perfect square trinomial	

Summative Assessment
Study Guide and Review
Practice Test

Assessment

SE = Student Edition, **TE** = Teacher Edition

	Diagnosis	Prescription
DIAGNOSTIC ASSESSMENT	**Beginning Chapter 7**	
	Get Ready for Chapter 7 **SE**	Response to Intervention **TE**
	Beginning Every Lesson	
	Then, Now, Why? **SE**	Chapter 0 **SE**
FORMATIVE ASSESSMENT	**During/After Every Lesson**	
	Guided Practice **SE**, every example Check Your Understanding **SE** H.O.T. Problems **SE** Spiral Review **SE** Additional Examples **TE** Watch Out! **TE** Step 4, Assess **TE**	Differentiated Instruction **TE** Differentiated Homework Options **TE**
	Mid-Chapter	
	Mid-Chapter Quiz **SE**	
SUMMATIVE ASSESSMENT	**Before Chapter Test**	
	Chapter Study Guide and Review **SE** Practice Test **SE** Standardized Test Practice **SE**	

CHAPTER 7

Quadratic Expressions and Equations

Differentiated Instruction

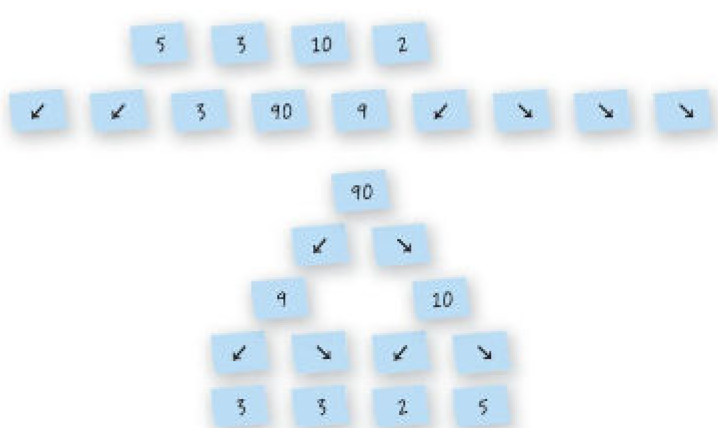
Option 1 Reaching All Learners AL OL BL

Visual/Spatial As students learn the rules for factoring trinomials, encourage them to use algebra tiles to confirm their results. Students should soon realize that the greater the values of *b* and *c* in the trinomials, the more cumbersome the algebra tiles become, which should then reinforce the importance of learning to factor using the methods in the text.

Auditory Ask groups to create a mnemonic device that will help them remember how to factor one of the types of trinomials studied in this chapter. Then write an example of a trinomial on the board and have a volunteer say his or her mnemonic device as he or she factors the trinomial.

Option 2 Approaching Level AL

In large print, write a number and each step of its prime factorization (using the factor tree-method) on sticky notes. Each number in the steps should be on a separate note. On other notes draw some arrows. Stick the notes randomly on the board. Ask volunteers to arrange the factor and arrow notes so that they show the prime factorization of the number.



Make a set of “problem” cards showing a monomial times a polynomial, such as $-3x(5x^2 - 2x + 3)$. Make a set of “simplification” cards showing the simplification of the problems on the “problem” cards. Make enough cards so that students from half the class gets one card from the “problem” set and the other students get cards from the “simplification” set. Ask each student holding a card from the “problem” set to find the classmate holding the “simplification” card that corresponds to his or her problem.

Option 3 Beyond Level BL

Tell students that some polynomials are prime when considering the set of integers, but can be factored when the set of real numbers is considered. For example, $x^2 - 7$ is a prime polynomial. However, it can be factored if the set of real numbers is considered. That is, $x^2 - 7 = (x + \sqrt{7})(x - \sqrt{7})$. Ask students to factor other prime polynomials such as $3x^2 - 2$ when the set of real numbers is considered. $(x\sqrt{3} + \sqrt{2})(x\sqrt{3} - \sqrt{2})$

Challenge students to a competition. Have pairs go to the board. Have one student multiply two 2-digit numbers using the FOIL method while the other student multiplies the same numbers using the multiplication algorithm. Repeat with other numbers. Which method appears to be faster? Then, extend the competition to include a mixed number times a mixed number.

Focus on Mathematical Content

Vertical Alignment

Before Chapter 7

Related Topics

- write prime factorizations using exponents
- identify the greatest common factor of a set of positive integers

Previous Topics

- use the Distributive Property to simplify algebraic expressions

Chapter 7

Related Topics

- add, subtract, and multiply polynomials
- factor as necessary in problem situations
- solve quadratic equations using concrete models, tables, graphs, and algebraic methods

After Chapter 7

Preparation

- use tools including factoring to simplify expressions and to transform and solve equations
- determine reasonable domain and range values of quadratic functions
- analyze situations involving quadratic functions and formulate quadratic equations to solve problems
- solve quadratic equations using graphs, tables, and algebraic methods

Lesson-by-Lesson Preview

7-1 Adding and Subtracting Polynomials

A *polynomial* is a monomial or a sum or difference of monomials. Polynomials can be added or subtracted.

- To add polynomials, add the coefficients of like terms using the rules for adding real numbers.
- To subtract polynomials, first replace each term of the polynomial being subtracted with its additive inverse. Then combine the like terms.

To aid in adding and subtracting polynomials, like terms can be grouped using a horizontal or vertical format.

7-2 Multiplying a Polynomial by a Monomial

The Distributive Property can be used to find the product of a polynomial and a monomial.

- Each term of the polynomial is multiplied by the monomial using the rules for monomial multiplication.
- Apply the rules for multiplying real numbers if the monomial is negative.
- Simplify the product by combining like terms.

Equations often contain polynomials that must be added, subtracted, or multiplied before they can be solved. To solve such equations, first simplify each side. Then apply the rules for solving multi-step equations and equations with variables on each side.

7-3 Multiplying Polynomials

When multiplying two binomials, use the Distributive Property in either a vertical or horizontal format.

- Multiply the terms of the first binomial by one term of the second binomial.
- Then, multiply the terms of the first binomial by the other term of the second binomial. Combine like terms.

A shortcut, called the FOIL method, can be used to multiply two binomials. To use this method, find the sum of the products of the First terms (F), the Outer terms (O), the Inner terms (I), and the Last terms (L).

F L I O

Product of First terms | Product of Outer terms | Product of Inner terms | Product of Last terms

$$(x+3)(x-2) = (x)(x) + (-2)(x) + (3)(x) + (3)(-2)$$
$$= x^2 - 2x + 3x - 6$$
$$= x^2 + x - 6$$

The Distributive Property can be used to multiply any two polynomials. Products are not in simplest form until all like terms have been combined.

7-4 Special Products

Some binomials have products that follow a specific pattern. Some patterns are the

- *square of a sum or a difference:* $(a + b)^2 = a^2 + 2ab + b^2$ or $(a - b)^2 = a^2 - 2ab + b^2$
- *product of a sum and a difference* of the same two terms: $(a + b)(a - b) = a^2 - b^2$

Being able to identify and use these patterns can make it easier to simplify these special products.

7-5 Using the Distributive Property

The Distributive Property is used to find the product of a monomial and a polynomial. Reverse the process to factor a polynomial whose terms have a GCF. To factor a polynomial,

- First, find the GCF of all its terms.
- Then, rewrite each term as the product of the GCF and its remaining factors.
- Finally, use the Distributive Property to factor out the GCF.

Or, use factoring by grouping if the polynomial has four or more terms.

Factoring can be used to solve some equations containing polynomials. According to the Zero Product Property, if the product of two factors is 0, then at least one of the factors must be 0. The Zero Product Property can be used to solve equations that can be written in the form $ab = 0$. Each factor is set equal to 0 and the resulting equations are solved.

7-6 Solving $x^2 + bx + c = 0$

Some trinomials of the form $x^2 + bx + c$ can be factored into two binomials.

- To factor a trinomial of the form $x^2 + bx + c$, find two integers, m and p, whose sum is b and whose product is c. The factors of the trinomial are the two binomials $(x + m)$ and $(x + p)$, when $b = m + p$ and $c = mp$.
- Determining whether m and p are positive or negative depends on b and c. If b is negative and c is negative, then m and p must have different signs.
- First, factor the trinomial. Then set each factor equal to 0. Solve the resulting equations.

7-7 Solving $ax^2 + bx + c = 0$

In this lesson the trinomials are of the form $ax^2 + bx + c$ with $a \neq 1$. To factor $ax^2 + bx + c$,

- first factor out the GCF of the terms,
- then, if the new trinomial has $a = 1$, use the method learned in Lesson 7-6 to complete the factorization.
- If, in the new trinomial, a still does not equal 1, then find two factors, m and p, such that $ac = mp$ and $b = m + p$. Rewrite the trinomial, replacing bx with $mx + px$, forming the polynomial $ax^2 + mx + px + c$. In this form, the grouping technique used in Lesson 7-5 can be used to factor the polynomial into two binomial factors.
- Any polynomial that cannot be factored is prime.
- Equations of the form $ax^2 + bx + c = 0$ can be solved by using the method above to factor the trinomial and then applying the Zero Product Property.

7-8 Differences of Squares

To factor the difference of squares, find a (the square root of the first term) and b (the square root of the last term). The two binomial factors are the sum of the square roots and difference of the square roots. In the factored form, $a^2 - b^2 = (a + b)(a - b)$.

If the terms of the original expression have a GCF, factor it out before applying any other factoring technique.

7-9 Perfect Squares

Products that result from squaring a binomial, $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$, are known as *perfect square trinomials*.

Three conditions must be satisfied for a trinomial to be a perfect square trinomial.

- The first term must be a perfect square.
- The last term must be a perfect square.
- The middle term must be twice the product of the square roots of the first and last terms.

To factor a perfect square trinomial with a plus sign before the middle term, use the pattern $a^2 + 2ab + b^2 = (a + b)^2$.

To factor a perfect square trinomial with a minus sign before the middle term, use the pattern $a^2 - 2ab + b^2 = (a - b)^2$.

The Square Root Property can be used to solve any equation that is in the form $x^2 = n$, where $n \geq 0$, or that can be written in that form. This property lets you take the square root of each side of the equation as long as both $\sqrt{n}$ and $-\sqrt{n}$ are considered.

7-10 Roots and Zeros

According to the *Fundamental Theorem of Algebra*, every polynomial equation with a degree greater than 0 has at least one root in the set of complex numbers. And according to a corollary of that theorem, the number of complex roots of a polynomial equation is the same as the degree of the equation.

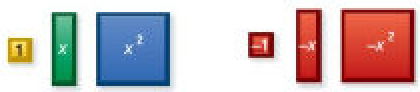
Descartes' Rule of Signs says that there is a relationship between the signs of the coefficients of a polynomial function and the number of positive and negative real zeros. By the *Complex Conjugates Theorem*, if an imaginary number is a zero of a function, its conjugate is also a zero.

EXPLORE 7-1 Algebra Lab
Adding and Subtracting Polynomials

Algebra tiles can be used to model polynomials. A polynomial is a monomial or the sum of monomials. The diagram below shows the models.

Polynomial Models

- Polynomials are modeled using three types of tiles.
- Each tile has an opposite.

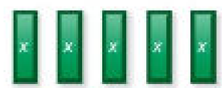


Activity 1 Model Polynomials

Use algebra tiles to model each polynomial.

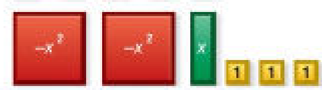
- $5x$

  To model this polynomial, you will need 5 green x-tiles.



- $-2x^2 + x + 3$

  To model this polynomial, you will need 2 red $-x^2$-tiles, 1 green x-tile, and 3 yellow 1-tiles.



Monomials such as $3x$ and $-2x$ are called *like terms* because they have the same variable to the same power.

Polynomial Models

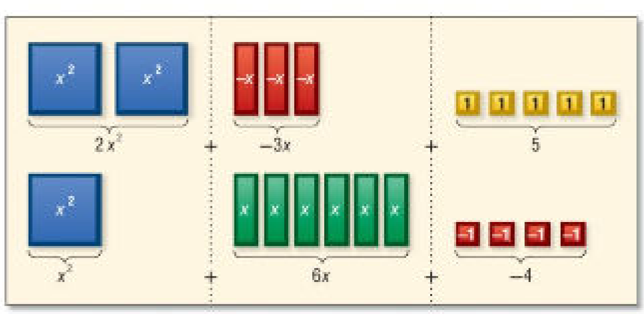
- Like terms are represented by tiles that have the same shape and size.
- A *zero pair* may be formed by pairing one tile with its opposite. You can remove or add zero pairs without changing the polynomial.

Activity 2 Add Polynomials

Use algebra tiles to find $(2x^2 - 3x + 5) + (x^2 + 6x - 4)$.

Step 1

Model each polynomial.



(continued on the next page)

1 Focus

Objective Use algebra tiles to add and subtract polynomials.

Materials for Each Student

- algebra tiles

Tips for New Teachers

Zero Pairs Prior to Activities 2 and 3, discuss the concept of a zero pair. Have students form zero pairs using 1-tiles, x-tiles, and x^2-tiles and their opposites.

2 Teach

Working in Cooperative Groups

Put students in groups of two or three, mixing abilities. Have groups complete Activity 1 and Exercises 1–3.

- Make sure students understand that the number of x-tiles and x^2-tiles represent the coefficients of x and x^2, respectively. The number of 1-tiles represents the constant in the expression.
- Tell students to be careful to use the tiles with the correct colors. It is easy to incorrectly substitute an x-tile for a $-x$-tile.
- Talk about like terms in the context of the tiles. Tiles with the same shape and size represent like terms.
- For Activity 2, tell students that it is easier to model the polynomials if they arrange the tiles in the same order as the monomials within each polynomial. In this case, the monomials are arranged in descending order of degree. Therefore, students should arrange the tiles in descending order from left to right.

- After groups have completed Activity 2, write the addition of the two polynomials vertically so students can see that the coefficients of like terms are added.
- For Activity 3, explain that adding a zero pair to the polynomial does not change its value because the zero pair is equal to zero.
- After groups complete Activity 3, write the difference vertically so students can see that coefficients of like terms are subtracted.

Practice Have students complete Exercises 6 and 7.

3 Assess

Formative Assessment

Use Exercise 8 to assess whether students can use models to compare polynomials.

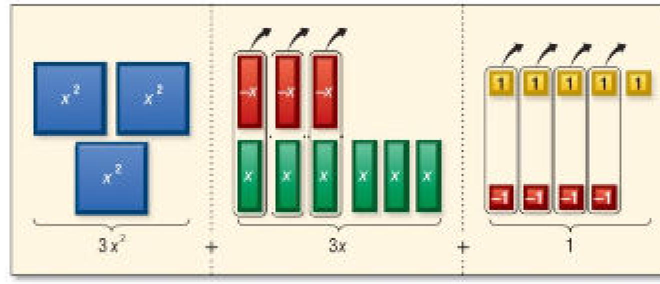
From Concrete to Abstract

Write a polynomial addition or subtraction problem on the board. Have students determine the sum or difference without using tiles. If they answer incorrectly, have them use their tiles to help them find their errors.

Algebra Lab

Adding and Subtracting Polynomials *Continued*

Step 2 Combine like terms and remove zero pairs.

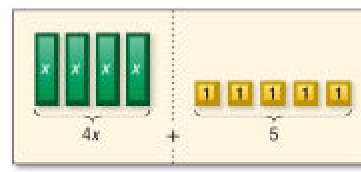


Step 3 Write the polynomial. $(2x^2 - 3x + 5) + (x^2 + 6x - 4) = 3x^2 + 3x + 1$

Activity 3 Subtract Polynomials

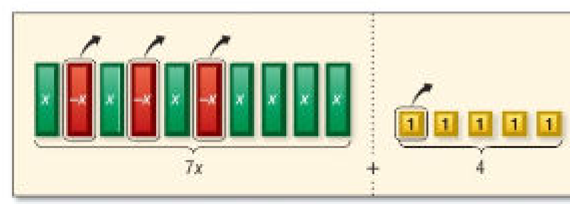
Use algebra tiles to find $(4x + 5) - (-3x + 1)$.

Step 1 Model the polynomial $4x + 5$.



Step 2 To subtract $-3x + 1$, remove 3 red $-x$-tiles and 1 yellow 1-tile. You can remove the 1-tile, but there are no $-x$-tiles. Add 3 zero pairs of x-tiles. Then remove the 3 red $-x$-tiles.

Step 3 Write the polynomial.
$(4x + 5) - (-3x + 1) = 7x + 4$

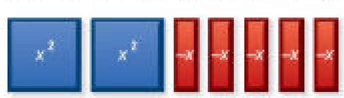


Model and Analyze

Use algebra tiles to model each polynomial. Then draw a diagram of your model. **1–3. See margin.**

1. $-2x^2$
2. $5x - 4$
3. $x^2 - 4x$

Write an algebraic expression for each model.

4.  **$2x^2 - 5x$**

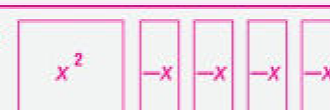
5.  **$-3x^2 + 2x + 1$**

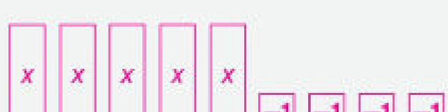
Use algebra tiles to find each sum or difference. **6. $4x^2 + 3x + 4$ 7. $x^2 + 12x + 3$ 8. $-5x^2 - 4x$**

6. $(x^2 + 5x - 2) + (3x^2 - 2x + 6)$
7. $(2x^2 + 8x + 1) - (x^2 - 4x - 2)$
8. $(-4x^2 + x) - (x^2 + 5x)$

Additional Answers

1. 

3. 

2. 

LESSON 7-1 Adding and Subtracting Polynomials

Then

- You identified monomials and their characteristics.

Now

1. Write polynomials in standard form.
2. Add and subtract polynomials.

Why?

- In 2017, sales of digital audio players are expected to reach record numbers. The sales data can be modeled by the equation $U = -2.7t^2 + 49.4t + 128.7$, where U is the number of units shipped in millions and t is the number of years since 2005.

The expression $-2.7t^2 + 49.4t + 128.7$ is an example of a polynomial. Polynomials can be used to model situations.

NewVocabulary
polynomial
binomial
trinomial
degree of a monomial
degree of a polynomial
standard form of a polynomial
leading coefficient

Mathematical Practices
Construct viable arguments and critique the reasoning of others.

1 Polynomials in Standard Form

A **polynomial** is a monomial or the sum of monomials, each called a *term* of the polynomial. Some polynomials have special names. A **binomial** is the sum of *two* monomials, and a **trinomial** is the sum of *three* monomials.

Monomial	Binomial	Trinomial
$5x$	$2x^2 + 7$	$x^3 - 10x + 1$

The **degree of a monomial** is the sum of the exponents of all its variables. A nonzero constant term has degree 0, and zero has no degree.

The **degree of a polynomial** is the greatest degree of any term in the polynomial. You can find the degree of a polynomial by finding the degree of each term. Polynomials are named based on their degree.

Degree	Name
0	constant
1	linear
2	quadratic
3	cubic
4	quartic
5	quintic
6 or more	6th degree, 7th degree, and so on

Example 1 Identify Polynomials

Determine whether each expression is a polynomial. If it is a polynomial, find the degree and determine whether it is a *monomial*, *binomial*, or *trinomial*.

Expression	Is it a polynomial?	Degree	Monomial, binomial, or trinomial?
a. $4y - 5xz$	Yes; $4y - 5xz$ is the sum of $4y$ and $-5xz$.	2	binomial
b. -6.5	Yes; -6.5 is a real number.	0	monomial
c. $7a^{-3} + 9b$	No; $7a^{-3} = \frac{7}{a^3}$, which is not a monomial.	—	—
d. $6x^3 + 4x + x + 3$	Yes; $6x^3 + 4x + x + 3 = 6x^3 + 5x + 3$, the sum of three monomials.	3	trinomial

GuidedPractice

1A. x yes; 1; monomial
1B. $-3y^2 - 2y + 4y - 1$ yes; 2; trinomial
1C. $5rx + 7tuv$ yes; 3; binomial
1D. $10x^{-4} - 8x^a$

1D. No; $10x^{-4} = \frac{10}{x^4}$, which is not a monomial, and $8x^a$ has a variable exponent.

1 Focus

VerticalAlignment

Before Lesson 7-1 Work with square roots and simplifying a radical.

Lesson 7-1 Find the degree of a polynomial. Write polynomials in standard form. Add and subtract polynomials.

After Lesson 7-1 Simplify the product of a polynomial by a monomial.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What is the value of t for the year 2007? 2
- What would be the value of t for the year 2010? 5
- Using the equation, find the value of U for the year 2007. 216.7
- How many monomials make up the expression that equals U? 3
- What are they? $-2.7t^2$; $49.4t$; 128.7

1 Degree of a Polynomial

Example 1 shows how to determine whether an expression is a polynomial and how to find the degree of a polynomial. **Example 2** shows how to write a polynomial in standard form and identify the leading coefficient.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Determine whether each expression is a polynomial. If it is a polynomial, find the degree and determine whether it is a *monomial*, *binomial*, or *trinomial*.

a. $6x - 4$ yes; 1; binomial

b. $x^2 + 2xy - 7$ yes; 2; trinomial

c. $\frac{14d + 19e^3}{5d^4}$ no

d. $26b^2$ yes; 2; monomial

2 Write each polynomial in standard form. Identify the leading coefficient.

a. $9x^2 + 3x^6 - 4x$
$3x^6 + 9x^2 - 4x$; 3

b. $12 + 5y + 6xy + 8xy^2$
$8xy^2 + 6xy + 5y + 12$; 8

2 Add and Subtract Polynomials

Example 3 shows how to add polynomials by grouping like terms. **Example 4** shows how to subtract a polynomial by adding its additive inverse. **Example 5** shows how to use addition or subtraction of polynomials to model a real-world situation.

The terms of a polynomial can be written in any order. However, polynomials in one variable are usually written in standard form. The **standard form of a polynomial** has the terms in order from greatest to least degree. In this form, the coefficient of the first term is called the **leading coefficient**.

leading coefficient → 4; greatest degree → 3

Standard form: $4x^3 - 5x^2 + 2x + 7$

Example 2 Standard Form of a Polynomial

Write each polynomial in standard form. Identify the leading coefficient.

a. $3x^2 + 4x^5 - 7x$

Find the degree of each term.

Degree: 2, 5, 1

Polynomial: $3x^2 + 4x^5 - 7x$

The greatest degree is 5. Therefore, the polynomial can be rewritten as $4x^5 + 3x^2 - 7x$, with a leading coefficient of 4.

b. $5y - 9 - 2y^4 - 6y^3$

Find the degree of each term.

Degree: 1, 0, 4, 3

Polynomial: $5y - 9 - 2y^4 - 6y^3$

The greatest degree is 4. Therefore, the polynomial can be rewritten as $-2y^4 - 6y^3 + 5y - 9$, with a leading coefficient of -2.

Guided Practice **2A.** $4x^4 - 2x^2 - 3x + 8$; 4 **2B.** $-7y^6 + 5y^3 - 2y^2 + y + 10$; -7

2A. $8 - 2x^2 + 4x^4 - 3x$

2B. $y + 5y^3 - 2y^2 - 7y^6 + 10$

2 Add and Subtract Polynomials

Adding polynomials involves adding like terms. You can group like terms by using a horizontal or vertical format.

Example 3 Add Polynomials

Find each sum.

a. $(2x^2 + 5x - 7) + (3 - 4x^2 + 6x)$

Horizontal Method

Group and combine like terms.

$(2x^2 + 5x - 7) + (3 - 4x^2 + 6x)$

$= [2x^2 + (-4x^2)] + [5x + 6x] + [-7 + 3]$ Group like terms.

$= -2x^2 + 11x - 4$ Combine like terms.

b. $(3y + y^3 - 5) + (4y^2 - 4y + 2y^3 + 8)$

Vertical Method

Align like terms in columns and combine.

$$\begin{array}{r} y^3 + 0y^2 + 3y - 5 \\ (+)\ 2y^3 + 4y^2 - 4y + 8 \\ \hline 3y^3 + 4y^2 - \ \ y + 3 \end{array}$$

Insert a placeholder to help align the terms.
Align and combine like terms.

Study Tip

Vertical Method Notice that the polynomials are written in standard form with like terms aligned. Since there is no y^2-term in the first polynomial, $0y^2$ is used as a placeholder.

Guided Practice

3A. $(5x^2 - 3x + 4) + (6x - 3x^2 - 3)$ $2x^2 + 3x + 1$

3B. $(y^4 - 3y + 7) + (2y^3 + 2y - 2y^4 - 11)$ $-y^4 + 2y^3 - y - 4$

Teach with Tech

Interactive Whiteboard Write numbers 1 through 10 on the board. Then write several polynomials on the board. For each polynomial, ask students to identify the degree of the polynomial. Grab the correct number and drag it underneath that polynomial.

Focus on Mathematical Content

Degree of a Polynomial The degree of a polynomial should not be confused with the number of terms. For example, $x^3 + 1$ is a binomial, but the degree is 3, not 2, because the greatest degree of any of the terms is 3.

StudyTip

Additive Inverse When finding the additive inverse of a polynomial, you are multiplying every term by −1.

You can subtract a polynomial by adding its additive inverse. To find the additive inverse of a polynomial, write the opposite of each term, as shown.

$$-(3x^2 + 2x - 6) = \underbrace{-3x^2 - 2x + 6}_{\text{Additive Inverse}}$$

Example 4 Subtract Polynomials

Find each difference.

a. $(3 - 2x + 2x^2) - (4x - 5 + 3x^2)$

Horizontal Method

Subtract $4x - 5 + 3x^2$ by adding its additive inverse.

$(3 - 2x + 2x^2) - (4x - 5 + 3x^2)$

$= (3 - 2x + 2x^2) + (-4x + 5 - 3x^2)$ — The additive inverse of $4x - 5 + 3x^2$ is $-4x + 5 - 3x^2$.

$= [2x^2 + (-3x^2)] + [(-2x) + (-4x)] + [3 + 5]$ — Group like terms.

$= -x^2 - 6x + 8$ — Combine like terms.

b. $(7p + 4p^3 - 8) - (3p^2 + 2 - 9p)$

Vertical Method

Align like terms in columns and subtract by adding the additive inverse.

$$\begin{array}{r} 4p^3 + 0p^2 + 7p - 8 \\ (-) \quad\quad 3p^2 - 9p + 2 \\ \hline \end{array} \quad \xrightarrow{\text{Add the opposite.}} \quad \begin{array}{r} 4p^3 + 0p^2 + 7p - 8 \\ (+) \quad -3p^2 + 9p - 2 \\ \hline 4p^3 - 3p^2 + 16p - 10 \end{array}$$

GuidedPractice

4A. $(4x^3 - 3x^2 + 6x - 4) - (-2x^3 + x^2 - 2)$ $6x^3 - 4x^2 + 6x - 2$

4B. $(8y - 10 + 5y^2) - (7 - y^3 + 12y)$ $y^3 + 5y^2 - 4y - 17$

Adding or subtracting integers results in an integer, so the set of integers is closed under addition and subtraction. Similarly, adding or subtracting polynomials results in a polynomial, so the set of polynomials is closed under addition and subtraction.

Real-WorldLink

Sales of digital cameras recently increased by 42% in one year. Sales are expected to increase by at least 15% each year as consumers upgrade their cameras.

Source: Big Planet Marketing Company

Real-World Example 5 Add and Subtract Polynomials

ELECTRONICS The equations $P = 7m + 137$ and $C = 4m + 78$ represent the number of cell phones P and digital cameras C sold in m months at an electronics store. Write an equation for the total monthly sales T of phones and cameras. Then predict the number of phones and cameras sold in 10 months.

To write an equation that represents the total sales T, add the equations that represent the number of cell phones P and digital cameras C.

$T = 7m + 137 + 4m + 78$

$= 11m + 215$

Substitute 10 for m to predict the number of phones and cameras sold in 10 months.

$T = 11(10) + 215$

$= 110 + 215$ or 325

Therefore, a total of 325 cell phones and digital cameras will be sold in 10 months.

GuidedPractice

5. Use the information above to write an equation that represents the difference in the monthly sales of cell phones and the monthly sales of digital cameras. Use the equation to predict the difference in monthly sales in 24 months. $D(m) = 3m - 59$; 13

Additional Examples

3 Find each sum.

a. $(7y^2 + 2y - 3) + (2 - 4y + 5y^2)$
$12y^2 - 2y - 1$

b. $(4x^2 - 2x + 7) + (3x - 7x^2 - 9)$
$-3x^2 + x - 2$

4 Find each difference.

a. $(6y^2 + 8y^4 - 5y) - (9y^4 - 7y + 2y^2)$
$-y^4 + 4y^2 + 2y$

b. $(6n^2 + 11n^3 + 2n) - (4n - 3 + 5n^2)$
$11n^3 + n^2 - 2n + 3$

WatchOut!

Preventing Errors Some students may find it helpful to mark through like terms as they mentally combine them. This saves time spent on rewriting to group like terms.

Focus on Mathematical Content

Number of Terms When adding or subtracting polynomials, the number of terms in the sum or difference may or may not be the same number as in the polynomials being added or subtracted. Students may assume that having more or fewer terms in the sum or difference must be the sign of an error or an unsimplified answer. Use an example to explain why this is faulty thinking.

Additional Example

5 VIDEO GAMES The total amount of toy sales T (in billions of dirhams) consists of two groups: sales of video games V and sales of traditional toys R. In recent years, the sales of traditional toys and total sales could be represented by the following equations, where n is the number of years since 2000.

$R = 0.46n^3 - 1.9n^2 + 3n + 19$

$T = 0.45n^3 - 1.85n^2 + 4.4n + 22.6$

a. Write an equation that represents the sales of video games V. $V = -0.01n^3 + 0.05n^2 + 1.4n + 3.6$

b. Use the equation to predict the amount of video game sales in the year 2012. 10.32 billion dirhams

Check Your Understanding

Example 1 Determine whether each expression is a polynomial. If it is a polynomial, find the degree and determine whether it is a *monomial, binomial,* or *trinomial.*

1. $7ab + 6b^2 - 2a^3$ **yes; 3; trinomial**
2. $2y - 5 + 3y^2$ **yes; 2; trinomial**
3. $3x^2$ **yes; 2; monomial**
4. $\frac{4m}{3p}$ **No; a monomial cannot have a variable in the denominator.**
5. $5m^2p^3 + 6$ **yes; 5; binomial**
6. $5q^{-4} + 6q$ **No; $5q^{-4} = \frac{5}{q^4}$, and a monomial cannot have a variable in the denominator.**

Example 2 Write each polynomial in standard form. Identify the leading coefficient.

7. $2x^5 - 12 + 3x$ **$2x^5 + 3x - 12$; 2**
8. $-4d^4 + 1 - d^2$ **$-4d^4 - d^2 + 1$; -4**
9. $4z - 2z^2 - 5z^4$ **$-5z^4 - 2z^2 + 4z$; -5**
10. $2a + 4a^3 - 5a^2 - 1$ **$4a^3 - 5a^2 + 2a - 1$; 4**

Examples 3–4 Find each sum or difference. **13. $-a^2 + 6a - 3$ 15. $-8z^3 - 3z^2 - 2z + 13$ 16. $-2d^2 + 6d - 20$**

11. $(6x^3 - 4) + (-2x^3 + 9)$ **$4x^3 + 5$**
12. $(g^3 - 2g^2 + 5g + 6) - (g^2 + 2g)$ **$g^3 - 3g^2 + 3g + 6$**
13. $(4 + 2a^2 - 2a) - (3a^2 - 8a + 7)$
14. $(8y - 4y^2) + (3y - 9y^2)$ **$-13y^2 + 11y$**
15. $(-4z^3 - 2z + 8) - (4z^3 + 3z^2 - 5)$
16. $(-3d^2 - 8 + 2d) + (4d - 12 + d^2)$
17. $(y + 5) + (2y + 4y^2 - 2)$ **$4y^2 + 3y + 3$**
18. $(3n^3 - 5n + n^2) - (-8n^2 + 3n^3)$ **$9n^2 - 5n$**

Example 5

19. **SENSE-MAKING** The total number of students T who traveled for spring break consists of two groups: students who flew to their destinations F and students who drove to their destination D. The number (in thousands) of students who flew and the total number of students who flew or drove can be modeled by the following equations, where n is the number of years since 1995.

$$T = 14n + 21 \qquad F = 8n + 7$$

a. Write an equation that models the number of students who drove to their destination for this time period. **$D(n) = 6n + 14$**

b. Predict the number of students who will drive to their destination in 2018. **152,000 students**

c. How many students will drive or fly to their destination in 2020? **164,000 students**

Practice and Problem Solving

Example 1 Determine whether each expression is a polynomial. If it is a polynomial, find the degree and determine whether it is a *monomial, binomial,* or *trinomial.*

20. $\frac{5y^3}{x^2} + 4x$ **No; a monomial cannot have a variable in the denominator.**
21. 21 **yes; 0; monomial**
22. $c^4 - 2c^2 + 1$ **yes; 4; trinomial**
23. $d + 3d^c$ **No; the exponent is a variable.**
24. $a - a^2$ **yes; 2; binomial**
25. $5n^3 + nq^3$ **yes; 4; binomial**

Example 2 Write each polynomial in standard form. Identify the leading coefficient.

26. $5x^2 - 2 + 3x$ **$5x^2 + 3x - 2$; 5**
27. $8y + 7y^3$ **$7y^3 + 8y$; 7**
28. $4 - 3c - 5c^2$ **$-5c^2 - 3c + 4$; -5**
29. $-y^3 + 3y - 3y^2 + 2$ **$-y^3 - 3y^2 + 3y + 2$; -1**
30. $11t + 2t^2 - 3 + t^5$ **$t^5 + 2t^2 + 11t - 3$; 1**
31. $2 + r - r^3$ **$-r^3 + r + 2$; -1**
32. $\frac{1}{2}x - 3x^4 + 7$ **$-3x^4 + \frac{1}{2}x + 7$; -3**
33. $-9b^2 + 10b - b^6$ **$-b^6 - 9b^2 + 10b$; -1**

3 Practice

Formative Assessment

Use Exercises 1–19 to check for understanding.

Use the chart at the bottom of the next page to customize assignments for your students.

Teach with Tech

Interactive Whiteboard Write an expression on the board to add or subtract two polynomials. Drag the like terms to group them together. Then combine like terms and simplify the expression.

Differentiated Instruction AL OL

Interpersonal Learners Have students work in pairs to find the sums and differences in Exercises 11–18. Then compare answers and steps used to get their answers. When they differ, pairs should consult with another pair of students. Students should offer constructive reinforcement to each other.

Examples 3–4 **Find each sum or difference.**

34. $(2c^2 + 6c + 4) + (5c^2 - 7)$ $7c^2 + 6c - 3$

35. $(2x + 3x^2) - (7 - 8x^2)$ $11x^2 + 2x - 7$

36. $(3c^3 - c + 11) - (c^2 + 2c + 8)$ $3c^3 - c^2 - 3c + 3$

37. $(z^2 + z) + (z^2 - 11)$ $2z^2 + z - 11$

38. $(2x - 2y + 1) - (3y + 4x)$ $-2x - 5y + 1$

39. $(4a - 5b^2 + 3) + (6 - 2a + 3b^2)$ $-2b^2 + 2a + 9$

40. $(x^2y - 3x^2 + y) + (3y - 2x^2y)$ $-x^2y - 3x^2 + 4y$

41. $(-8xy + 3x^2 - 5y) + (4x^2 - 2y + 6xy)$ $7x^2 - 2xy - 7y$

42. $(5n - 2p^2 + 2np) - (4p^2 + 4n)$ $-6p^2 + 2np + n$

43. $(4rxt - 8r^2x + x^2) - (6rx^2 + 5rxt - 2x^2)$ $3x^2 - rxt - 8r^2x - 6rx^2$

Example 5

44. PETS From 1999 through 2009, the number of birds D and the number of cats C (in hundreds) adopted from animal shelters in a country are modeled by the equations $D = 2n + 3$ and $C = n + 4$, where n is the number of years since 1999.

a. Write a function that models the total number T of birds and cats adopted in hundreds for this time period. $T(n) = 3n + 7$

b. If this trend continues, how many birds and cats will be adopted in 2013? 4900 birds and cats

B **Classify each polynomial according to its degree and number of terms.**

45. $4x - 3x^2 + 5$ quadratic trinomial

46. $11z^3$ cubic monomial

47. $9 + y^4$ quartic binomial

48. $3x^3 - 7$ cubic binomial

49. $-2x^5 - x^2 + 5x - 8$ quintic polynomial

50. $10t - 4t^2 + 6t^3$ cubic trinomial

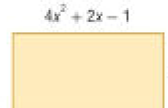
51. ENROLLMENT In a rapidly growing school system, the numbers (in hundreds) of total students is represented by N and the number of students in Kindergarten through 5th grade is represented by P. The equations $N = 1.25t\ 2 - t + 7.5$ and $P = 0.7t\ 2 - 0.95t + 3.8$, model the number of students enrolled from 2000 to 2009, where t is the number of years since 2000.

a. Write an equation modeling the number of students S in grades 6 through 12 enrolled for this time period. $S = 0.55t^2 - 0.05t + 3.7$

b. How many students were enrolled in grades 6 through 12 in the school system in 2007? 3030

52. REASONING The perimeter of the triangle can be represented by the expression $3x^2 - 7x + 2$. Write a polynomial that represents the measure of the third side. $4x$

53. GEOMETRY Consider the rectangle.



a. What does $(4x^2 + 2x - 1)(2x^2 - x + 3)$ represent?

b. What does $2(4x^2 + 2x - 1) + 2(2x^2 - x + 3)$ represent?

53a. the area of the rectangle 53b. the perimeter of the rectangle

Find each sum or difference.

54. $(4x + 2y - 6z) + (5y - 2z + 7x) + (-9z - 2x - 3y)$ $9x + 4y - 17z$

55. $(5a^2 - 4) + (a^2 - 2a + 12) + (4a^2 - 6a + 8)$ $10a^2 - 8a + 16$

56. $(3c^2 - 7) + (4c + 7) - (c^2 + 5c - 8)$ $2c^2 - c + 8$

57. $(3n^3 + 3n - 10) - (4n^2 - 5n) + (4n^3 - 3n^2 - 9n + 4)$ $7n^3 - 7n^2 - n - 6$

58. FOOTBALL The National Football League is divided into two conferences, the American A and the National N. From 2002 through 2009, the total attendance T (in thousands) for both conferences and for the American Conference games can be modeled by the following equations, where x is the number of years since 2002.

$T = -0.69x^3 + 55.83x^2 + 643.31x + 10{,}538$ $A = -3.78x^3 + 58.96x^2 + 265.96x + 5257$

Estimate how many people attended National Conference football games in 2009.
8,829,000 people

Teaching the Mathematical Practices

Sense-Making Mathematically proficient students start by explaining to themselves the meaning of a problem. In Exercise 19, ask students to explain how the given polynomials relate to the number of students who drove.

Reasoning Mathematically proficient students abstract a given situation and represent it symbolically and manipulate the representing symbols. In Exercise 52, tell students to start by writing an equation for perimeter.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	20–44, 60–62, 65–91	21–43 odd, 60, 66–69	20–44 even, 61, 62, 65, 70–91
OL Core	21–49 odd, 58–69	20–33, 66–69	34–44, 58–65, 70–91
BL Advanced	44–83, (optional: 84–91)		

Multiple Representations

In Exercise 60, students explore perimeter and area of rectangles graphically and analyze their results to determine the length and width that produces the largest area possible.

59. **CAR RENTAL** The cost to rent a car for a day is AED 15 plus 15 fils for each kilometer driven.

a. Write a polynomial that represents the cost of renting a car for m kilometers. **$15 + 0.15m$**

b. If a car is driven 145 kilometers, how much would it cost to rent? **AED 36.75**

c. If a car is driven 105 kilometers each day for four days, how much would it cost to rent a car? **AED 123**

d. If a car is driven 220 kilometers each day for seven days, how much would it cost to rent a car? **AED 336**

60. **MULTIPLE REPRESENTATIONS** In this problem, you will explore perimeter and area.

a. **Geometric** Draw three rectangles that each have a perimeter of 400 centimeters. **See Ch. 7 Answer Appendix.**

b. **Tabular** Record the width and length of each rectangle in a table like the one shown below. Find the area of each rectangle.

Rectangle	Length	Width	Area
1	100 m	**100 m**	**10,000 m^2**
2	50 m	**150 m**	**7500 m^2**
3	75 m	**125 m**	**9375 m^2**
4	x m	**$(200 - x)$ m**	**$x(200 - x)$ m^2**

c. **Graphical** On a coordinate system, graph the area of rectangle 4 in terms of the length, x. Use the graph to determine the largest area possible. **See Ch. 7 Answer Appendix.**

d. **Analytical** Determine the length and width that produce the largest area.
The length and width of the rectangle must be 100 centimeters each to have the largest area.

H.O.T. Problems Use Higher-Order Thinking Skills

61. **CRITIQUE** Majed and Mazen are finding $(2x^2 - x) - (3x + 3x^2 - 2)$. Is either of them correct? Explain your reasoning.

Neither; neither of them found the additive inverse correctly. All terms should have been multiplied by −1.

Majed

$(2x^2 - x) - (3x + 3x^2 - 2)$
$= (2x^2 - x) + (-3x + 3x^2 - 2)$
$= 5x^2 - 4x - 2$

Mazen

$(2x^2 - x) - (3x + 3x^2 - 2)$
$= (2x^2 - x) + (-3x - 3x^2 - 2)$
$= -x^2 - 4x - 2$

62. **REASONING** Determine whether each of the following statements is *true* or *false*. Explain your reasoning.

a. A binomial can have a degree of zero. **62a. False; sample answer: a binomial must have at least one monomial term with degree greater than zero.**

b. The order in which polynomials are subtracted does not matter. **False; sample answer: $(2x - 3) - (4x - 3) = -2x$, but $(4x - 3) - (2x - 3) = 2x$.**

63. **CHALLENGE** Write a polynomial that represents the sum of an odd integer $2n + 1$ and the next two consecutive odd integers. **$6n + 9$**

64. **WRITING IN MATH** Why would you add or subtract equations that represent real-world situations? Explain.

64. Sample answer: When you add or subtract two or more polynomial equations, like terms are combined, which reduces the number of terms in the resulting equation. This could help minimize the number of operations performed when using the equations.

65. **WRITING IN MATH** Describe how to add and subtract polynomials using both the vertical and horizontal formats. **See margin.**

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Standardized Test Practice

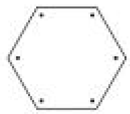
66. Three consecutive integers can be represented by x, $x + 1$, and $x + 2$. What is the sum of these three integers? **C**

A $x(x + 1)(x + 2)$ C $3x + 3$

B $x^3 + 3$ D $x + 3$

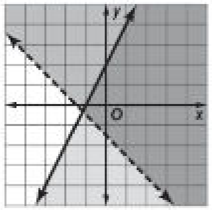
67. **SHORT RESPONSE** What is the perimeter of a square with sides that measure $2x + 3$ units? **8x + 12 units**

68. Jamal cuts a board in the shape of a regular hexagon and pounds in a nail at each vertex, as shown. How many rubber bands will he need to stretch a rubber band across every possible pair of nails? **F**



F 15 G 14 H 12 J 9

69. Which ordered pair is in the solution set of the system of inequalities shown in the graph? **C**



A $(-3, 0)$ C $(5, 0)$

B $(0, -3)$ D $(0, 5)$

Spiral Review

Simplify. (Lesson 0-10)

70. $\sqrt{72}$ $6\sqrt{2}$

71. $\sqrt{18} \cdot \sqrt{14}$ $6\sqrt{7}$

72. $\sqrt{44x^4y^3}$ $2x^2y\sqrt{11y}$

73. $\frac{3}{\sqrt{18}}$ $\frac{\sqrt{2}}{2}$

74. $\sqrt{\frac{28}{75}}$ $\frac{2\sqrt{21}}{15}$

75. $\frac{\sqrt{8a^6}}{\sqrt{108}}$ $\frac{|a^3|\sqrt{6}}{9}$

76. $\frac{5}{4 - \sqrt{2}}$ $\frac{20 + 5\sqrt{2}}{14}$

77. $\frac{4\sqrt{3}}{2 + \sqrt{5}}$ $4\sqrt{15} - 8\sqrt{3}$

78. **FINANCIAL LITERACY** Suppose you buy 3 shirts and 2 pairs of slacks on sale at a clothing store for AED 72. The next day, a friend buys 2 shirts and 4 pairs of slacks for AED 96. If the shirts you each bought were all the same price and the slacks were also all the same price, then what was the cost of each shirt and each pair of slacks? (Lesson 0-9) **shirt: AED 12; slacks: AED 18**

Graph the following points, and connect them in order to form a figure. (Lesson 0-8) **79–80. See margin.**

79. $A(-5, 3)$, $B(3, -4)$, and $C(-2, -3)$

80. $P(-2, 1)$, $Q(3, 4)$, $R(5, 1)$, and $S(0, -2)$

GROCERIES **Find an approximate metric weight for each item.** (Lesson 0-2)

81.

Net Wt: 15 oz **424.5 g**

82.

Net Wt: 8.2 oz **232.5 g**

83.

Net Wt: 2.5 lb **1.25 kg**

Skills Review

Simplify.

84. $t(t^5)(t^7)$ t^{13}

85. $n^3(n^2)(-2n^3)$ $-2n^8$

86. $(5t^5v^2)(10t^3v^4)$ $50t^8v^6$

87. $(-8u^4z^5)(5uz^4)$ $-40u^5z^9$

88. $[(3)^2]^3$ 729

89. $[(2)^3]^2$ 64

90. $(2m^4k^3)^2(-3mk^2)^3$ $-108m^{11}k^{12}$

91. $(6xy^2)^2(2x^2y^2z^2)^3$ $288x^8y^{10}z^6$

Differentiated Instruction OL BL

Extension Tell students the equations for the monthly unit sales of CDs C and DVDs D are $C = 7m + 87$ and $D = 9m + 152$, where m represents time in months since a store opened. Suppose the total monthly sales of CDs, DVDs, and videos is represented by $T = 15m + 248$. Write an equation that can be used to calculate monthly video sales V. How many videos did the store sell in the sixth month when $m = 5$? $V = (-1)m + 9$; 4

Teaching the Mathematical Practices

Critique Mathematically proficient students distinguish correct logic or reasoning from that which is flawed, and explain any flaws in an argument. In Exercise 61, have students check each step of each solution. Remind students that both need to add the additive inverse of each term in the polynomial being subtracted.

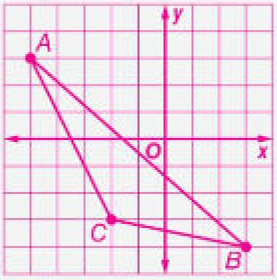
4 Assess

Ticket Out the Door Make several copies each of five polynomial expressions. Give one expression to each student. As the students leave the room, ask them to tell you the degree of their expressions.

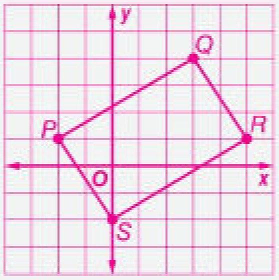
Additional Answers

65. Sample answer: To add polynomials in a horizontal format, you combine like terms. For the vertical format, you write the polynomials in standard form, align like terms in columns, and combine like terms. To subtract polynomials in a horizontal format you find the additive inverse of the polynomial you are subtracting, and then combine like terms. For the vertical format you write the polynomials in standard form, align like terms in columns, and subtract by adding the additive inverse.

79.



80.



1 Focus

VerticalAlignment

Before Lesson 7-2 Multiply monomials.

Lesson 7-2 Multiply a polynomial by a monomial.
Solve equations involving the products of monomials and polynomials.

After Lesson 7-2 Multiply binomials and polynomials.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What is the formula for finding the area of a rectangle? $A = \ell w$, where ℓ is the length and w is the width.
- What are ℓ and w for the expression shown? ℓ is $(3w + 8)$ and w is w.
- Which of the dimensions is a monomial? w
- Describe how you would find the area of the room if the width is 20 meters. $20(3 \cdot 20 + 8) = 20(60 + 8) = 1200 + 160 = 1360 \text{ m}^2$

LESSON 7-2 Multiplying a Polynomial by a Monomial

:: Then	:: Now	:: Why?
● You multiplied monomials.	● 1 Multiply a polynomial by a monomial. 2 Solve equations involving the products of monomials and polynomials.	● Sindiyya is opening a fitness club. She tells the contractor that the length of the fitness room should be three times the width plus 8 meters. To cover the floor with mats for exercise classes, Sindiyya needs to know the area of the floor. So she multiplies the width times the length, $w(3w + 8)$.

Mathematical Practices
Use appropriate tools strategically.

1 Polynomial Multiplied by Monomial

To find the product of a polynomial and a monomial, you can use the Distributive Property.

Example 1 Multiply a Polynomial by a Monomial

Find $-3x^2(7x^2 - x + 4)$.

Horizontal Method

$-3x^2(7x^2 - x + 4)$	Original expression
$= -3x^2(7x^2) - (-3x^2)(x) + (-3x^2)(4)$	Distributive Property
$= -21x^4 - (-3x^3) + (-12x^2)$	Multiply.
$= -21x^4 + 3x^3 - 12x^2$	Simplify.

Vertical Method

$$\begin{array}{r} 7x^2 - x + 4 \\ (\times) \qquad -3x^2 \\ \hline -21x^4 + 3x^3 - 12x^2 \end{array}$$

Distributive Property
Multiply.

GuidedPractice

Find each product.

1A. $5a^2(-4a^2 + 2a - 7)$ $-20a^4 + 10a^3 - 35a^2$

1B. $-6d^3(3d^4 - 2d^3 - d + 9)$ $-18d^7 + 12d^6 + 6d^4 - 54d^3$

We can use this same method more than once to simplify large expressions.

Example 2 Simplify Expressions

Simplify $2p(-4p^2 + 5p) - 5(2p^2 + 20)$.

$2p(-4p^2 + 5p) - 5(2p^2 + 20)$	Original expression
$= (2p)(-4p^2) + (2p)(5p) + (-5)(2p^2) + (-5)(20)$	Distributive Property
$= -8p^3 + 10p^2 - 10p^2 - 100$	Multiply.
$= -8p^3 + (10p^2 - 10p^2) - 100$	Commutative and Associative Properties
$= -8p^3 - 100$	Combine like terms.

GuidedPractice

Simplify each expression.

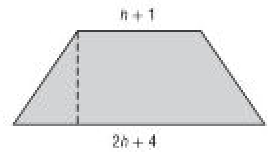
2A. $3(5x^2 + 2x - 4) - x(7x^2 + 2x - 3)$ **2B.** $15t(10y^3t^5 + 5y^2t) - 2y(yt^2 + 4y^2)$

2A. $-7x^3 + 13x^2 + 9x - 12$
2B. $150y^3t^6 + 73y^2t^2 - 8y^3$

We can use the Distributive Property to multiply monomials by polynomials and solve real world problems.

Standardized Test Example 3 Write and Evaluate a Polynomial Expression

GRIDDED RESPONSE The theme for a school event is "Solid Gold." For one decoration, Sumayya is covering a trapezoid-shaped piece of poster board with metallic gold paper to look like a bar of gold. If the height of the poster board is 18 centimeters, how much metallic paper will Sumayya need in square centimeters?



Read the Test Item

The question is asking you to find the area of the trapezoid with a height of h and bases of $h + 1$ and $2h + 4$.

Solve the Test Item

Write an equation to represent the area of the trapezoid.
Let $b_1 = h + 1$, let $b_2 = 2h + 4$ and let h = height of the trapezoid.

$A = \frac{1}{2}h(b_1 + b_2)$ Area of a trapezoid

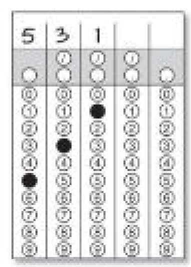
$= \frac{1}{2}h[(h + 1) + (2h + 4)]$ $b_1 = h + 1$ and $b_2 = 2h + 4$

$= \frac{1}{2}h(3h + 5)$ Add and simplify.

$= \frac{3}{2}h^2 + \frac{5}{2}h$ Distributive Property

$= \frac{3}{2}(18)^2 + \frac{5}{2}(18)$ $h = 18$

$= 531$ Simplify.



Sumayya will need 531 square centimeters of metallic paper.
Grid in your response of 531.

Test-TakingTip

Tools Many standardized tests provide formula sheets with commonly used formulas. If you are unsure of the correct formula, check the sheet before beginning to solve the problem.

Real-WorldLink

In a recent year, the pet supply business hit an estimated AED 25.9 billion in sales. This business ranges from gourmet food to rhinestone tiaras, pearl collars, and cashmere coats.

Source: *Entrepreneur Magazine*

GuidedPractice

3. SHORT RESPONSE Fatema is making triangular bandanas for the cats in her pet club. The base of the bandana is the length of the collar with 4 centimeters added to each end to tie it on. The height is $\frac{1}{2}$ of the collar length.

A. If Fatema's cat has a collar length of 12 centimeters, how much fabric does she need in square centimeters? **60**

B. If Fatema makes a bandana for her friend's cat with a 6-centimeter collar, how much fabric does Fatema need in square centimeters? **21**

1 Polynomial Multiplied by Monomial

Example 1 shows how the Distributive Property can be used to multiply a polynomial by a monomial. **Example 2** shows how to use the Distributive Property more than once to simplify large expressions. **Example 3** shows how to write and evaluate a polynomial expression for a real-world problem.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Find $6y(4y^2 - 9y - 7)$. $24y^3 - 54y^2 - 42y$

2 Simplify $3(2t^2 - 4t - 15) + 6t(5t + 2)$. $36t^2 - 45$

3 **GRIDDED RESPONSE** Admission to the Super Fun Amusement Park is AED10. Once in the park, super rides are an additional AED3 each, and regular rides are an additional AED2. Wafa goes to the park and rides 15 rides, of which s of those 15 are super rides. Find the cost in dirhams if Wafa rode 9 super rides. **49**

Teach with Tech

Document Camera Display a square photograph. Label each side as p. Discuss how to find the area of the photo. Then add a portion of a frame to the top of the photo. Discuss the new length of the photo and frame, and how to find the new area of the view.

Teaching the Mathematical Practices

Tools Mathematically proficient are able to use relevant external mathematical resources. Point out that there are different ways to represent the same formula, and formula sheets may show a different representation than one they learned. For example, the area of a trapezoid may also be shown as

$$A = \frac{h}{2}(b_1 + b_2) \text{ or } A = h\left(\frac{b_1 + b_2}{2}\right).$$

Focus on Mathematical Content

Order of Operations When simplifying expressions involving products of monomials and polynomials, the order of operations must be followed. Multiplication precedes addition, unless parentheses indicate otherwise.

2 Solve Equations with Polynomial Expressions

Example 4 shows how to solve equations that contain polynomials on both sides.

Additional Example

4 Solve $b(12 + b) - 7 = 2b + b(-4 + b)$. $\frac{1}{2}$

3 Practice

Formative Assessment

Use Exercises 1–17 to check for understanding.

Use the chart at the bottom of the next page to customize assignments for your students.

Tips for New Teachers

Multiplying by a Negative Monomial If students are having difficulty multiplying by a monomial with a negative sign, you may want to have them apply the negative first (by multiplying all terms by −1) and then multiply by the rest of the monomial.

2 Solve Equations with Polynomial Expressions

We can use the Distributive Property to solve equations that involve the products of monomials and polynomials.

StudyTip

Combining Like Terms When simplifying a long expression, it may be helpful to put a circle around one set of like terms, a rectangle around another set, a triangle around another set, and so on.

Example 4 Equations with Polynomials on Both Sides

Solve $2a(5a - 2) + 3a(2a + 6) + 8 = a(4a + 1) + 2a(6a - 4) + 50$.

$2a(5a - 2) + 3a(2a + 6) + 8 = a(4a + 1) + 2a(6a - 4) + 50$	Original equation
$10a^2 - 4a + 6a^2 + 18a + 8 = 4a^2 + a + 12a^2 - 8a + 50$	Distributive Property
$16a^2 + 14a + 8 = 16a^2 - 7a + 50$	Combine like terms.
$14a + 8 = -7a + 50$	Subtract $16a^2$ from each side.
$21a + 8 = 50$	Add $7a$ to each side.
$21a = 42$	Subtract 8 from each side.
$a = 2$	Divide each side by 21.

CHECK

$2a(5a - 2) + 3a(2a + 6) + 8 = a(4a + 1) + 2a(6a - 4) + 50$	
$2(2)[5(2) - 2] + 3(2)[2(2) + 6] + 8 \stackrel{?}{=} 2[4(2) + 1] + 2(2)[6(2) - 4] + 50$	
$4(8) + 6(10) + 8 \stackrel{?}{=} 2(9) + 4(8) + 50$	Simplify.
$32 + 60 + 8 \stackrel{?}{=} 18 + 32 + 50$	Multiply.
$100 = 100$ ✓	Add and subtract.

GuidedPractice

Solve each equation.

4A. $2x(x + 4) + 7 = (x + 8) + 2x(x + 1) + 12$ $2\frac{3}{5}$

4B. $d(d + 3) - d(d - 4) = 9d - 16$ 8

1. $-15w^3 + 10w^2 - 20w$ 2. $18g^5 + 24g^4 + 60g^3 - 6g^2$ 3. $32k^2m^4 + 8k^3m^3 + 20k^2m^2$

Check Your Understanding

Example 1 Find each product. 5. $14a^5b^3 + 2a^6b^2 - 4a^2b$

1. $5w(-3w^2 + 2w - 4)$
2. $6g^2(3g^3 + 4g^2 + 10g - 1)$
3. $4km^2(8km^2 + 2k^2m + 5k)$
4. $-3p^4r^3(2p^2r^4 - 6p^6r^3 - 5)$ $-6p^6r^7 + 18p^{10}r^6 + 15p^4r^3$
5. $2ab(7a^4b^2 + a^5b - 2a)$
6. $c^2d^3(5cd^7 - 3c^3d^2 - 4d^3)$ $5c^3d^{10} - 3c^5d^5 - 4c^2d^6$

Example 2 Simplify each expression. 7. $4t^3 + 15t^2 - 8t + 4$

7. $t(4t^2 + 15t + 4) - 4(3t - 1)$
8. $x(3x^2 + 4) + 2(7x - 3)$ $3x^3 + 18x - 6$
9. $-2d(d^3c^2 - 4dc^2 + 2d^2c) + c^2(dc^2 - 3d^4)$ $-5d^4c^2 + 8d^2c^2 - 4d^3c + dc^4$
10. $-5w^2(8w^2x - 11wx^2) + 6x(9wx^4 - 4w - 3x^2)$ $-40w^4x + 55w^3x^2 + 54wx^5 - 24wx - 18x^3$

Example 3

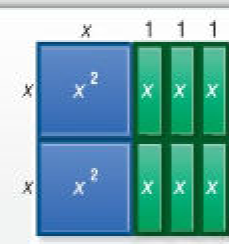
11. **GRIDDED RESPONSE** Mohammed is buying a new plasma television. The height of the screen of the television is one half the width plus 12.7 centimeters. The width is 76.2 centimeters. Find the height of the screen in centimeters. 50.8

Example 4 Solve each equation.

12. $-6(11 - 2c) = 7(-2 - 2c)$ 2
13. $t(2t + 3) + 20 = 2t(t - 3)$ $-\frac{20}{9}$
14. $-2(w + 1) + w = 7 - 4w$ 3
15. $3(y - 2) + 2y = 4y + 14$ 20
16. $a(a + 3) + a(a - 6) + 35 = a(a - 5) + a(a + 7)$ 7
17. $n(n - 4) + n(n + 8) = n(n - 13) + n(n + 1) + 16$ 1

DifferentiatedInstruction AL OL

Visual/Spatial Learners Have students group algebra tiles to form a rectangle with a wide of $2x$ and a length of $x + 3$ using 2 blue x^2-tiles and 6 green x-tiles. Ask students to use their models to write an expression for the area of the rectangle. Then ask students to use the formula for area to calculate the area. $2x^2 + 6x$; $2x(x + 3) = 2x^2 + 6x$



Practice and Problem Solving

Example 1 Find each product. 18–23. See margin.

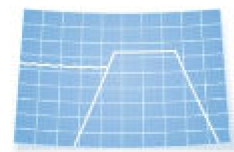
18. $b(b^2 - 12b + 1)$
19. $f(f^2 + 2f + 25)$
20. $-3m^3(2m^3 - 12m^2 + 2m + 25)$
21. $2j^2(5j^3 - 15j^2 + 2j + 2)$
22. $2pr^2(2pr + 5p^2r - 15p)$
23. $4t^3u(2t^2u^2 - 10tu^4 + 2)$

Example 2 Simplify each expression. 24. $-13x^2 - 9x - 27$ 26. $-20d^3 + 55d + 35$

24. $-3(5x^2 + 2x + 9) + x(2x - 3)$
25. $a(-8a^2 + 2a + 4) + 3(6a^2 - 4)$ $-8a^3 + 20a^2 + 4a - 12$
26. $-4d(5d^2 - 12) + 7(d + 5)$
27. $-9g(-2g + g^2) + 3(g^2 + 4)$ $-9g^3 + 21g^2 + 12$
28. $2j(7j^2k^2 + jk^2 + 5k) - 9k(-2j^2k^2 + 2k^2 + 3j)$ $14j^3k^2 + 2j^2k^2 - 17jk + 18j^2k^3 - 18k^3$
29. $4n(2n^3p^2 - 3np^2 + 5n) + 4p(6n^2p - 2np^2 + 3p)$
$8n^4p^2 + 12n^2p^2 + 20n^2 - 8np^3 + 12p^2$

Example 3

30. **DAMS** A new dam being built has the shape of a trapezoid. The length of the base at the bottom of the dam is 2 times the height. The length of the base at the top of the dam is $\frac{1}{5}$ times the height minus 9.15 meters.



a. Write an expression to find the area of the trapezoidal cross section of the dam. $\frac{11}{10}h^2 - 15h$

b. If the height of the dam is 54.9 meters, find the area of this cross section. 3060.23 m^2

Example 4 Solve each equation.

31. $7(t^2 + 5t - 9) + t = t(7t - 2) + 13$ 2
32. $w(4w + 6) + 2w = 2(2w^2 + 7w - 3)$ 1
33. $5(4z + 6) - 2(z - 4) = 7z(z + 4) - z(7z - 2) - 48$ $\frac{43}{6}$
34. $9c(c - 11) + 10(5c - 3) = 3c(c + 5) + c(6c - 3) - 30$ 0
35. $2f(5f - 2) - 10(f^2 - 3f + 6) = -8f(f + 4) + 4(2f^2 - 7f)$ $\frac{30}{43}$
36. $2k(-3k + 4) + 6(k^2 + 10) = k(4k + 8) - 2k(2k + 5)$ -6

B Simplify each expression. 37. $20np^4 + 6n^3p^3 - 8np^2$ 38. $6r^5t + 3r^3t^4 + 9r^2t^3$

37. $\frac{2}{3}np^2(30p^2 + 9n^2p - 12)$
38. $\frac{3}{5}r^2t(10r^3 + 5rt^3 + 15t^2)$
39. $-5q^2w^3(4q + 7w) + 4qw^2(7q^2w + 2q) - 3qw(3q^2w^2 + 9)$ 39. $-q^3w^3 - 35q^2w^4 + 8q^2w^2 - 27qw$
40. $-x^2z(2z^2 + 4xz^3) + xz^2(xz + 5x^3z) + x^2z^3(3x^2z + 4xz)$ $-x^2z^3 + 5x^4z^3 + 3x^4z^4$

41. **PARKING** A parking garage charges AED 30 per month plus 50 fils per daytime hour and 25 fils per hour during nights and weekends. Suppose Tarek parks in the garage for 47 hours in January and h of those are night and weekend hours.

a. Find an expression for Tarek's January bill. $53.50 - 0.25h$

b. Find the cost if Tarek had 12 hours of night and weekend hours. AED 50.50

42. **MODELING** Hamad is building a house for his new kitten. The upper face of the house is a trapezoid. If the height of the trapezoid is 30.5 centimeters, find the area of the face of this piece of the house. 2051.6 m^2

Tips for New Teachers

Multiplication Facts In Exercises 18 through 29, some students may prefer using the horizontal method for multiplying a polynomial by a monomial. Others may prefer the vertical method. Since these two methods are equivalent, either may be used.

Exercise Alert

Formula For Exercises 30 and 42, suggest students refer to Example 3 to help them with the formula for the area of a trapezoid.

Teaching the Mathematical Practices

Modeling Mathematically proficient students routinely interpret their mathematical results in the context of the situation. In Exercise 41, ask students what their result means and why it might be useful to Trent.

Additional Answers

18. $b^3 - 12b^2 + b$
19. $f^3 + 2f^2 + 25f$
20. $-6m^6 + 36m^5 - 6m^4 - 75m^3$
21. $10j^5 - 30j^4 + 4j^3 + 4j^2$
22. $4p^2r^3 + 10p^3r^3 - 30p^2r^2$
23. $8t^5u^3 - 40t^4u^5 + 8t^3u$

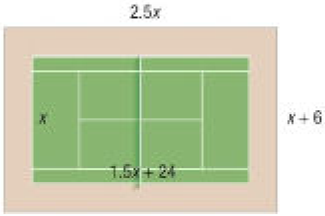
Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	18–36, 45, 48–71	19–35 odd, 51–54	18–36 even, 45, 48–50, 55–71
OL Core	19–39 odd, 41–45, 48–71	18–36, 51–54	37–45, 48–50, 55–71
BL Advanced	37–69, (optional: 70–71)		

Multiple Representations

In Exercise 44, students use algebraic expressions and a table of values to show the degree of a product of a monomial and a polynomial.

43. **TENNIS** The tennis club is building a new tennis court with a path around it.



a. Write an expression for the area of the tennis court. **$1.5x^2 + 24x$**

b. Write an expression for the area of the path. **$x^2 - 9x$**

c. If $x = 11$ meters, what is the perimeter of the outside of the path? **80.5 m**

44. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the degree of the product of a monomial and a polynomial.

a. **Tabular** Write three monomials of different degrees and three polynomials of different degrees. Determine the degree of each monomial and polynomial. Multiply the monomials by the polynomials. Determine the degree of each product. Record your results in a table like the one shown below.

Monomial	Degree	Polynomial	Degree	Product of Monomial and Polynomial	Degree
$2x$	1	$x^2 - 1$	2	$2x^3 - 2x$	3
$3x^2$	2	$x^5 + 1$	5	$3x^7 + 3x^2$	7
$4x^3$	3	$x^6 + 1$	6	$4x^9 + 4x^3$	9

b. **Verbal** Make a conjecture about the degree of the product of a monomial and a polynomial. What is the degree of the product of a monomial of degree a and a polynomial of degree b? **The degree of the product is the sum of the degree of the monomial and the degree of the polynomial; $a + b$.**

H.O.T. Problems Use Higher-Order Thinking Skills

45. **ERROR ANALYSIS** Obaid and Adnan both worked on this problem. Is either of them correct? Explain your reasoning. **Adnan; Obaid used the Distributive Property incorrectly.**

Obaid

$2x^2(3x^2 + 4x + 2)$

$6x^4 + 8x^2 + 4x^2$

$6x^4 + 12x^2$

Adnan

$2x^2(3x^2 + 4x + 2)$

$6x^4 + 8x^3 + 4x^2$

50. Sample answer: To multiply a polynomial by a monomial, use the Distributive Property. Multiply each term of the polynomial by the monomial. Then simplify by multiplying the coefficients together and using the Product of Powers Property for the variables.

46. **PERSEVERANCE** Find p such that $3x^p(4x^{2p+3} + 2x^{3p-2}) = 12x^{12} + 6x^{10}$. **3**

47. **CHALLENGE** Simplify $4x^{-3}y^2(2x^5y^{-4} + 6x^{-7}y^6 - 4x^0y^{-2})$. **$8x^2y^{-2} + 24x^{-10}y^8 - 16x^{-3}$**

48. **REASONING** Is there a value for x that makes the statement $(x + 2)^2 = x^2 + 2^2$ true? If so, find a value for x. Explain your reasoning. **Yes; 0; when 0 is substituted in for x in the equation, both sides are 2^2 or 4, which makes the equation true.**

49. **OPEN ENDED** Write a monomial and a polynomial using n as the variable. Find their product. **Sample answer: $3n$, $4n + 1$; $12n^2 + 3n$**

50. **WRITING IN MATH** Describe the steps to multiply a polynomial by a monomial.

404 | Lesson 7-2 | Multiplying a Polynomial by a Monomial

Standardized Test Practice

51. Every week a store sells j jeans and t T-shirts. The store makes AED 8 for each T-shirt and AED 12 for each pair of jeans. Which of the following expressions represents the total amount of money, in dirhams, the store makes every week? **B**

A $8j + 12t$ C $20(j + t)$
B $12j + 8t$ D $96jt$

52. If $a = 5x + 7y$ and $b = 2y - 3x$, what is $a + b$? **H**

F $2x - 9y$ H $2x + 9y$
G $3y + 4x$ J $2x - 5y$

53. **GEOMETRY** A triangle has sides of length 5 centimeters and 8.5 centimeters. Which of the following cannot be the length of the third side? **A**

A 3.5 centimeters
B 4 centimeters
C 5.5 centimeters
D 12 centimeters

54. **SHORT RESPONSE** Write an equation in which x varies directly as the cube of y and inversely as the square of z. **Sample answer: $x = \frac{y^3}{z^2}$**

Spiral Review

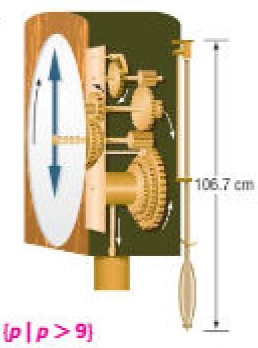
Find each sum or difference. (Lesson 7-1) **56. $4z^2 + 2z - 7$ 57. $-9a^2 + 4a + 7$**

55. $(2x^2 - 7) + (8 - 5x^2)$ **$-3x^2 + 1$**
56. $(3z^2 + 2z - 1) + (z^2 - 6)$
57. $(2a - 4a^2 + 1) - (5a^2 - 2a - 6)$
58. $(a^3 - 3a^2 + 4) - (4a^2 + 7)$ **$a^3 - 7a^2 - 3$**
59. $(2ab - 3a + 4b) + (5a + 4ab)$ **$6ab + 2a + 4b$**
60. $(8c^3 - 3c^2 + c - 2) - (3c^3 + 9)$ **$5c^3 - 3c^2 + c - 11$**

61. **CLOCKS** The period of a pendulum is the time required for it to make one complete swing back and forth. The formula of the period P in seconds of a pendulum is $P = 2\pi\sqrt{\frac{\ell}{9.8}}$, where ℓ is the length of the pendulum in meters. (Lesson 0-10)

a. What is the period of the pendulum in the clock shown to the nearest tenth of a second? **about 2.1 seconds**

b. About how many centimeters long should the pendulum be in order for it to have a period of 1 second? **about 9.7 cm**



Solve each inequality. (Lesson 0-6)

62. $-14n \geq 42$ **$\{n \mid n \leq -3\}$**
63. $p + 6 > 15$ **$\{p \mid p > 9\}$**
64. $-2a - 5 < 20$ **$\{a \mid a > -12.5\}$**
65. $5x \leq 3x - 26$ **$\{x \mid x \leq -13\}$**

Skills Review

Simplify.

66. $b(b^2)(b^3)$ **b^6**
67. $2y(3y^2)$ **$6y^3$**
68. $-y^4(-2y^3)$ **$2y^7$**
69. $-3z^3(-5z^4 + 2z)$ **$15z^7 - 6z^4$**
70. $2m(-4m^4) - 3(-5m^3)$ **$-8m^5 + 15m^3$**
71. $4p^2(-2p^3) + 2p^4(5p^6)$ **$-8p^5 + 10p^{10}$**

WatchOut!

Error Analysis For Exercise 45, point out to students that Pearl's method should draw a critical eye because her final polynomial has only two terms. When a monomial is multiplied by a polynomial with three different degree terms, the result will be a polynomial with three terms.

Teaching the Mathematical Practices

Perseverance Mathematically proficient students consider analogous problems. In Exercise 46, students may be intimidated by the variables in the exponents. Encourage them to think of how they treat numbers as exponents.

4 Assess

Crystal Ball Ask students to write a sentence predicting how learning to multiply a polynomial by a monomial will help them to learn to multiply polynomials by other polynomials in the next lesson.

DifferentiatedInstruction OL BL

Extension Give students this problem: Nate multiplied a polynomial by a monomial and got $6x^8 - 3x^4 + 9x^2$. If the polynomial factor was $2x^6 - x^2 + 3$, what was the monomial factor? **$3x^2$**

1 Focus

Objective Use algebra tiles to multiply polynomials.

Materials for Each Student

- algebra tiles
- product mat

Teaching Tip

Some students may benefit from laying tiles along the top and side of the product mat to model each expression. Have them remove the two factors before determining their final product.

2 Teach

Working in Cooperative Groups

Put students in groups of two or three, mixing abilities. Have groups complete Activities 1–3 and Exercise 1.

- For Activity 1, make sure groups mark the dimensions properly on the product mat. Since x-tiles are rectangular, remind students that the long side is the correct side to use to mark a value of x on the mat.
- When students are filling in the mats with the tiles, remind them to look carefully at the horizontal and vertical dimensions of each tile on the product mat. If both dimensions have a value of x, then use an x^2-tile. If one dimension is x and the other is 1, then use an x-tile. If both dimensions are 1, then use a 1-tile.

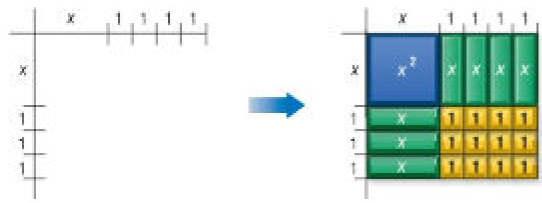
EXPLORE 7-3 Algebra Lab
Multiplying Polynomials

You can use algebra tiles to find the product of two binomials.

Activity 1 Multiply Binomials

Use algebra tiles to find $(x + 3)(x + 4)$.

The rectangle will have a width of $x + 3$ and a length of $x + 4$. Use algebra tiles to mark off the dimensions on a product mat. Then complete the rectangle with algebra tiles.

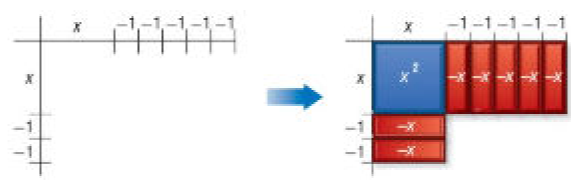


The rectangle consists of 1 blue x^2-tile, 7 green x-tiles, and 12 yellow 1-tiles. The area of the rectangle is $x^2 + 7x + 12$. So, $(x + 3)(x + 4) = x^2 + 7x + 12$.

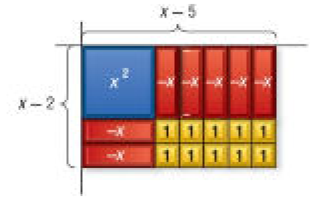
Activity 2 Multiply Binomials

Use algebra tiles to find $(x - 2)(x - 5)$.

Step 1 The rectangle will have a width of $x - 2$ and a length of $x - 5$. Use algebra tiles to mark off the dimensions on a product mat. Then begin to make the rectangle with algebra tiles.



Step 2 Determine whether to use 10 yellow 1-tiles or 10 red -1-tiles to complete the rectangle. The area of each yellow tile is the product of -1 and -1. Fill in the space with 10 yellow 1-tiles to complete the rectangle.



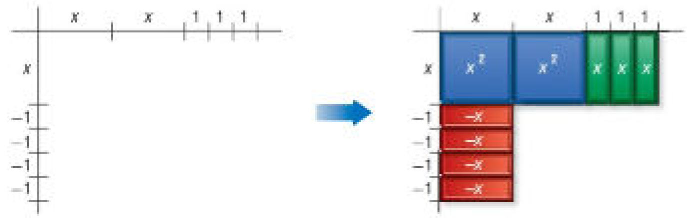
The rectangle consists of 1 blue x^2-tile, 7 red $-x$-tiles, and 10 yellow 1-tiles. The area of the rectangle is $x^2 - 7x + 10$. So, $(x - 2)(x - 5) = x^2 - 7x + 10$.

406 | Explore 7-3 | Algebra Lab: Multiplying Polynomials

Activity 3 Multiply Binomials

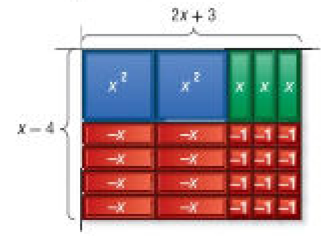
Use algebra tiles to find $(x-4)(2x+3)$.

Step 1 The rectangle will have a width of $x-4$ and a length of $2x+3$. Use algebra tiles to mark off the dimensions on a product mat. Then begin to make the rectangle with algebra tiles.



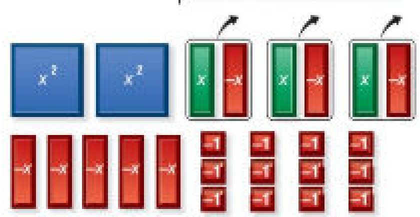
Step 2 Determine what color x-tiles and what color 1-tiles to use to complete the rectangle. The area of each red x-tile is the product of x and -1. The area of each red -1-tile is represented by $1(-1)$ or -1.

Complete the rectangle with 4 red x-tiles and 12 red -1-tiles.



Step 3 Rearrange the tiles to simplify the polynomial you have formed. Notice that a 3 zero pair are formed by three positive and three negative x-tiles.

There are 2 blue x^2-tiles, 5 red $-x$-tiles, and 12 red -1-tiles left. In simplest form, $(x-4)(2x+3) = 2x^2 - 5x - 12$.



Model and Analyze

Use algebra tiles to find each product.

1. $(x+1)(x+4)$ **x^2+5x+4**
2. $(x-3)(x-2)$ **x^2-5x+6**
3. $(x+5)(x-1)$ **x^2+4x-5**
4. $(x+2)(2x+3)$ **$2x^2+7x+6$**
5. $(x-1)(2x-1)$ **$2x^2-3x+1$**
6. $(x+4)(2x-5)$ **$2x^2+3x-20$**

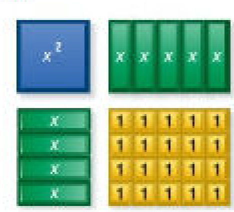
7–8. See margin for drawings.

Is each statement *true* or *false*? Justify your answer with a drawing of algebra tiles.

7. $(x-4)(x-2) = x^2-6x+8$ **true**
8. $(x+3)(x+5) = x^2+15$ **false**

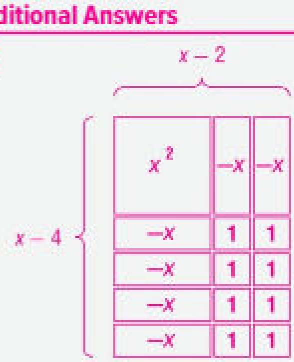
9. **WRITING IN MATH** You can also use the Distributive Property to find the product of two binomials. The figure at the right shows the model for $(x+4)(x+5)$ separated into four parts. Write a sentence or two explaining how this model shows the use of the Distributive Property.

By the Distributive Property, $(x+4)(x+5) = x(x+5) + 4(x+5)$. The top row represents $x(x+5)$ or x^2+5x. The bottom row represents $4(x+5)$ or $4x+20$.

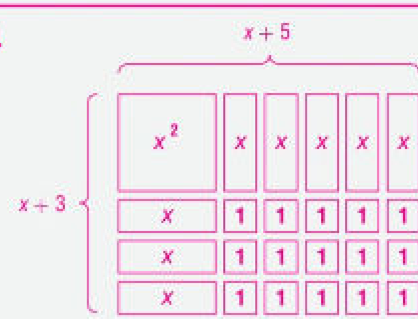


Additional Answers

7.



8.



- For Activity 2, Step 2, have students pay close attention to whether the dimensions for each tile are positive or negative, as this affects which tile to use. If both dimensions are positive, then the tile is positive. If one is positive and the other is negative, the tile is negative. If both are negative, then the tile is positive.
- For Activity 3, as an alternative to removing zero pairs, have students write the expression based on the tiles without removing zero pairs. They can then simplify the expression by combining like terms.

Practice Have students complete Exercises 2–9.

3 Assess

Formative Assessment

Use Exercise 9 to assess whether students can model a product correctly.

From Concrete to Abstract

After students have completed Exercise 9, help them to see that when using the Distributive Property to multiply polynomials, each term from the first polynomial is multiplied by each term from the second polynomial.

Extending the Concept

Ask students to model $(x-3)(x+2)$ using algebra tiles. Then ask students to write the expression based on the tiles without removing zero pairs.
$x^2+2x-3x-6$

Finally, ask students to find the sum of the product of the two first terms, outer terms, inner terms, and last terms of $(x-3)(x+2)$ and then compare the results to the expression they wrote.
The two expressions are the same.

1 Focus

VerticalAlignment

Before Lesson 7-3 Multiply polynomials by monomials.

Lesson 7-3 Multiply binomials by using the FOIL method. Multiply polynomials by using the Distributive Property.

After Lesson 7-3 Find squares of binomials involving sums and differences.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What expression would you get if you multiplied the first term in $(h - 81.3)$ times $\left(\frac{1}{2}h + 28\right)$? $\frac{1}{2}h^2 + 28h$
- What expression would you get if you multiplied the second term in $(h - 81.3)$ by $\left(\frac{1}{2}h + 28\right)$? $-40.65h - 2276.4$
- What expression do you get when you add these two answers together? $\frac{1}{2}h^2 - 12.65h - 2276.4$

LESSON 7-3 Multiplying Polynomials

:: Then

- You multiplied polynomials by monomials.

:: Now

1. Multiply binomials by using the FOIL method.
2. Multiply polynomials by using the Distributive Property.

:: Why?

- Bodyboards, which are used to ride waves, are made of foam and are more rectangular than surfboards. A bodyboard's dimensions are determined by the height and skill level of the user.

The length of Omar's bodyboard should be Omar's height h minus 81.3 centimeters or $h - 81.3$. The board's width should be half of Omar's height plus 28 centimeters or $\frac{1}{2}h + 28$. To approximate the area of the bodyboard, you need to find $(h - 81.3)\left(\frac{1}{2}h + 28\right)$.

NewVocabulary
FOIL method
quadratic expression

Mathematical Practices
Look for and make use of structure.

1 Multiply Binomials

To multiply two binomials such as $h - 81.3$ and $\frac{1}{2}h + 28$, the Distributive Property is used. Binomials can be multiplied horizontally or vertically.

Example 1 The Distributive Property

Find each product.

a. $(2x + 3)(x + 5)$

Vertical Method

Multiply by 5.	Multiply by x.	Combine like terms.
$\begin{array}{r} 2x + 3 \\ (\times)\ x + 5 \\ \hline 10x + 15 \end{array}$	$\begin{array}{r} 2x + 3 \\ (\times)\ x + 5 \\ \hline 10x + 15 \\ 2x^2 + 3x \end{array}$	$\begin{array}{r} 2x + 3 \\ (\times)\ x + 5 \\ \hline 10x + 15 \\ 2x^2 + 3x \\ \hline 2x^2 + 13x + 15 \end{array}$
$5(2x + 3) = 10x + 15$	$x(2x + 3) = 2x^2 + 3x$	

Horizontal Method

$(2x + 3)(x + 5) = 2x(x + 5) + 3(x + 5)$ — Rewrite as the sum of two products.
$= 2x^2 + 10x + 3x + 15$ — Distributive Property
$= 2x^2 + 13x + 15$ — Combine like terms.

b. $(x - 2)(3x + 4)$

Vertical Method

Multiply by 4.	Multiply by $3x$.	Combine like terms.
$\begin{array}{r} x - 2 \\ (\times)\ 3x + 4 \\ \hline 4x - 8 \end{array}$	$\begin{array}{r} x - 2 \\ (\times)\ 3x + 4 \\ \hline 4x - 8 \\ 3x^2 - 6x \end{array}$	$\begin{array}{r} x - 2 \\ (\times)\ 3x + 4 \\ \hline 4x - 8 \\ 3x^2 - 6x \\ \hline 3x^2 - 2x - 8 \end{array}$
$4(x - 2) = 4x - 8$	$3x(x - 2) = 3x^2 - 6x$	

Horizontal Method

$(x - 2)(3x + 4) = x(3x + 4) - 2(3x + 4)$ — Rewrite as the difference of two products.
$= 3x^2 + 4x - 6x - 8$ — Distributive Property
$= 3x^2 - 2x - 8$ — Combine like terms.

GuidedPractice

1A. $(3m + 4)(m + 5)$ $3m^2 + 19m + 20$ **1B.** $(5y - 2)(y + 8)$ $5y^2 + 38y - 16$

A shortcut version of the Distributive Property for multiplying binomials is called the **FOIL method**.

KeyConcept FOIL Method

Words To multiply two binomials, find the sum of the products of ***F*** the *First* terms, ***O*** the *Outer* terms, ***I*** the *Inner* terms, ***L*** and the *Last* terms.

Example

	Product of First Terms		Product of Outer Terms		Product of Inner Terms		Product of Last Terms
$(x + 4)(x - 2) =$	$(x)(x)$	+	$(x)(-2)$	+	$(4)(x)$	+	$(4)(-2)$

$= x^2 - 2x + 4x - 8$

$= x^2 + 2x - 8$

ReadingMath

Polynomials as Factors The expression $(x + 4)(x - 2)$ is read *the quantity x plus 4 times the quantity x minus 2.*

Example 2 FOIL Method

Find each product.

a. $(2y - 7)(3y + 5)$

$(2y - 7)(3y + 5) = (2y)(3y) + (2y)(5) + (-7)(3y) + (-7)(5)$ FOIL method

$= 6y^2 + 10y - 21y - 35$ Multiply.

$= 6y^2 - 11y - 35$ Combine like terms.

b. $(4a - 5)(2a - 9)$

$(4a - 5)(2a - 9)$

$= (4a)(2a) + (4a)(-9) + (-5)(2a) + (-5)(-9)$ FOIL method

$= 8a^2 - 36a - 10a + 45$ Multiply.

$= 8a^2 - 46a + 45$ Combine like terms.

GuidedPractice

2A. $(x + 3)(x - 4)$ $x^2 - x - 12$ **2B.** $(4b - 5)(3b + 2)$ $12b^2 - 7b - 10$

2C. $(2y - 5)(y - 6)$ $2y^2 - 17y + 30$ **2D.** $(5a + 2)(3a - 4)$ $15a^2 - 14a - 8$

Notice that when two linear expressions are multiplied, the result is a quadratic expression. A **quadratic expression** is an expression in one variable with a degree of 2. When three linear expressions are multiplied, the result has a degree of 3.

The FOIL method can be used to find an expression that represents the area of a rectangular object when the lengths of the sides are given as binomials.

1 Multiply Binomials

Example 1 shows how to multiply two binomials using the Distributive Property. **Example 2** shows how to multiply two binomials using a shortcut of the Distributive Property called the FOIL method. **Example 3** shows how to use the FOIL method to solve a real-world problem.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Find each product.

a. $(y + 8)(y - 4)$ $y^2 + 4y - 32$

b. $(2x + 1)(x + 6)$ $2x^2 + 13x + 6$

2 Find each product.

a. $(z - 6)(z - 12)$ $z^2 - 18z + 72$

b. $(5x - 4)(2x + 8)$ $10x^2 + 32x - 32$

Focus on Mathematical Content

Multiplying Other Polynomials The FOIL method only works for multiplying two binomials. To multiply any other polynomials, the Distributive Property must be used.

Tips for New Teachers

FOIL Point out that FOIL is a memory tool. The order in which the terms are multiplied is not important, as long as all four products are found.

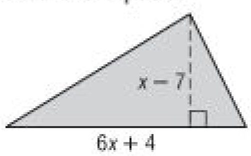
DifferentiatedInstruction OL BL

If students are less familiar with the Distributive Property,

Then they may wish to use the vertical method for multiplying binomials because it is similar to multiplying two-digit numbers. Suggest that students use the method with which they are most comfortable.

Additional Example

3 PATIO A patio in the shape of the triangle shown below is being built in Laila's backyard. The dimensions given are in meters. The area A of the triangle is one half the height h times the base b. Write an expression for the area of the patio.



$(3x^2 - 19x - 14)\ m^2$

2 Multiply Polynomials

Example 4 shows how the Distributive Property can be used to multiply any two polynomials.

Additional Example

4 Find each product.

a. $(3a + 4)(a^2 - 12a + 1)$ $3a^3 - 32a^2 - 45a + 4$

b. $(2b^2 + 7b + 9)(b^2 + 3b - 1)$ $2b^4 + 13b^3 + 28b^2 + 20b - 9$

Teach with Tech

Interactive Whiteboard Work through an example multiplying two binomials. Write the example on the board and use a different color for each of the four terms. Then create a table with four columns, one for each of the four parts of FOIL. In each column, write the expression for the two terms to be multiplied using the color coding from the original expression.

Real-WorldLink

The cost of a swimming pool depends on many factors, including the size of the pool, whether the pool is an above-ground or an in-ground pool, and the material used.

Source: American Dream Homes

Real-World Example 3 FOIL Method

SWIMMING POOL **A contractor is building a deck around a rectangular swimming pool. The deck is x meters from every side of the pool. Write an expression for the total area of the pool and deck.**

Understand We need to find an expression for the total area of the pool and deck.

Plan Find the product of the length and width of the pool with the deck.

Solve Since the deck is the same distance from every side of the pool, the length and width of the pool are $2x$ longer. So, the length can be represented by $2x + 20$ and the width can be represented by $2x + 15$.

Area = length · width	Area of a rectangle
$= (2x + 20)(2x + 15)$	Substitution
$= (2x)(2x) + (2x)(15) + (20)(2x) + (20)(15)$	FOIL Method
$= 4x^2 + 30x + 40x + 300$	Multiply.
$= 4x^2 + 70x + 300$	Combine like terms.

So, the total area of the deck and pool is $4x^2 + 70x + 300$.

Check Choose a value for x. Substitute this value into $(2x + 20)(2x + 15)$ and $4x^2 + 70x + 300$. The result should be the same for both expressions.

GuidedPractice $4x^2 + 90x + 500$

3. If the pool is 25 meters long and 20 meters wide, find the area of the pool and deck.

2 Multiply Polynomials

The Distributive Property can also be used to multiply any two polynomials.

Example 4 The Distributive Property

Find each product.

a. $(6x + 5)(2x^2 - 3x - 5)$

$(6x + 5)(2x^2 - 3x - 5)$

$= 6x(2x^2 - 3x - 5) + 5(2x^2 - 3x - 5)$	Distributive Property
$= 12x^3 - 18x^2 - 30x + 10x^2 - 15x - 25$	Multiply.
$= 12x^3 - 8x^2 - 45x - 25$	Combine like terms.

b. $(2y^2 + 3y - 1)(3y^2 - 5y + 2)$

$(2y^2 + 3y - 1)(3y^2 - 5y + 2)$

$= 2y^2(3y^2 - 5y + 2) + 3y(3y^2 - 5y + 2) - 1(3y^2 - 5y + 2)$	Distributive Property
$= 6y^4 - 10y^3 + 4y^2 + 9y^3 - 15y^2 + 6y - 3y^2 + 5y - 2$	Multiply.
$= 6y^4 - y^3 - 14y^2 + 11y - 2$	Combine like terms.

StudyTip

Multiplying Polynomials If a polynomial with c terms and a polynomial with d terms are multiplied together, there will be $c \cdot d$ terms before simplifying. In Example 4a, there are $2 \cdot 3$ or 6 terms before simplifying.

GuidedPractice $6x^3 + 11x^2 - 59x + 40$ $4m^4 + m^3 - 21m^2 + 31m - 15$

4A. $(3x - 5)(2x^2 + 7x - 8)$

4B. $(m^2 + 2m - 3)(4m^2 - 7m + 5)$

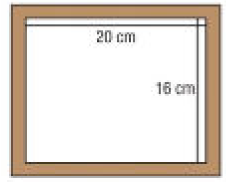
Check Your Understanding

Examples 1–2 Find each product.

1. $(x+5)(x+2)$ $x^2+7x+10$
2. $(y-2)(y+4)$ y^2+2y-8
3. $(b-7)(b+3)$ $b^2-4b-21$
4. $(4n+3)(n+9)$ $4n^2+39n+27$
5. $(8h-1)(2h-3)$ $16h^2-26h+3$
6. $(2a+9)(5a-6)$ $10a^2+33a-54$

Example 3

7. **FRAME** Hussam is designing a frame as shown at the right. The frame has a width of x centimeters all the way around. Write an expression that represents the total area of the picture and frame. $4x^2+72x+320$



Example 4 Find each product.

8. $(2a-9)(3a^2+4a-4)$ $6a^3-19a^2-44a+36$
9. $(4y^2-3)(4y^2+7y+2)$ $16y^4+28y^3-4y^2-21y-6$
10. $(x^2-4x+5)(5x^2+3x-4)$ $5x^4-17x^3+9x^2+31x-20$
11. $(2n^2+3n-6)(5n^2-2n-8)$ $10n^4+11n^3-52n^2-12n+48$

16. $15y^2-17y+4$
17. $24d^2-62d+35$
18. $6m^2+19m+15$
19. $49n^2-84n+36$

Practice and Problem Solving

Examples 1–2 Find each product. 12. $3c^2+4c-15$ 13. $2g^2+15g-50$ 15. $24x^2+18x+3$

12. $(3c-5)(c+3)$
13. $(g+10)(2g-5)$
14. $(6a+5)(5a+3)$ $30a^2+43a+15$
15. $(4x+1)(6x+3)$
16. $(5y-4)(3y-1)$
17. $(6d-5)(4d-7)$
18. $(3m+5)(2m+3)$
19. $(7n-6)(7n-6)$
20. $(12t-5)(12t+5)$ $144t^2-25$
21. $(5r+7)(5r-7)$ $25r^2-49$
22. $(8w+4x)(5w-6x)$ $40w^2-28wx-24x^2$
23. $(11z-5y)(3z+2y)$ $33z^2+7yz-10y^2$

Example 3

24. **GARDEN** A walkway surrounds a rectangular garden. The width of the garden is 8 meters, and the length is 6 meters. The width x of the walkway around the garden is the same on every side. Write an expression that represents the total area of the garden and walkway. $4x^2+28x+48$

Example 4 Find each product. **25–30. See margin.**

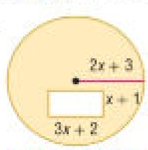
25. $(2y-11)(y^2-3y+2)$
26. $(4a+7)(9a^2+2a-7)$
27. $(m^2-5m+4)(m^2+7m-3)$
28. $(x^2+5x-1)(5x^2-6x+1)$
29. $(3b^3-4b-7)(2b^2-b-9)$
30. $(6z^2-5z-2)(3z^3-2z-4)$

B Simplify.

31. $(m+2)[(m^2+3m-6)+(m^2-2m+4)]$ $2m^3+5m^2-4$
32. $[(t^2+3t-8)-(t^2-2t+6)](t-4)$ $5t^2-34t+56$

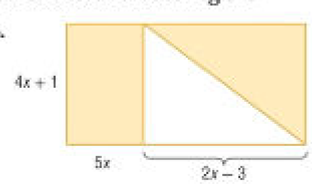
STRUCTURE Find an expression to represent the area of each shaded region.

33.



$4\pi x^2+12\pi x+9\pi-3x^2-5x-2$

34.



$24x^2-\frac{3}{2}$

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	12–30, 45, 47–66	13–29 odd, 50–53	12–30 even, 45, 47–49, 54–66
OL Core	13–33 odd, 35, 36, 37–41 odd, 43–45, 47–66	12–30, 50–53	31–45, 47–49, 54–66
BL Advanced	31–62, (optional: 63–66)		

3 Practice

Formative Assessment

Use Exercises 1–11 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

WatchOut!

Common Errors When students multiply polynomials horizontally, they often try to combine terms that are not like terms. For students who are having difficulty finding the product in Exercises 12–23, suggest that they try multiplying the polynomials in vertical form, aligning like terms.

Exercise Alert

Formula For Exercise 33, students will need to know the formula for the area of a circle, $A=\pi r^2$.

Teaching the Mathematical Practices

Structure Mathematically proficient students can see complicated things as single objects or as being composed of several objects. In Exercises 33 and 34, ask students what shapes they see in each diagram, and how the area of each shape can be represented.

Additional Answers

25. $2y^3-17y^2+37y-22$
26. $36a^3+71a^2-14a-49$
27. $m^4+2m^3-34m^2+43m-12$
28. $5x^4+19x^3-34x^2+11x-1$
29. $6b^5-3b^4-35b^3-10b^2+43b+63$
30. $18z^5-15z^4-18z^3-14z^2+24z+8$

Multiple Representations

In Exercise 44, students use a table of values and analysis to express the results of squaring a sum of the form $(a + b)$.

35. **VOLLEYBALL** The dimensions of a sand volleyball court are represented by a width of $6y - 5$ meters and a length of $3y + 4$ meters.

a. Write an expression that represents the area of the court. **$18y^2 + 9y - 20$**

b. The length of a sand volleyball court is 9.5 meters. Find the area of the court. **141 meter**

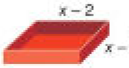
36. **GEOMETRY** Write an expression for the area of a triangle with a base of $2x + 3$ and a height of $3x - 1$. **$3x^2 + \frac{7}{2}x - \frac{3}{2}$**

Find each product. **37–42. See margin.**

37. $(a - 2b)^2$ **38.** $(3c + 4d)^2$ **39.** $(x - 5y)^2$

40. $(2r - 3t)^3$ **41.** $(5g + 2h)^3$ **42.** $(4y + 3z)(4y - 3z)^2$

43. **CONSTRUCTION** A sandbox kit allows you to build a square sandbox or a rectangular sandbox as shown.



a. What are the possible values of x? Explain. **See margin.**

b. Which shape has the greater area? **square**

c. What is the difference in areas between the two? **4 ft²**

44. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the square of a sum.

a. **Tabular** Copy and complete the table for each sum.

Expression	(Expression)²
$x + 5$	$x^2 + 10x + 25$
$3y + 1$	$9y^2 + 6y + 1$
$z + q$	$z^2 + 2zq + q^2$

b. **Verbal** Make a conjecture about the terms of the square of a sum. **See margin.**

c. **Symbolic** For a sum of the form $a + b$, write an expression for the square of the sum. **$a^2 + 2ab + b^2$**

45. Always; by grouping two adjacent terms a trinomial can be written as a binomial, a sum of two quantities, and apply the FOIL method. For example, $(2x + 3)(x^2 + 5x + 7) = (2x + 3)[x^2 + (5x + 7)] = 2x(x^2) + 2x(5x + 7) + 3(x^2) + 3(5x + 7)$. Then use the Distributive Property and simplify.

H.O.T. Problems Use Higher-Order Thinking Skills

45. **REASONING** Determine if the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

The FOIL method can be used to multiply a binomial and a trinomial.

46. **CHALLENGE** Find $(x^m + x^p)(x^{m-1} - x^{1-p} + x^p)$.
$x^{2m-1} - x^{m-p+1} + x^{m+p} + x^{m+p-1} - x + x^{2p}$

47. **OPEN ENDED** Write a binomial and a trinomial involving a single variable. Then find their product. **Sample answer: $x - 1$, $x^2 - x - 1$; $(x - 1)(x^2 - x - 1) = x^3 - 2x^2 + 1$**

48. **REGULARITY** Compare and contrast the procedure used to multiply a trinomial by a binomial using the vertical method with the procedure used to multiply a three-digit number by a two-digit number. **See Ch. 7 Answer Appendix.**

49. **WRITING IN MATH** Summarize the methods that can be used to multiply polynomials. **See Ch. 7 Answer Appendix.**

Standardized Test Practice

50. What is the product of $2x - 5$ and $3x + 4$? **B**

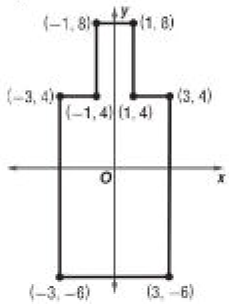
A $5x - 1$

B $6x^2 - 7x - 20$

C $6x^2 - 20$

D $6x^2 + 7x - 20$

51. Which statement is correct about the symmetry of this design? **F**



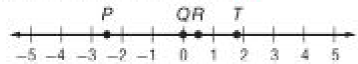
F The design is symmetrical only about the y-axis.

G The design is symmetrical only about the x-axis.

H The design is symmetrical about both the y- and the x-axes.

J The design has no symmetry.

52. Which point on the number line represents a number that, when cubed, will result in a number greater than itself? **D**



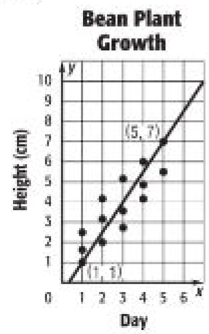
A P

B Q

C R

D T

53. SHORT RESPONSE For a science project, Nabila selected three bean plants of equal height. Then, for five days, she measured their heights in centimeters and plotted the values on the graph below.



She drew a line of best fit on the graph. What is the slope of the line that she drew? $\frac{3}{2}$

Spiral Review

54. SAVINGS Noura has AED 6,000 to invest. She puts x dirhams of this money into a savings account that earns 2% interest per year. She uses the rest of the money to purchase a certificate of deposit that earns 4% interest. Write an equation for the amount of money that Noura will have in one year. (Lesson 7-2) $T = 1.02x + 1.04(6000 - x)$

Find each sum or difference. (Lesson 7-1) **57.** $3n^3 - 6n^2 + 10$

55. $(7a^2 - 5) + (-3a^2 + 10)$ $4a^2 + 5$

56. $(8n - 2n^2) + (4n - 6n^2)$ $12n - 8n^2$

57. $(4 + n^3 + 3n^2) + (2n^3 - 9n^2 + 6)$

58. $(-4u^2 - 9 + 2u) + (6u + 14 + 2u^2)$ $-2u^2 + 8u + 5$

59. $(b + 4) + (c + 3b - 2)$ $4b + c + 2$

60. $(3a^3 - 6a) - (3a^3 + 5a)$ $-11a$

61. $(-4m^3 - m + 10) - (3m^3 + 3m^2 - 7)$ $-7m^3 - 3m^2 - m + 17$

62. $(3a + 4ab + 3b) - (2b + 5a + 8ab)$ $-2a - 4ab + b$

Skills Review

Simplify.

63. $(-2t^4)^3 - 3(-2t^3)^4$ $-56t^{12}$

64. $(-3h^2)^3 - 2(-h^3)^2$ $-29h^6$

65. $2(-5y^3)^2 + (-3y^3)^3$ $50y^6 - 27y^9$

66. $3(-6n^4)^2 + (-2n^2)^2$ $108n^8 + 4n^4$

Teaching the Mathematical Practices

Regularity Mathematically proficient students look both for general methods and for shortcuts. In Exercise 48, tell students to start by multiplying a three-digit number and a two-digit number to analyze the procedure.

4 Assess

Name the Math Ask students to name the mathematical procedures they use when multiplying two binomials with the FOIL method.

Additional Answers

37. $a^2 - 4ab + 4b^2$

38. $9c^2 + 24cd + 16d^2$

39. $x^2 - 10xy + 25y^2$

40. $8r^3 - 36r^2t + 54rt^2 - 27t^3$

41. $125g^3 + 150g^2h + 60gh^2 + 8h^3$

42. $64y^3 - 48y^2z - 36yz^2 + 27z^3$

43a. $x > 4$; If $x = 4$ the width of the rectangular sandbox would be zero and if $x < 4$ the width of the rectangular sandbox would be negative.

44b. The first term of the square of a sum is the first term of the sum squared. The middle term of the sum is two times the first term of the sum multiplied by the last term of the sum. The third term of the square of the sum is the last term of the sum squared.

Differentiated Instruction OL BL

Extension Tell students that one way to multiply 25 and 18 mentally is to multiply $(20 + 5)$ by $(20 - 2)$. Have them show how the FOIL method can be used to find each product.

a. 35(19) $35(19) = (30 + 5)(10 + 9) = (30)(10) + (30)(9) + 5(10) + 5(9) = 300 + 270 + 50 + 45 = 665$

b. 67(102) $67(102) = (60 + 7)(100 + 2) = (60)(100) + (60)(2) + 7(100) + 7(2) = 6000 + 120 + 700 + 14 = 6834$

1 Focus

VerticalAlignment

Before Lesson 7-4 Multiply binomials by using the FOIL method.

Lesson 7-4 Find squares of sums and differences.
Find the product of a sum and a difference.

After Lesson 7-4 Factor a polynomial using the Distributive Property.

2 Teach

Scaffolding Questions

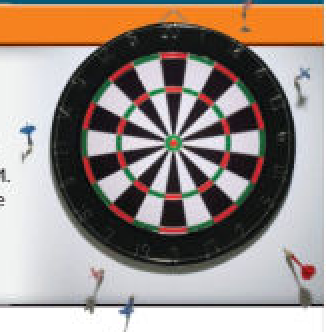
Have students read the **Why?** section of the lesson.

Ask:

- What two factors equal $(2r + 24)^2$ when multiplied? $(2r + 24)(2r + 24)$
- Use the FOIL method to find $(2r + 24)^2$. $4r^2 + 96r + 576$
- If $2r = a$ and $24 = b$, does $(2r + 24)^2$ equal $a^2 + 2ab + b^2$? Explain. Yes; $(2r + 24)^3 = (2r)^2 + 2(2r)(24) + 24^2 = 4r^2 + 96r + 576$.

LESSON 7-4 Special Products

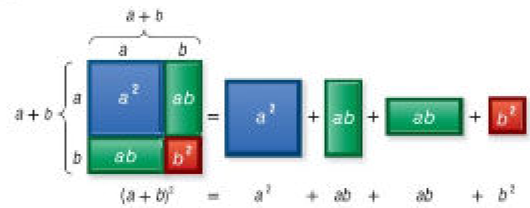
:: Then	:: Now	:: Why?
● You multiplied binomials by using the FOIL method.	● **1** Find squares of sums and differences. **2** Find the product of a sum and a difference.	● Faleh wants to attach a dartboard to a square piece of corkboard. If the radius of the dartboard is $r + 12$, how the square corkboard need to be? Faleh knows that the diameter of the dartboard is $2(r + 12)$ or $2r + 24$. Each side of the square also measures $2r + 24$. To find how much corkboard is needed, Faleh must find the area of the square: $A = (2r + 24)^2$.



Mathematical Practices
Look for and express regularity in repeated reasoning.

1 Squares of Sums and Differences

Some pairs of binomials, such as squares like $(2r + 24)^2$, have products that follow a specific pattern. Using the pattern can make multiplying easier. The square of a sum, $(a + b)^2$ or $(a + b)(a + b)$, is one of those products.



KeyConcept Square of a Sum

Words The square of $a + b$ is the square of a plus twice the product of a and b plus the square of b.

Symbols $(a + b)^2 = (a + b)(a + b)$
$= a^2 + 2ab + b^2$

Example $(x + 4)^2 = (x + 4)(x + 4)$
$= x^2 + 8x + 16$

Example 1 Square of a Sum

Find $(3x + 5)^2$.

$(a + b)^2 = a^2 + 2ab + b^2$ — Square of a sum

$(3x + 5)^2 = (3x)^2 + 2(3x)(5) + 5^2$ — $a = 3x$, $b = 5$

$= 9x^2 + 30x + 25$ — Simplify. Use FOIL to check your solution.

GuidedPractice

Find each product.

1A. $(8c + 3d)^2$ $64c^2 + 48cd + 9d^2$

1B. $(3x + 4y)^2$ $9x^2 + 24xy + 16y^2$

There is also a pattern for the *square of a difference*. Write $a - b$ as $a + (-b)$ and square it using the square of a sum pattern.

$$
\begin{aligned}
(a-b)^2 &= [a+(-b)]^2 \\
&= a^2 + 2(a)(-b) + (-b)^2 && \text{Square of a sum} \\
&= a^2 - 2ab + b^2 && \text{Simplify.}
\end{aligned}
$$

KeyConcept Square of a Difference

Words The square of $a - b$ is the square of a minus twice the product of a and b plus the square of b.

Symbols $(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$ **Example** $(x-3)^2 = (x-3)(x-3) = x^2 - 6x + 9$

WatchOut!

Regularity Remember that $(x-7)^2$ does not equal $x^2 - 7^2$, or $x^2 - 49$.

$(x-7)^2$
$= (x-7)(x-7)$
$= x^2 - 14x + 49$

Example 2 Square of a Difference

Find $(2x - 5y)^2$.

$$
\begin{aligned}
(a-b)^2 &= a^2 - 2ab + b^2 && \text{Square of a difference} \\
(2x-5y)^2 &= (2x)^2 - 2(2x)(5y) + (5y)^2 && a = 2x \text{ and } b = 5y \\
&= 4x^2 - 20xy + 25y^2 && \text{Simplify.}
\end{aligned}
$$

GuidedPractice

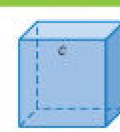
Find each product.

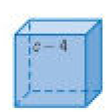
2A. $(6p-1)^2$ $36p^2 - 12p + 1$ **2B.** $(a-2b)^2$ $a^2 - 4ab + 4b^2$

The product of the square of a sum or the square of a difference is called a *perfect square trinomial*. We can use these to find patterns to solve real-world problems.

Real-World Example 3 Square of a Difference

PHYSICAL SCIENCE **Each edge of a cube of aluminum is 4 centimeters less than each edge of a cube of copper. Write an equation to model the surface area of the aluminum cube.**





Let c = the length of each edge of the cube of copper. So, each edge of the cube of aluminum is $c - 4$.

$$
\begin{aligned}
SA &= 6s^2 && \text{Formula for surface area of a cube} \\
SA &= 6(c-4)^2 && \text{Replace } s \text{ with } c - 4. \\
SA &= 6[c^2 - 2(4)(c) + 4^2] && \text{Square of a difference} \\
SA &= 6(c^2 - 8c + 16) && \text{Simplify.}
\end{aligned}
$$

GuidedPractice

3. GARDENING Ahmed has a garden that is g meters long and g meters wide. He wants to add 3 meters to the length and the width.

A. Show how the new area of the garden can be modeled by the square of a binomial. $(g+3)^2$

B. Find the square of this binomial. $g^2 + 6g + 9$

1 Squares of Sums and Differences

Example 1 shows how to follow a specific pattern to find the square of a sum. **Example 2** shows how to find the square of a difference. **Example 3** shows how to use the square of a difference to write an expression that models a real–world situation.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Find $(7z+2)^2$. $49z^2 + 28z + 4$

2 Find $(3c-4)^2$. $9c^2 - 24c + 16$

3 **GEOMETRY** Write an expression that represents the area of a square that has a side length of $3x + 12$ units. $(9x^2 + 72x + 144)$ units^2

Tips for New Teachers

Alternative Method Even though it is important to learn the special products, point out to students that they can always find these products using methods from previous lessons in the chapter.

Teaching the Mathematical Practices

Regularity Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Encourage students to look for patterns in the Examples and step through finding each pattern before you present the formulas in the Key Concepts.

Focus on Mathematical Content

Squares of Sums and Differences Since the square of a sum and the square of a difference are the same except for the sign of the middle term, the risk of making a careless mistake when finding the sum or difference of a square is high. Tell students to pay close attention to the signs when finding squares of sums or differences.

Teach with Tech

Wiki On your secure classroom wiki have students write a real world situation that uses their choice of the square of sums, the square of differences, or the product of a sum and difference. These situations can be used as a review for a chapter test.

2 Product of a Sum and a Difference

Example 4 shows how to use the pattern for the product of the sum and difference of the same two terms.

Additional Example

4 Find $(9d + 4)(9d - 4)$.
$81d^2 - 16$

Focus on Mathematical Content

Difference of Squares The product of a sum and difference of the same two terms, $(a + b)(a - b)$, is $a^2 - b^2$. This resulting product has a special name, the *difference of squares.*

3 Practice

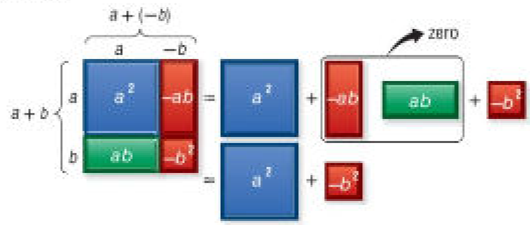
Formative Assessment

Use Exercises 1–11 to check for understanding.

Use the chart at the bottom of the next page to customize assignments for your students.

2 Product of a Sum and a Difference

Now we will see what the result is when we multiply a sum and a difference, or $(a + b)(a - b)$. Recall that $a - b$ can be written as $a + (-b)$.



Notice that the middle terms are opposites and add to a zero pair.
So $(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$.

StudyTip

Patterns When using any of these patterns, a and b can be numbers, variables, or expressions with numbers and variables.

KeyConcept Product of a Sum and a Difference

Words The product of $a + b$ and $a - b$ is the square of a minus the square of b.

Symbols $(a + b)(a - b) = (a - b)(a + b)$
$= a^2 - b^2$

Example 4 Product of a Sum and a Difference

Find $(2x^2 + 3)(2x^2 - 3)$.

$(a + b)(a - b) = a^2 - b^2$	Product of a sum and a difference
$(2x^2 + 3)(2x^2 - 3) = (2x^2)^2 - (3)^2$	$a = 2x^2$ and $b = 3$
$= 4x^4 - 9$	Simplify.

GuidedPractice

Find each product.

4A. $(3n + 2)(3n - 2)$ $9n^2 - 4$

4B. $(4c - 7d)(4c + 7d)$ $16c^2 - 49d^2$

Check Your Understanding

Examples 1–2 Find each product. **4.** $9m^2 - 24m + 16$ **5.** $g^2 - 8gh + 16h^2$ **6.** $9c^2 + 36cd + 36d^2$

1. $(x + 5)^2$ $x^2 + 10x + 25$
2. $(11 - a)^2$ $121 - 22a + a^2$
3. $(2x + 7y)^2$ $4x^2 + 28xy + 49y^2$
4. $(3m - 4)(3m - 4)$
5. $(g - 4h)(g - 4h)$
6. $(3c + 6d)^2$

Example 3

7. GENETICS The color of a cat's fur is genetic. Dark genes D are dominant over yellow genes y. A cat with genes DD or Dy will have dark fur. A cat with genes yy will have yellow fur. Pepper's genes for fur color are Dy, and Ramiro's are yy.

	D	y
D	DD	Dy
y	Dy	yy

a. Write an expression for the possible fur colors of Pepper's and Ramiro's kittens. $0.5Dy + 0.5y^2$

b. What is the probability that a kitten will have yellow fur? 50%

DifferentiatedInstruction AL

If students have trouble remembering the pattern for special products studied in this lesson,

Then have them write the symbols for and examples of each Key Concept in this lesson on separate index cards. They can use their note cards for a quick reminder on how to proceed when they are finding products of squares of sums or differences or the product of a sum and a difference.

Example 4 Find each product.

8. $(a-3)(a+3)$ a^2-9
9. $(x+5)(x-5)$ x^2-25
10. $(6y-7)(6y+7)$ $36y^2-49$
11. $(9t+6)(9t-6)$ $81t^2-36$

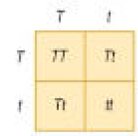
Practice and Problem Solving

Examples 1–2 Find each product.

12. $(a+10)(a+10)$ $a^2+20a+100$
13. $(b-6)(b-6)$ $b^2-12b+36$
14. $(h+7)^2$ $h^2+14h+49$
15. $(x+6)^2$ $x^2+12x+36$
16. $(8-m)^2$ $64-16m+m^2$
17. $(9-2y)^2$ $81-36y+4y^2$
18. $(2b+3)^2$ $4b^2+12b+9$
19. $(5t-2)^2$ $25t^2-20t+4$
20. $(8h-4n)^2$ $64h^2-64hn+16n^2$

Example 3

21. **GENETICS** The ability to roll your tongue is inherited genetically from parents if either parent has the dominant trait *T*. Children of two parents without the trait will not be able to roll their tongues.



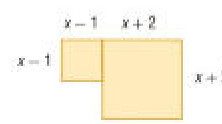
a. Show how the combinations can be modeled by the square of a sum. $(T+t)^2=T^2+2Tt+t^2$

b. Predict the percent of children that will have both dominant genes, one dominant gene, and both recessive genes. *TT*: 25%; *Tt*: 50%; *tt*: 25%

Example 4 Find each product. **22–44. See margin.**

22. $(u+3)(u-3)$
23. $(b+7)(b-7)$
24. $(2+x)(2-x)$
25. $(4-x)(4+x)$
26. $(2q+5r)(2q-5r)$
27. $(3a^2+7b)(3a^2-7b)$
28. $(5y+7)^2$
29. $(8-10a)^2$
30. $(10x-2)(10x+2)$
31. $(3t+12)(3t-12)$
32. $(a+4b)^2$
33. $(3q-5r)^2$
34. $(2c-9d)^2$
35. $(g+5h)^2$
36. $(6y-13)(6y+13)$
37. $(3a^4-b)(3a^4+b)$
38. $(5x^2-y^2)^2$
39. $(8a^2-9b^3)(8a^2+9b^3)$
40. $\left(\frac{3}{4}k+8\right)^2$
41. $\left(\frac{2}{5}y-4\right)^2$
42. $(7z^2+5y^2)(7z^2-5y^2)$
43. $(2m+3)(2m-3)(m+4)$
44. $(r+2)(r-5)(r-2)(r+5)$

45. **SENSE-MAKING** Write a polynomial that represents the area of the figure at the right. $2x^2+2x+5$



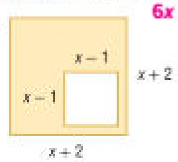
46. **FLYING DISKS** A flying disk shaped like a circle has a radius of $x+3$ centimeters.

a. Write an expression representing the area of the flying disk. $\pi x^2+6\pi x+9\pi$

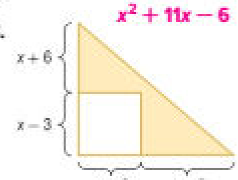
b. A hole with a radius of $x-1$ centimeters is cut in the center of the disk. Write an expression for the remaining area. $8\pi x+8\pi$

GEOMETRY Find the area of each shaded region.

47. $6x+3$



48. $x^2+11x-6$



Teaching the Mathematical Practices

Sense-Making Mathematically proficient students can explain the correspondences between equations, verbal descriptions, and diagrams. In Exercise 45, ask students to explain how to write and simplify the polynomial based on the diagram.

Exercise Alert

Formula For Exercises 46 and 55, students will need to know the formula for the area of a circle, $A=\pi r^2$.

Additional Answers

22. u^2-9
23. b^2-49
24. $4-x^2$
25. $16-x^2$
26. $4q^2-25r^2$
27. $9a^4-49b^2$
28. $25y^2+70y+49$
29. $64-160a+100a^2$
30. $100x^2-4$
31. $9t^2-144$
32. $a^2+8ab+16b^2$
33. $9q^2-30qr+25r^2$
34. $4c^2-36cd+81d^2$
35. $g^2+10gh+25h^2$
36. $36y^2-169$
37. $9a^8-b^2$
38. $25x^4-10x^2y^2+y^4$
39. $64a^4-81b^6$
40. $\frac{9}{16}k^2+12k+64$
41. $\frac{4}{25}y^2-\frac{16}{5}y+16$
42. $49z^4-25y^4$
43. $4m^3+16m^2-9m-36$
44. r^4-29r^2+100

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	12–28, 56, 57, 59–90	13–27 odd, 62–65	12–28 even, 56, 57, 59–61, 66–90
OL Core	13–53 odd, 55–57, 59–90	12–45, 62–65	46–57, 59–61, 66–90
BL Advanced	46–87, (optional: 86–90)		

Multiple Representations

In Exercise 56, students use algebra and a physical model to represent the difference of two squares.

Find each product. **49.** $c^3 + 3c^2d + 3cd^2 + d^3$ **50.** $8a^3 - 12a^2b + 6ab^2 - b^3$

49. $(c + d)(c + d)(c + d)$

50. $(2a - b)^3$

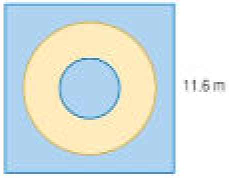
51. $(f + g)(f - g)(f + g)$ $f^3 + f^2g - fg^2 - g^3$

52. $(k - m)(k + m)(k - m)$ $k^3 - k^2m - km^2 + m^3$

53. $(n - p)^2(n + p)$ $n^3 - n^2p - np^2 + p^3$

54. $(q + r)^2(q - r)$ $q^3 + q^2r - qr^2 - r^3$

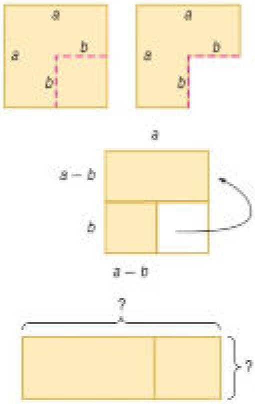
55. WRESTLING A high school wrestling mat must be a square with 11.6-meter sides and contain two circles as shown. Suppose the inner circle has a radius of r meters, and the radius of the outer circle is 2.7 meters longer than the inner circle.



a. Write an expression for the area of the larger circle. **55a. about $(3.14r^2 + 56.52r + 254.34)$ ft²**

b. Write an expression for the area of the portion of the square outside the larger circle. **about $(1189.66 - 3.14r^2 - 56.52r)$ m²**

56. MULTIPLE REPRESENTATIONS In this problem, you will investigate a pattern. Begin with a square piece of construction paper. Label each edge of the paper a. In any of the corners, draw a smaller square and label the edges b.



a. **Numerical** Find the area of each of the squares. **See margin.**

b. **Concrete** Cut the smaller square out of the corner. What is the area of the shape? $a^2 - b^2$

c. **Analytical** Remove the smaller rectangle on the bottom. Turn it and slide it next to the top rectangle. What is the length of the new arrangement? What is the width? What is the area? $a + b$, $a - b$, $(a + b)(a - b)$

d. **Analytical** What pattern does this verify? $(a + b)(a - b) = a^2 - b^2$

H.O.T. Problems Use Higher-Order Thinking Skills

57. WHICH ONE DOESN'T BELONG? Which expression does not belong? Explain. **See margin.**

$(2c - d)(2c - d)$	$(2c + d)(2c - d)$	$(2c + d)(2c + d)$	$(c + d)(c + d)$

58. STRUCTURE Does a pattern exist for the cube of a sum, $(a + b)^3$?

a. Investigate this question by finding the product $(a + b)(a + b)(a + b)$. $a^3 + 3a^2b + 3ab^2 + b^3$

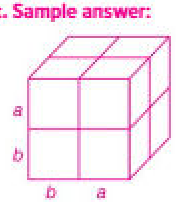
b. Use the pattern you discovered in part **a** to find $(x + 2)^3$. $x^3 + 6x^2 + 12x + 8$

c. Draw a diagram of a geometric model for $(a + b)^3$.

d. What is the pattern for the cube of a difference, $(a - b)^3$? $a^3 - 3a^2b + 3ab^2 - b^3$

59. REASONING Find c that makes $25x^2 - 90x + c$ a perfect square trinomial. **81**

58c. Sample answer:

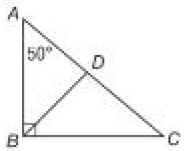


60. OPEN ENDED Write two binomials with a product that is a binomial. Then write two binomials with a product that is not a binomial. **Sample answer: $(x - 2)(x + 2) = x^2 - 4$ and $(x - 2)(x - 2) = x^2 - 4x + 4$**

61. WRITING IN MATH Describe how to square the sum of two quantities, square the difference of two quantities, and how to find the product of a sum of two quantities and a difference of two quantities. **See margin.**

Standardized Test Practice

62. **GRIDDED RESPONSE** In the right triangle, $\overline{DB}$ bisects $\angle B$. What is the measure of $\angle ADB$ in degrees? **85**



63. What is the product of $(2a-3)$ and $(2a-3)$? **D**

A $4a^2+12a+9$
B $4a^2+9$
C $4a^2-12a-9$
D $4a^2-12a+9$

64. Mazen can drive 4 kilometers in m minutes. At this rate, how many minutes will it take him to drive 19 kilometers? **G**

F $76m$
G $\frac{19m}{4}$
H $\frac{4m}{19}$
J $\frac{4}{19m}$

65. What property is illustrated by the equation $2x+0=2x$? **C**

A Commutative Property of Addition
B Additive Inverse Property
C Additive Identity Property
D Associative Property of Addition

Spiral Review

Find each product. (Lesson 7-3)

66. $(y-4)(y-2)$ y^2-6y+8
67. $(2c-1)(c+3)$ $2c^2+5c-3$
68. $(d-9)(d+5)$ $d^2-4d-45$
69. $(4h-3)(2h-7)$ $8h^2-34h+21$
70. $(3x+5)(2x+3)$ $6x^2+19x+15$
71. $(5m+4)(8m+3)$ $40m^2+47m+12$

Simplify. (Lesson 7-2)

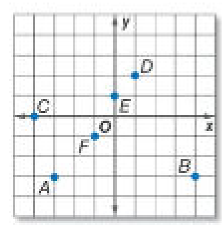
72. $x(2x-7)+5x$ $2x^2-2x$
73. $c(c-8)+2c(c+3)$ $3c^2-2c$
74. $8y(-3y+7)-11y^2$ $-35y^2+56y$
75. $-2d(5d)-3d(d+6)$ $-13d^2-18d$
76. $5m(2m^3+m^2+8)+4m$ $10m^4+5m^3+44m$
77. $3p(6p-4)+2\left(\frac{1}{2}p^2-3p\right)$ $19p^2-18p$

Simplify. (Lesson 0-10)

78. $\sqrt{48}$ $4\sqrt{3}$
79. $\sqrt{162}$ $9\sqrt{2}$
80. $\sqrt{25a^6b^4}$ $5a^3b^2$
81. $\sqrt{45xy^8}$ $3y^4\sqrt{5x}$

Write the ordered pair for each point shown. (Lesson 0-8)

82. A $(-3,-3)$
83. B $(4,-3)$
84. C $(-4,0)$
85. D $(1,2)$
86. E $(0,1)$
87. F $(-1,-1)$



Skills Review

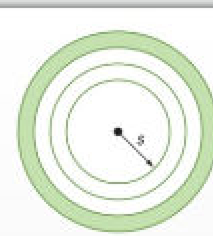
Write each polynomial in standard form. Identify the leading coefficient.

88. $2x^2-x^4-8+x$
89. $-5p^4+p^2+12+2p^5$
90. $-10+a^3-a+6a^2$

88. $-x^4+2x^2+x-8$; -1
89. $2p^5-5p^4+p^2+12$; 2
90. a^3+6a^2-a-10; 1

Differentiated Instruction OL BL

Extension A diagram of the Gwennap Pit is shown here. Tell students the historical Gwennap Pit, an outdoor amphitheater in southern England, consists of a circular stage surrounded by circular levels used for seating. Each seating level is about 1 meter wide. Suppose the radius of the stage is s meters.



a. Find binomial representations for the radii of the second and third seating levels. $s+2$, $s+3$

b. Find the area of the shaded region representing the third seating level. about $(6.3s+15.7)$ m^2

Exercise Alerts

Construction Paper For Exercise 56, students will need a square piece of construction paper.

Isometric Dot Paper For Exercise 58, students may want to use isometric dot paper to draw their models for the cube of a sum.

Teaching the Mathematical Practices

Structure Mathematically proficient students look closely to discern a pattern or structure. In Exercise 58a, point out that $(a+b)(a+b)(a+b)$ can be rewritten as $(a+b)(a^2+2ab+b^2)$.

4 Assess

Ticket Out the Door Make several copies each of five squares of sums that need to be multiplied. Give one expression to each student. As the students leave the room, ask them to tell you the products of the expressions.

Additional Answers

56a. The area of the larger square is a^2. The area of the smaller square is b^2.

57. Sample answer: $(2c+d)\cdot(2c-d)$; The product of these binomials is a difference of two squares and does not have a middle term. The other three do.

61. Sample answer: To find the square of a sum, apply the FOIL method or apply the pattern. The square of the sum of two quantities is the first quantity squared plus two times the product of the two quantities plus the second quantity squared. The square of the difference of two quantities is the first quantity squared minus two times the product of the two quantities plus the second quantity squared. The product of the sum and difference of two quantities is the square of the first quantity minus the square of the second quantity.

Formative Assessment

Use the Mid-Chapter Quiz to assess students' progress in the first half of the chapter.

For problems answered incorrectly, have students review the lessons indicated in parentheses.

FOLDABLES StudyOrganizer

Dinah Zike's Foldables®

Before students complete the Mid-Chapter Quiz, encourage them to review the information for Lessons 7-1 through 7-4 in their Foldables.

Additional Answer

7c. 30.07 people/mi^2; Since 2020 is 90 years after 1930, substitute 90 for y in the equation.; 38.3 people/mi^2; Since 2030 is 100 years after 1930, substitute 100 for y in the equation.

CHAPTER 7

Mid-Chapter Quiz

Lessons 7-1 through 7-4

Determine whether each expression is a polynomial. If it is a polynomial, find the degree and determine whether it is a *monomial*, *binomial*, or *trinomial*. (Lesson 7-1)

1. $3y^2 - 2$ **yes; 2; binomial**
2. $4t^5 + 3t^2 + t$ **yes; 5; trinomial**
3. $\frac{3x}{5y}$ **not a polynomial**
4. ax^{-3} **not a polynomial**
5. $3b^2$ **yes; 2; monomial**
6. $2x^{-3} - 4x + 1$ **not a polynomial**

7. **POPULATION** The table shows the population density for a country for various years. (Lesson 7-1)

Year	Years Since 1930	People/ Square kilometer
1930	0	0.8
1960	30	2.6
1980	50	7.3
1990	60	10.9
2000	70	18.2

a. The population density d of a country from 1930 to 2000 can be modeled by $d = 0.005n^2 - 0.127n + 1$, where n represents the number of years since 1930. Identify the type of polynomial for $0.005n^2 - 0.127n + 1$. **quadratic trinomial**

b. What is the degree of the polynomial? **2**

c. Predict the population density of the country for 2020 and for 2030. Explain your method. **See margin.**

Find each sum or difference. (Lesson 7-1)

8. $(y^2 + 2y + 3) + (y^2 + 3y - 1)$ $\mathbf{2y^2 + 5y + 2}$
9. $(3n^3 - 2n + 7) - (n^2 - 2n + 8)$ $\mathbf{3n^3 - n^2 - 1}$
10. $(5d + d^2) - (4 - 4d^2)$ $\mathbf{5d^2 + 5d - 4}$
11. $(x + 4) + (3x + 2x^2 - 7)$ $\mathbf{2x^2 + 4x - 3}$
12. $(3a - 3b + 2) - (4a + 5b)$ $\mathbf{-a - 8b + 2}$
13. $(8x - y^2 + 3) + (9 - 3x + 2y^2)$ $\mathbf{5x + y^2 + 12}$

Find each product. (Lesson 7-2)

14. $6y(y^2 + 3y + 1)$ $\mathbf{6y^3 + 18y^2 + 6y}$
15. $3n(n^2 - 5n + 2)$ $\mathbf{3n^3 - 15n^2 + 6n}$
16. $d^2(-4 - 3d + 2d^2)$ $\mathbf{-4d^2 - 3d^3 + 2d^4}$
17. $-2xy(3x^2 + 2xy - 4y^2)$ $\mathbf{-6x^3y - 4x^2y^2 + 8xy^3}$
18. $ab^2(12a + 5b - ab)$ $\mathbf{12a^2b^2 + 5ab^3 - a^2b^3}$
19. $x^2y^4(3xy^2 - x + 2y^2)$ $\mathbf{3x^3y^6 - x^3y^4 + 2x^2y^6}$

20. **MULTIPLE CHOICE** Simplify $x(4x + 5) + 3(2x^2 - 4x + 1)$. (Lesson 7-2) **B**

A $10x^2 + 17x + 3$
B $10x^2 - 7x + 3$
C $2x^2 - 7x + 3$
D $2x^2 + 17x + 3$

Find each product. (Lesson 7-3)

21. $(x + 2)(x + 5)$ $\mathbf{x^2 + 7x + 10}$
22. $(3b - 2)(b - 4)$ $\mathbf{3b^2 - 14b + 8}$
23. $(n - 5)(n + 3)$ $\mathbf{n^2 - 2n - 15}$
24. $(4c - 2)(c + 2)$ $\mathbf{4c^2 + 6c - 4}$
25. $(k - 1)(k - 3k^2)$ $\mathbf{-3k^3 + 4k^2 - k}$
26. $(8d - 3)(2d^2 + d + 1)$ $\mathbf{16d^3 + 2d^2 + 5d - 3}$

27. **MANUFACTURING** A company is designing a box for dry pasta in the shape of a rectangular prism. The length is 2 centimeters more than twice the width, and the height is 3 centimeters more than the length. Write an expression in terms of the width for the volume of the box. (Lesson 7-3) $\mathbf{4w^3 + 14w^2 + 10w}$

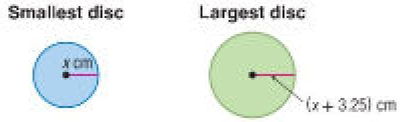
Find each product. (Lesson 7-4)

28. $(x + 2)^2$ $\mathbf{x^2 + 4x + 4}$
29. $(n - 11)^2$ $\mathbf{n^2 - 22n + 121}$
30. $(4b - 2)^2$ $\mathbf{16b^2 - 16b + 4}$
31. $(6c + 3)^2$ $\mathbf{36c^2 + 36c + 9}$
32. $(5d - 3)(5d + 3)$ $\mathbf{25d^2 - 9}$
33. $(9k + 1)(9k - 1)$ $\mathbf{81k^2 - 1}$

34a. $\pi(x + 3.25)^2$ in^2, $(\pi x^2 + 6.5\pi x + 10.5625\pi)$ in^2

34. **DISC GOLF** The discs approved for use in disc golf vary in size. (Lesson 7-4)

Smallest disc   Largest disc



a. Write two different expressions for the area of the largest disc.

b. If x is 26.7, what are the areas of the smallest and largest discs? **2350.9 cm^2; 3832.25 cm^2**

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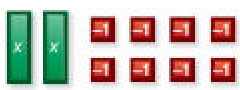
EXPLORE 7-5 Algebra Lab
Factoring Using the Distributive Property

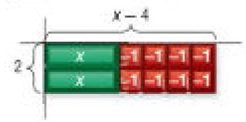
When two or more numbers are multiplied, these numbers are *factors* of the product. Sometimes you know the product of binomials and are asked to find the factors. This is called factoring. You can use algebra tiles and a product mat to factor binomials.

Activity 1 Use Algebra Tiles to Factor $2x - 8$

Step 1 Model $2x - 8$.



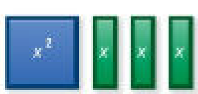
Step 2 Arrange the tiles into a rectangle. The total area of the rectangle represents the product, and its length and width represent the factors.



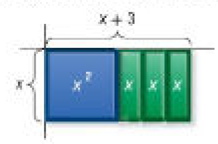
The rectangle has a width of 2 and a length of $x - 4$. Therefore, $2x - 8 = 2(x - 4)$.

Activity 2 Use Algebra Tiles to Factor $x^2 + 3x$

Step 1 Model $x^2 + 3x$.



Step 2 Arrange the tiles into a rectangle.



The rectangle has a width of x and a length of $x + 3$. Therefore, $x^2 + 3x = x(x + 3)$.

Model and Analyze

Use algebra tiles to factor each binomial.

9. Sample answer: Binomials can be factored if they can be represented by a rectangle. Examples: $3x + 3$ can be factored and $3x + 2$ cannot be factored.

1. $4x + 12$ **$4(x + 3)$**
2. $4x - 6$ **$2(2x - 3)$**
3. $3x^2 + 4x$ **$x(3x + 4)$**
4. $10 - 2x$ **$2(5 - x)$**

Determine whether each binomial can be factored. Justify your answer with a drawing.

5. $6x - 9$ **yes**
6. $5x - 4$ **no**
7. $4x^2 + 7$ **no**
8. $x^2 + 3x$ **yes**

5–8. See Ch. 8 Answer Appendix for drawings.

9. **WRITING IN MATH** Write a paragraph that explains how you can use algebra tiles to determine whether a binomial can be factored. Include an example of one binomial that can be factored and one that cannot.

1 Focus

Objective Use algebra tiles to model using the Distributive Property to factor binomials.

Materials for Each Student

- algebra tiles and product mats

2 Teach

Working in Cooperative Groups

Put students in groups of two or three, mixing abilities. Have groups complete Activities 1 and 2 and Exercises 1–4.

- For Exercises 1–4, students should recognize that they must arrange the tiles into a rectangle with a width greater than 1 in order to find the factors of the polynomial.
- Point out that the area of the rectangle represents the polynomial, and the length and width represent the factors of the polynomial.

Practice Have students complete Exercises 5–9.

- For Exercises 5–9, emphasize that if a binomial can only be modeled with a width of 1, it cannot be factored.

3 Assess

Formative Assessment

Use Exercise 3 to assess whether students can use algebra tiles to factor a binomial.

From Concrete to Abstract

Write $x^2 + 5x$ on the board. Have students factor the binomial without using tiles. If they answer incorrectly, have them use their tiles to help them find their errors.

1 Focus

VerticalAlignment

Before Lesson 7-5 Use the Distributive Property to evaluate expressions.

Lesson 7-5 Use the Distributive Property to factor polynomials. Solve quadratic equations of the form $ax^2 + bx = 0$.

After Lesson 7-5 Solve equations of the form $ax^2 + bx + c = 0$.

2 Teach

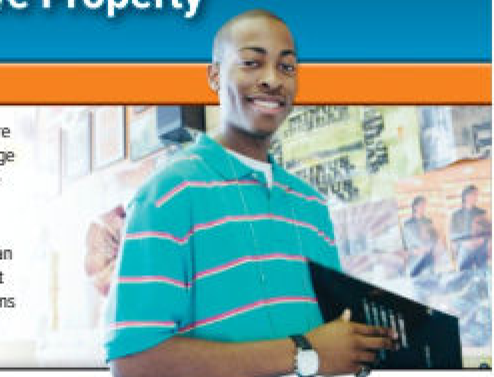
Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What is the formula for the area of a rectangle? $A = \ell \times w$
- What do you multiply w by to get $1.6w^2 + 6w$? $1.6w + 6$
- What is the area for Mr. Eissa's store expressed as a monomial times a polynomial? $w(1.6w + 6)$
- What is the area when $w = 50$? $4300\ m^2$

LESSON 7-5 Using the Distributive Property



:: Then	:: Now	:: Why?
● Used the Distributive Property to evaluate expressions.	● 1 Use the Distributive Property to factor polynomials. 2 Solve equations of the form $ax^2 + bx = 0$.	● The cost of rent for Mr. Eissa's store is determined by the square meterage of the space. The area of the store can be modeled by the equation $A = 1.6w^2 + 6w$, where w is the width of the store in meters. We can use factoring and the Zero Product Property to find possible dimensions of the store.

NewVocabulary
factoring
factoring by grouping
Zero Product Property

Mathematical Practices
Reason abstractly and quantitatively.

1 Use the Distributive Property to Factor

You have used the Distributive Property to multiply a monomial by a polynomial. You can work backward to express a polynomial as the product of a monomial factor and a polynomial factor.

$$1.6w^2 + 6w = 1.6w(w) + 6(w)$$
$$= w(1.6w + 6)$$

So, $w(1.6w + 6)$ is the *factored form* of $1.6w^2 + 6w$. **Factoring** a polynomial involves finding the *completely* factored form.

Example 1 Use the Distributive Property

Use the Distributive Property to factor each polynomial.

a. $27y^2 + 18y$

Find the GCF of each term.

$27y^2 = 3 \cdot 3 \cdot 3 \cdot y \cdot y$ — Factor each term.

$18y = 2 \cdot 3 \cdot 3 \cdot y$ — Circle common factors.

$\text{GCF} = 3 \cdot 3 \cdot y$ or $9y$

Write each term as the product of the GCF and the remaining factors. Use the Distributive Property to *factor out* the GCF.

$27y^2 + 18y = 9y(3y) + 9y(2)$ — Rewrite each term using the GCF.

$= 9y(3y + 2)$ — Distributive Property

b. $-4a^2b - 8ab^2 + 2ab$

$-4a^2b = -1 \cdot 2 \cdot 2 \cdot a \cdot a \cdot b$ — Factor each term.

$-8ab^2 = -1 \cdot 2 \cdot 2 \cdot 2 \cdot a \cdot b \cdot b$ — Circle common factors.

$2ab = 2 \cdot a \cdot b$

$\text{GCF} = 2 \cdot a \cdot b$ or $2ab$

$-4a^2b - 8ab^2 + 2ab = 2ab(-2a) - 2ab(4b) + 2ab(1)$ — Rewrite each term using the GCF.

$= 2ab(-2a - 4b + 1)$ — Distributive Property

GuidedPractice

1A. $15w - 3v$ $3(5w - v)$

1B. $7u^2t^2 + 21ut^2 - ut$ $ut(7ut + 21t - 1)$

Using the Distributive Property to factor polynomials with four or more terms is called **factoring by grouping** because terms are put into groups and then factored. The Distributive Property is then applied to a common binomial factor.

KeyConcept Factoring by Grouping

Words A polynomial can be factored by grouping only if all of the following conditions exist.

- There are four or more terms.
- Terms have common factors that can be grouped together.
- There are two common factors that are identical or additive inverses of each other.

Symbols
$$ax + bx + ay + by = (ax + bx) + (ay + by)$$
$$= x(a + b) + y(a + b)$$
$$= (x + y)(a + b)$$

Example 2 Factor by Grouping

Factor $4qr + 8r + 3q + 6$.

$4qr + 8r + 3q + 6$ — Original expression

$= (4qr + 8r) + (3q + 6)$ — Group terms with common factors.

$= 4r(q + 2) + 3(q + 2)$ — Factor the GCF from each group.

Notice that $(q + 2)$ is common in both groups, so it becomes the GCF.

$= (4r + 3)(q + 2)$ — Distributive Property

GuidedPractice

Factor each polynomial.

2A. $rn + 5n - r - 5$ $(r + 5)(n - 1)$

2B. $3np + 15p - 4n - 20$ $(n + 5)(3p - 4)$

It can be helpful to recognize when binomials are additive inverses of each other. For example $6 - a = -1(a - 6)$.

StudyTip

Check To check your factored answers, multiply your factors out. You should get your original expression as a result.

Example 3 Factor by Grouping with Additive Inverses

Factor $2mk - 12m + 42 - 7k$.

$2mk - 12m + 42 - 7k$

$= (2mk - 12m) + (42 - 7k)$ — Group terms with common factors.

$= 2m(k - 6) + 7(6 - k)$ — Factor the GCF from each group.

$= 2m(k - 6) + 7[(-1)(k - 6)]$ — $6 - k = -1(k - 6)$

$= 2m(k - 6) - 7(k - 6)$ — Associative Property

$= (2m - 7)(k - 6)$ — Distributive Property

GuidedPractice

Factor each polynomial.

3A. $c - 2cd + 8d - 4$

3B. $3p - 2p^2 - 18p + 27$

3A. $(-c + 4)(2d - 1)$ or $(c - 4)(1 - 2d)$
3B. $(p + 9)(3 - 2p)$ or $(-p - 9)(2p - 3)$

1 Use the Distributive Property to Factor

Example 1 shows how to use the Distributive Property to factor a polynomial. **Example 2** shows how to use grouping to factor a polynomial. **Example 3** shows how to use grouping with additive inverses to factor a polynomial.

Formative Assessment

Use the Guided Practice exercises after each Example to determine students' understanding of concepts.

Additional Examples

1 Use the Distributive Property to factor each polynomial.

a. $15x + 25x^2$ $5x(3 + 5x)$

b. $12xy + 24xy^2 - 30x^2y^4$ $6xy(2 + 4y - 5xy^3)$

2 Factor $2xy + 7x - 2y - 7$. $(x - 1)(2y + 7)$

3 Factor $15a - 3ab + 4b - 20$. $(-3a + 4)(b - 5)$ or $(3a - 4)(5 - b)$

Focus on Mathematical Content

Greatest Common Factor

Factoring using the Distributive Property requires expressing a polynomial as the product of the greatest common monomial factor of the polynomial's terms and a polynomial factor. To find the nonmonomial factor, each term of the polynomial is divided by the common monomial factor.

Tips for New Teachers

Reasoning Sometimes students find the monomial that is the GCF of the terms of the polynomial but do not know how to get the other polynomial factor. One way to find the remaining factor is to divide each term of the polynomial by the GCF. Tell students to check their answers by multiplying their factors using the Distributive Property.

Teach with Tech

Video Recording Separate the class into groups and have each group create a video showing how to factor by grouping. Assign a different expression to each group, and share each group's video with the class.

2 Solve Equations by Factoring

Example 4 shows how to solve two different quadratic equations—one in factored form and one that must be factored. **Example 5** shows how to solve a real-world problem using the Zero Product Property.

Additional Example

4 Solve each equation. Check your solutions.

a. $(x - 2)(4x - 1) = 0$ $2, \frac{1}{4}$

b. $4y = 12y^2$ $0, \frac{1}{3}$

Focus on Mathematical Content

Zero Product Property Quadratic equations can be solved by using the Zero Product Property: If the product of two factors is zero, then one of the factors is zero. To solve equations using this property, write the equation with the terms in factored form on one side of the equation and zero on the other side. Each factor is then set equal to zero, and the resulting equations are solved to arrive at the solutions.

2 Solve Equations by Factoring

Some equations can be solved by factoring. Consider the following.

$3(0) = 0$ $\quad 0(2 - 2) = 0$ $\quad -312(0) = 0$ $\quad 0(0.25) = 0$

Notice that in each case, at least one of the factors is 0. These examples are demonstrations of the **Zero Product Property**.

KeyConcept Zero Product Property

Words	If the product of two factors is 0, then at least one of the factors must be 0.
Symbols	For any real numbers a and b, if $ab = 0$, then $a = 0$, $b = 0$, or both a and b equal zero.

Recall that a solution or root of an equation is any value that makes the equation true.

WatchOut!

Unknown Value It may be tempting to solve an equation by dividing each side by the variable. However, the variable has an unknown value, so you may be dividing by 0, which is undefined.

Example 4 Solve Equations

Solve each equation. Check your solutions.

a. $(2d + 6)(3d - 15) = 0$

$(2d + 6)(3d - 15) = 0$	Original equation
$2d + 6 = 0$ or $3d - 15 = 0$	Zero Product Property
$2d = -6$ $\quad 3d = 15$	Solve each equation.
$d = -3$ $\quad d = 5$	Divide.

The roots are −3 and 5.

CHECK Substitute −3 and 5 for d in the original equation.

$(2d + 6)(3d - 15) = 0$	$(2d + 6)(3d - 15) = 0$
$[2(-3) + 6][3(-3) - 15] \stackrel{?}{=} 0$	$[2(5) + 6][3(5) - 15] \stackrel{?}{=} 0$
$(-6 + 6)(-9 - 15) \stackrel{?}{=} 0$	$(10 + 6)(15 - 15) \stackrel{?}{=} 0$
$(0)(-24) \stackrel{?}{=} 0$	$16(0) \stackrel{?}{=} 0$
$0 = 0$ ✓	$0 = 0$ ✓

b. $c^2 = 3c$

$c^2 = 3c$	Original equation
$c^2 - 3c = 0$	Subtract 3c from each side to get 0 on one side of the equation.
$c(c - 3) = 0$	Factor by using the GCF to get the form $ab = 0$.
$c = 0$ or $c - 3 = 0$	Zero Product Property
$c = 3$	Solve each equation.
The roots are 0 and 3.	Check by substituting 0 and 3 for c.

GuidedPractice

4A. $3n(n + 2) = 0$ $0, -2$ **4B.** $8b^2 - 40b = 0$ $0, 5$ **4C.** $x^2 = -10x$ $0, -10$

DifferentiatedInstruction AL OL

If students have trouble solving quadratic equations like Example 4b,

Then you may wish to allow students to use algebra tiles. They can use the methods from Explore 8-5 to factor the quadratic to solve the equation.

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Real-WorldLink

Dog agility tests a person's skills as a trainer and handler. Competitors race through an obstacle course that includes hurdles, tunnels, a see-saw, and line poles.

Source: United States Dog Agility Association

Real-World Example 5 Use Factoring

AGILITY A famous dog competes with her trainer in the agility course. Within the course, the dog must leap over a hurdle. The dog's jump can be modeled by the equation $h = -16t^2 + 20t$, where h is the height of the leap in inches at t seconds. Find the values of t when $h = 0$.

$h = -16t^2 + 20t$	Original equation
$0 = -16t^2 + 20t$	Substitution, $h = 0$
$0 = 4t(-4t + 5)$	Factor by using the GCF.
$4t = 0$ or $-4t + 5 = 0$	Zero Product Property
$t = 0$ $\quad -4t = -5$	Solve each equation.
$t = \frac{5}{4}$ or 1.25	Divide each side by −4.

The dog's height is 0 inches at 0 seconds and 1.25 seconds into the jump.

GuidedPractice

5. **KANGAROOS** The hop of a kangaroo can be modeled by $h = 24t - 16t^2$ where h represents the height of the hop in feet and t is the time in seconds. Find the values of t when $h = 0$. **0, 1.5**

Check Your Understanding

Example 1 Use the Distributive Property to factor each polynomial.

1. $21b - 15a$ **3(7b − 5a)**
2. $14c^2 + 2c$ **2c(7c + 1)**
3. $10g^2h^2 + 9gh^2 - g^2h$ **gh(10gh + 9h − g)**
4. $12jk^2 + 6j^2k + 2j^2k^2$ **2jk(6k + 3j + jk)**

Examples 2–3 Factor each polynomial.

5. $np + 2n + 8p + 16$ **(n + 8)(p + 2)**
6. $xy - 7x + 7y - 49$ **(x + 7)(y − 7)**
7. $3bc - 2b - 10 + 15c$ **(b + 5)(3c − 2)**
8. $9fg - 45f - 7g + 35$ **(9f − 7)(g − 5)**

Example 4 Solve each equation. Check your solutions.

9. $3k(k + 10) = 0$ **0, −10**
10. $(4m + 2)(3m - 9) = 0$ **$-\frac{1}{2}$, 3**
11. $20p^2 - 15p = 0$ **0, $\frac{3}{4}$**
12. $r^2 = 14r$ **0, 14**

Example 5

13. **SPIDERS** Jumping spiders can commonly be found in homes and barns throughout the United States. A jumping spider's jump can be modeled by the equation $h = 33.3t - 16t^2$, where t represents the time in seconds and h is the height in feet.
    a. When is the spider's height at 0 feet? **0 seconds and 2.08125 seconds**
    b. What is the spider's height after 1 second? after 2 seconds? **17.3 ft, 2.6 ft**

14. **REASONING** At a national celebration, a rocket is launched straight up with an initial velocity of 125 feet per second. The height h of the rocket in feet above sea level is modeled by the formula $h = 125t - 16t^2$, where t is the time in seconds after the rocket is launched.
    a. What is the height of the rocket when it returns to the ground? **0 ft**
    b. Let $h = 0$ in the equation and solve for t. **0, 7.8125**
    c. How many seconds will it take for the rocket to return to the ground? **about 7.8 s**

Additional Example

5 **FOOTBALL** A football is kicked into the air. The height of the football can be modeled by the equation $h = -16x^2 + 16x$, where h is the height reached by the ball after x seconds. Find the values of x when $h = 0$.
0 seconds, 1 second

3 Practice

Formative Assessment

Use Exercises 1–14 to check for understanding.

Use the chart at the bottom of the next page to customize assignments for your students.

Teaching the Mathematical Practices

Reasoning Mathematically proficient attend to the meaning of quantities, not just how to compute them. In Exercise 14, tell students that the path of a projectile or other body moving through space is called a trajectory. Quadratic functions are always used to represent motion of a trajectory.

Follow-up

Students have explored modeling using quadratic functions.

Ask:

- What are the advantages of using quadratic functions for modeling? Sample answer: They have well-known and understood properties; they can be used to model situations that have both increasing and decreasing behavior; the computation that is done to make predictions is relatively easy to perform.

Teaching the Mathematical Practices

Sense-Making Mathematically proficient students check their answers to problems using a different method. In Exercise 45, have students check their answers by substituting values for a and b and finding the areas directly and by using their expressions.

Exercise Alert

Grid Paper For Exercise 47, students will need grid paper.

Multiple Representations

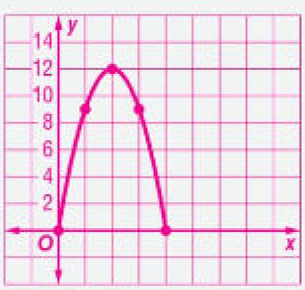
In Exercise 51, students use a diagram and analysis to factor an algebraic expression.

Additional Answers

47a.

x	0	1	2	3	4
y	0	9	12	9	0

47b.



51b.

x^2	$+3x$
$-2x$	-6

51c.

	x	$+3$
x	x^2	$+3x$
-2	$-2x$	-6

$(x + 3)(x - 2)$

53. If $a = 0$ and $b = 0$, then all real numbers are solutions. If $a \neq 0$, then the solutions are $-\frac{b}{a}$ and $\frac{b}{a}$.

56. Rewrite the equation to have zero on one side of the equals sign. Then factor the other side. Set each factor equal to zero, and then solve each equation.

Practice and Problem Solving

Example 1 Use the Distributive Property to factor each polynomial.

15. $16t - 40y$ **$8(2t - 5y)$**
16. $30v + 50x$ **$10(3v + 5x)$**
17. $2k^2 + 4k$ **$2k(k + 2)$**
18. $5z^2 + 10z$ **$5z(z + 2)$**
19. $4a^2b^2 + 2a^2b - 10ab^2$ **$2ab(2ab + a - 5b)$**
20. $5c^2v - 15c^2v^2 + 5c^2v^3$ **$5c^2v(1 - 3v + v^2)$**

Examples 2–3 Factor each polynomial. **25. $(9q - 10)(5p - 3)$ 35. $3cd(9d - 6cd + 1)$ 37. $2(8u - 15)(3t + 2)$**

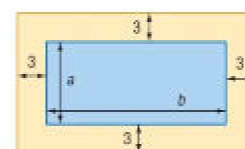
21. $fg - 5g + 4f - 20$ **$(g + 4)(f - 5)$**
22. $a^2 - 4a - 24 + 6a$ **$(a - 4)(a + 6)$**
23. $hj - 2h + 5j - 10$ **$(h + 5)(j - 2)$**
24. $xy - 2x - 2 + y$ **$(x + 1)(y - 2)$**
25. $45pq - 27q - 50p + 30$
26. $24ty - 18t + 4y - 3$ **$(6t + 1)(4y - 3)$**
27. $3dt - 21d + 35 - 5t$ **$(3d - 5)(t - 7)$**
28. $8r^2 + 12r$ **$4r(2r + 3)$**
29. $21th - 3t - 35h + 5$ **$(3t - 5)(7h - 1)$**
30. $vp + 12v + 8p + 96$ **$(v + 8)(p + 12)$**
31. $5br - 25b + 2r - 10$ **$(r - 5)(5b + 2)$**
32. $2nu - 8u + 3n - 12$ **$(2u + 3)(n - 4)$**
33. $5gf^2 + g^2f + 15gf$ **$gf(5f + g + 15)$**
34. $rp - 9r + 9p - 81$ **$(r + 9)(p - 9)$**
35. $27cd^2 - 18c^2d^2 + 3cd$
36. $18r^3t^2 + 12r^2t^2 - 6r^2t$ **$6r^2t(3rt + 2t - 1)$**
37. $48tu - 90t + 32u - 60$
38. $16gh + 24g - 2h - 3$ **$(8g - 1)(2h + 3)$**

Example 4 Solve each equation. Check your solutions.

39. $3b(9b - 27) = 0$ **0, 3**
40. $2n(3n + 3) = 0$ **0, −1**
41. $(8z + 4)(5z + 10) = 0$ **$-\frac{1}{2}, -2$**
42. $(7x + 3)(2x - 6) = 0$ **$-\frac{3}{7}, 3$**
43. $b^2 = -3b$ **0, −3**
44. $a^2 = 4a$ **0, 4**

Example 5

45. **SENSE-MAKING** Use the drawing at the right.



a. Write an expression in factored form to represent the area of the blue section. **ab**

b. Write an expression in factored form to represent the area of the region formed by the outer edge. **$(a + 6)(b + 6)$**

c. Write an expression in factored form to represent the yellow region. **$6(a + b + 6)$**

46. **FIREWORKS** A ten-centimeter fireworks shell is fired from ground level. The height of the shell in meters is given by the formula $h = 80t - 4.9t^2$, where t is the time in seconds after launch.

a. Write the expression that represents the height in factored form. **$t(263 - 16t)$**

b. At what time will the height be 0? Is this answer practical? Explain.

c. What is the height of the shell 8 seconds and 10 seconds after being fired? **1080 m; 1030 m**

d. At 10 seconds, is the shell rising or falling? **The shell has begun to fall.**

46b. 0 and 16.4375 seconds; Yes, the shell starts at ground level and is in the air for 16.4375 seconds before landing on the ground again.

47. **ARCHITECTURE** The frame of a castle doorway is an arch that can be modeled by the graph of the equation $y = -3x^2 + 12x$, where x and y are measured in meters. On a coordinate plane, the floor is represented by the x-axis. **a–b. See margin.**

a. Make a table of values for the height of the arch if $x = 0, 1, 2, 3$, and 4 meters.

b. Plot the points from the table on a coordinate plane and connect the points to form a smooth curve to represent the arch.

c. How high is the doorway? **12 m**

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	15–45, 52, 54–80	15–45 odd, 57–60	16–44 even, 52, 54–56, 61–80
OL Core	15–45 odd, 46–52, 54–80	15–45, 57–60	46–52, 54–56, 61–80
BL Advanced	46–80		

48. **RIDES** Suppose the height of a rider after being dropped can be modeled by $h = -4.9t^2 - 29t + 48.8$, where h is the height in meters and t is time in seconds.

a. Write an expression to represent the height in factored form. **$16(-t^2 - 6t + 10)$**

b. From what height is the rider initially dropped? **160 m**

c. At what height will the rider be after 3 seconds of falling? Is this possible? Explain. **−272 m; No, the rider cannot be a negative number of meters in the air.**

49. **ARCHERY** The height h in meters of an arrow can be modeled by the equation $h = 64t - 16t^2$, where t is time in seconds. Ignoring the height of the archer, how long after the arrow is released does it hit the ground? **4 s**

50. **TENNIS** A tennis player hits a tennis ball upward with an initial velocity of 24.4 meters per second. The height h in meters of the tennis ball can be modeled by the equation $h = 24.4t - 16t^2$, where t is time in seconds. Ignoring the height of the tennis player, how long does it take the ball to hit the ground? **5 s**

51. **MULTIPLE REPRESENTATIONS** In this problem, you will explore the *box method* of factoring. To factor $x^2 + x - 6$, write the first term in the top left-hand corner of the box, and then write the last term in the lower right-hand corner.

	?	?
?	x^2	?
?	?	−6

a. **Analytical** Determine which two factors have a product of −6 and a sum of 1. **3 and −2**

b. **Symbolic** Write each factor in an empty square in the box. Include the positive or negative sign and variable. **See margin.**

c. **Analytical** Find the factor for each row and column of the box. What are the factors of $x^2 + x - 6$? **See margin.**

d. **Verbal** Describe how you would use the box method to factor $x^2 - 3x - 40$.

51d. Sample answer: Place x^2 in the top left-hand corner and place −40 in the lower right-hand corner. Then determine which two factors have a product of −40 and a sum of −3. Then place these factors in the box. Then find the factor of each row and column. The factors will be listed on the very top and far left of the box.

H.O.T. Problems Use Higher-Order Thinking Skills

52. **CRITIQUE** Faleh and Mansour are solving $2m^2 = 4m$. Is either of them correct? Explain your reasoning. **Mansour; the equation first must have 0 on one side.**

Faleh

$2m^2 = 4m$

$\frac{2m^2}{m} = \frac{4m^2}{2m}$

$2m = 2$

$m = 1$

Mansour

$2m^2 = 4m$

$2m^2 - 4m = 0$

$2m(m - 2) = 0$

$2m = 0$ or $m - 2 = 0$

$m = 0$ or 2

53. **CHALLENGE** Given the equation $(ax + b)(ax - b) = 0$, solve for x. What do we know about the values of a and b? **See margin.**

54. **OPEN ENDED** Write a four-term polynomial that can be factored by grouping. Then factor the polynomial. **Sample answers: $x^2 + 2xy + 3x + 6y$, $(x + 3)(x + 2y)$**

55. **REASONING** Given the equation $c = a^2 - ab$, for what values of a and b does $c = 0$? **Sample answer: $a = 0$ or $a = b$ for any real values of a and b.**

56. **WRITING IN MATH** Explain how to solve a quadratic equation by using the Zero Product Property. **See margin.**

Teaching the Mathematical Practices

Critique Mathematically proficient students compare the effectiveness of two plausible arguments. In Exercise 52, remind students that the quickest way to check a solution is to substitute the value back into the original equation.

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4 Assess

Yesterday's News Have students write how yesterday's concept of finding the GCF of a set of monomials helped them with today's new material.

Standardized Test Practice

57. Which is a factor of $6z^2 - 3z - 2 + 4z$? **D**

A $2z + 1$ C $z + 2$
B $3z - 2$ D $2z - 1$

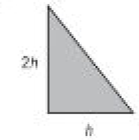
58. **PROBABILITY** Hala has 10 blocks: 2 red, 4 blue, 3 yellow, and 1 green. What is the probability that a randomly chosen block will be either red or yellow? **H**

F $\frac{3}{10}$ H $\frac{1}{2}$
G $\frac{1}{5}$ J $\frac{7}{10}$

59. **GRIDDED RESPONSE** Najla is making a 140-centimeter by 160-centimeter quilt with quilt squares that measure 8 centimeter on each side. How many will be needed to make the quilt? **350**

60. **GEOMETRY** The area of the right triangle shown below is $5h$ square centimeters. What is the height of the triangle? **D**

A 2 cm
B 5 cm
C 8 cm
D 10 cm



Spiral Review

61. **GENETICS** Brown genes B are dominant over blue genes b. A person with genes BB or Bb has brown eyes. Someone with genes bb has blue eyes. Manal has brown eyes with Bb genes, and Ali has blue eyes. Write an expression for the possible eye coloring of Manal and Ali's children. Determine the probability that their child would have blue eyes. (Lesson 7-4) **$0.5Bb + 0.5b^2$; $\frac{1}{2}$**

Find each product. (Lesson 7-2)

62. $n(n^2 - 4n + 3)$ **$n^3 - 4n^2 + 3n$**
63. $2b(b^2 + b - 5)$ **$2b^3 + 2b^2 - 10b$**
64. $-c(4c^2 + 2c - 2)$ **$-4c^3 - 2c^2 + 2c$**
65. $-4x(x^3 + x^2 + 2x - 1)$ **$-4x^4 - 4x^3 - 8x^2 + 4x$**
66. $2ab(4a^2b + 2ab - 2b^2)$ **$8a^3b^2 + 4a^2b^2 - 4ab^3$**
67. $-3xy(x^2 + xy + 2y^2)$ **$-3x^3y - 3x^2y^2 - 6xy^3$**

68. **CLASS TRIP** Mr. Mahmoud's History class will take taxis from their school in Dubai, to the Etihad Museum. The fare is AED 2.75 for the first kilometer and AED 1.25 for each additional kilometer. If the distance is m kilometers and t taxis are needed, write an expression for the cost to transport the group. (Lesson 7-2) **$2.75t + 1.25t(m - 1)$**

Find the degree of each polynomial. (Lesson 7-1)

69. 2 **0**
70. $-3a$ **1**
71. $5x^2 + 3x$ **2**
72. $d^4 - 6c^2$ **4**
73. $2x^3 - 4z + 8xz$ **3**
74. $3d^4 + 5d^3 - 4c^2 + 1$ **4**

Skills Review

Find each product.

75. $(a + 2)(a + 5)$ **$a^2 + 7a + 10$**
76. $(d + 4)(d + 10)$ **$d^2 + 14d + 40$**
77. $(z - 1)(z - 8)$ **$z^2 - 9z + 8$**
78. $(c + 9)(c - 3)$ **$c^2 + 6c - 27$**
79. $(x - 7)(x - 6)$ **$x^2 - 13x + 42$**
80. $(g - 2)(g + 11)$ **$g^2 + 9g - 22$**

Differentiated Instruction BL

Extension Write the following polynomial on the board: $c^2xy - c^3 - x^2y + cx$. Ask students to factor it by grouping.
$(c^2 - x)(xy - c)$, or $(x - c^2)(c - xy)$

428 | Lesson 7-5 | Using the Distributive Property

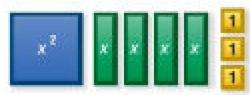
EXPLORE 7-6 Algebra Lab
Factoring Trinomials

You can use algebra tiles to factor trinomials. If a polynomial represents the area of a rectangle formed by algebra tiles, then the rectangle's length and width are *factors* of the area. If a rectangle cannot be formed to represent the trinomial, then the trinomial is not factorable.

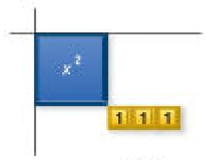
Activity 1 Factor $x^2 + bx + c$

Use algebra tiles to factor $x^2 + 4x + 3$.

Step 1 Model $x^2 + 4x + 3$.



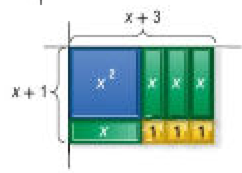
Step 2 Place the x^2-tile at the corner of the product mat. Arrange the 1-tiles into a rectangular array. Because 3 is prime, the 3 tiles can be arranged in a rectangle in one way, a 1-by-3 rectangle.



Step 3 Complete the rectangle with the x-tiles.

The rectangle has a width of $x + 1$ and a length of $x + 3$.

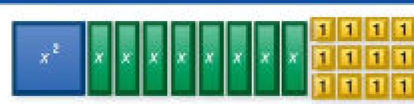
Therefore, $x^2 + 4x + 3 = (x + 1)(x + 3)$.



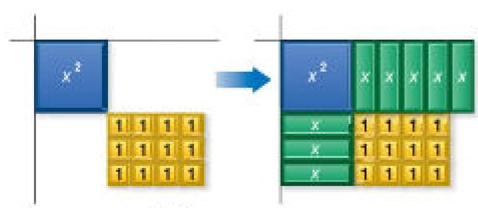
Activity 2 Factor $x^2 + bx + c$

Use algebra tiles to factor $x^2 + 8x + 12$.

Step 1 Model $x^2 + 8x + 12$.



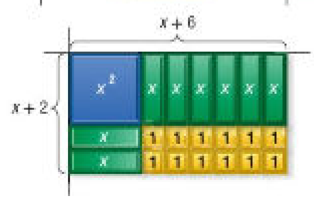
Step 2 Place the x^2-tile at the corner of the product mat. Arrange the 1-tiles into a rectangular array. Since $12 = 3 \times 4$, try a 3-by-4 rectangle. Try to complete the rectangle. Notice that there is an extra x-tile.



Step 3 Arrange the 1-tiles into a 2-by-6 rectangular array. This time you can complete the rectangle with the x-tiles.

The rectangle has a width of $x + 2$ and a length of $x + 6$.

Therefore, $x^2 + 8x + 12 = (x + 2)(x + 6)$.



1 Focus

Objective Use algebra tiles to model factoring trinomials.

Materials for Each Group

- algebra tiles
- product mat

Teaching Tip

You may want to remind students that the area of the rectangle represents the polynomial, and the length and width of the rectangle represent the factors of the polynomial.

2 Teach

Working in Cooperative Groups

Place students in groups of two or three, mixing abilities. Have groups complete Activities 1–4.

- Ask students to name the shape they must form with the tiles in order to factor a polynomial. rectangle
- In Activity 1, remind students to read the width of the tiles along the edge of the rectangle. The x^2-tiles have a width of x and the x-tiles have a width of one.
- For Activity 2, encourage students to try several different arrangements until they can form a rectangle. While the x^2-tile should be in the corner, there is more than one correct way to arrange the tiles into a rectangle.

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- As students work through Activity 3, remind them to pay close attention to the sign of each tile.
- As students work through Activity 4, remind them to be careful to add one x-tile and one $-x$-tile when they add a zero pair.

Practice Have students complete Exercises 5–13.

3 Assess

Formative Assessment

Use Exercises 7 and 8 to assess whether students can use algebra tiles to factor trinomials.

From Concrete to Abstract

After students complete Exercises 1–8, ask them whether they noticed a correlation between the need to use zero pairs to factor the trinomial and the appearance of the resulting factors. Sample answer: When zero pairs are used, the signs of the constant terms of the factors are opposite. When zero pairs are not used, the signs of the constant terms of factors are the same.

Extending the Concept

Ask students what they notice about the sum of the constant terms in the factors of the trinomials in Exercises 1–8. Sample answer: Their sum equals the coefficient of the middle term of the trinomial.

Additional Answers

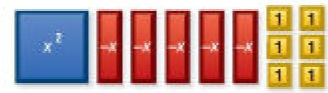
1. $(x + 1)(x + 2)$
2. $(x + 2)(x + 4)$
3. $(x - 1)(x + 4)$
4. $(x - 3)(x - 4)$
5. $(x + 2)(x + 5)$
6. $(x - 1)(x - 1)$
7. $(x + 4)(x - 3)$
8. $(x - 3)(x - 5)$

Algebra Lab

Factoring Trinomials *Continued*

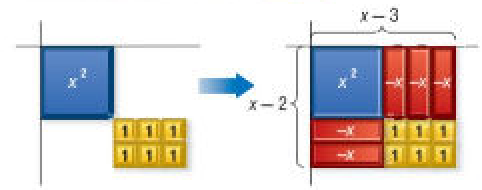
Activity 3 Factor $x^2 - bx + c$

Use algebra tiles to factor $x^2 - 5x + 6$.



Step 1 Model $x^2 - 5x + 6$.

Step 2 Place the x^2-tile at the corner of the product mat. Arrange the 1-tiles into a 2-by-3 rectangular array as shown.

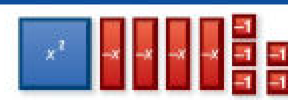


Step 3 Complete the rectangle with the x-tiles. The rectangle has a width of $x - 2$ and a length of $x - 3$.

Therefore, $x^2 - 5x + 6 = (x - 2)(x - 3)$.

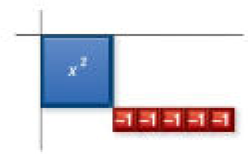
Activity 4 Factor $x^2 - bx - c$

Use algebra tiles to factor $x^2 - 4x - 5$.



Step 1 Model $x^2 - 4x - 5$.

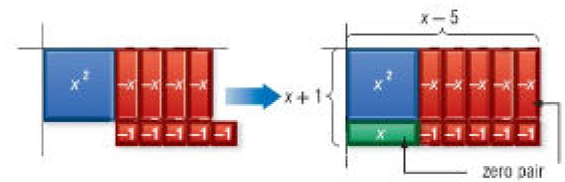
Step 2 Place the x^2-tile at the corner of the product mat. Arrange the 1-tiles into a 1-by-5 rectangular array as shown.



Step 3 Place the x-tile as shown. Recall that you can add zero pairs without changing the value of the polynomial. In this case, add a zero pair of x-tiles.

The rectangle has a width of $x + 1$ and a length of $x - 5$.

Therefore, $x^2 - 4x - 5 = (x + 1)(x - 5)$.



Model and Analyze

13. Trinomials can be factored if they can be represented by a rectangle. Sample answers: $x^2 + 4x + 4$ can be factored, and $x^2 + 6x + 4$ cannot be factored.

Use algebra tiles to factor each trinomial. 1–8. See margin.

1. $x^2 + 3x + 2$
2. $x^2 + 6x + 8$
3. $x^2 + 3x - 4$
4. $x^2 - 7x + 12$
5. $x^2 + 7x + 10$
6. $x^2 - 2x + 1$
7. $x^2 + x - 12$
8. $x^2 - 8x + 15$

Tell whether each trinomial can be factored. Justify your answer with a drawing.

9. $x^2 + 3x + 6$ no
10. $x^2 - 5x - 6$ yes
11. $x^2 - x - 4$ no
12. $x^2 - 4$ yes

9–12. See Ch. 8 Answer Appendix for drawings.

13. **WRITING IN MATH** How can you use algebra tiles to determine whether a trinomial can be factored?

Teaching the Mathematical Practices

Tools Mathematically proficient students consider the available tools when solving a mathematical problem. Encourage students to use algebra tiles as they progress in the chapter. Provide tiles or paper tiles that they can keep in their notebooks so they are always available.

LESSON 7-6 Solving $x^2 + bx + c = 0$

Then

- You multiplied binomials by using the FOIL method.

Now

1. Factor trinomials of the form $x^2 + bx + c$.
2. Solve equations of the form $x^2 + bx + c = 0$.

Why?

- Huda is having a rectangular in-ground swimming pool installed and she wants to include a 24-meter fence around the pool. The pool requires a space of 36 square meters. What dimensions should the pool have?

To solve this problem, the landscape architect needs to find two numbers that have a product of 36 and a sum of 12, half the perimeter of the pool.

NewVocabulary
quadratic equation

Mathematical Practices
Look for and make use of structure.
Look for and express regularity in repeated reasoning.

1 Factor $x^2 + bx + c$

You have learned how to multiply two binomials using the FOIL method. Each of the binomials was a factor of the product. The pattern for multiplying two binomials can be used to factor certain types of trinomials.

$$(x + 3)(x + 4) = x^2 + 4x + 3x + 3 \cdot 4 \quad \text{Use the FOIL method.}$$
$$= x^2 + (4 + 3)x + 3 \cdot 4 \quad \text{Distributive Property}$$
$$= x^2 + 7x + 12 \quad \text{Simplify.}$$

Notice that the coefficient of the middle term, $7x$, is the sum of 3 and 4, and the last term, 12, is the product of 3 and 4.

Observe the following pattern in this multiplication.

$$(x + 3)(x + 4) = x^2 + (4 + 3)x + (3 \cdot 4)$$
$$(x + m)(x + p) = x^2 + (p + m)x + mp \quad \text{Let } 3 = m \text{ and } 4 = p.$$
$$= x^2 + (m + p)x + mp \quad \text{Commutative (+)}$$
$$x^2 + bx + c \quad b = m + p \text{ and } c = mp$$

Notice that the coefficient of the middle term is the sum of m and p, and the last term is the product of m and p. This pattern can be used to factor trinomials of the form $x^2 + bx + c$.

KeyConcept Factoring $x^2 + bx + c$

Words	To factor trinomials in the form $x^2 + bx + c$, find two integers, m and p, with a sum of b and a product of c. Then write $x^2 + bx + c$ as $(x + m)(x + p)$.
Symbols	$x^2 + bx + c = (x + m)(x + p)$ when $m + p = b$ and $mp = c$.
Example	$x^2 + 6x + 8 = (x + 2)(x + 4)$, because $2 + 4 = 6$ and $2 \cdot 4 = 8$.

When c is positive, its factors have the same signs. Both of the factors are positive or negative based upon the sign of b. If b is positive, the factors are positive. If b is negative, the factors are negative.

1 Focus

VerticalAlignment

Before Lesson 7-6 Multiply binomials by using the FOIL method.

Lesson 7-6 Factor trinomials of the form $x^2 + bx + c$. Solve equations of the form $x^2 + bx + c = 0$.

After Lesson 7-6 Factor trinomials into two binomials.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- Why do you need to find two numbers that have a product of 36 to find the dimensions of the pool? The pool is a rectangle, so the area is equal to the length times the width. Since the pool's area is 36 m^2, the length and width must be two numbers with a product of 36.
- What pairs of integers have a product of 36? 1 and 36; 2 and 18; 3 and 12; 4 and 9; 6 and 6
- Which pair has a sum of 12? 6 and 6
- What are the dimensions of the pool? 6 m by 6 m

431

1 Factor $x^2 + bx + c$

Examples 1–3 show how to factor trinomials of the form $x^2 + bx + c$, when b and c are positive, when b is negative and c is positive, and when c is negative.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Factor $x^2 + 7x + 12$. $(x + 3)(x + 4)$

2 Factor $x^2 - 12x + 27$. $(x - 3)(x - 9)$

WatchOut!

Preventing Errors If students use a graphing calculator to check their factoring, make sure they clear all other functions from the [Y=] list and clear all other drawings from the draw menu. Caution students that while two graphs may appear to coincide in the standard viewing window, they may not. Suggest that students use the [TABLE] feature to verify identical y-values.

Tips for New Teachers

Calculators Have students change the line type on the graph of the second equation so they can see it as it is graphed on the first graph.

Problem-Solving Tip

Guess and Check When factoring a trinomial, make an educated guess, check for reasonableness, and then adjust the guess until the correct answer is found.

Example 1 b and c are Positive

Factor $x^2 + 9x + 20$.

In this trinomial, $b = 9$ and $c = 20$. Since c is positive and b is positive, you need to find two positive factors with a sum of 9 and a product of 20. Make an organized list of the factors of 20, and look for the pair of factors with a sum of 9.

Factors of 20	Sum of Factors
1, 20	21
2, 10	12
4, 5	9

The correct factors are 4 and 5.

$x^2 + 9x + 20 = (x + m)(x + p)$ — Write the pattern.

$= (x + 4)(x + 5)$ — $m = 4$ and $p = 5$

CHECK You can check this result by multiplying the two factors. The product should be equal to the original expression.

$(x + 4)(x + 5) = x^2 + 5x + 4x + 20$ — FOIL Method

$= x^2 + 9x + 20$ ✓ — Simplify.

Guided Practice

Factor each polynomial.

1A. $d^2 + 11x + 24$ $(d + 3)(d + 8)$ **1B.** $9 + 10t + t^2$ $(t + 9)(t + 1)$

When factoring a trinomial in which b is negative and c is positive, use what you know about the product of binomials to narrow the list of possible factors.

Study Tip

Regularity Once the correct factors are found, it is not necessary to test any other factors. In Example 2, −2 and −6 are the correct factors, so −3 and −4 do not need to be tested.

Example 2 b is Negative and c is Positive

Factor $x^2 - 8x + 12$. Confirm your answer using a graphing calculator.

In this trinomial, $b = -8$ and $c = 12$. Since c is positive and b is negative, you need to find two negative factors with a sum of −8 and a product of 12.

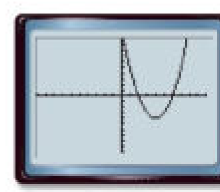
Factors of 12	Sum of Factors
−1, −12	−13
−2, −6	−8
−3, −4	−7

The correct factors are −2 and −6.

$x^2 - 8x + 12 = (x + m)(x + p)$ — Write the pattern.

$= (x - 2)(x - 6)$ — $m = -2$ and $p = -6$

CHECK Graph $y = x^2 - 8x + 12$ and $y = (x - 2)(x - 6)$ on the same screen. Since only one graph appears, the two graphs must coincide. Therefore, the trinomial has been factored correctly. ✓



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Guided Practice

Factor each polynomial.

2A. $21 - 22m + m^2$ $(m - 1)(m - 21)$ **2B.** $w^2 - 11w + 28$ $(w - 7)(w - 4)$

Differentiated Instruction AL OL

If the concept of factoring trinomials seems somewhat abstract to some students,

Then whenever you introduce abstract concepts, it is a good idea to reinforce them with concrete examples. After introducing factoring trinomials, refer students to the lesson-opener problem. Ask students to describe any similarities they notice between finding the dimensions of the pool and factoring a trinomial. Some students may benefit from modeling some problems using algebra tiles. They can then use this method to solve quadratic equations.

432 | Lesson 7-6 | Solving $x^2 + bx + c = 0$

ReviewVocabulary

absolute value the distance a number is from zero on a number line, written $|n|$

When c is negative, its factors have opposite signs. To determine which factor is positive and which is negative, look at the sign of b. The factor with the greater absolute value has the same sign as b.

Example 3 c is Negative

Factor each polynomial. Confirm your answers using a graphing calculator.

a. $x^2 + 2x - 15$

In this trinomial, $b = 2$ and $c = -15$. Since c is negative, the factors m and p have opposite signs. So either m or p is negative, but not both. Since b is positive, the factor with the greater absolute value is also positive.

List the factors of -15, where one factor of each pair is negative. Look for the pair of factors with a sum of 2.

Factors of −15	Sum of Factors
−1, 15	14
−3, 5	2

The correct factors are −3 and 5.

$x^2 + 2x - 15 = (x + m)(x + p)$ — Write the pattern.

$= (x - 3)(x + 5)$ — $m = -3$ and $p = 5$

CHECK $(x - 3)(x + 5) = x^2 + 5x - 3x - 15$ — FOIL Method

$= x^2 + 2x - 15$ ✓ — Simplify.

b. $x^2 - 7x - 18$

In this trinomial, $b = -7$ and $c = -18$. Either m or p is negative, but not both. Since b is negative, the factor with the greater absolute value is also negative.

List the factors of -18, where one factor of each pair is negative. Look for the pair of factors with a sum of -7.

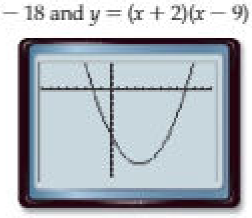
Factors of −18	Sum of Factors
1, −18	−17
2, −9	−7
3, −6	−3

The correct factors are 2 and −9.

$x^2 - 7x - 18 = (x + m)(x + p)$ — Write the pattern.

$= (x + 2)(x - 9)$ — $m = 2$ and $p = -9$

CHECK Graph $y = x^2 - 7x - 18$ and $y = (x + 2)(x - 9)$ on the same screen.



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The graphs coincide. Therefore, the trinomial has been factored correctly. ✓

GuidedPractice

3A. $y^2 + 13y - 48$ $(y - 3)(y + 16)$ **3B.** $r^2 - 2r - 24$ $(r + 4)(r - 6)$

Additional Example

3 Factor each polynomial.

a. $x^2 + 3x - 18$
$(x + 6)(x - 3)$

b. $x^2 - x - 20$
$(x - 5)(x + 4)$

Focus on Mathematical Content

Factoring Trinomials A trinomial of the form $x^2 + bx + c$ may or may not be factorable into binomial factors. If the trinomial is factorable, then the factors of c must be two integers, m and p, such that $m + p = b$ and $mp = c$.

Teach with Tech

Web Page Develop a class Web page about how to factor different types of trinomials. Update the page with information from the most recent lesson, such as notes from class, video clips, and additional resources.

Teaching the Mathematical Practices

Regularity Mathematically proficient students maintain oversight of the process, while attending to the details of solving a problem. Stress that exactly one pair of factors correctly factors a trinomial of the form $x^2 + bx + c$ that can be factored.

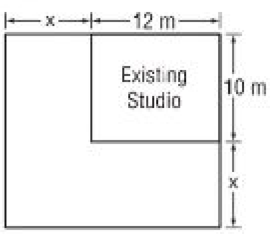
2 Solve Equations by Factoring

Example 4 shows how to solve an equation of the form $x^2 + bx + c = 0$ by factoring. **Example 5** shows how to solve a real-world problem by factoring.

Additional Examples

4 Solve $x^2 + 2x = 15$. Check your solutions. $-5, 3$

5 **ARCHITECTURE** Maha wants to build a new art studio that has three times the area of her old studio by increasing the length and width of the old studio by the same amount. What should be the dimensions of the new studio?



The dimensions of the new studio should be 18 m by 20 m.

WatchOut!

Common Errors Students often are not careful when rewriting equations so that one side equals zero. Remind them that they must perform the same operation on both sides of the equation and pay attention to the signs in the resulting equation.

2 Solve Equations by Factoring

A **quadratic equation** can be written in the standard form $ax^2 + bx + c = 0$, where $a \neq 0$. Some equations of the form $x^2 + bx + c = 0$ can be solved by factoring and then using the Zero Product Property.

StudyTip

Solving an Equation By Factoring Remember to get 0 on one side of the equation before factoring.

Example 4 Solve an Equation by Factoring

Solve $x^2 + 6x = 27$. Check your solutions.

$x^2 + 6x = 27$	Original equation
$x^2 + 6x - 27 = 0$	Subtract 27 from each side.
$(x - 3)(x + 9) = 0$	Factor.
$x - 3 = 0$ or $x + 9 = 0$	Zero Product Property
$x = 3$ $\quad$ $x = -9$	Solve each equation.

The roots are 3 and −9.

CHECK Substitute 3 and −9 for x in the original equation.

$$x^2 + 6x = 27 \qquad x^2 + 6x = 27$$
$$(3)^2 + 6(3) \stackrel{?}{=} 27 \qquad (-9)^2 + 6(-9) \stackrel{?}{=} 27$$
$$9 + 18 \stackrel{?}{=} 27 \qquad 81 - 54 \stackrel{?}{=} 27$$
$$27 = 27 \checkmark \qquad 27 = 27 \checkmark$$

GuidedPractice

Solve each equation. Check your solutions.

4A. $z^2 - 3z = 70$ $-7, 10$ $\qquad$ **4B.** $x^2 + 3x - 18 = 0$ $3, -6$

Factoring can be useful when solving real-world problems.

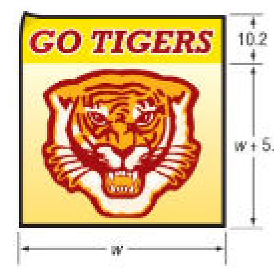
Real-WorldLink

A company that produces event signs recommends foamcore boards for event signs that will be used only once. For signs used more than once, use a stronger type of foamcore board.

Source: MegaPrint Inc.

Real-World Example 5 Solve a Problem by Factoring

DESIGN **Najat is designing a poster. The top of the poster is 10.2 centimeters long and the rest of the poster is 5.1 centimeters longer than the width. If the poster requires 3974.2 square centimeters of poster board, find the width w of the poster.**



Understand You want to find the width of the poster.

Plan Since the poster is a rectangle, width • length = area.

Solve Let w = the width of the poster.
The length is $w + 5.1 + 10.2$ or $w + 15.3$.

$w(w + 15.3) = 3974.2$	Write the equation.
$w^2 + 15.3w = 3974.2$	Multiply.
$w^2 + 15.3w - 3974.2 = 0$	Subtract 616 from each side.
$(w + 71.1)(w - 56) = 0$	Factor.
$w + 71.1 = 0$ or $w - 56 = 0$	Zero Product Property
$w = -71.1$ $\quad$ $w = 56$	Solve each equation.

Since dimensions cannot be negative, the width is 56 centimeters.

Check If the width is 56 centimeters, then the area of the poster is $56 \cdot (56 + 15.3)$ or 3974.2 square centimeters, which is the amount the poster requires. ✓

GuidedPractice

5. **GEOMETRY** The height of a parallelogram is 18 centimeters less than its base. If the area is 175 square centimeters, what is its height? 7 cm

DifferentiatedInstruction OL BL

Extension Write the trinomials $x^2 + x - 6$ and $x^2 - x - 6$ on the board. Ask students to compare the two trinomials. How are the trinomials related? How are their factors related? The trinomials are the same except for the sign of the middle term. When factored, $x^2 + x - 6$ is $(x + 3)(x - 2)$, while $x^2 - x - 6$ is $(x - 3)(x + 2)$. The factors have opposite signs in the constant term.

Check Your Understanding

Examples 1–3 Factor each polynomial. Confirm your answers using a graphing calculator.

1. $x^2 + 14x + 24$ **$(x + 2)(x + 12)$**
2. $y^2 - 7y - 30$ **$(y - 10)(y + 3)$**
3. $n^2 + 4n - 21$ **$(n + 7)(n - 3)$**
4. $m^2 - 15m + 50$ **$(m - 5)(m - 10)$**

Example 4 Solve each equation. Check your solutions.

5. $x^2 - 4x - 21 = 0$ **$-3, 7$**
6. $n^2 - 3n + 2 = 0$ **$1, 2$**
7. $x^2 - 15x + 54 = 0$ **$6, 9$**
8. $x^2 + 12x = -32$ **$-4, -8$**
9. $x^2 - x - 72 = 0$ **$-8, 9$**
10. $x^2 - 10x = -24$ **$4, 6$**

Example 5

11. **FRAMING** Lamis bought a frame for a photo, but the photo is too big for the frame. Lamis needs to reduce the width and length of the photo by the same amount. The area of the photo should be reduced to half the original area. If the original photo is 12 centimeters by 16 centimeters, what will be the dimensions of the smaller photo? **8 cm by 12 cm**

Practice and Problem Solving

Examples 1–3 Factor each polynomial. Confirm your answers using a graphing calculator.

12. $x^2 + 17x + 42$ **$(x + 3)(x + 14)$**
13. $y^2 - 17y + 72$ **$(y - 9)(y - 8)$**
14. $a^2 + 8a - 48$ **$(a - 4)(a + 12)$**
15. $n^2 - 2n - 35$ **$(n - 7)(n + 5)$**
16. $44 + 15h + h^2$ **$(h + 4)(h + 11)$**
17. $40 - 22x + x^2$ **$(x - 2)(x - 20)$**
18. $-24 - 10x + x^2$ **$(x + 2)(x - 12)$**
19. $-42 - m + m^2$ **$(m + 6)(m - 7)$**

Example 4 Solve each equation. Check your solutions.

20. $x^2 - 7x + 12 = 0$ **$3, 4$**
21. $y^2 + y = 20$ **$4, -5$**
22. $x^2 - 6x = 27$ **$-3, 9$**
23. $a^2 + 11a = -18$ **$-2, -9$**
24. $c^2 + 10c + 9 = 0$ **$-1, -9$**
25. $x^2 - 18x = -32$ **$2, 16$**
26. $n^2 - 120 = 7n$ **$-8, 15$**
27. $d^2 + 56 = -18d$ **$-4, -14$**
28. $y^2 - 90 = 13y$ **$-5, 18$**
29. $h^2 + 48 = 16h$ **$4, 12$**

Example 5

30. **GEOMETRY** A triangle has an area of 36 square meters. If the height of the triangle is 6 meters more than its base, what are its height and base? **12 m, 6 m**

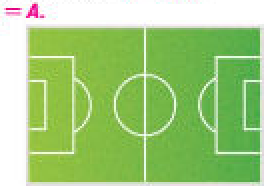
31. **GEOMETRY** A rectangle has an area represented by $x^2 - 4x - 12$ square meters. If the length is $x + 2$ meters, what is the width of the rectangle? **$(x - 6)$ m**

32a. Let ℓ = length, A = area of the field, $\ell(\ell - 45) = A$.

B 32. **FOOTBALL** The width of a high school football field is 41.2 meters shorter than its length.

a. Define a variable, and write an expression for the area of the field.

b. The area of the field is 7525.2 square meters. Find the dimensions. **68.6 m by 109.8 m**



STRUCTURE Factor each polynomial.

33. $q^2 + 11qr + 18r^2$ **$(q + 2r)(q + 9r)$**
34. $x^2 - 14xy - 51y^2$ **$(x + 3y)(x - 17y)$**
35. $x^2 - 6xy + 5y^2$ **$(x - y)(x - 5y)$**
36. $a^2 + 10ab - 39b^2$ **$(a + 13b)(a - 3b)$**

3 Practice

Formative Assessment

Use Exercises 1–11 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

WatchOut!

Student Misconceptions For Exercises 1–4, students may need to be reminded that the order in which they record the factors does not matter. So, $(x + m)(x + p)$ and $(x + p)(x + m)$ are both correct.

Exercise Alert

Formula For Exercise 30, students will need to know that the formula for the area of a triangle is $A = \frac{1}{2}bh$.

Teaching the Mathematical Practices

Structure Mathematically proficient students look closely to discern a pattern or structure. In Exercises 33–36, point out that the same patterns that apply to polynomials that contain numbers apply to variables.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	12–31, 41, 46, 47, 49–66	13–31 odd, 50–53	12–30 even, 41, 46, 47, 49, 54–66
OL Core	13–31 odd, 32, 33, 35, 37, 39–41, 46, 47, 49–66	12–31, 50–53	32–41, 46, 47, 49, 54–66
BL Advanced	32–66		

Multiple Representations

In Exercise 40, students use information organized in a table and algebra to factor polynomials.

37. **SWIMMING** The length of a rectangular reservoir is 20 meters greater than its width. The area of the reservoir is 525 square meters.

a. Define a variable and write an equation for the area of the reservoir. Sample answer: Let w = width; $(w + 20)w = 525$.

b. Solve the equation. $-35, 15$

c. Interpret the solutions. Do both solutions make sense? Explain.

37c. The solution of 15 means that the width is 15 m. The solution −35 does not make sense because length cannot be negative.

GEOMETRY **Find an expression for the perimeter of a rectangle with the given area.**

38. $A = x^2 + 24x - 81$ $4x + 48$

39. $A = x^2 + 13x - 90$ $4x + 26$

40. **MULTIPLE REPRESENTATIONS** In this problem, you will explore factoring when the leading coefficient is not 1.

a. **Tabular** Copy and complete the table below.

Product of Two Binomials	$ax^2 + mx + px + c$	$ax^2 + bx + c$	$m \times p$	$a \times c$
$(2x + 3)(x + 4)$	$2x^2 + 8x + 3x + 12$	$2x^2 + 11x + 12$	24	24
$(x + 1)(3x + 5)$	$3x^2 + 5x + 3x + 5$	$3x^2 + 8x + 5$	15	15
$(2x - 1)(4x + 1)$	$8x^2 + 2x - 4x - 1$	$8x^2 - 2x - 1$	−8	−8
$(3x + 5)(4x - 2)$	$12x^2 - 6x + 20x - 10$	$12x^2 + 14x - 10$	−120	−120

b. **Analytical** How are m and p related to a and c? $mp = ac$

c. **Analytical** How are m and p related to b? $m + p = b$

d. **Verbal** Describe a process you can use for factoring a polynomial of the form $ax^2 + bx + c$. Look for two integers, m and p, for which $mp = ac$ and $m + p = b$.

41. Abdalla; Abdulaziz answer once multiplied is $x^2 - 6x - 16$. The middle term should be positive.

H.O.T. Problems Use Higher-Order Thinking Skills

41. **ERROR ANALYSIS** Abdulaziz and Abdalla have factored $x^2 + 6x - 16$. Is either of them correct? Explain your reasoning.

Abdulaziz	Abdalla
$x^2 + 6x - 16 = (x + 2)(x - 8)$	$x^2 + 6x - 16 = (x - 2)(x + 8)$

48. $(4y - 5)^2 + 3(4y - 5) - 70 = [(4y - 5) + 10] \cdot [(4y - 5) - 7] = (4y + 5)(4y - 12) = 4(4y + 5)(y - 3)$

ARGUMENTS **Find all values of k so that each polynomial can be factored using integers.**

42. $x^2 + kx - 19$ $-18, 18$

43. $x^2 + kx + 14$ $-15, -9, 9, 15$

44. $x^2 - 8x + k, k > 0$ $7, 12, 15, 16$

45. $x^2 - 5x + k, k > 0$ $4, 6$

46. **REASONING** For any factorable trinomial, $x^2 + bx + c$, will the absolute value of b *sometimes*, *always*, or *never* be less than the absolute value of c? Explain. See margin.

47. **OPEN ENDED** Give an example of a trinomial that can be factored using the factoring techniques presented in this lesson. Then factor the trinomial. Sample answer: $x^2 + 19x - 20$; $(x - 1)(x + 20)$

48. **CHALLENGE** Factor $(4y - 5)^2 + 3(4y - 5) - 70$.

49. **WRITING IN MATH** Explain how to factor trinomials of the form $x^2 + bx + c$ and how to determine the signs of the factors of c. See margin.

Standardized Test Practice

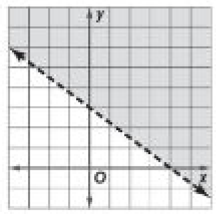
50. Which inequality is shown in the graph below? **C**

A $y \leq -\frac{3}{4}x + 3$

B $y < -\frac{3}{4}x + 3$

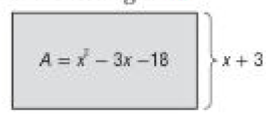
C $y > -\frac{3}{4}x + 3$

D $y \geq -\frac{3}{4}x + 3$



51. SHORT RESPONSE Amani must earn more than AED 254 from selling candy bars in order to go on a trip with the Community Society. If each candy bar is sold for AED 1.25, what is the fewest candy bars she must sell? **204**

52. GEOMETRY Which expression represents the length of the rectangle? **H**



F $x + 5$

G $x + 6$

H $x - 6$

J $x - 5$

53. The difference of 21 and a number n is 6. Which equation shows the relationship? **A**

A $21 - n = 6$ C $21n = 6$

B $21 + n = 6$ D $6n = -21$

Spiral Review

Factor each polynomial. (Lesson 7-5)

54. $10a^2 + 40a$ **$10a(a + 4)$**

55. $11x + 44x^2y$ **$11x(1 + 4xy)$**

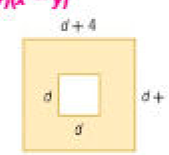
56. $2m^3p^2 - 16mp^2 + 8mp$ **$2mp(m^2p - 8p + 4)$**

57. $2ax + 6xc + ba + 3bc$ **$(2x + b)(a + 3c)$**

58. $8ac - 2ad + 4bc - bd$ **$(2a + b)(4c - d)$**

59. $x^2 - xy - xy + y^2$ **$(x - y)(x - y)$**

60. Write a polynomial that represents the area of the shaded region in the figure at the right. (Lesson 7-4) **$8d + 16$**



Refer to the conversion charts inside the back cover of your textbook and in Lesson 0-2. (Lesson 0-2)

61. RUNNING Alia is participating in a 5-kilometer charity run next weekend. About how many kilometers is the race? **about 3 km**

62. NATURE An African elephant weighs about 9 tons. About how many kilograms is this? **about 9000 kg**

63. SPORTS A football field is 100 yards long from one end zone to the other. How many meters long is a football field? **91.44 m**

Skills Review

Factor each polynomial. **64. $(2m + r)(3x - 2)$** **65. $(3x - 4)(a - 2b)$** **66. $2(d^2 + f)(g + 2h)$**

64. $6mx - 4m + 3rx - 2r$

65. $3ax - 6bx + 8b - 4a$

66. $2d^2g + 2fg + 4d^2h + 4fh$

WatchOut!

Error Analysis For Exercise 41, remind students that they can check the correctness of the factors by multiplying them to see if the result is the original polynomial. They could also use their graphing calculators for the procedure in Example 2.

Teaching the Mathematical Practices

Arguments Mathematically proficient students make conjectures and build a logical progression of statements to explore the truth of their conjectures. In Exercises 42–45, tell students to start by thinking about how they would factor each polynomial if k is given. How is k related to the other constants in the polynomial?

4 Assess

Crystal Ball Tell students that the next lesson they will study is titled *Quadratic Equations*: $ax^2 + bx + c = 0$. Ask them to write how they think what they learned today will connect with the next lesson.

Additional Answers

46. Sometimes; Sample answer: The trinomial $x^2 + 10x + 9 = (x + 1)(x + 9)$ and $10 > 9$. The trinomial $x^2 + 7x + 10 = (x + 2)(x + 5)$ and $7 < 10$.

49. Sample answer: Find factors m and n such that $m + n = b$ and $mn = c$. If b and c are positive, then m and n are positive. If b is negative and c is positive, then m and n are negative. When c is negative, m and n have different signs and the factor with the greatest absolute value has the same sign as b.

1 Focus

VerticalAlignment

Before Lesson 7-7 Factor trinomials of the form $x^2 + bx + c$.

Lesson 7-7 Factor trinomials of the form $ax^2 + bx + c$. Solve equations of the form $ax^2 + bx + c = 0$.

After Lesson 7-7 Factor binomials that are the difference of squares.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- How is the trinomial $16t^2 - 5t + 120$ different from those you learned how to factor in Lesson 1-6? The coefficient of the t^2 term of the trinomial is an integer greater than 1, while the coefficients of the x^2 terms of the trinomials in Lesson 1-6 were always 1.
- What trinomial does the product of $(3x + 1)(2x + 5)$ give? $6x^2 + 17x + 5$
- How does the coefficient of the x^2 term relate to the coefficients of the x term in each factor? The coefficient of the x^2 term equals their product. $3 \times 2 = 6$

LESSON 7-7 Solving $ax^2 + bx + c = 0$

Then

- You factored trinomials of the form $x^2 + bx + c$.

Now

1. Factor trinomials of the form $ax^2 + bx + c$.
2. Solve equations of the form $ax^2 + bx + c = 0$.

Why?

- The path of a rider on the amusement park ride shown at the right can be modeled by $16t^2 - 5t + 120$.

Factoring this expression can help the ride operators determine how long a rider rides on the initial swing.

NewVocabulary
prime polynomial

Mathematical Practices
Model with mathematics.

1 Factor $ax^2 + bx + c$ In the last lesson, you factored quadratic expressions of the form $ax^2 + bx + c$, where $a = 1$. In this lesson, you will apply the factoring methods to quadratic expressions in which a is not 1.

The dimensions of the rectangle formed by the algebra tiles are the factors of $2x^2 + 5x + 3$. The factors of $2x^2 + 5x + 3$ are $x + 1$ and $2x + 3$.

You can also use the method of factoring by grouping to solve this expression.

Step 1 Apply the pattern: $2x^2 + 5x + 3 = 2x^2 + mx + px + 3$.

Step 2 Find two numbers that have a product of $2 \cdot 3$ or 6 and a sum of 5.

Factors of 6	Sum of Factors
1, 6	7
2, 3	5

Step 3 Use grouping to find the factors.

$2x^2 + 5x + 3 = 2x^2 + mx + px + 3$ Write the pattern.
$= 2x^2 + 2x + 3x + 3$ $m = 2$ and $p = 3$
$= (2x^2 + 2x) + (3x + 3)$ Group terms with common factors.
$= 2x(x + 1) + 3(x + 1)$ Factor the GCF.
$= (2x + 3)(x + 1)$ $x + 1$ is the common factor.

Therefore, $2x^2 + 5x + 3 = (2x + 3)(x + 1)$.

KeyConcept Factoring $ax^2 + bx + c$

Words To factor trinomials of the form $ax^2 + bx + c$, find two integers, m and p, with a sum of b and a product of ac. Then write $ax^2 + bx + c$ as $ax^2 + mx + px + c$, and factor by grouping.

Example $5x^2 - 13x + 6 = 5x^2 - 10x - 3x + 6$ $m = -10$ and $p = -3$
$= 5x(x - 2) + (-3)(x - 2)$
$= (5x - 3)(x - 2)$

Example 1 Factor $ax^2 + bx + c$

Factor each trinomial.

a. $7x^2 + 29x + 4$

In this trinomial, $a = 7$, $b = 29$, and $c = 4$. You need to find two numbers with a sum of 29 and a product of $7 \cdot 4$ or 28. Make a list of the factors of 28 and look for the pair of factors with the sum of 29.

Factors of 28	Sum of Factors	
1, 28	29	The correct factors are 1 and 28.

$7x^2 + 29x + 4 = 7x^2 + mx + px + 4$ — Write the pattern.

$= 7x^2 + 1x + 28x + 4$ — $m = 1$ and $p = 28$

$= (7x^2 + 1x) + (28x + 4)$ — Group terms with common factors.

$= x(7x + 1) + 4(7x + 1)$ — Factor the GCF.

$= (x + 4)(7x + 1)$ — $7x + 1$ is the common factor.

b. $3x^2 + 15x + 18$

The GCF of the terms $3x^2$, $15x$, and 18 is 3. Factor this first.

$3x^2 + 15x + 18 = 3(x^2 + 5x + 6)$ — Distributive Property

$= 3(x + 3)(x + 2)$ — Find two factors of 6 with a sum of 5.

StudyTip

Greatest Common Factor Always look for a GCF of the terms of a polynomial before you factor.

GuidedPractice

1A. $5x^2 + 13x + 6$ **$(5x + 3)(x + 2)$** **1B.** $6x^2 + 22x - 8$ **$2(3x - 1)(x + 4)$**

Real-WorldCareer

Urban Planner Urban planners design the layout of an area. They take into consideration the available land and geographical and environmental factors to design an area that benefits the community the most. City planners have a bachelor's degree in planning and almost half have a master's degree.

Sometimes the coefficient of the x-term is negative.

Example 2 Factor $ax^2 - bx + c$

Factor $3x^2 - 17x + 20$.

In this trinomial, $a = 3$, $b = -17$, and $c = 20$. Since b is negative, $m + p$ will be negative. Since c is positive, mp will be positive.

To determine m and p, list the negative factors of ac or 60. The sum of m and p should be -17.

Factors of 60	Sum of Factors	
−2, −30	−32	
−3, −20	−23	
−4, −15	−19	
−5, −12	−17	The correct factors are −5 and −12.

$3x^2 - 17x + 20 = 3x^2 - 12x - 5x + 20$ — $m = -12$ and $p = -5$

$= (3x^2 - 12x) + (-5x + 20)$ — Group terms with common factors.

$= 3x(x - 4) + (-5)(x - 4)$ — Factor the GCF.

$= (3x - 5)(x - 4)$ — Distributive Property

GuidedPractice

2A. $2n^2 - n - 1$ **$(n - 1)(2n + 1)$** **2B.** $10y^2 - 35y + 30$ **$5(2y - 3)(y - 2)$**

1 Factor $ax^2 + bx + c$

Example 1 shows how to factor a trinomial of the form $ax^2 + bx + c$. **Example 2** shows how to factor a trinomial of the form $ax^2 - bx + c$. **Example 3** shows how to determine whether a polynomial is prime.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Factor each trinomial.

a. $5x^2 + 27x + 10$ $(5x + 2)(x + 5)$

b. $4x^2 + 24x + 32$ $4(x + 2)(x + 4)$

2 Factor $24x^2 - 22x + 3$. $(4x - 3)(6x - 1)$

WatchOut!

Preventing Errors Many students forget to include the GCF that they factored from the trinomial. Remind students to put the GCF in front of the other two factors.

DifferentiatedInstruction AL

If some students have trouble factoring polynomials,

Then place students in groups to factor polynomials such as those in Example 1. Depending on the number of factors and number of students in each group, have each student find one or two factors for mp. By dividing the labor, students should be able to find the factors for mp that sum to $m + p$ quickly. Once they have found the factors, have students complete the factoring as a group.

WatchOut!

Preventing Errors Make sure students list all possible factors of *mp*, including both positive and negative factors, before they decide the polynomial is prime.

Additional Example

3 Factor $3x^2 + 7x - 5$. If the polynomial cannot be factored using integers, write *prime*.
prime

2 Solve Equations by Factoring

Example 4 shows how to solve a real-world problem by writing an equation of the form $ax^2 + bx + c = 0$ and then solving it.

Additional Example

4 **MODEL ROCKETS** Mr. Zayed's science class built a model rocket. They launched the rocket outside. It cleared the top of a 18.3-meter high pole and then landed in a nearby tree. If the launch pad was 0.6 meters above the ground, the initial velocity of the rocket was 19.5 meters per second, and the rocket landed 9.1 meters above the ground, how long was the rocket in flight? Use the equation $h = -1$ $6t^2 + vt + h_0$.
about 3.5 seconds

Teach with Tech

Interactive Whiteboard Create a template that shows the multiplication of two binomials, such as (□±□)(□±□). Demonstrate finding the factors of a quadratic expression. Drag each factor into the spaces until the correct one is found.

A polynomial that cannot be written as a product of two polynomials with integral coefficients is called a **prime polynomial**.

Example 3 **Determine Whether a Polynomial is Prime**

Factor $4x^2 - 3x + 5$, if possible. If the polynomial cannot be factored using integers, write *prime*.

In this trinomial, $a = 4$, $b = -3$, and $c = 5$. Since b is negative, $m + p$ is negative. Since c is positive, mp is positive. So, m and p are both negative. Next, list the factors of 20. Look for the pair with a sum of -3.

Factors of 20	Sum of Factors
−20, −1	−21
−4, −5	−9
−2, −10	−12

There are no factors with a sum of -3. So the quadratic expression cannot be factored using integers. Therefore, $4x^2 - 3x + 5$ is prime.

GuidedPractice

Factor each polynomial, if possible. If the polynomial cannot be factored using integers, write *prime*.

3A. $4r^2 - r + 7$ **prime**

3B. $2x^2 + 3x - 5$ **$(2x + 5)(x - 1)$**

2 Solve Equations by Factoring

A model for the height of a projectile is given by $h = -16t^2 + vt + h_0$, where h is the height in feet, t is the time in seconds, v is the initial upward velocity in feet per second, and h_0 is the initial height in feet. Equations of the form $ax^2 + bx + c = 0$ can be solved by factoring and by using the Zero Product Property.

Real-World Example 4 **Solve Equations by Factoring**

WILDLIFE **Suppose a cheetah pouncing on an antelope leaps with an initial upward velocity of 5.8 feet per second. How long is the cheetah in the air if it lands on the antelope's hind quarter, 0.9 feet from the ground?**

$h = -16t^2 + vt + h_0$	Equation for height
$0.9 = -16t^2 + 5.8t + 0$	$h = 3$, $v = 19$, and $h_0 = 0$
$0 = -16t^2 + 5.8t - 0.9$	Subtract 3 from each side.
$0 = 16t^2 - 5.8t + 0.9$	Multiply each side by −1.
$0 = (16t - 0.9)(t - 1)$	Factor $16t^2 - 5.8t + 0.9$.
$16t - 0.9 = 0$ or $t - 1 = 0$	Zero Product Property
$16t = 3$ $\quad$ $t = 1$	Solve each equation.
$t = \frac{3}{16}$	

The solutions are $\frac{3}{16}$ and 1 seconds. It takes the cheetah $\frac{3}{16}$ second to reach a height of 0.9 feet on his way up. It takes the cheetah 1 second to reach a height of 0.9 feet on his way down. So, the cheetah is in the air 1 second before he catches the antelope.

Real-WorldLink

Cheetahs are the fastest land animals in the world, reaching speeds of up to 70 mph. It can accelerate from 0 to 40 mph in 3 strides. It takes just seconds for the cheetah to reach the full speed of 70 mph.

Source: Cheetah Conservation Fund

WatchOut!

Keep the −1 Do not forget to carry the −1 that was factored out through the rest of the steps or multiply both sides by −1.

GuidedPractice

4. **PHYSICAL SCIENCE** A person throws a ball upward from a 506-feet tall building. The ball's height h in feet after t seconds is given by the equation $h = -16t^2 + 48t + 506$. The ball lands on a balcony that is 218 feet above the ground. How many seconds was it in the air? **6 s**

Focus on Mathematical Content

Factoring $ax^2 + bx + c$ To factor trinomials by grouping where x^2 has a coefficient other than 1, express the trinomial as four terms, $ax^2 + mx + px + c$, where $m + p = b$ and $mp = ac$. Make a table of possible factors for mp and the sum of these factors for b. Replace a, m, p, and c with their values. Group terms with common factors. Factor out the GCF from each grouping, then factor using the Distributive Property.

Check Your Understanding

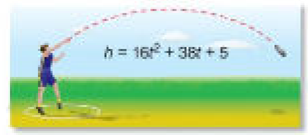
Examples 1–3 **Factor each polynomial, if possible. If the polynomial cannot be factored using integers, write *prime*.**

1. $3x^2 + 17x + 10$ **$(3x + 2)(x + 5)$**
2. $2x^2 + 22x + 56$ **$2(x + 4)(x + 7)$**
3. $5x^2 - 3x + 4$ **prime**
4. $3x^2 - 11x - 20$ **$(3x + 4)(x - 5)$**

Example 4 **Solve each equation. Confirm your answers using a graphing calculator.**

5. $2x^2 + 9x + 9 = 0$ **$-\frac{3}{2}, -3$**
6. $3x^2 + 17x + 20 = 0$ **$-\frac{5}{3}, -4$**
7. $3x^2 - 10x + 8 = 0$ **$\frac{4}{3}, 2$**
8. $2x^2 - 17x + 30 = 0$ **$\frac{5}{2}, 6$**

9. **MODELING** Khalid throws the discus at a school meet.
   a. What is the initial height of the discus? **5 ft**
   b. After how many seconds does the discus hit the ground? **2.5 seconds**



10. $(5x + 4)(x + 6)$ 11. $(2x + 3)(x + 8)$ 13. $2(2x + 5)(x + 7)$ 14. $(2x + 3)(x - 3)$

Practice and Problem Solving

Examples 1–3 **Factor each polynomial, if possible. If the polynomial cannot be factored using integers, write *prime*.**

10. $5x^2 + 34x + 24$
11. $2x^2 + 19x + 24$
12. $4x^2 + 22x + 10$ **$2(2x + 1)(x + 5)$**
13. $4x^2 + 38x + 70$
14. $2x^2 - 3x - 9$
15. $4x^2 - 13x + 10$ **$(4x - 5)(x - 2)$**
16. $2x^2 + 3x + 6$ **prime**
17. $5x^2 + 3x + 4$ **prime**
18. $12x^2 + 69x + 45$ **$3(4x + 3)(x + 5)$**
19. $4x^2 - 5x + 7$ **prime**
20. $5x^2 + 23x + 24$ **$(5x + 8)(x + 3)$**
21. $3x^2 - 8x + 15$ **prime**

Example 4

22. **SHOT PUT** A catapult throws a shot put with an initial upward velocity of 29 feet per second and from an initial height of 6 feet.
   a. Write an equation that models the height of the shot put in feet with respect to time in seconds. **$h = -16t^2 + 29t + 6$**
   b. After how many seconds will the shot put hit the ground? **2 seconds**

Solve each equation. Confirm your answers using a graphing calculator.

23. $2x^2 + 9x - 18 = 0$ **$\frac{3}{2}, -6$**
24. $4x^2 + 17x + 15 = 0$ **$-\frac{5}{4}, -3$**
25. $-3x^2 + 26x = 16$ **$\frac{2}{3}, 8$**
26. $-2x^2 + 13x = 15$ **$\frac{3}{2}, 5$**
27. $-3x^2 + 5x = -2$ **$-\frac{1}{3}, 2$**
28. $-4x^2 + 19x = -30$ **$-\frac{5}{4}, 6$**

B 29. **BASKETBALL** When Jaber shoots a free throw, the ball is 6 feet from the floor and has an initial upward velocity of 20 feet per second. The hoop is 10 feet from the floor.
   a. Use the vertical motion model to determine an equation that models Jaber's free throw. **$10 = -16t^2 + 20t + 6$**
   b. How long is the basketball in the air before it reaches the hoop? **1 second**
   c. Raed shoots a free throw that is 5 feet 9 inches from the floor with the same initial upward velocity. Will the ball be in the air more or less time? Explain. **Less; sample answer: It starts closer to the ground so the shot will not have as far to fall.**

30. **DIVING** Bilal dives from a 36 feet platform. The equation $h = -16t^2 + 14t + 36$ models the dive. How long will it take Bilal to reach the water? **2 seconds**

3 Practice

Formative Assessment

Use Exercises 1–9 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Exercise Alert

Scissors For Exercise 39, students will need a pair of scissors.

Multiple Representations

In Exercise 39, students use a concrete model and algebraic reasoning to represent the difference of two squares.

4 Assess

Yesterday's News Have students write how yesterday's lesson on quadratic equations helped them with today's lesson on quadratic equations.

Teaching the Mathematical Practices

Modeling Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life. In Exercise 9, point out that the equation for the height of a projectile applies. Tell students that this is used frequently and encourage them to learn the equation given before Example 4.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	10–28, 40, 41, 43–73	11–27 odd, 45–48	10–28 even, 40, 41, 43, 44, 49–73
OL Core	11–27 odd, 29–31, 33–37 odd, 39–41, 43–73	10–28, 45–48	29–41, 43, 44, 49–73
BL Advanced	29–68, (optional: 69–73)		

Teaching the Mathematical Practices

Critique Mathematically proficient students respond to the arguments of others. For Exercise 40, tell students to think about the first step in solving an equation by factoring: *write the equation in standard form*. You might also remind them that while at least one factor in a zero product must be 0, no such principle holds for any other number.

31. **NUMBER THEORY** Six times the square of a number x plus 11 times the number equals 2. What are possible values of x? -2 or $\frac{1}{6}$

Factor each polynomial, if possible. If the polynomial cannot be factored using integers, write *prime*.

32. $-6x^2 - 23x - 20$ $-(2x + 5)(3x + 4)$
33. $-4x^2 - 15x - 14$ $-(x + 2)(4x + 7)$
34. $-5x^2 + 18x + 8$ $-(x - 4)(5x + 2)$
35. $-6x^2 + 31x - 35$ $-(2x - 7)(3x - 5)$
36. $-4x^2 + 5x - 12$ prime
37. $-12x^2 + x + 20$ $-(3x - 4)(4x + 5)$
38. **URBAN PLANNING** The city has commissioned the building of a rectangular park. The area of the park can be expressed as $660x^2 + 524x + 85$. Factor this expression to find binomials with integer coefficients that represent possible dimensions of the park. If $x = 8$, what is a possible perimeter of the park?
$(22x + 5)(30x + 17)$; 876 units

39. **MULTIPLE REPRESENTATIONS** In this problem, you will explore factoring a special type of polynomial.
    a. **Geometric** Draw a square and label the sides a. Within this square, draw a smaller square that shares a vertex with the first square. Label the sides b. What are the areas of the two squares? a^2 and b^2
    b. **Geometric** Cut and remove the small square. What is the area of the remaining region? $a^2 - b^2$
    c. **Analytical** Draw a diagonal line between the inside corner and outside corner of the figure, and cut along this line to make two congruent pieces. Then rearrange the two pieces to form a rectangle. What are the dimensions? width: $a - b$, length: $a + b$
    d. **Analytical** Write the area of the rectangle as the product of two binomials. $(a - b)(a + b)$
    e. **Verbal** Complete this statement: $a^2 - b^2 = ...$ Why is this statement true?
    $(a - b)(a + b)$; the figure with area $a^2 - b^2$ and the rectangle with area $(a - b)(a + b)$ have the same area, so $a^2 - b^2 = (a - b)(a + b)$.

H.O.T. Problems Use Higher-Order Thinking Skills

40. **CRITIQUE** Hamad and Saeed are solving $6x^2 - x = 12$. Is either of them correct? Explain your reasoning.

Hamad

$6x^2 - x = 12$
$x(6x - 1) = 12$
$x = 12$ or $6x - 1 = 12$
$6x = 13$
$x = \frac{13}{6}$

Saeed

$6x^2 - x = 12$
$6x^2 - x - 12 = 0$
$(2x - 3)(3x + 4) = 0$
$2x - 3 = 0$ or $3x + 4 = 0$
$x = \frac{3}{2}$ $x = -\frac{4}{3}$

40. Saeed; sample answer: He rewrote the equation to have zero on one side. Then he factored and used the Zero Product Property.

44. Sample answer: Find two numbers, m and p, with a product of ac and a sum of b.

41. **REASONING** A square has an area of $9x^2 + 30xy + 25y^2$ square centimeters. The dimensions are binomials with positive integer coefficients. What is the perimeter of the square? Explain. See margin.

42. **CHALLENGE** Find all values of k so that $2x^2 + kx + 12$ can be factored as two binomials using integers. $\pm 25, \pm 14, \pm 11, \pm 10$

43. **WRITING IN MATH** What should you consider when solving a quadratic equation that models a real-world situation? See margin.

44. **WRITING IN MATH** Explain how to determine which values should be chosen for m and p when factoring a polynomial of the form $ax^2 + bx + c$.

442 | Lesson 7-7 | Solving $ax^2 + bx + c = 0$

Standardized Test Practice

45. **Gridded Response** Suhaila has two sisters. One sister is 8 years older than her and the other sister is 2 years younger than her. The product of Suhaila's sisters' ages is 56. How old is Suhaila? **6**

46. What is the product of $\frac{2}{3}a^3b^5$ and $\frac{3}{5}a^5b^2$? **A**

A $\frac{2}{5}a^8b^7$

B $\frac{2}{5}a^2b^3$

C $\frac{2}{5}a^8b^3$

D $\frac{2}{5}a^2b^7$

47. What is the solution set of $x^2 + 2x - 24 = 0$? **J**

F $\{-4, 6\}$ H $\{-3, 8\}$

G $\{3, -8\}$ J $\{4, -6\}$

48. Which is the solution set of $x \geq -2$? **C**

A 

B 

C 

D 

Spiral Review

Factor each polynomial. (Lesson 7-6)

49. $x^2 - 9x + 14$ **$(x - 2)(x - 7)$**

50. $n^2 - 8n + 15$ **$(n - 3)(n - 5)$**

51. $x^2 - 5x - 24$ **$(x + 3)(x - 8)$**

52. $z^2 + 15z + 36$ **$(z + 12)(z + 3)$**

53. $r^2 + 3r - 40$ **$(r + 8)(r - 5)$**

54. $v^2 + 16v + 63$ **$(v + 9)(v + 7)$**

Solve each equation. Check your solutions. (Lesson 7-5)

55. $a(a - 9) = 0$ **0, 9**

56. $(2y + 6)(y - 1) = 0$ **−3, 1**

57. $10x^2 - 20x = 0$ **0, 2**

58. $8b^2 - 12b = 0$ **0, 1.5**

59. $15a^2 = 60a$ **0, 4**

60. $33x^2 = -22x$ **$-\frac{2}{3}$, 0**

Name an appropriate method to solve each system of equations. Then solve the system. (Lesson 0-9)

61. $-5x + 2y = 13$
$2x + 3y = -9$
elimination; $x = -3, y = -1$

62. $y = -5x + 7$
$y = 3x - 17$
graphing; $x = 3, y = -8$

63. $x - 8y = 16$
$7x - 4y = -18$
substitution; $x = -4, y = -2.5$

Complete each sentence. (Lesson 0-1)

64. 54 cm = __?__ m **4.5**

65. 275 mm = __?__ m **0.275**

66. 7 gal = __?__ pt **56**

67. **TRUCKS** A sport-utility vehicle has a maximum load limit of 75 kilograms for its roof. You want to place a 38-kilogram cargo carrier and 4 pieces of luggage on top of the roof. Write and solve an inequality to find the average allowable weight for each piece of luggage. (Lesson 0-6) **$4x + 38 \leq 75$; 9.25 kg or less**

Skills Review

Find the principal square root of each number.

68. 16 **4**

69. 36 **6**

70. 64 **8**

71. 81 **9**

72. 121 **11**

73. 100 **10**

Additional Answers

41. $(12x + 20y)$ cm; The area of the square equals $(3x + 5y) \cdot (3x + 5y)$ cm^2, so the length of one side is $(3x + 5y)$ cm The perimeter is $4(3x + 5y)$ or $(12x + 20y)$ cm

43. Sample answer: A quadratic equation may have zero, one, or two solutions. If there are two solutions, you must consider the context of the situation to determine whether one or both solutions answer the given question.

Differentiated Instruction OL BL

Extension Have students write a quadratic equation in the form $ax^2 + bx + c = 0$. Tell them that a, b, and c must be integers, and the solutions to the equation must be $\frac{1}{2}$ and 4. Sample answer: $2x^2 - 9x + 4 = 0$

1 Focus

VerticalAlignment

Before Lesson 7-8 Factor trinomials into two binomials.

Lesson 7-8 Factor binomials that are the difference of squares. Use the difference of squares to solve equations.

After Lesson 7-8 Factor perfect square trinomials.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- Find $(x + 3)(x - 3)$. $x^2 - 9$
- Find $(x - 6)(x + 6)$. $x^2 - 36$
- Find $(a - b)(a + b)$. $a^2 - b^2$

Teach with Tech

Interactive Whiteboard Use the template from the previous lesson to demonstrate why the difference of squares is factorable, but the difference of sums is not.

LESSON 7-8 Differences of Squares

Then

- You factored trinomials into two binomials.

Now

- 1 Factor binomials that are the difference of squares.
- 2 Use the difference of squares to solve equations.

Why?

- Computer graphics designers use a combination of art and mathematics skills to design images and videos. They use equations to form shapes and lines on computers. Factoring can help to determine the dimensions and shapes of the figures.

NewVocabulary
difference of two squares

Mathematical Practices
Make sense of problems and persevere in solving them.

1 Factor Differences of Squares

You have previously learned about the product of the sum and difference of two quantities. This resulting product is referred to as the **difference of two squares**. So, the factored form of the difference of squares is called the product of the sum and difference of the two quantities.

KeyConcept Difference of Squares

Symbols $a^2 - b^2 = (a + b)(a - b)$ or $(a - b)(a + b)$

Examples $x^2 - 25 = (x + 5)(x - 5)$ or $(x - 5)(x + 5)$

$t^2 - 64 = (t + 8)(t - 8)$ or $(t - 8)(t + 8)$

Example 1 Factor Differences of Squares

Factor each polynomial.

a. $16h^2 - 9a^2$

$16h^2 - 9a^2 = (4h)^2 - (3a)^2$ Write in the form of $a^2 - b^2$.

$= (4h + 3a)(4h - 3a)$ Factor the difference of squares.

b. $121 - 4b^2$

$121 - 4b^2 = (11)^2 - (2b)^2$ Write in the form of $a^2 - b^2$.

$= (11 - 2b)(11 + 2b)$ Factor the difference of squares.

c. $27g^3 - 3g$

Because the terms have a common factor, factor out the GCF first. Then proceed with other factoring techniques.

$27g^3 - 3g = 3g(9g^2 - 1)$ Factor out the GCF of $3g$.

$= 3g[(3g)^2 - (1)^2]$ Write in the form $a^2 - b^2$.

$= 3g(3g - 1)(3g + 1)$ Factor the difference of squares.

GuidedPractice

1A. $81 - c^2$ $(9 + c)(9 - c)$

1B. $64g^2 - h^2$ $(8g + h)(8g - h)$

1C. $9x^3 - 4x$ $x(3x + 2)(3x - 2)$

1D. $-4y^3 + 9y$ $-y(2y + 3)(2y - 3)$

To factor a polynomial completely, a technique may need to be applied more than once. This also applies to the difference of squares pattern.

Example 2 Apply a Technique More than Once

Factor each polynomial.

a. $b^4 - 16$

$b^4 - 16 = (b^2)^2 - (4)^2$ — Write $b^4 - 16$ in $a^2 - b^2$ form.

$= (b^2 + 4)(b^2 - 4)$ — Factor the difference of squares.

$= (b^2 + 4)(b^2 - 2^2)$ — $b^2 - 4$ is also a difference of squares.

$= (b^2 + 4)(b + 2)(b - 2)$ — Factor the difference of squares.

b. $625 - x^4$

$625 - x^4 = (25)^2 - (x^2)^2$ — Write $625 - x^4$ in $a^2 - b^2$ form.

$= (25 + x^2)(25 - x^2)$ — Factor the difference of squares.

$= (25 + x^2)(5^2 - x^2)$ — Write $25 - x^2$ in $a^2 - b^2$ form.

$= (25 + x^2)(5 - x)(5 + x)$ — Factor the difference of squares.

WatchOut!

Sum of Squares The sum of squares, $a^2 + b^2$, does not factor into $(a + b)(a + b)$. The sum of squares is a prime polynomial and cannot be factored.

GuidedPractice

2A. $y^4 - 1$ $(y^2 + 1)(y + 1)(y - 1)$

2B. $4a^4 - b^4$ $(2a^2 + b^2)(2a^2 - b^2)$

2C. $81 - x^4$ $(3 - x)(3 + x)(9 + x^2)$

2D. $16y^4 - 1$ $(4y^2 + 1)(2y - 1)(2y + 1)$

Sometimes more than one factoring technique needs to be applied to ensure that a polynomial is factored completely.

Example 3 Apply Different Techniques

Factor each polynomial.

a. $5x^5 - 45x$

$5x^5 - 45x = 5x(x^4 - 9)$ — Factor out GCF.

$= 5x[(x^2)^2 - (3)^2]$ — Write $x^4 - 9$ in the form $a^2 - b^2$.

$= 5x(x^2 - 3)(x^2 + 3)$ — Factor the difference of squares.

$x^2 - 3$ is not a difference of squares because 3 is not a perfect square.

b. $7x^3 + 21x^2 - 7x - 21$

$7x^3 + 21x^2 - 7x - 21$ — Original expression

$= 7(x^3 + 3x^2 - x - 3)$ — Factor out GCF.

$= 7[(x^3 + 3x^2) - (x + 3)]$ — Group terms with common factors.

$= 7[x^2(x + 3) - 1(x + 3)]$ — Factor each grouping.

$= 7(x + 3)(x^2 - 1)$ — $x + 3$ is the common factor.

$= 7(x + 3)(x + 1)(x - 1)$ — Factor the difference of squares.

GuidedPractice

3A. $2y^4 - 50$ $2(y^2 - 5)(y^2 + 5)$

3B. $6x^4 - 96$ $6(x - 2)(x + 2)(x^2 + 4)$

3C. $2m^3 + m^2 - 50m - 25$ $(2m + 1)(m + 5)(m - 5)$

3D. $r^3 + 6r^2 + 11r + 66$ $(r^2 + 11)(r + 6)$

1 Factor Differences of Squares

Example 1 shows how to factor the differences of squares. **Examples 2** and **3** show how to apply a factoring technique more than once to factor a polynomial completely.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Factor each polynomial.

a. $m^2 - 64$ $(m + 8)(m - 8)$

b. $16y^2 - 81z^2$ $(4y + 9z)(4y - 9z)$

c. $3b^3 - 27b$ $3b(b + 3)(b - 3)$

2 Factor each polynomial.

a. $y^4 - 625$ $(y^2 + 25)(y + 5)(y - 5)$

b. $256 - n^4$ $(16 + n^2)(4 - n)(4 + n)$

3 Factor each polynomial.

a. $9x^5 - 36x$ $9x(x^2 - 2)(x^2 + 2)$

b. $6x^3 + 30x^2 - 24x - 120$ $6(x + 2)(x - 2)(x + 5)$

WatchOut!

Preventing Errors Students should always check their factoring by multiplying the factors using the FOIL method.

Focus on Mathematical Content

Factoring Differences of Squares The binomial $a^2 - b^2$ is the difference of two squares, a^2 and b^2. One of the binomial factors, $(a + b)$, is the sum of the principal square roots of a^2 and b^2, and the other binomial factor, $(a - b)$, is the difference of their principal square roots.

WatchOut!

Preventing Errors Students should notice that after the GCF has been factored out and the difference of squares factoring technique has been applied once, one of the factors should be prime.

2 Solve Equations by Factoring

Example 4 shows how to answer a multiple-choice test item on solving equations by factoring.

Additional Example

4 STANDARDIZED TEST PRACTICE

In the equation $y = q^2 - \frac{4}{25}$, which is a value of q when $y = 0$? **D**

A $\frac{2}{25}$ C 0

B $\frac{4}{25}$ D $-\frac{2}{5}$

Tips for New Teachers

Sense-Making Students may not be used to thinking of fractions as perfect squares. Remind them that if both the numerator and denominator are perfect squares, then the fraction is a perfect square.

Teaching the Mathematical Practices

Sense-Making Mathematically proficient students check their answers to problems using a different method. Point out that substituting the choices in a multiple choice question is sometimes a faster solution method.

2 Solve Equations by Factoring

After factoring, you can apply the Zero Product Property to an equation that is written as the product of factors set equal to 0.

Test-Taking Tip

Sense-Making Another method that can be used to solve this equation is to substitute each answer choice into the equation.

Standardized Test Example 4 Solve an Equation by Factoring

In the equation $y = x^2 - \frac{9}{16}$, which is a value of x when $y = 0$?

A $-\frac{9}{4}$ B 0 C $\frac{3}{4}$ D $\frac{9}{4}$

Read the Test Item

Replace y with 0 and then solve.

Solve the Test Item

$y = x^2 - \frac{9}{16}$ Original equation

$0 = x^2 - \frac{9}{16}$ Replace y with 0.

$0 = x^2 - \left(\frac{3}{4}\right)^2$ Write in the form $a^2 - b^2$.

$0 = \left(x + \frac{3}{4}\right)\left(x - \frac{3}{4}\right)$ Factor the difference of squares.

$0 = x + \frac{3}{4}$ or $0 = x - \frac{3}{4}$ Zero Product Property

$x = -\frac{3}{4}$ $x = \frac{3}{4}$ The correct answer is C.

Guided Practice

4. Which are the solutions of $18x^3 = 50x$? **H**

F $0, \frac{5}{3}$ G $-\frac{5}{3}, \frac{5}{3}$ H $-\frac{5}{3}, \frac{5}{3}, 0$ J $-\frac{5}{3}, \frac{5}{3}, 1$

Check Your Understanding

Examples 1–3 Factor each polynomial. **9. $(c + 1)(c - 1)(2c + 3)$ 11. $(t + 4)(t - 4)(3t + 2)$**

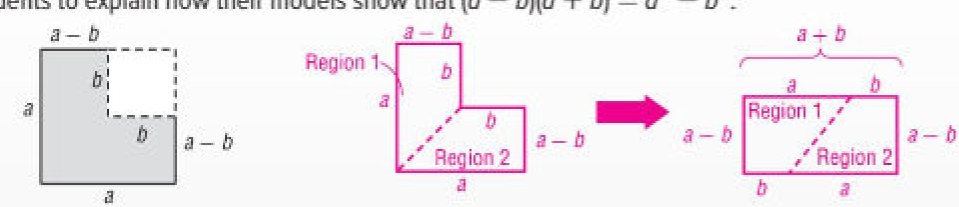
1. $x^2 - 9$ **$(x + 3)(x - 3)$**
2. $4a^2 - 25$ **$(2a + 5)(2a - 5)$**
3. $9m^2 - 144$ **$9(m + 4)(m - 4)$**
4. $2p^3 - 162p$ **$2p(p + 9)(p - 9)$**
5. $u^4 - 81$ **$(u + 3)(u - 3)(u^2 + 9)$**
6. $2d^4 - 32f^4$ **$2(d^2 + 4f^2)(d + 2f)(d - 2f)$**
7. $20r^4 - 45n^4$ **$5(2r^2 - 3n^2)(2r^2 + 3n^2)$**
8. $256n^4 - c^4$ **$(16n^2 + c^2)(4n + c)(4n - c)$**
9. $2c^3 + 3c^2 - 2c - 3$
10. $f^3 - 4f^2 - 9f + 36$ **$(f + 3)(f - 3)(f - 4)$**
11. $3t^3 + 2t^2 - 48t - 32$
12. $w^3 - 3w^2 - 9w + 27$ **$(w - 3)(w + 3)(w - 3)$**

Example 4 **EXTENDED RESPONSE** **During an accident, skid marks may result from sudden breaking. The formula $\frac{1}{24}s^2 = d$ approximates a vehicle's speed s in miles per hour given the length d in feet of the skid marks on dry concrete.**

13. If skid marks on dry concrete are 54 feet long, how fast was the car traveling when the brakes were applied? **36 mph**
14. If the skid marks on dry concrete are 150 feet long, how fast was the car traveling when the brakes were applied? **60 mph**

Differentiated Instruction OL BL

Visual/Spatial Learners Draw the geometric model shown below on the board. Ask students to use their own paper square and scissors to make the model for a^2 when the b^2 square is removed. Then ask students to explain how their models show that $(a - b)(a + b) = a^2 - b^2$.



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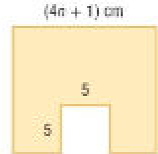
Practice and Problem Solving

Examples 1–3 Factor each polynomial. **25–43. See margin.**

15. $q^2 - 121$ **$(q + 11)(q - 11)$**
16. $r^4 - k^4$ **$(r^2 + k^2)(r + k)(r - k)$**
17. $6n^4 - 6$ **$6(n^2 + 1)(n + 1)(n - 1)$**
18. $w^4 - 625$ **$(w^2 + 25)(w + 5)(w - 5)$**
19. $r^2 - 9t^2$ **$(r + 3t)(r - 3t)$**
20. $2c^2 - 32d^2$ **$2(c + 4d)(c - 4d)$**
21. $h^3 - 100h$ **$h(h + 10)(h - 10)$**
22. $h^4 - 256$ **$(h^2 + 16)(h + 4)(h - 4)$**
23. $2x^3 - x^2 - 162x + 81$ **$(x + 9)(x - 9)(2x - 1)$**
24. $x^2 - 4y^2$ **$(x + 2y)(x - 2y)$**
25. $7h^4 - 7p^4$
26. $3c^3 + 2c^2 - 147c - 98$
27. $6k^2h^4 - 54k^4$
28. $5a^3 - 20a$
29. $f^3 + 2f^2 - 64f - 128$
30. $3r^3 - 192r$
31. $10q^3 - 1210q$
32. $3xn^4 - 27x^3$
33. $p^3r^5 - p^3r$
34. $8c^3 - 8c$
35. $r^3 - 5r^2 - 100r + 500$
36. $3t^3 - 7t^2 - 3t + 7$
37. $a^2 - 49$
38. $4m^3 + 9m^2 - 36m - 81$
39. $3m^4 + 243$
40. $3x^3 + x^2 - 75x - 25$
41. $12a^3 + 2a^2 - 192a - 32$
42. $x^4 + 6x^3 - 36x^2 - 216x$
43. $15m^3 + 12m^2 - 375m - 300$

Example 4

44. **GEOMETRY** The drawing at the right is a square with a square cut out of it.



a. Write an expression that represents the area of the shaded region. **$(4n + 1)^2 - 5^2$**

b. Find the dimensions of a rectangle with the same area as the shaded region in the drawing. Assume that the dimensions of the rectangle must be represented by binomials with integral coefficients. **$(4n + 6)$ by $(4n - 4)$**

B

45. **DECORATIONS** An arch decorated with balloons was used to decorate a castle's entrance. The shape of the arch can be modeled by the equation $y = -0.5x^2 + 4.5x$, where x and y are measured in meters and the x-axis represents the floor.

a. Write the expression that represents the height of the arch in factored form. **$-0.5x(x - 9)$**

b. How far apart are the two points where the arch touches the floor? **9 m**

c. Graph this equation on your calculator. What is the highest point of the arch? **10.125 m**

46. **SENSE-MAKING** Hiyam is building a deck in her backyard. The plans for the deck show that it is to be 24 meters by 24 meters. Hiyam wants to reduce one dimension by a number of meters and increase the other dimension by the same number of meters. If the area of the reduced deck is 512 square meters, what are the dimensions of the deck? **16 m by 32 m**

47. **SALES** The sales of a particular CD can be modeled by the equation $S = -25m^2 + 125m$, where S is the number of CDs sold in thousands, and m is the number of months that it is on the market.

a. In what month should the music store expect the CD to stop selling? **5**

b. In what month will CD sales peak? **2.5**

c. How many copies will the CD sell at its peak? **156,250**

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	15–44, 57, 60–89	15–43 odd, 64–67	16–44 even, 57, 60–63, 68–89
OL Core	15–43 odd, 44–47, 49–55 odd, 56–57, 60–89	15–44, 64–67	45–57, 60–63, 68–89
BL Advanced	45–81, (optional: 82–89)		

3 Practice

Formative Assessment

Use Exercises 1–14 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Exercise Alert

Graphing Calculator For Exercise 45, students will need a graphing calculator.

Teaching the Mathematical Practices

Sense-Making Mathematically proficient students draw diagrams of important features and relationships. In Exercise 46, point out that a diagram can help students visualize the situation.

Additional Answers

25. $7(h^2 + p^2)(h + p)(h - p)$
26. $(c + 7)(c - 7)(3c + 2)$
27. $6k^2(h^2 + 3k)(h^2 - 3k)$
28. $5a(a + 2)(a - 2)$
29. $(f + 8)(f - 8)(f + 2)$
30. $3r(r + 8)(r - 8)$
31. $10q(q + 11)(q - 11)$
32. $3x(n^2 + 3x)(n^2 - 3x)$
33. $p^3r(r + 1)(r - 1)(r^2 + 1)$
34. $8c(c + 1)(c - 1)$
35. $(r + 10)(r - 10)(r - 5)$
36. $(t + 1)(t - 1)(3t - 7)$
37. $(a + 7)(a - 7)$
38. $(m + 3)(m - 3)(4m + 9)$
39. $3(m^4 + 81)$
40. $(x + 5)(x - 5)(3x + 1)$
41. $2(a + 4)(a - 4)(6a + 1)$
42. $x(x + 6)(x - 6)(x + 6)$
43. $3(m + 5)(m - 5)(5m + 4)$

Multiple Representations

In Exercise 56, students use algebra and information organized in a table to explore the format of a perfect-square trinomial.

Solve each equation by factoring. Confirm your answers using a graphing calculator.

48. $36w^2 = 121$ $\frac{11}{6}, -\frac{11}{6}$

49. $100 = 25x^2$ $2, -2$

50. $64x^2 - 1 = 0$ $\frac{1}{8}, -\frac{1}{8}$

51. $4y^2 - \frac{9}{16} = 0$ $\frac{3}{8}, -\frac{3}{8}$

52. $\frac{1}{4}b^2 = 16$ $-8, 8$

53. $81 - \frac{1}{25}x^2 = 0$ $-45, 45$

54. $9d^2 - 81 = 0$ $3, -3$

55. $4a^2 = \frac{9}{64}$ $\frac{3}{16}, -\frac{3}{16}$

56. MULTIPLE REPRESENTATIONS In this problem, you will investigate perfect square trinomials. **56c.** $(a + b)(a + b) = a^2 + 2ab + b^2$ and $(a - b)(a - b) = a^2 - 2ab + b^2$

a. **Tabular** Copy and complete the table below by factoring each polynomial. Then write the first and last terms of the given polynomials as perfect squares.

b. **Analytical** Write the middle term of each polynomial using the square roots of the perfect squares of the first and last terms. **See table.**

Polynomial	Factored Polynomial	First Term	Last Term	Middle Term
$4x^2 + 12x + 9$	$(2x + 3)(2x + 3)$	$4x^2 = (2x)^2$	$9 = 3^2$	$12x = 2 \cdot 2x \cdot 3$
$9x^2 - 24x + 16$	$(3x - 4)(3x - 4)$	$9x^2 = (3x)^2$	$16 = 4^2$	$-24x = -2 \cdot 3x \cdot 4$
$4x^2 - 20x + 25$	$(2x - 5)(2x - 5)$	$4x^2 = (2x)^2$	$25 = 5^2$	$-20x = -2 \cdot 2x \cdot 5$
$16x^2 + 24x + 9$	$(4x + 3)(4x + 3)$	$16x^2 = (4x)^2$	$9 = 3^2$	$24x = 2 \cdot 4x \cdot 3$
$25x^2 + 20x + 4$	$(5x + 2)(5x + 2)$	$25x^2 = (5x)^2$	$4 = 2^2$	$20x = 2 \cdot 5x \cdot 2$

c. **Algebraic** Write the pattern for a perfect square trinomial.

d. **Verbal** What conditions must be met for a trinomial to be classified as a perfect square trinomial? **The first and last terms must be perfect squares and the middle term must be 2 times the square roots of the first and last terms.**

H.O.T. Problems Use Higher-Order Thinking Skills

57. ERROR ANALYSIS Najat and Najla are factoring an expression. Is either of them correct? Explain your reasoning.

57. Najla; sample answer: Checking Najat's answer gives us $16x^2 - 25y^2$. The exponent on x in the final product should be 4.

Najat	Najla
$16x^4 - 25y^2 =$	$16x^4 - 25y^2 =$
$(4x - 5y)(4x + 5y)$	$(4x^2 - 5y)(4x^2 + 5y)$

63. When the difference of squares pattern is multiplied together using the FOIL method, the outer and inner terms are opposites of each other. When these terms are added together, the sum is zero.

58. CHALLENGE Factor and simplify $9 - (k + 3)^2$, a difference of squares. $[3 + (k + 3)][3 - (k + 3)] = (k + 6)(-k) = -k^2 - 6k$

59. PERSEVERANCE Factor $x^{16} - 81$. $(x^4 - 3)(x^4 + 3)(x^8 + 9)$

60. REASONING Write and factor a binomial that is the difference of two perfect squares and that has a greatest common factor of $5mk$.
Sample answer: $5mka^2 - 5mkb^2 = 5mk(a^2 - b^2) = 5mk(a - b)(a + b)$

61. REASONING Determine whether the following statement is *true* or *false*. Give an example or counterexample to justify your answer. false; $a^2 + b^2$

All binomials that have a perfect square in each of the two terms can be factored.

62. OPEN ENDED Write a binomial in which the difference of squares pattern must be repeated to factor it completely. Then factor the binomial.
Sample answer: $x^4 - 16$; $(x - 2)(x + 2)(x^2 + 4)$

63. WRITING IN MATH Describe why the difference of squares pattern has no middle term with a variable.

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Standardized Test Practice

64. One of the roots of $2x^2 + 13x = 24$ is -8. What is the other root? **B**

A $-\frac{3}{2}$ C $\frac{2}{3}$

B $-\frac{2}{3}$ D $\frac{3}{2}$

65. Which of the following is the sum of both solutions of the equation $x^2 + 3x = 54$? **G**

F -21 H 3

G -3 J 21

66. What are the x-intercepts of the graph of $y = -3x^2 + 7x + 20$? **C**

A $\frac{5}{3}, -4$ C $-\frac{5}{3}, 4$

B $-\frac{5}{3}, -4$ D $\frac{5}{3}, 4$

67. **EXTENDED RESPONSE** Two cars leave City A at the same time from different parts of the city and both drive to City B. The distance in miles of the cars from the center of City A can be represented by the two equations below, where t represents the time in hours.

Car A: $65t + 15$ Car B: $60t + 25$

a. Car A, because it is traveling at 65 mph, and Car B is traveling at 60 mph.

a. Which car is faster? Explain.

b. Find an expression that models the distance between the two cars. **$5t - 10$**

c. How far apart are the cars after $2\frac{1}{2}$ hours? **2.5 miles**

Spiral Review

Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*. (Lesson 7-7)

68. $5x^2 - 17x + 14$ **$(5x - 7)(x - 2)$**
69. $5a^2 - 3a + 15$ **prime**
70. $10x^2 - 20xy + 10y^2$ **$10(x - y)(x - y)$**

Solve each equation. Check your solutions. (Lesson 7-6)

71. $n^2 - 9n = -18$ **{3, 6}**
72. $10 + a^2 = -7a$ **{−5, −2}**
73. $22x - x^2 = 96$ **{6, 16}**

Solve each equation. Check the solut ions. (Lesson 7-4)

74. $2x^2 = 32$ **$-4, 4$**
75. $(x - 4)^2 = 25$ **$-1, 9$**
76. $4x^2 - 4x + 1 = 16$ **$-\frac{3}{2}, \frac{5}{2}$**
77. $2x^2 + 16x = -32$ **-4**
78. $(x + 3)^2 = 5$ **$-3 \pm \sqrt{5}$**
79. $4x^2 - 12x = -9$ **$\frac{3}{2}$**

Find each sum or difference. (Lesson 7-1)

82. $-3b^4 + 2b^3 - 9b^2 + 13$
83. $6h^4 + 2h^3 + h^2 + 5$

80. $(3n^2 - 3) + (4 + 4n^2)$ **$7n^2 + 1$**
81. $(2d^2 - 7d - 3) - (4d^2 + 7)$ **$-2d^2 - 7d - 10$**
82. $(2b^3 - 4b^2 + 4) - (3b^4 + 5b^2 - 9)$
83. $(8 - 4h^2 + 6h^4) + (5h^2 - 3 + 2h^3)$

Skills Review

Find each product.

84. $(x - 6)^2$ **$x^2 - 12x + 36$**
85. $(x - 2)(x - 2)$ **$x^2 - 4x + 4$**
86. $(x + 3)(x + 3)$ **$x^2 + 6x + 9$**
87. $(2x - 5)^2$ **$4x^2 - 20x + 25$**
88. $(6x - 1)^2$ **$36x^2 - 12x + 1$**
89. $(4x + 5)(4x + 5)$ **$16x^2 + 40x + 25$**

WatchOut!

Error Analysis For Exercise 57, make sure students can explain what Elizabeth did wrong. Stress that her error is a common one. Ask students what they can do to avoid making the same mistake themselves.

Teaching the Mathematical Practices

Perseverance Mathematically proficient students analyze givens, constraints, relationships, and goals. In Exercise 59, tell students to make sure their final polynomial is completely factored.

4 Assess

Ticket Out the Door Have students write the factors of $18x^2 - 50$.
$2(3x + 5)(3x - 5)$

Differentiated Instruction OL BL

Extension Have students solve the equation $x^3 - 4x = 12 - 3x^2$ by factoring. Ask them to check their solutions. $-3, -2, 2$

1 Focus

VerticalAlignment

Before Lesson 7-9 Find the product of the sum and difference of two quantities.

Lesson 7-9 Factor perfect square trinomials. Solve equations involving perfect squares.

After Lesson 7-9 Determine the number and type of roots for a polynomial equation.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- If the initial height is 64 meters, what equation would you write to determine how long it takes an object to hit the ground? $0 = -16t^2 + 64$
- Solve the equation for t. $-16(t-2)(t+2) = 0$; 2 seconds
- How could you use the definition of square root to solve the equation? $0 = -16t^2 + 64$; $-64 = -16t^2$; $4 = t^2$; $t = \pm\sqrt{4}$ or ± 2; Since a negative answer is not reasonable in this situation, the solution is $t = 2$.

LESSON 7-9 Perfect Squares

Then

- You found the product of a sum and difference.

Now

1. Factor perfect square trinomials.
2. Solve equations involving perfect squares.

Why?

- In a vacuum, a feather and a piano would fall at the same speed, or velocity. To find about how long it takes an object to hit the ground if it is dropped from an initial height of h_0 feet above ground, you would need to solve the equation $0 = -16t^2 + h_0$, where t is time in seconds after the object is dropped.

NewVocabulary
perfect square trinomial

Mathematical Practices
Attend to precision.

1 Factor Perfect Square Trinomials

You have learned the patterns for the products of the binomials $(a + b)^2$ and $(a - b)^2$. Recall that these are special products that follow specific patterns.

$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = (a-b)(a-b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$$

These products are called **perfect square trinomials**, because they are the squares of binomials. The above patterns can help you factor perfect square trinomials.

For a trinomial to be factorable as a perfect square, the first and last terms must be perfect squares and the middle term must be two times the square roots of the first and last terms.

The trinomial $16x^2 + 24x + 9$ is a perfect square trinomial, as illustrated below.

$$16x^2 + 24x + 9$$

- Is the first term a perfect square? Yes, because $16x^2 = (4x)^2$.
- Is the middle term twice the product of the square roots of the first and last terms? Yes, because $24x = 2(4x)(3)$.
- Is the last term a perfect square? Yes, because $9 = 3^2$.

KeyConcept Factoring Perfect Square Trinomials

Symbols	$a^2 + 2ab + b^2 = (a+b)(a+b) = (a+b)^2$
	$a^2 - 2ab + b^2 = (a-b)(a-b) = (a-b)^2$
Examples	$x^2 + 8x + 16 = (x+4)(x+4)$ or $(x+4)^2$
	$x^2 - 6x + 9 = (x-3)(x-3)$ or $(x-3)^2$

StudyTip

Recognizing Perfect Square Trinomials If the constant term of the trinomial is negative, the trinomial is not a perfect square trinomial, so it is not necessary to check the other conditions.

Example 1 Recognize and Factor Perfect Square Trinomials

Determine whether each trinomial is a perfect square trinomial. Write *yes* or *no*. If so, factor it.

a. $4y^2 + 12y + 9$

1. Is the first term a perfect square? — Yes, $4y^2 = (2y)^2$.
2. Is the last term a perfect square? — Yes, $9 = 3^2$.
3. Is the middle term equal to $2(2y)(3)$? — Yes, $12y = 2(2y)(3)$

Since all three conditions are satisfied, $4y^2 + 12y + 9$ is a perfect square trinomial.

$4y^2 + 12y + 9 = (2y)^2 + 2(2y)(3) + 3^2$ — Write as $a^2 + 2ab + b^2$.

$= (2y + 3)^2$ — Factor using the pattern.

b. $9x^2 - 6x + 4$

1. Is the first term a perfect square? — Yes, $9x^2 = (3x)^2$.
2. Is the last term a perfect square? — Yes, $4 = 2^2$.
3. Is the middle term equal to $-2(3x)(2)$? — No, $-6x \neq -2(3x)(2)$.

Since the middle term does not satisfy the required condition, $9x^2 - 6x + 4$ is not a perfect square trinomial.

GuidedPractice

1A. $9y^2 + 24y + 16$ yes; $(3y + 4)(3y + 4)$ **1B.** $2a^2 + 10a + 25$ no

A polynomial is completely factored when it is written as a product of prime polynomials. More than one method might be needed to factor a polynomial completely. When completely factoring a polynomial, the Concept Summary can help you decide where to start.

Remember, if the polynomial does not fit any pattern or cannot be factored, the polynomial is prime.

ConceptSummary Factoring Methods

Steps	Number of Terms	Examples
Step 1 Factor out the GCF.	any	$4x^3 + 2x^2 - 6x = 2x(2x^2 + x - 3)$
Step 2 Check for a difference of squares or a perfect square trinomial.	2 or 3	$9x^2 - 16 = (3x + 4)(3x - 4)$ $16x^2 + 24x + 9 = (4x + 3)^2$
Step 3 Apply the factoring patterns for $x^2 + bx + c$ or $ax^2 + bx + c$ (general trinomials), or factor by grouping.	3 or 4	$x^2 - 8x + 12 = (x - 2)(x - 6)$ $2x^2 + 13x + 6 = (2x + 1)(x + 6)$ $12y^2 + 9y + 8y + 6$ $= (12y^2 + 9y) + (8y + 6)$ $= 3y(4y + 3) + 2(4y + 3)$ $= (4y + 3)(3y + 2)$

1 Factor Perfect Square Trinomials

Example 1 shows how to determine whether a trinomial is a perfect square trinomial and, if it is, how to factor it. **Example 2** shows how to use various factoring methods to factor a polynomial completely.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Example

1 Determine whether each trinomial is a perfect square trinomial. Write *yes* or *no*. If it is a perfect square, factor it.

a. $25x^2 - 30x + 9$ yes; $(5x - 3)^2$

b. $49y^2 + 42y + 36$ no

WatchOut!

Preventing Errors Students should be reminded to look closely at the coefficient of the second term of a perfect square trinomial. Its sign determines whether the factors are in the form $(a + b)$ or $(a - b)$.

Teach with Tech

Blog On your secure classroom blog have students create a blog entry explaining the inverse relationship between squaring a binomial and factoring a perfect square trinomial.

WatchOut!

Common Errors Students often fail to factor polynomials completely. Point out that $4x^2 - 36$ is a difference of squares and can be factored as $(2x - 6)(2x + 6)$, but remind students that a polynomial is not considered completely factored if the terms of any of its factors have a GCF greater than 1.

Student Misconceptions Students have been taught that second-degree equations will have two solutions, so they may be confused when an equation involving a perfect square trinomial has only one solution. Explain that perfect square trinomials have a repeated factor, so that both solutions are the same number. Thus, only one solution is listed.

Additional Example

2 Factor each polynomial if possible. If the polynomial cannot be factored, write *prime*.

a. $6x^2 - 96$ $6(x + 4)(x - 4)$

b. $16y^2 + 8y - 15$ $(4y + 5)(4y - 3)$

2 Solve Equations with Perfect Squares

Example 3 shows how to solve equations with repeated factors. **Examples 4** and **5** show how to solve equations using the Square Root Property.

Additional Example

3 Solve $4x^2 + 36x = -81$. $-\frac{9}{2}$

Example 2 Factor Completely

Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*.

a. $5x^2 - 80$

Step 1 The GCF of $5x^2$ and -80 is 5, so factor it out.

Step 2 Since there are two terms, check for a difference of squares.

$5x^2 - 80 = 5(x^2 - 16)$ — 5 is the GCF of the terms.

$= 5(x^2 - 4^2)$ — $x^2 = x \cdot x$ and $16 = 4 \cdot 4$

$= 5(x - 4)(x + 4)$ — Factor the difference of squares.

b. $9x^2 - 6x - 35$

Step 1 The GCF of $9x^2$, $-6x$, and -35 is 1.

Step 2 Since 35 is not a perfect square, this is not a perfect square trinomial.

Step 3 Factor using the pattern $ax^2 + bx + c$. Are there two numbers with a product of $9(-35)$ or -315 and a sum of -6? Yes, the product of 15 and -21 is -315, and the sum is -6.

$9x^2 - 6x - 35 = 9x^2 + mx + px - 35$ — Write the pattern.

$= 9x^2 + 15x - 21x - 35$ — $m = 15$ and $n = -21$

$= (9x^2 + 15x) + (-21x - 35)$ — Group terms with common factors.

$= 3x(3x + 5) - 7(3x + 5)$ — Factor out the GCF from each grouping.

$= (3x + 5)(3x - 7)$ — $3x + 5$ is the common factor.

StudyTip

Check Your Answer You can check your answer by:

- Using the FOIL method.
- Using the Distributive Property.
- Graphing the original expression and factored expression and comparing the graphs.

If the product of the factors does not match the original expression exactly, the answer is incorrect.

GuidedPractice

2A. $2x^2 - 32$ $2(x - 4)(x + 4)$

2B. $12x^2 + 5x - 25$ $(4x - 5)(3x + 5)$

2 Solve Equations with Perfect Squares When solving equations involving repeated factors, it is only necessary to set one of the repeated factors equal to zero.

Example 3 Solve Equations with Repeated Factors

Solve $9x^2 - 48x = -64$.

$9x^2 - 48x = -64$ — Original equation

$9x^2 - 48x + 64 = 0$ — Add 64 to each side.

$(3x)^2 - 2(3x)(8) + (8)^2 = 0$ — Recognize $9x^2 - 48x + 64$ as a perfect square trinomial.

$(3x - 8)^2 = 0$ — Factor the perfect square trinomial.

$(3x - 8)(3x - 8) = 0$ — Write $(3x - 8)^2$ as two factors.

$3x - 8 = 0$ — Set the repeated factor equal to zero.

$3x = 8$ — Add 8 to each side.

$x = \frac{8}{3}$ — Divide each side by 3.

GuidedPractice

Solve each equation. Check your solutions.

3A. $a^2 + 12a + 36 = 0$ -6

3B. $y^2 - \frac{4}{3}y + \frac{4}{9} = 0$ $\frac{2}{3}$

Focus on Mathematical Content

Factor Perfect Square Trinomials Once a trinomial has been determined to be a perfect square trinomial, it can be factored into two identical binomials or expressed as a binomial squared. The binomial factors are the sum or difference, depending on the sign of the middle term of the trinomial, of the principal square roots of the first term and last term of the trinomial.

You have solved equations like $x^2 - 16 = 0$ by factoring. You can also use the definition of a square root to solve the equation.

$$x^2 - 16 = 0 \quad \text{Original equation}$$
$$x^2 = 16 \quad \text{Add 16 to each side.}$$
$$x = \pm\sqrt{16} \quad \text{Take the square root of each side.}$$

ReadingMath
Square Root Solutions $\pm\sqrt{16}$ is read as plus or minus the square root of 16.

Remember that there are two square roots of 16, namely 4 and −4. Therefore, the solution set is $\{-4, 4\}$. You can express this as $\{\pm 4\}$.

KeyConcept Square Root Property

Words	To solve a quadratic equation in the form $x^2 = n$, take the square root of each side.
Symbols	For any number $n \geq 0$, if $x^2 = n$, then $x = \pm\sqrt{n}$.
Example	$x^2 = 25$ $x = \pm\sqrt{25}$ or ± 5

In the equation $x^2 = n$, if n is not a perfect square, you need to approximate the square root. Use a calculator to find an approximation. If n is a perfect square, you will have an exact answer.

StudyTip
Solving by Inspection Equations involving square roots can often be solved mentally. For $x^2 = n$, think: *The square of what number is n?* When n is a perfect square, x is rational. Otherwise, x is irrational.

Example 4 Use the Square Root Property

Solve each equation. Check your solutions.

a. $(y - 6)^2 = 81$

$(y - 6)^2 = 81$ — Original equation

$y - 6 = \pm\sqrt{81}$ — Square Root Property

$y - 6 = \pm 9$ — $81 = 9 \cdot 9$

$y = 6 \pm 9$ — Add 6 to each side.

$y = 6 + 9$ or $y = 6 - 9$ — Separate into two equations.

$= 15$ $\quad = -3$ — Simplify.

The roots are 15 and −3. — Check in the original equation.

b. $(x + 6)^2 = 12$

$(x + 6)^2 = 12$ — Original equation

$x + 6 = \pm\sqrt{12}$ — Square Root Property

$x = -6 \pm \sqrt{12}$ — Subtract 6 from each side.

The roots are $-6 \pm \sqrt{12}$ or $-6 + \sqrt{12}$ and $-6 - \sqrt{12}$.

Using a calculator, $-6 + \sqrt{12} \approx -2.54$ and $-6 - \sqrt{12} \approx -9.46$.

GuidedPractice

4A. $(a - 10)^2 = 121$ **21, −1**

4B. $(z + 3)^2 = 26$ **$-3 \pm \sqrt{26}$ or about 2.1 and −8.1**

Additional Example

4 Solve each equation. Check the solutions.

a. $(b - 7)^2 = 36$ **1, 13**

b. $(x + 9)^2 = 8$ **$-9 \pm 2\sqrt{2}$**

Tips for New Teachers

Solutions to Second-Degree Equations If students do not understand how a second-degree equation can have only one solution, suggest that they graph a perfect square trinomial on a graphing calculator. The graph will immediately reveal how this is possible. The vertex of the graph of a perfect square trinomial equation lies on the x-axis, and therefore the equation has only one solution.

Additional Example

5 **PHYSICAL SCIENCE** A book falls from a shelf that is 5 meters above the floor. A model for the height h in meters if an object dropped from an initial height of h_0 meters is $h = -16t^2 + h_0$, where t is the time in seconds after the object is dropped. Use this model to determine approximately how long it took for the book to reach the ground. 0.56 s

Tips for New Teachers

Formulas Students are naturally curious about *why* certain formulas work. Encourage this curiosity and ask students to develop a way to test the formula $h = -16t^2 + h_0$ from Example 5. What assumptions must be made for the formula to hold true?

3 Practice

Formative Assessment

Use Exercises 1–11 to check for understanding.

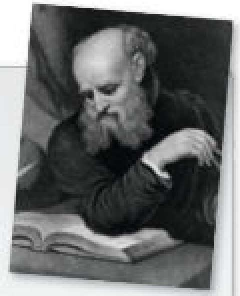
Use the chart at the bottom of this page to customize assignments for your students.

WatchOut!

Factoring Remind students that any of the factoring methods they have studied thus far can be used in the exercises.

Teaching the Mathematical Practices

Reasoning Mathematically proficient students make sense of quantities and their relationships in problem situations. In Exercise 11, ask students how to represent the height of the paintbrush when it hits the floor.



Math History Link

Galileo Galilei (1564–1642) Galileo was the first person to prove that objects of different weights fall at the same velocity by dropping two objects of different weights from the top of the Leaning Tower of Pisa in 1589.

Real-World Example 5 Solve an Equation

PHYSICAL SCIENCE **During an experiment, a ball is dropped from a height of 205 feet. The formula $h = -16t^2 + h_0$ can be used to approximate the number of seconds t it takes for the ball to reach height h from an initial height of h_0 in feet. Find the time it takes the ball to reach the ground.**

At ground level, $h = 0$ and the initial height is 205, so $h_0 = 205$.

$h = -16t^2 + h_0$	Original Formula
$0 = -16t^2 + 205$	Replace h with 0 and h_0 with 205.
$-205 = -16t^2$	Subtract 205 from each side.
$12.8125 = t^2$	Divide each side by −16.
$\pm 3.6 \approx t$	Use the Square Root Property.

Since a negative number does not make sense in this situation, the solution is 3.6. It takes about 3.6 seconds for the ball to reach the ground.

Guided Practice

5. Find the time it takes a ball to reach the ground if it is dropped from a bridge that is half as high as the one described above. about 2.5 seconds

Check Your Understanding

Example 1 Determine whether each trinomial is a perfect square trinomial. Write *yes* or *no*. If so, factor it.

1. $25x^2 + 60x + 36$ yes; $(5x + 6)^2$
2. $6x^2 + 30x + 36$ no

Example 2 Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*.

3. $2x^2 - x - 28$ $(x - 4)(2x + 7)$
4. $6x^2 - 34x + 48$ $2(x - 3)(3x - 8)$
5. $4x^2 + 64$ $4(x^2 + 16)$
6. $4x^2 + 9x - 16$ prime

Examples 3–4 Solve each equation. Confirm your answers using a graphing calculator.

7. $4x^2 = 36$ ± 3
8. $25a^2 - 40a = -16$ $\frac{4}{5}$
9. $64y^2 - 48y + 18 = 9$ $\frac{3}{8}$
10. $(z + 5)^2 = 47$ $-5 \pm \sqrt{47}$ or about −11.86 and 1.86

Example 5

11. **REASONING** While painting his bedroom, Hassan drops his paintbrush off his ladder from a height of 6 feet. Use the formula $h = -16t^2 + h_0$ to approximate the number of seconds it takes for the paintbrush to hit the floor. 0.6 second

Practice and Problem Solving

Example 1 Determine whether each trinomial is a perfect square trinomial. Write *yes* or *no*. If so, factor it.

12. $4x^2 - 42x + 110$ no
13. $16x^2 - 56x + 49$ yes; $(4x - 7)^2$
14. $81x^2 - 90x + 25$ yes; $(9x - 5)^2$
15. $x^2 + 26x + 168$ no

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	12–47, 53, 55–89	13–47 odd, 61–64	12–46 even, 53, 55–60, 65–89
OL Core	13–47 odd, 48–53, 55–89	12–47, 61–64	48–53, 55–60, 65–89
BL Advanced	48–79, (optional: 80–89)		

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Example 2 Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*. 28. $(x + 2y)(x - 2)(x + 2)$ 30. $(r - 6)(r + 6)(2r - 1)$ 32. $2cd(c^2 + d^2)(2c - 5)$

16. $24d^2 + 39d - 18$ $3(8d - 3)(d + 2)$
17. $8x^2 + 10x - 21$ prime
18. $2b^2 + 12b - 24$ $2(b^2 + 6b - 12)$
19. $8y^2 - 200z^2$ $8(y - 5z)(y + 5z)$
20. $16a^2 - 121b^2$ $(4a - 11b)(4a + 11b)$
21. $12m^3 - 22m^2 - 70m$ $2m(2m - 7)(3m + 5)$
22. $8c^2 - 88c + 242$ $2(2c - 11)^2$
23. $12x^2 - 84x + 147$ $3(2x - 7)^2$
24. $w^4 - w^2$ $w^2(w - 1)(w + 1)$
25. $12p^3 - 3p$ $3p(2p + 1)(2p - 1)$
26. $16q^3 - 48q^2 + 36q$ $4q(2q - 3)^2$
27. $4t^3 + 10t^2 - 84t$ $2t(t + 6)(2t - 7)$
28. $x^3 + 2x^2y - 4x - 8y$
29. $2a^2b^2 - 2a^2 - 2ab^3 + 2ab$ $2a(a - b)(b + 1)(b - 1)$
30. $2r^3 - r^2 - 72r + 36$
31. $3k^3 - 24k^2 + 48k$ $3k(k - 4)(k - 4)$
32. $4c^4d - 10c^3d + 4c^2d^3 - 10cd^3$
33. $g^2 + 2g - 3h^2 + 4h$ prime

Examples 3–4 Solve each equation. Confirm your answers using a graphing calculator.

34. $4m^2 - 24m + 36 = 0$ 3
35. $(y - 4)^2 = 7$ $4 \pm \sqrt{7}$
36. $a^2 + \frac{10}{7}a + \frac{25}{49} = 0$ $-\frac{5}{7}$
37. $x^2 - \frac{3}{2}x + \frac{9}{16} = 0$ $\frac{3}{4}$
38. $x^2 + 8x + 16 = 25$ 1, −9
39. $5x^2 - 60x = -180$ 6
40. $4x^2 = 80x - 400$ 10
41. $9 - 54x = -81x^2$ $\frac{1}{3}$
42. $4c^2 + 4c + 1 = 15$ $\frac{-1 \pm \sqrt{15}}{2}$
43. $x^2 - 16x + 64 = 6$ $8 \pm \sqrt{6}$

44. **PHYSICAL SCIENCE** For an experiment in physics class, a water balloon is dropped from the window of the school building. The window is 40 feet high. How long does it take until the balloon hits the ground? Round to the nearest hundredth. 1.58 seconds

45. **SCREENS** The area A in square meters of a projected picture on a movie screen can be modeled by the equation $A = 0.25d^2$, where d represents the distance from a projector to a movie screen. At what distance will the projected picture have an area of 100 square meters? 20 m

Example 5

46. **GEOMETRY** The area of a square is represented by $9x^2 - 42x + 49$. Find the length of each side. $|3x - 7|$

47. **GEOMETRY** The area of a square is represented by $16x^2 + 40x + 25$. Find the length of each side. $|4x + 5|$

B 48. **GEOMETRY** The volume of a rectangular prism is represented by the expression $8y^3 + 40y^2 + 50y$. Find the possible dimensions of the prism if the dimensions are represented by polynomials with integer coefficients. Sample answer: $2y$, $2y + 5$, $2y + 5$

49. **POOLS** Eissa wants to buy an above-ground swimming pool for his yard. Model A is 106.7 centimeters deep and holds 49.6 cubic meters of water. The length of the rectangular pool is 1.5 meters more than the width.

a. What is the surface area of the water? $500\ m^2$

b. What are the dimensions of the pool? 20 m by 25 m by 42 cm

c. Model B pool holds twice as much water as Model A. What are some possible dimensions for this pool? Sample answer: 20 m by 50 m by 42 cm

d. Model C has length and width that are both twice as long as Model A, but the height is the same. What is the ratio of the volume of Model A to Model C? 1:4

Teaching the Mathematical Practices

Precision Mathematically proficient students are careful about specifying units of measure. In Exercise 51, make sure students remember that volume is measured in cubic units and length is in linear units.

WatchOut!

Error Analysis For Exercise 53, point out to students that they must always take a final look at the factors to be sure they are all factored completely.

Teaching the Mathematical Practices

Regularity Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. In Exercise 57, ask students to write their answers as steps that can be used to factor any polynomial.

Additional Answers

56. Sample answer: The equation $x^3 + x^2 + x + 1 = 0$ only has one solution. The polynomial factors to $(x^2 + 1)(x + 1)$ and setting those factors equal to zero gives us only one solution $x = -1$ since $x^2 + 1 = 0$ has no real solutions.

57. First look for a GCF in all the terms and factor the GCF out of all the terms. Then, if the polynomial has two terms, check if the terms are the differences of squares and factor if so. If the polynomial has three terms, check if the polynomial if a perfect square polynomial and factor if so. If the polynomial has four or more terms, factor by grouping. If the polynomial does not have a GCF and cannot be factored, the polynomial is a prime polynomial.

60. Determine if the first and last terms are perfect squares. Then determine if the middle term must be equal to ± 2 times the principal square roots of the first and last terms. If these three criteria are met, the trinomial is a perfect square trinomial.

50. GEOMETRY Use the rectangular prism at the right.

a. Write an expression for the height and width of the prism in terms of the length, ℓ. **height = $\ell - 6$; width = $\ell - 10$**

b. Write a polynomial for the volume of the prism in terms of the length. **$V = \ell^3 - 16\ell^2 + 60\ell$**

51. PRECISION A zoo has an aquarium shaped like a rectangular prism. It has a volume of 5 cubic meters. The height of the aquarium is 2.7 meters taller than the width, and the length is 1.2 meter shorter than the width. What are the dimensions of the aquarium? **6 meters wide by 2 meters long by 15 meters high**

52. ELECTION Faleh is building a box shown with a volume of 96 cubic centimeters. What are the dimensions of the voting box? **4 cm high by 12 cm long by 2 cm wide**

H.O.T. Problems Use Higher-Order Thinking Skills

53. ERROR ANALYSIS Huda and Hana are factoring the expression $x^8 - x^4$ completely. Is either of them correct? Explain your reasoning.

Huda	Hana
$x^8 - x^4 = x^4(x^2 + 1)(x^2 - 1)$	$x^8 - x^4 = x^4(x^2 + 1)(x - 1)(x + 1)$

Hana; Huda did not factor the expression completely.

54. CHALLENGE Factor $x^{n+6} + x^{n+2} + x^n$ completely. **$x^n(x^6 + x^2 + 1)$**

55. OPEN ENDED Write a perfect square trinomial equation in which the coefficient of the middle term is negative and the last term is a fraction. Solve the equation. **Sample answer: $x^2 - 3x + \frac{9}{4} = 0; \left\{\frac{3}{2}\right\}$**

56. REASONING A counterexample is a specific case in which a statement is false. Find a counterexample to the following statement. **See margin.**

A polynomial equation of degree three always has three real solutions.

57. REGULARITY Explain how to factor a polynomial completely. **See margin.**

58. WHICH ONE DOESN'T BELONG? Identify the trinomial that does not belong. Explain.

$4x^2 - 36x + 81$	$25x^2 + 10x + 1$	$4x^2 + 10x + 4$	$9x^2 - 24x + 16$

$4x^2 + 10x + 4$ because it is the only expression that is not a perfect square trinomial.

59. OPEN ENDED Write a binomial that can be factored using the difference of two squares twice. Set your binomial equal to zero and solve the equation. **Sample answer: $x^4 - 1$; 1, −1**

60. WRITING IN MATH Explain how to determine whether a trinomial is a perfect square trinomial. **See margin.**

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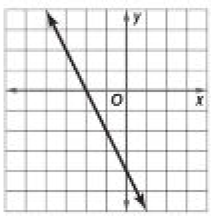
Standardized Test Practice

61. What is the solution set for the equation $(x - 3)^2 = 25$? **B**

A $\{-8, 2\}$ C $\{4, 14\}$
B $\{-2, 8\}$ D $\{-4, 14\}$

62. SHORT RESPONSE Write an equation in slope-intercept form for the graph shown below. $y = -2x - 4$



63. At an amphitheater, the price of 2 lawn seats and 2 pavilion seats is AED 120. The price of 3 lawn seats and 4 pavilion seats is AED 225. How much do lawn and pavilion seats cost? **H**

F AED 20 and AED 41.25
G AED 10 and AED 50
H AED 15 and AED 45
J AED 30 and AED 30

64. GEOMETRY The circumference of a circle is $\frac{6\pi}{5}$ units. What is the area of the circle? **A**

A $\frac{9\pi}{25}$ units^2 C $\frac{6\pi}{5}$ units^2
B $\frac{3\pi}{5}$ units^2 D $\frac{12\pi}{5}$ units^2

Spiral Review

Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*. (Lesson 7-8)

65. $x^2 - 16$ $(x - 4)(x + 4)$
66. $4x^2 - 81y^2$ $(2x - 9y)(2x + 9y)$
67. $1 - 100p^2$ $(1 - 10p)(1 + 10p)$
68. $3a^2 - 20$ prime
69. $25n^2 - 1$ $(5n - 1)(5n + 1)$
70. $36 - 9c^2$ $9(2 - c)(2 + c)$

Solve each equation. Confirm your answers using a graphing calculator. (Lesson 7-7)

71. $4x^2 - 8x - 32 = 0$ $\{-2, 4\}$
72. $6x^2 - 48x + 90 = 0$ $\{3, 5\}$
73. $14x^2 + 14x = 28$ $\{-2, 1\}$
74. $2x^2 - 10x = 48$ $\{-3, 8\}$
75. $5x^2 - 25x = -30$ $\{2, 3\}$
76. $8x^2 - 16x = 192$ $\{-4, 6\}$

77. AMUSEMENT RIDE The height h in feet of a car above the exit ramp of a free-fall ride can be modeled by $h(t) = -16t^2 + s$. t is the time in seconds after the car drops, and s is the starting height of the car in feet. If the designer wants the ride to last 3 seconds, what should be the starting height in feet? (Lesson 7-9) 144 m

Factor each polynomial. If the polynomial cannot be factored, write *prime*. (Lesson 7-8)

78. $x^2 - 81$ $(x - 9)(x + 9)$
79. $a^2 - 121$ $(a - 11)(a + 11)$
80. $n^2 + 100$ prime
81. $-25 + 4y^2$ $(2y - 5)(2y + 5)$
82. $p^4 - 16$ $(p - 2)(p + 2)(p^2 + 4)$
83. $4t^4 - 4$ $4(t - 1)(t + 1)(t^2 + 1)$

Skills Review

Find the slope of the line that passes through each pair of points.

84. $(5, 7), (-2, -3)$ $\frac{10}{7}$
85. $(2, -1), (5, -3)$ $-\frac{2}{3}$
86. $(-4, -1), (-3, -3)$ -2
87. $(-3, -4), (5, -1)$ $\frac{3}{8}$
88. $(-2, 3), (8, 3)$ 0
89. $(-5, 4), (-5, -1)$ undefined

4 Assess

Name the Math Have each student tell a partner or write how to determine if a trinomial is a perfect square trinomial.

Follow-up

Students have explored factoring.

Ask:

- What are the limitations of mathematical modeling? Sample answer: Not all situations can be modeled. For those that can, the domain of the model may need to be limited. For example, with a model that represents the height of an object over time, height and time cannot be negative. Also, model may only produce approximate answers. Therefore, after a model is created, it should be carefully analyzed before it is used to make predictions/decisions.

Differentiated Instruction OL BL

Extension Write the following polynomial on the board for students to factor completely: $(m - p)m^2 - 2m(m - p) + (m - p)$

$(m - p)(m - 1)(m - 1)$ or $(m - p)(m - 1)^2$

1 Focus

Vertical Alignment

Before Lesson 7-10 Factor perfect square trinomials.

Lesson 7-10 Determine the number and type of roots for a polynomial equation. Find the zeros of a polynomial function.

After Lesson 7-10 Analyze the characteristics of graph of quadratic functions.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- According to the equation, what was the average price of a liter of petrol in 1990? about AED1.37
- Would the given equation be valid for negative values of x? Possibly; negative values of x would correspond to years *before* 1990.

LESSON 7-10 Roots and Zeros

Then

- You used complex numbers to describe solutions of quadratic equations.

Now

1. Determine the number and type of roots for a polynomial equation.
2. Find the zeros of a polynomial function.

Why?

- The function $g(x) = 1.384x^4 - 0.003x^3 + 0.28x^2 - 0.078x + 1.365$ can be used to model the average price of a gallon of petrol in a given year if x is the number of years since 1990. To find the average price of petrol in a specific year, you can use the roots of the related polynomial equation.

Mathematical Practices
Attend to precision.

1 Synthetic Types of Roots Previously, you learned that a zero of a function $f(x)$ is any value c such that $f(c) = 0$. When the function is graphed, the real zeros of the function are the x-intercepts of the graph.

Concept Summary Zeros, Factors, Roots, and Intercepts

Words Let $P(x) = a_nx^n + \cdots + a_1x + a_0$ be a polynomial function. Then the following statements are equivalent.

- c is a zero of $P(x)$.
- c is a root or solution of $P(x) = 0$.
- $x - c$ is a factor of $a_nx^n + \cdots + a_1x + a_0$.
- If c is a real number, then $(c, 0)$ is an x-intercept of the graph of $P(x)$.

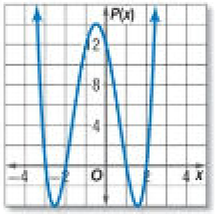
Example Consider the polynomial function $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$.

The zeros of $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$ are $-3, -2, 1$, and 2.

The roots of $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ are $-3, -2, 1$, and 2.

The factors of $x^4 + 2x^3 - 7x^2 - 8x + 12$ are $(x + 3), (x + 2), (x - 1)$, and $(x - 2)$.

The x-intercepts of the graph of $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$ are $(-3, 0), (-2, 0), (1, 0)$, and $(2, 0)$.



When solving a polynomial equation with degree greater than zero, there may be one or more real roots or no real roots (the roots are imaginary numbers). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero will have at least one root in the set of complex numbers. This is the **Fundamental Theorem of Algebra**.

Key Concept Fundamental Theorem of Algebra

Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.

Example 1 Determine Number and Type of Roots

Solve each equation. State the number and type of roots.

a. $x^2 + 6x + 9 = 0$

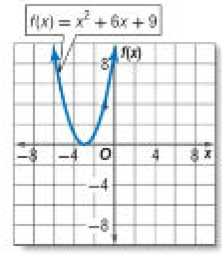
$x^2 + 6x + 9 = 0$ Original equation

$(x + 3)^2 = 0$ Factor.

$x + 3 = 0$ Take the root of each side.

$x = -3$ Solve for x.

Because $(x + 3)$ is twice a factor of $x^2 + 6x + 9$, -3 is a double root. Thus, the equation has one real repeated root, -3.



CHECK The graph of the equation touches the x-axis at $x = -3$. Since -3 is a double root, the graph does not cross the axis. ✓

ReadingMath

Repeated Roots Polynomial equations can have double roots, triple roots, quadruple roots, and so on. In general, these are referred to as *multiple roots*.

b. $x^3 + 25x = 0$

$x^3 + 25x = 0$ Original equation

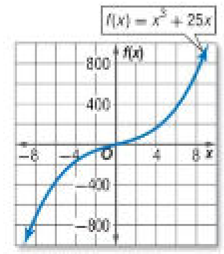
$x(x^2 + 25) = 0$ Factor.

$x = 0$ or $x^2 + 25 = 0$

$x^2 = -25$

$x = \pm\sqrt{-25}$ or $\pm 5i$

This equation has one real root, 0, and two imaginary roots, $5i$ and $-5i$.



CHECK The graph of this equation crosses the x-axis at only one place, $x = 0$. ✓

GuidedPractice **1A.** $0, i\sqrt{2}, -i\sqrt{2}$; 1 real, 2 imaginary

1A. $x^3 + 2x = 0$

1B. $x^4 - 16 = 0$ $2, -2, 2i, -2i$; 2 real, 2 imaginary

1C. $x^3 + 4x^2 - 7x - 10 = 0$ **1C.** $-1, 2, -5$; 3 real

1D. $3x^3 - x^2 + 9x - 3 = 0$

1D. $\frac{1}{3}, i\sqrt{3}, -i\sqrt{3}$; 1 real, 2 imaginary

Examine the solutions for each equation in Example 1. Notice that the number of solutions for each equation is the same as the degree of each polynomial. The following corollary to the Fundamental Theorem of Algebra describes this relationship between the degree and the number of roots of a polynomial equation.

KeyConcept Corollary to the Fundamental Theorem of Algebra

Words	A polynomial equation of degree n has exactly n roots in the set of complex numbers, including repeated roots.		
Example	$x^3 + 2x^2 + 6$ 3 roots	$4x^4 - 3x^3 + 5x - 6$ 4 roots	$-2x^5 - 3x^2 + 8$ 5 roots

Similarly, an nth degree polynomial function has exactly n zeros.

Additionally, French mathematician René Descartes discovered a relationship between the signs of the coefficients of a polynomial function and the number of positive and negative real zeros.

1 Synthetic Types of Roots

Example 1 shows how to determine the number and type of roots of a polynomial equation. **Example 2** shows how to determine the possible number of positive real zeros, negative real zeros, and imaginary zeros of a polynomial function.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Example

1 Solve each equation. State the number and type of roots.

a. $x^2 + 2x - 48 = 0$ This equation has two real roots, 6 and -8.

b. $y^4 - 256 = 0$ This equation has two real roots, 4 and -4, and two imaginary roots, $4i$ and $-4i$.

Focus on Mathematical Content

Zeros The real zeros of a polynomial function f are the x-intercepts of the graph of f. They are also the real solutions of the polynomial equation $f(x) = 0$. A polynomial function cannot have more zeros than its degree.

Teach with Tech

Interactive Whiteboard Write a polynomial function on the board. Show students how to apply Descartes' Rule of Signs by highlighting each change in signs in the coefficients.

Additional Example

2 State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $p(x) = -x^6 + 4x^3 - 2x^2 - x - 1$. The function has either 2 or 0 positive zeros, 2 or 0 negative zeros, and 6, 4, or 2 imaginary zeros.

WatchOut!

Common Misconceptions Point out to students the method for determining the number of imaginary zeros for a polynomial function. In Example 2, the polynomial has degree 6, so it has a maximum of 6 real zeros. You find the numbers of positive and negative real zeros, and subtract the sum of these two numbers from 6 to find the number of imaginary zeros. Remind students that imaginary zeros come in conjugate pairs, so the number of imaginary zeros must be an even number.

StudyTip

Zero at the Origin If a zero of a function is at the origin, the sum of the number of positive real zeros, negative real zeros, and imaginary zeros is reduced by how many times 0 is a zero of the function.

KeyConcept Descartes' Rule of Signs

Let $P(x) = a_nx^n + \cdots + a_1x + a_0$ be a polynomial function with real coefficients. Then

- the number of positive real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and
- the number of negative real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this by an even number.

Example 2 Find Numbers of Positive and Negative Zeros

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = x^6 + 3x^5 - 4x^4 - 6x^3 + x^2 - 8x + 5$.

Because $f(x)$ has degree 6, it has six zeros, either real or imaginary. Use Descartes' Rule of Signs to determine the possible number and type of *real* zeros.

Count the number of changes in sign for the coefficients of $f(x)$.

$$f(x) = x^6 + 3x^5 - 4x^4 - 6x^3 + x^2 - 8x + 5$$

no (+ to +), yes (+ to −), no (− to −), yes (− to +), yes (+ to −), yes (− to +)

There are 4 sign changes, so there are 4, 2, or 0 positive real zeros.

Count the number of changes in sign for the coefficients of $f(-x)$.

$$f(-x) = (-x)^6 + 3(-x)^5 - 4(-x)^4 - 6(-x)^3 + (-x)^2 - 8(-x) + 5$$
$$= x^6 - 3x^5 - 4x^4 + 6x^3 + x^2 + 8x + 5$$

yes (+ to −), no (− to −), yes (− to +), no (+ to +), no (+ to +), no (+ to +)

There are 2 sign changes, so there are 2, or 0 negative real zeros.
Make a chart of the possible combinations of real and imaginary zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros	Total Number of Zeros
4	2	0	$4 + 2 + 0 = 6$
4	0	2	$4 + 0 + 2 = 6$
2	2	2	$2 + 2 + 2 = 6$
2	0	4	$2 + 0 + 4 = 6$
0	2	4	$0 + 2 + 4 = 6$
0	0	6	$0 + 0 + 6 = 6$

GuidedPractice

2. State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $h(x) = 2x^5 + x^4 + 3x^3 - 4x^2 - x + 9$. 2 or 0; 3 or 1; 4, 2, or 0

2 Find Zeros You can use the various strategies and theorems you have learned to find all of the zeros of a function.

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Example 3 Use Synthetic Substitution to Find Zeros

Find all of the zeros of $f(x) = x^4 - 18x^2 + 12x + 80$.

Step 1 Determine the total number of zeros.

Since $f(x)$ has degree 4, the function has 4 zeros.

Step 2 Determine the type of zeros.

Examine the number of sign changes for $f(x)$ and $f(-x)$.

$f(x) = x^4 - 18x^2 + 12x + 80$ — yes, yes, no

$f(-x) = x^4 - 18x^2 - 12x + 80$ — yes, no, yes

Because there are 2 sign changes for the coefficients of $f(x)$, the function has 2 or 0 positive real zeros. Because there are 2 sign changes for the coefficients of $f(-x)$, $f(x)$ has 2 or 0 negative real zeros. Thus, $f(x)$ has 4 real zeros, 2 real zeros and 2 imaginary zeros, or 4 imaginary zeros.

Step 3 Determine the real zeros.

List some possible values, and then use synthetic substitution to evaluate $f(x)$ for real values of x.

x	1	0	−18	12	80
−3	1	−3	−9	39	−37
−2	1	−2	−14	40	0
−1	1	−1	−17	29	51
0	1	0	−18	12	80
1	1	1	−17	−5	75
2	1	2	−14	−2	76

Each row shows the coefficients of the depressed polynomial and the remainder.

From the table, we can see that one zero occurs at $x = -2$. Since there are 2 negative real zeros, use synthetic substitution with the depressed polynomial function $f(x) = x^3 - 2x^2 - 14x + 40$ to find a second negative zero.

x	1	−2	−14	40
−4	1	−6	10	0
−5	1	−7	21	−65
−6	1	−8	34	−164

A second negative zero is at $x = -4$. Since the depressed polynomial $x^2 - 6x + 10$ is quadratic, use the Quadratic Formula to find the remaining zeros of $f(x) = x^2 - 6x + 10$.

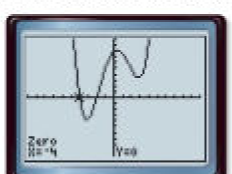
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic Formula

$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}$ Replace a with 1, b with −6, and c with 10.

$= 3 \pm i$ Simplify.

The function has zeros at −4, −2, $3 + i$, and $3 - i$.

CHECK Graph the function on a graphing calculator. The graph crosses the x-axis two times, so there are two real zeros. Use the zero function under the CALC menu to locate each zero. The two real zeros are −4 and −2.



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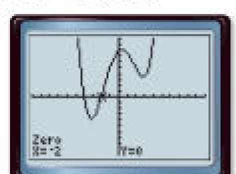

[−10, 10] scl: 1 by [−100, 100] scl: 10

StudyTip

Testing for Zeros If a value is not a zero for a polynomial, then it will not be a zero for the depressed polynomial either, so it does not need to be checked again.

StudyTip

Locating Zeros Refer to Lesson 4-2 on how to use the CALC menu to locate a zero on your calculator.

GuidedPractice

3. Find all of the zeros of $h(x) = x^3 + 2x^2 + 9x + 18$. **−2, 3*i*, −3*i***

2 Find Zeros

Example 3 shows how to find the zeros of a polynomial function using synthetic substitution. **Example 4** shows how to use zeros to write a polynomial function.

Additional Example

3 Find all of the zeros of $f(x) = x^3 - x^2 + 2x + 4$. $-1, 1 + i\sqrt{3}, 1 - i\sqrt{3}$.

DifferentiatedInstruction AL OL BL

If students sometimes make mistakes in mathematics exercises because they cannot read their own handwriting,

Then stress that throughout this course, students must work using neat and careful handwriting. It is extremely easy to misread coefficients and exponents, or misread ***i*** as the number 1.

Additional Example

4 Write a polynomial function of least degree with integral coefficients, the zeros of which include 4 and $4 - i$.
$f(x) = x^3 - 12x^2 + 49x - 68$

ReviewVocabulary
complex conjugates two complex numbers of the form $a + bi$ and $a - bi$

In Chapter 4, you learned that the product of complex conjugates is always a real number and that complex roots always come in conjugate pairs. For example, if one root of $x^2 - 8x + 52 = 0$ is $4 + 6i$, then the other root is $4 - 6i$.

This applies to the zeros of polynomial functions as well. For any polynomial function with real coefficients, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the **Complex Conjugates Theorem**.

KeyConcept Complex Conjugates Theorem

Words	Let a and b be real numbers, and $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.
Example	If $3 + 4i$ is a zero of $f(x) = x^3 - 4x^2 + 13x + 50$, then $3 - 4i$ is also a zero of the function.

When you are given all of the zeros of a polynomial function and are asked to determine the function, convert the zeros to factors and then multiply all of the factors together. The result is the polynomial function.

Example 4 Use Zeros to Write a Polynomial Function

Write a polynomial function of least degree with integral coefficients, the zeros of which include −1 and 5 − *i*.

Understand If $5 - i$ is a zero, then $5 + i$ is also a zero according to the Complex Conjugates Theorem. So, $x + 1$, $x - (5 - i)$, and $x - (5 + i)$ are factors of the polynomial.

Plan Write the polynomial function as a product of its factors.

$P(x) = (x + 1)[x - (5 - i)][x - (5 + i)]$

Solve Multiply the factors to find the polynomial function.

$P(x) = (x + 1)\,[x - (5 - i)][x - (5 + i)]$	Write the equation.
$= (x + 1)\,[(x - 5) + i][(x - 5) - i]$	Regroup terms.
$= (x + 1)\,[(x - 5)^2 - i^2]$	Difference of squares
$= (x + 1)\,[(x^2 - 10x + 25 - (-1)]$	Square terms.
$= (x + 1)\,(x^2 - 10x + 26)$	Simplify.
$= x^3 - 10x^2 + 26x + x^2 - 10x + 26$	Multiply.
$= x^3 - 9x^2 + 16x + 26$	Combine like terms.

Check Because there are 3 zeros, the degree of the polynomial function must be 3, so $P(x) = x^3 - 9x^2 + 16x + 26$ is a polynomial function of least degree with integral coefficients and zeros of -1, $5 - i$, and $5 + i$.

GuidedPractice

4. Write a polynomial function of least degree with integral coefficients having zeros that include -1 and $1 + 2i$. $f(x) = x^3 - x^2 + 3x + 5$

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Check Your Understanding

Example 1 Solve each equation. State the number and type of roots.

1. $x^2 - 3x - 10 = 0$ **−2, 5; 2 real**
2. $x^3 + 12x^2 + 32x = 0$ **−8, −4, 0; 3 real**
3. $16x^4 - 81 = 0$ **$-\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}i, \frac{3}{2}i$; 2 real, 2 imaginary**
4. $0 = x^3 - 8$ **$2, -1 + i\sqrt{3}, -1 - i\sqrt{3}$; 1 real, 2 imaginary**

Example 2 State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function. **6. 1; 1 or 3; 0 or 2**

5. $f(x) = x^3 - 2x^2 + 2x - 6$ **3 or 1; 0; 0 or 2**
6. $f(x) = 6x^4 + 4x^3 - x^2 - 5x - 7$
7. $f(x) = 3x^5 - 8x^3 + 2x - 4$ **1 or 3; 0 or 2; 0, 2, or 4**
8. $f(x) = -2x^4 - 3x^3 - 2x - 5$ **0; 0 or 2; 2 or 4**

Example 3 Find all of the zeros of each function.

9. $f(x) = x^3 + 9x^2 + 6x - 16$ **−8, −2, 1**
10. $f(x) = x^3 + 7x^2 + 4x + 28$ **$-7, -2i, 2i$**
11. $f(x) = x^4 - 2x^3 - 8x^2 - 32x - 384$ **$-4, 6, -4i, 4i$**
12. $f(x) = x^4 - 6x^3 + 9x^2 + 6x - 10$ **$-1, 1, 3 - i, 3 + i$**

Example 4 Write a polynomial function of least degree with integral coefficients that have the given zeros.

13. $4, -1, 6$ **$f(x) = x^3 - 9x^2 + 14x + 24$**
14. $3, -1, 1, 2$ **$f(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$**
15. $-2, 5, -3i$ **$f(x) = x^4 - 3x^3 - x^2 - 27x - 90$**
16. $-4, 4 + i$ **$f(x) = x^3 - 4x^2 - 15x + 68$**

Practice and Problem Solving

Example 1 Solve each equation. State the number and type of roots. **17–26. See margin.**

17. $2x^2 + x - 6 = 0$
18. $4x^2 + 1 = 0$
19. $x^3 + 1 = 0$
20. $2x^2 - 5x + 14 = 0$
21. $-3x^2 - 5x + 8 = 0$
22. $8x^3 - 27 = 0$
23. $16x^4 - 625 = 0$
24. $x^3 - 6x^2 + 7x = 0$
25. $x^5 - 8x^3 + 16x = 0$
26. $x^5 + 2x^3 + x = 0$

Example 2 State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

27. $f(x) = x^4 - 5x^3 + 2x^2 + 5x + 7$ **0 or 2; 0 or 2; 0, 2, or 4**
28. $f(x) = 2x^3 - 7x^2 - 2x + 12$ **0 or 2; 1; 0 or 2**
29. $f(x) = -3x^5 + 5x^4 + 4x^2 - 8$ **0 or 2; 1; 2 or 4**
30. $f(x) = x^4 - 2x^2 - 5x + 19$ **0 or 2; 0 or 2; 0, 2, or 4**
31. $f(x) = 4x^6 - 5x^4 - x^2 + 24$ **0 or 2; 0 or 2; 2, 4, or 6**
32. $f(x) = -x^5 + 14x^3 + 18x - 36$ **0 or 2; 1; 2 or 4**

Example 3 Find all of the zeros of each function.

33. $f(x) = x^3 + 7x^2 + 4x - 12$ **−6, −2, 1**
34. $f(x) = x^3 + x^2 - 17x + 15$ **−5, 1, 3**
35. $f(x) = x^4 - 3x^3 - 3x^2 - 75x - 700$ **$-4, 7, -5i, 5i$**
36. $f(x) = x^4 + 6x^3 + 73x^2 + 384x + 576$ **$-3, -3, -8i, 8i$**
37. $f(x) = x^4 - 8x^3 + 20x^2 - 32x + 64$ **37. $4, 4, -2i, 2i$**
38. $f(x) = x^5 - 8x^3 - 9x$ **$-3, 0, 3, -i, i$**

Example 4 Write a polynomial function of least degree with integral coefficients that have the given zeros. **39–44. See margin.**

39. $5, -2, -1$
40. $-4, -3, 5$
41. $-1, -1, 2i$
42. $-3, 1, -3i$
43. $0, -5, 3 + i$
44. $-2, -3, 4 - 3i$

B 45. **REASONING** A computer manufacturer determines that the profit for producing x computers per day is $P(x) = -0.006x^4 + 0.15x^3 - 0.05x^2 - 1.8x$.

a. How many positive real zeros, negative real zeros, and imaginary zeros exist? **2 or 0; 1; 1 or 3**

b. What is the meaning of the zeros in this situation?

45b. Nonnegative roots represent numbers of computers produced per day which lead to no profit for the manufacturer.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	17–48, 56, 58–75	17–47 odd, 61–64	18–48 even, 56, 58–60, 65–75
OL Core	17–49 odd, 53, 56, 58–75	17–48, 61–64	49–56, 58–60, 65–75
BL Advanced	49–73, (optional: 74–75)		

3 Practice

Formative Assessment

Use Exercises 1–16 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Teaching the Mathematical Practices

Reasoning Mathematically proficient students make sense of quantities and their relationships in problem situations. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Additional Answers

17. $-2, \frac{3}{2}$; 2 real
18. $-\frac{1}{2}i, \frac{1}{2}i$; 2 imaginary
19. $-1, \frac{1 \pm i\sqrt{3}}{2}$; 1 real, 2 imaginary
20. $\frac{5 \pm i\sqrt{87}}{4}$; 2 imaginary
21. $-\frac{8}{3}, 1$; 2 real
22. $\frac{3}{2}, \frac{-3 \pm 3i\sqrt{3}}{4}$; 1 real, 2 imaginary
23. $-\frac{5}{2}, \frac{5}{2}, -\frac{5}{2}i, \frac{5}{2}i$; 2 real, 2 imaginary
24. $0, 3 + \sqrt{2}, 3 - \sqrt{2}$; 3 real
25. $-2, -2, 0, 2, 2$; 5 real
26. $0, -i, -i, i, i$; 1 real, 4 imaginary
39. $f(x) = x^3 - 2x^2 - 13x - 10$
40. $f(x) = x^3 + 2x^2 - 23x - 60$
41. $f(x) = x^4 + 2x^3 + 5x^2 + 8x + 4$
42. $f(x) = x^4 + 2x^3 + 6x^2 + 18x - 27$
43. $f(x) = x^4 - x^3 - 20x^2 + 50x$
44. $f(x) = x^4 - 3x^3 - 9x^2 + 77x + 150$

Teaching the Mathematical Practices

Arguments Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. And they are able to analyze situations by breaking them into cases, and can recognize and use counterexamples.

Sketch the graph of each function using its zeros. See Chapter 7 Answer Appendix.

46. $f(x) = x^3 - 5x^2 - 2x + 24$

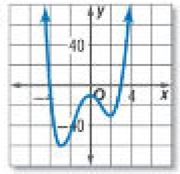
47. $f(x) = 4x^3 + 2x^2 - 4x - 2$

48. $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$

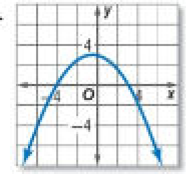
49. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

Match each graph to the given zeros.

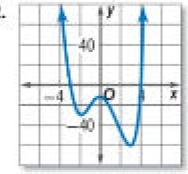
a. $-3, 4, i, -i$ b. $-4, 3$ c. $-4, 3, i, -i$

50. 

c

51. 

b

52. 

a

53c. 23.8; Sample answer: According to the model, the music hall will not earn any money after 2026.

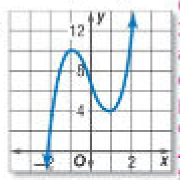
53. **CONCERTS** The amount of money a Music Hall took in from 2003 to 2010 can be modeled by $M(x) = -2.03x^3 + 50.1x^2 - 214x + 4020$, where x is the years since 2003.

a. How many positive real zeros, negative real zeros, and imaginary zeros exist? 3 or 1; 0; 2 or 0

b. Graph the function using your calculator. See Chapter 7 Answer Appendix.

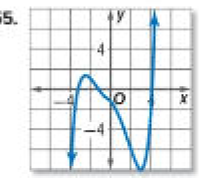
c. Approximate all real zeros to the nearest tenth. What is the significance of each zero in the context of the situation?

Determine the number of positive real zeros, negative real zeros, and imaginary zeros for each function. Explain your reasoning.

54. 

degree: 3

0 positive, 1 negative, 2 imaginary; Sample answer: The graph does not cross the positive x-axis, and crosses the negative x-axis once. Because the degree of the polynomial is 3, there are 3 − 1 or 2 imaginary zeros.

55. 

degree: 5

1 positive, 2 negative, 2 imaginary; Sample answer: The graph crosses the positive x-axis once, and crosses the negative x-axis twice. Because the degree of the polynomial is 5, there are 5 − 3 or 2 imaginary zeros.

H.O.T. Problems Use Higher-Order Thinking Skills

56. **OPEN ENDED** Sketch the graph of a polynomial function with: a–c. See Chapter 7 Answer Appendix.

a. 3 real, 2 imaginary zeros b. 4 real zeros c. 2 imaginary zeros

57. **CHALLENGE** Write an equation in factored form of a polynomial function of degree 5 with 2 imaginary zeros, 1 nonintegral zero, and 2 irrational zeros. Explain. See margin.

58. **ARGUMENTS** Determine which equation is not like the others. Explain. See margin.

$r^4 + 1 = 0$	$r^3 + 1 = 0$	$r^2 - 1 = 0$	$r^3 - 8 = 0$

59. **REASONING** Provide a counterexample for each statement.

a. All polynomial functions of degree greater than 2 have at least 1 negative real root. Sample answer: $f(x) = x^4 + 4x^2 + 4$

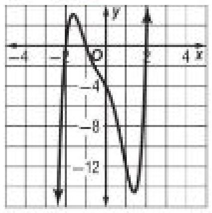
b. All polynomial functions of degree greater than 2 have at least 1 positive real root. Sample answer: $f(x) = x^3 + 6x^2 + 9x$

60. **WRITING IN MATH** Explain to a friend how you would use Descartes' Rule of Signs to determine the number of possible positive real roots and the number of possible negative roots of the polynomial function $f(x) = x^4 - 2x^3 + 6x^2 + 5x - 12$. See Chapter 7 Answer Appendix.

Standardized Test Practice

61. Use the graph of the polynomial function below. Which is not a factor of the polynomial $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$? **C**

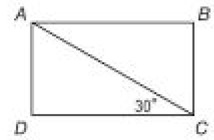
A $x - 2$
B $x + 2$
C $x - 1$
D $x + 1$



62. **SHORT RESPONSE** A structure is in the shape of an equilateral triangle. Each side of the triangle is 8 meters long. The structure is divided in half by a support from one vertex to the midpoint of the side of the triangle opposite the vertex. Approximately how long is the support? **6.9 meters**

63. **GEOMETRY** In rectangle $ABCD$, $\overline{AD}$ is 8 units long. What is the length of $\overline{AB}$? **H**

F 4 units
G 8 units
H $8\sqrt{3}$ units
J 16 units



64. **SAT/ACT** The total area of a rectangle is $25a^4 - 16b^2$ square units. Which factors could represent the length and width? **B**

A $(5a^2 + 4b)$ units and $(5a^2 + 4b)$ units
B $(5a^2 + 4b)$ units and $(5a^2 - 4b)$ units
C $(5a^2 - 4b)$ units and $(5a^2 - 4b)$ units
D $(5a - 4b)$ units and $(5a - 4b)$ units
E $(5a + 4b)$ units and $(5a - 4b)$ units

Spiral Review

65. **GENETICS** Brown genes B are dominant over blue genes b. A person with genes BB or Bb has brown eyes. Someone with genes bb has blue eyes. Yasmin has brown eyes with genes Bb, and Ayoub has blue eyes. Write an expression for the possible eye coloring of their children. Then find the probability that a child would have blue eyes. (Lesson 7-4) **$0.5Bb + 0.5b^2$; $\frac{1}{2}$**

Factor each polynomial. (Lesson 7-6)

66. $x^2 - 4x - 21$ **$(x + 3)(x - 7)$**
67. $11x + x^2 + 30$ **$(x + 6)(x + 5)$**
68. $32 + x^2 - 12x$ **$(x - 8)(x - 4)$**
69. $-36 - 9x + x^2$ **$(x - 12)(x + 3)$**
70. $x^2 + 12x + 20$ **$(x + 10)(x + 2)$**
71. $-x + x^2 - 42$ **$(x - 7)(x + 6)$**

72. **MANUFACTURING** A company is designing a box in the shape of a rectangular prism. The length is 2 centimeters more than twice the width, and the height is 3 centimeters more than the length. Write an expression for the volume of the box. (Lesson 7-3) **$4w^3 + 14w^2 + 10w$**

Skills Review

Find all of the possible values of $\pm\frac{b}{a}$ for each replacement set. **73–75. See margin.**

73. $a = \{1, 2, 4\}$; $b = \{1, 2, 3, 6\}$
74. $a = \{1, 5\}$; $b = \{1, 2, 4, 8\}$
75. $a = \{1, 2, 3, 6\}$; $b = \{1, 7\}$

4 Assess

Formative Assessment

Crystal Ball Have students write how what they learned today about the Fundamental Theorem of Algebra and Descartes' Rule of Signs will help them with identifying all the rational zeros of a polynomial function in tomorrow's lesson.

Additional Answers

57. Sample answer: $f(x) = (x + 2i)(x - 2i)(3x + 5)(x + \sqrt{5})(x - \sqrt{5})$ Use conjugates for the imaginary and irrational values.

58. $r^4 + 1 = 0$; Sample answer: The equation has imaginary solutions and all of the others have real solutions.

73. $\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}$

74. $\pm1, \pm2, \pm4, \pm8, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{4}{5}, \pm\frac{8}{5}$

75. $\pm1, \pm7, \pm\frac{1}{2}, \pm\frac{7}{2}, \pm\frac{1}{3}, \pm\frac{7}{3}, \pm\frac{1}{6}, \pm\frac{7}{6}$

Differentiated Instruction OL BL

Extension Explain to students that if $(x - r)^k$ is a factor of a polynomial $f(x)$, then r is a zero of the polynomial function $f(x)$ and k is the degree of the factor that produced r. From this, it is said that r has a *multiplicity of k*. When $k = 1$, we say that the zero has a multiplicity of 1 and is often called a *simple zero*. Use the following example to explain multiplicity and simple zeros: $f(x) = x^7 - 5x^6 + 6x^5 + 4x^4 - 8x^3$. When factored, $f(x) = x^3(x - 2)^3(x + 1)$. This polynomial function has 3 zeros: $x = -1$ (a simple zero), $x = 0$ with a multiplicity of 3, and $x = 2$ with multiplicity of 3. Challenge students to find the zeros and their multiplicities for the polynomial $f(x) = (x^6 - x^4)(x + 1)$. This polynomial has 3 zeros: $x = 1$, which is a simple zero, $x = 0$ with a multiplicity of 4, and $x = -1$ with multiplicity of 2.

FOLDABLES StudyOrganizer

Dinah Zike's Foldables®

Have students look through the chapter to make sure they have included examples in their Foldables for each lesson of the chapter. Suggest that students keep their Foldables handy while completing the Study Guide and Review pages. Point out that their Foldables can serve as a quick review when studying for the chapter test.

CHAPTER 7 Study Guide and Review

Study Guide

KeyConcepts

Operations with Polynomials (Lessons 7-1 through 7-4)

- To add or subtract polynomials, add or subtract like terms.
- To multiply polynomials, use the Distributive Property.
- Special products: $(a + b)^2 = a^2 + 2ab + b^2$
  $(a - b)^2 = a^2 - 2ab + b^2$
  $(a + b)(a - b) = a^2 - b^2$

Factoring Using the Distributive Property (Lesson 7-5)

- Using the Distributive Property to factor polynomials with four or more terms is called factoring by grouping.
  $ax + bx + ay + by = x(a + b) + y(a + b)$
  $= (a + b)(x + y)$

Solving Quadratic Equations by Factoring (Lessons 7-6 through 7-8)

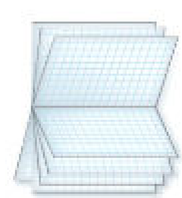
- To factor $x^2 + bx + c$, find m and p with a sum of b and a product of c. Then write $x^2 + bx + c$ as $(x + m)(x + p)$.
- To factor $ax^2 + bx + c$, find m and p with a sum of b and a product of ac. Then write as $ax^2 + mx + px + c$ and factor by grouping.
- $a^2 - b^2 = (a - b)(a + b)$

Perfect Squares and Factoring (Lesson 7-9)

- For a trinomial to be a perfect square, the first and last terms must be perfect squares, and the middle term must be twice the product of the square roots of the first and last terms.
- For any number $n \geq 0$, if $x^2 = n$, then $x = \pm\sqrt{n}$.

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.



KeyVocabulary

binomial	polynomial
degree of a monomial	prime polynomial
degree of a polynomial	quadratic equation
difference of two squares	quadratic expression
factoring	Square Root Property
factoring by grouping	standard form of a polynomial
FOIL method	trinomial
leading coefficient	Zero Product Property
perfect square trinomial	

VocabularyCheck

State whether each sentence is *true* or *false*. If *false*, replace the underlined phrase or expression to make a true sentence.

1. $x^2 + 5x + 6$ is an example of a prime polynomial. **false; sample answer: $x^2 + 5x + 7$**
2. $(x + 5)(x - 5)$ is the factorization of a difference of squares. **true**
3. $4x^2 - 2x + 7$ is a polynomial of degree 2. **true**
4. $(x + 5)(x - 2)$ is the factored form of $x^2 - 3x - 10$. **false; $(x - 5)(x + 2)$**
5. Expressions with four or more unlike terms can sometimes be factored by grouping. **true**
6. The Zero Product Property states that if $ab = 1$, then a or b is 1. **false; $ab = 0$, then a or b is 0, or both a and b are 0.**
7. $x^2 - 12x + 36$ is an example of a perfect square trinomial. **true**
8. The leading coefficient of $1 + 6a + 9a^2$ is 1. **false; 9**
9. $x^2 - 16$ is an example of a perfect square trinomial. **false; difference of squares**
10. The FOIL method is used to multiply two trinomials. **false; binomials**

7-1 Adding and Subtracting Polynomials

Write each polynomial in standard form. **11. $3x^2 + x + 2$**

11. $x + 2 + 3x^2$

12. $1 - x^4$ **$-x^4 + 1$**

13. $2 + 3x + x^2$ **$x^2 + 3x + 2$**

14. $3x^5 - 2 + 6x - 2x^2 + x^3$ **$3x^5 + x^3 - 2x^2 + 6x - 2$**

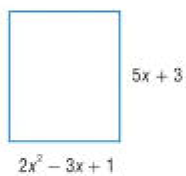
Find each sum or difference.

15. $(x^3 + 2) + (-3x^3 - 5)$ **$-2x^3 - 3$**

16. $a^2 + 5a - 3 - (2a^2 - 4a + 3)$ **$-a^2 + 9a - 6$**

17. $(4x - 3x^2 + 5) + (2x^2 - 5x + 1)$ **$-x^2 - x + 6$**

18. **PICTURE FRAMES** Noura is framing a painting that is a rectangle. What is the perimeter of the frame? **$4x^2 + 4x + 8$**



Example 1

Write $3 - x^2 + 4x$ in standard form.

Step 1 Find the degree of each term.

3:	degree 0
$-x^2$:	degree 2
$4x$:	degree 1

Step 2 Write the terms in descending order of degree.

$3 - x^2 + 4x = -x^2 + 4x + 3$

Example 2

Find $(8r^2 + 3r) - (10r^2 - 5)$.

$(8r^2 + 3r) - (10r^2 - 5)$

$= (8r^2 + 3r) + (-10r^2 + 5)$ Use the additive inverse.

$= (8r^2 - 10r^2) + 3r + 5$ Group like terms.

$= -2r^2 + 3r + 5$ Add like terms.

7-2 Multiplying a Polynomial by a Monomial

Solve each equation.

19. $x^2(x + 2) = x(x^2 + 2x + 1)$ **0**

20. $2x(x + 3) = 2(x^2 + 3)$ **1**

21. $2(4w + w^2) - 6 = 2w(w - 4) + 10$ **1**

22. **GEOMETRY** Find the area of the rectangle. **$3x^3 + 3x^2 - 21x$**

Example 3

Solve $m(2m - 5) + m = 2m(m - 6) + 16$.

$m(2m - 5) + m = 2m(m - 6) + 16$

$2m^2 - 5m + m = 2m^2 - 12m + 16$

$2m^2 - 4m = 2m^2 - 12m + 16$

$-4m = -12m + 16$

$8m = 16$

$m = 2$

7-3 Multiplying Polynomials

Find each product. **23. $x^2 + 4x - 21$ 24. $18a^2 + 3a - 10$**

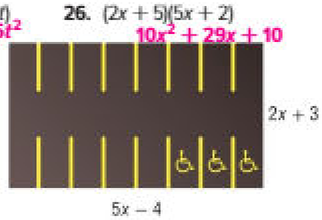
23. $(x - 3)(x + 7)$

24. $(3a - 2)(6a + 5)$

25. $(3r - 7t)(2r + 5t)$ **$6r^2 + rt - 35t^2$**

26. $(2x + 5)(5x + 2)$ **$10x^2 + 29x + 10$**

27. **PARKING LOT** The parking lot shown is to be paved. What is the area to be paved? **$10x^2 + 7x - 12$**



Example 4

Find $(6x - 5)(x + 4)$.

$(6x - 5)(x + 4)$

F O I L

$= (6x)(x) + (6x)(4) + (-5)(x) + (-5)(4)$

$= 6x^2 + 24x - 5x - 20$ Multiply.

$= 6x^2 + 19x - 20$ Combine like terms.

Lesson-by-Lesson Review

Intervention If the given examples are not sufficient to review the topics covered by the questions, remind students that the lesson references tell them where to review that topic in their textbooks.

CHAPTER 7 Study Guide and Review *Continued*

7-4 Special Products

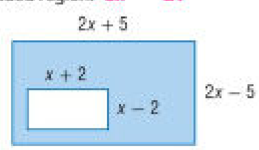
Find each product.

28. $(x + 5)(x - 5)$ $x^2 - 25$ **29.** $(3x - 2)^2$ $9x^2 - 12x + 4$

30. $(5x + 4)^2$ $25x^2 + 40x + 16$ **31.** $(2x - 3)(2x + 3)$ $4x^2 - 9$

32. $(2r + 5t)^2$ $4r^2 + 20rt + 25t^2$ **33.** $(3m - 2)(3m + 2)$ $9m^2 - 4$

34. **GEOMETRY** Write an expression to represent the area of the shaded region. $3x^2 - 21$



Example 5

Find $(x - 7)^2$.

$(a - b)^2 = a^2 - 2ab + b^2$	Square of a Difference
$(x - 7)^2 = x^2 - 2(x)(7) + (-7)^2$	$a = x$ and $b = 7$
$= x^2 - 14x + 49$	Simplify.

Example 6

Find $(5a - 4)(5a + 4)$.

$(a + b)(a - b) = a^2 - b^2$	Product of a Sum and Difference
$(5a - 4)(5a + 4) = (5a)^2 - (4)^2$	$a = 5a$ and $b = 4$
$= 25a^2 - 16$	Simplify.

7-5 Using the Distributive Property

Use the Distributive Property to factor each polynomial.

35. $12x + 24y$ $12(x + 2y)$

36. $14x^2y - 21xy + 35xy^2$ $7xy(2x - 3 + 5y)$

37. $8xy - 16x^3y + 10y$ $2y(4x - 8x^3 + 5)$

38. $a^2 - 4ac + ab - 4bc$ $(a + b)(a - 4c)$

39. $2x^2 - 3xz - 2xy + 3yz$ $(2x - 3z)(x - y)$

40. $24am - 9an + 40bm - 15bn$ $(3a + 5b)(8m - 3n)$

Solve each equation. Check your solutions.

41. $x(3x - 6) = 0$ 0, 2 **42.** $6x^2 = 12x$ 0, 2

43. $x^2 = 3x$ 0, 3 **44.** $3x^2 = 5x$ $0, \frac{5}{3}$

45. **GEOMETRY** The area of the rectangle shown is $x^3 - 2x^2 + 5x$ square units. What is the length? $x^2 - 2x + 5$

Example 7

Factor $12y^2 + 9y + 8y + 6$.

$12y^2 + 9y + 8y + 6$	
$= (12y^2 + 9y) + (8y + 6)$	Group terms with common factors.
$= 3y(4y + 3) + 2(4y + 3)$	Factor the GCF from each group.
$= (4y + 3)(3y + 2)$	Distributive Property

Example 8

Solve $x^2 - 6x = 0$. Check your solutions.

Write the equation so that it is of the form $ab = 0$.

$x^2 - 6x = 0$	Original equation
$x(x - 6) = 0$	Factor by using the GCF.
$x = 0$ or $x - 6 = 0$	Zero Product Property
$x = 6$	Solve.

The roots are 0 and 6. Check by substituting 0 and 6 for x in the original equation.

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7-6 Solving $x^2 + bx + c = 0$

Factor each trinomial. Confirm your answers using a graphing calculator.

46. $x^2 - 8x + 15$ **47.** $x^2 + 9x + 20$

48. $x^2 - 5x - 6$ **49.** $x^2 + 3x - 18$

Solve each equation. Check your solutions.

50. $x^2 + 5x - 50 = 0$ **−10, 5**

51. $x^2 - 6x + 8 = 0$ **2, 4**

52. $x^2 + 12x + 32 = 0$ **−8, −4**

53. $x^2 - 2x - 48 = 0$ **−6, 8**

54. $x^2 + 11x + 10 = 0$ **−10, −1**

46. $(x - 5)(x - 3)$
47. $(x + 5)(x + 4)$
48. $(x - 6)(x + 1)$
49. $(x + 6)(x - 3)$

55. **ART** An artist is working on a painting that is 3 centimeters longer than it is wide. The area of the painting is 154 square centimeters. What is the length of the painting? **14 cm**

Example 9

Factor $x^2 + 10x + 21$

$b = 10$ and $c = 21$, so $m + p$ is positive and mp is positive. Therefore, m and p must both be positive. List the positive factors of 21, and look for the pair of factors with a sum of 10.

Factors of 21	Sum of 10
1, 21	22
3, 7	10

The correct factors are 3 and 7.

$x^2 + 10x + 21 = (x + m)(x + p)$ Write the pattern.

$= (x + 3)(x + 7)$ $m = 3$ and $p = 7$

7-7 Solving $ax^2 + bx + c = 0$

Factor each trinomial, if possible. If the trinomial cannot be factored, write *prime*.

56. $12x^2 + 22x - 14$ **$2(2x - 1)(3x + 7)$**

57. $2y^2 - 9y + 3$ **prime**

58. $3x^2 - 6x - 45$ **$3(x - 5)(x + 3)$**

59. $2a^2 + 13a - 24$ **$(2a - 3)(a + 8)$**

Solve each equation. Confirm your answers using a graphing calculator.

60. $40x^2 + 2x = 24$ **$\frac{3}{4}, -\frac{4}{5}$**

61. $2x^2 - 3x - 20 = 0$ **$4, -\frac{5}{2}$**

62. $-16t^2 + 36t - 8 = 0$ **$2, \frac{1}{4}$**

63. $6x^2 - 7x - 5 = 0$ **$\frac{5}{3}, -\frac{1}{2}$**

64. **GEOMETRY** The area of the rectangle shown is $6x^2 + 11x - 7$ square units. What is the width of the rectangle? **$3x + 7$**

Example 10

Factor $12a^2 + 17a + 6$

$a = 12$, $b = 17$, and $c = 6$. Since b is positive, $m + p$ is positive. Since c is positive, mp is positive. So, m and p are both positive. List the factors of 12(6) or 72, where both factors are positive.

Factors of 72	Sum of 17
1, 72	73
2, 36	38
3, 24	27
4, 18	22
6, 12	18
8, 9	17

The correct factors are 8 and 9.

$12a^2 + 17a + 6 = 12a^2 + ma + pa + 6$
$= 12a^2 + 8a + 9a + 6$
$= (12a^2 + 8a) + (9a + 6)$
$= 4a(3a + 2) + 3(3a + 2)$
$= (3a + 2)(4a + 3)$

So, $12a^2 + 17a + 6 = (3a + 2)(4a + 3)$.

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CHAPTER 7 Study Guide and Review *Continued*

7-8 Differences of Squares

Factor each polynomial.

65. $y^2 - 81$ $(y + 9)(y - 9)$

66. $64 - 25x^2$ $(8 + 5x)(8 - 5x)$

67. $16a^2 - 21b^2$ prime

68. $3x^2 - 3$ $3(x + 1)(x - 1)$

Solve each equation by factoring. Confirm your answers using a graphing calculator.

69. $a^2 - 25 = 0$ $5, -5$

70. $9x^2 - 25 = 0$ $\frac{5}{3}, -\frac{5}{3}$

71. $81 - y^2 = 0$ $-9, 9$

72. $x^2 - 5 = 20$ $-5, 5$

73. **EROSION** A boulder falls down a mountain into water 64 feet below. The distance d that the boulder falls in t seconds is given by the equation $d = 16t^2$. How long does it take the boulder to hit the water? 2 seconds

Example 11

Solve $x^2 - 4 = 12$ by factoring.

$x^2 - 4 = 12$	Original equation
$x^2 - 16 = 0$	Subtract 12 from each side.
$x^2 - (4)^2 = 0$	$16 = 4^2$
$(x + 4)(x - 4) = 0$	Factor the difference of squares.
$x + 4 = 0$ or $x - 4 = 0$	Zero Product Property
$x = -4$ $\quad$ $x = 4$	Solve each equation.

The solutions are -4 and 4.

7-9 Perfect Squares

Factor each polynomial, if possible. If the polynomial cannot be factored write *prime*.

74. $x^2 + 12x + 36$ $(x + 6)^2$

75. $x^2 + 5x + 25$ prime

76. $9y^2 - 12y + 4$ $(3y - 2)^2$

77. $4 - 28a + 49a^2$ $(2 - 7a)^2$

78. $x^4 - 1$ $(x^2 + 1)(x + 1)(x - 1)$

79. $x^4 - 16x^2$ $x^2(x + 4)(x - 4)$

Solve each equation. Confirm your answers using a graphing calculator.

80. $(x - 5)^2 = 121$ $16, -6$

81. $4c^2 + 4c + 1 = 9$ $1, -2$

82. $4y^2 = 64$ $-4, 4$

83. $16d^2 + 40d + 25 = 9$ $-2, -\frac{1}{2}$

84. **LANDSCAPING** A sidewalk of equal width is being built around a square yard. What is the width of the sidewalk? 2.5 m

Example 12

Solve $(x - 9)^2 = 144$.

$(x - 9)^2 = 144$	Original equation
$x - 9 = \pm\sqrt{144}$	Square Root Property
$x - 9 = \pm 12$	$144 = 12 \cdot 12$
$x = 9 \pm 12$	Add 9 to each side.
$x = 9 + 12$ or $x = 9 - 12$	Zero Product Property
$x = 21$ $\quad$ $x = -3$	Solve.

CHECK

$(x - 9)^2 = 144$	$(x - 9)^2 = 144$
$(21 - 9)^2 \stackrel{?}{=} 144$	$(-3 - 9)^2 \stackrel{?}{=} 144$
$(12)^2 \stackrel{?}{=} 144$	$(-12)^2 \stackrel{?}{=} 144$
$144 = 144$ ✓	$144 = 144$ ✓

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7-10 Roots and Zeros

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

85. $f(x) = -2x^3 + 11x^2 - 3x + 2$ **85–89. See margin.**

86. $f(x) = -4x^4 - 2x^3 - 12x^2 - x - 23$

87. $f(x) = x^6 - 5x^3 + x^2 + x - 6$

88. $f(x) = -2x^5 + 4x^4 + x^2 - 3$

89. $f(x) = -2x^6 + 4x^4 + x^2 - 3x - 3$

Example 13

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = 3x^4 + 2x^3 - 2x^2 - 26x - 48$.

$f(x)$ has one sign change, so there is 1 positive real zero.

$f(-x)$ has 3 sign changes, so there are 3 or 1 negative real zeros.

There are 0 or 2 imaginary zeros.

Additional Answers

85. positive real zeros: 3 or 1
negative real zeros: 0
imaginary zeros: 2 or 0

86. positive real zeros: 0
negative real zeros: 4, 2, or 0
imaginary zeros: 4, 2, or 0

87. positive real zeros: 3 or 1
negative real zeros: 1
imaginary zeros: 4 or 2

88. positive real zeros: 2 or 0
negative real zeros: 1
imaginary zeros: 4 or 2

89. positive real zeros: 2 or 0
negative real zeros: 2 or 0
imaginary zeros: 6, 4, or 2

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Additional Answer

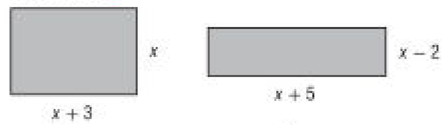
9a. AED4000 is the amount of the investment, 1 will add the amount of the investment to the interest, 0.05 is the interest rate as a decimal, and 2 is the number of the years of the investment.

CHAPTER 7 Practice Test

Find each sum or difference. 2. $3m - 6n^2 + 6n$

1. $(x + 5) + (x^2 - 3x + 7)$ $x^2 - 2x + 12$
2. $(7m - 8n^2 + 3n) - (-2n^2 + 4m - 3n)$

3. **MULTIPLE CHOICE** Abeer is carpeting two of the rooms in her house. The dimensions are shown. Which expression represents the total area to be carpeted? **B**



A $x^2 + 3x$
B $2x^2 + 6x - 10$
C $x^2 + 3x - 5$
D $8x + 12$

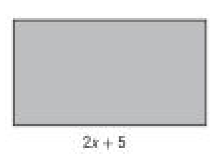
Find each product.

4. $a(a^2 + 2a - 10)$ $a^3 + 2a^2 - 10a$
5. $(2a - 5)(3a + 5)$ $6a^2 - 5a - 25$
6. $(x - 3)(x^2 + 5x - 6)$ $x^3 + 2x^2 - 21x + 18$
7. $(x + 3)^2$ $x^2 + 6x + 9$
8. $(2b - 5)(2b + 5)$ $4b^2 - 25$

9. **FINANCIAL LITERACY** Suppose you invest AED 4,000 in a 2-year certificate of deposit (CD).
   a. If the interest rate is 5% per year, the expression $4000(1 + 0.05)^2$ can be evaluated to find the total amount of money after two years. Explain the numbers in this expression. **See margin.**
   b. Find the amount at the end of two years. **AED 4,410**
   c. Suppose you invest AED 10,000 in a CD for 4 years at an annual rate of 6.25%. What is the total amount of money you will have after 4 years? **about AED 12,744**

10. **MULTIPLE CHOICE** The area of the rectangle shown below is $2x^2 - x - 15$ square units. What is the width of the rectangle? **H**

F $x - 5$
G $x + 3$
H $x - 3$
J $2x - 3$



Solve each equation.

11. $5(t^2 - 3t + 2) = t(5t - 2)$ $\frac{10}{13}$
12. $3x(x + 2) = 3(x^2 - 2)$ -1

Factor each polynomial.

13. $5xy - 10x$ $5x(y - 2)$
14. $7ab + 14ab^2 + 21a^2b$ $7ab(1 + 2b + 3a)$
15. $4x^2 + 8x + x + 2$ $(4x + 1)(x + 2)$
16. $10a^2 - 50a - a + 5$ $(10a - 1)(a - 5)$

Solve each equation. Confirm your answers using a graphing calculator.

17. $y(y - 14) = 0$ $0, 14$
18. $3x(x + 6) = 0$ $0, -6$
19. $a^2 = 12a$ $0, 12$

20. **MULTIPLE CHOICE** Amani is carpeting a room that has an area of $x^2 - 100$ square units. If the width of the room is $x - 10$ units, what is the length of the room? **B**

A $x - 10$ units
B $x + 10$ units
C $x - 100$ units
D 10 units

Factor each trinomial.

21. $x^2 + 7x + 6$ $(x + 6)(x + 1)$
22. $x^2 - 3x - 28$ $(x - 7)(x + 4)$
23. $10x^2 - x - 3$ $(5x - 3)(2x + 1)$
24. $15x^2 + 7x - 2$ $(3x + 2)(5x - 1)$
25. $x^2 - 25$ $(x + 5)(x - 5)$
26. $4x^2 - 81$ $(2x + 9)(2x - 9)$
27. $9x^2 - 12x + 4$ $(3x - 2)(3x - 2)$
28. $16x^2 + 40x + 25$ $(4x + 5)(4x + 5)$

Solve each equation. Confirm your answers using a graphing calculator.

29. $x^2 - 4x = 21$ $-3, 7$
30. $x^2 - 2x - 24 = 0$ $-4, 6$
31. $6x^2 - 5x - 6 = 0$ $-\frac{2}{3}, \frac{3}{2}$
32. $2x^2 - 13x + 20 = 0$ $4, \frac{5}{2}$

33. **MULTIPLE CHOICE** Which choice is a factor of $x^4 - 1$ when it is factored completely? **G**

F $x^2 - 1$
G $x - 1$
H x
J 1

CHAPTER 7 Preparing for Standardized Tests

Solve Multi-Step Problems

Some problems that you will encounter on standardized tests require you to solve multiple parts in order to come up with the final solution. Use this lesson to practice these types of problems.

Strategies for Solving Multi-Step Problems

Step 1

Read the problem statement carefully.

Ask yourself:

- What am I being asked to solve? What information is given?
- Are there any intermediate steps that need to be completed before I can solve the problem?

Step 2

Organize your approach.

- List the steps you will need to complete in order to solve the problem.
- Remember that there may be more than one possible way to solve the problem.

Step 3

Solve and check.

- Work as efficiently as possible to complete each step and solve.
- If time permits, check your answer.

Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

A florist has 80 roses, 50 tulips, and 20 lilies that he wants to use to create bouquets. He wants to create the maximum number of bouquets possible and use all of the flowers. Each bouquet should have the same number of each type of flower. How many roses will be in each bouquet?

A 4 roses **C** 10 roses

B 8 roses **D** 15 roses

1 Focus

Objective Use strategies for solving multi-step standardized test problems.

2 Teach

Scaffolding Questions

Ask:

- Have you ever had to write a research paper or complete some other school project? Answers will vary.
- Did you write the entire paper or complete the project in one session? Or did you use a number of steps to complete the job? Answers will vary.
- Could you have completed the steps in a different order? Could you have used a different set of steps? Answers will vary.

Additional Example

A geologist is preparing rock boxes for students. She has 90 igneous, 75 metamorphic, and 120 sedimentary rocks. She wants to make the maximum number of boxes possible and use all of the rocks. Each box should have the same number of each type of rock. How many metamorphic rocks will be in each box? A

A 5 metamorphic rocks

B 6 metamorphic rocks

C 8 metamorphic rocks

D 15 metamorphic rocks

3 Assess

Use Exercises 1–4 to assess students' understanding.

CHAPTER 7 Preparing for Standardized Tests *continued*

Read the problem carefully. You are given the number of roses, tulips, and lilies and told that bouquets will be made using the same number of flowers in each. You need to find the number of roses that will be in each bouquet.

Step 1 Find the GCF of the number of roses, tulips, and lilies.

Step 2 Use the GCF to determine how many bouquets will be made.

Step 3 Divide the total number of roses by the number of bouquets.

Step 1 Write the prime factorization of each number of flowers to find the GCF.

$80 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

$50 = 2 \cdot 5 \cdot 5$

$20 = 2 \cdot 2 \cdot 5$

$\text{GCF} = 2 \cdot 5 = 10$

Step 2 The GCF of the number of roses, tulips, and lilies tells you how many bouquets can be made because each bouquet will contain the same number of flowers. So, the florist can make a total of 10 bouquets.

Step 3 Divide the number of roses by the number of bouquets to find the number of roses in each bouquet.

$\frac{80}{10} = 8$

So, there will be 8 roses in each bouquet. The answer is B.

Exercises

Read each problem. Identify what you need to know. Then use the information in the problem to solve.

1. Which of the following values is not a solution to $x^3 - 3x^2 - 25x + 75 = 0$? C

   A $x = 5$ **C** $x = -3$

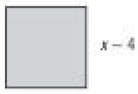
   B $x = 3$ **D** $x = -5$

2. There are 12 teachers, 90 students, and 36 parent volunteers going on a field trip. Mrs. Nisreen wants to divide everyone into equal groups with the same number of teachers, students, and parents in each group. If she makes as many groups as possible, how many students will be in each group? J

   F 6 **H** 12

   G 9 **J** 15

3. What is the area of the square? D



   A $x^2 + 16$

   B $4x - 16$

   C $x^2 - 8x - 16$

   D $x^2 - 8x + 16$

4. Students are selling magazines to raise money for a field trip. They make AED 2.75 for each magazine they sell. If they want to raise AED 600, what is the least amount of magazines they need to sell? J

   F 121 **H** 202

   G 177 **J** 219

8b. $t = 0$, 12.5 seconds; Sample answer: The rocket is on the ground at launch and takes 12.5 seconds to land again after its flight.

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CHAPTER 7

Standardized Test Practice

Cumulative, Chapter 7

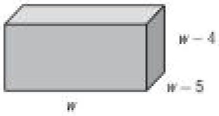
Multiple Choice

1. Which of the following is a solution to $x^2 + 6x - 112 = 0$? F
   - F -14
   - G -8
   - H 6
   - J 12

2. Which of the following polynomials is prime? D
   - A $5x^2 + 34x + 24$
   - B $4x^2 + 22x + 10$
   - C $4x^2 + 38x + 70$
   - D $5x^2 + 3x + 4$

3. Which of the following is not a factor of the polynomial $45a^2 - 80b^2$? H
   - F 5
   - G $3a - 4b$
   - H $2a - 5b$
   - J $3a + 4b$

4. A rectangular gift box has dimensions that can be represented as shown in the figure. The volume of the box is $56w$ cubic centimeters. Which of the following is *not* a dimension of the box? A



   - A 6 cm
   - B 7 cm
   - C 8 cm
   - D 12 cm

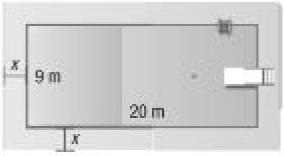
Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

5. The equation $h = -16t^2 + 40t + 3$ models the height h in feet of a soccer ball after t seconds. What is the height of the ball after 2 seconds? 19 ft

6. Factor $2x^4 - 32$ completely. $2(x - 2)(x + 2)(x^2 + 4)$

7. **GRIDDED RESPONSE** Huda is having a cement walkway installed around the perimeter of her swimming pool with the dimensions shown below. If $x = 3$ find the area, in square meters, of the pool and walkway. 390



Extended Response

Record your answers on a sheet of paper. Show your work.

8. The height in feet of a model rocket t seconds after being launched into the air is given by the function $h(t) = -16t^2 + 200t$.
   a. Write the expression that shows the height of the rocket in factored form. $h(t) = t(-16t + 200)$
   b. At what time(s) is the height of the rocket equal to zero feet above the ground? Describe the real world meaning of your answer. See margin.
   c. What is the greatest height reached by the model rocket? When does this occur? 625 ft; at $t = 6.25$ seconds

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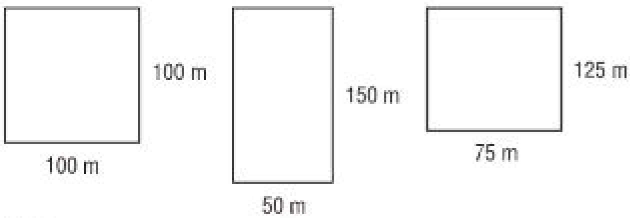
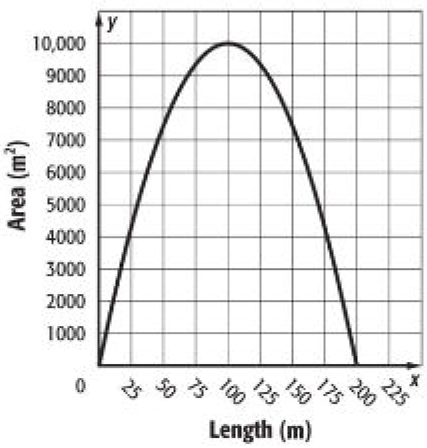
Lesson 7-1

60a. Sample answer:



60c.



The largest possible area is 10,000 square meters.

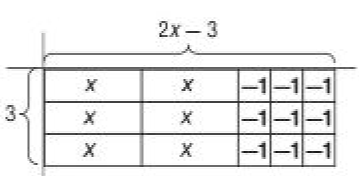
Lesson 7-3

48. The three monomials that make up the trinomial are similar to the three digits that make up the 3-digit number. The single monomial is similar to a 1-digit number. With each procedure you perform 3 multiplications. The difference is that polynomial multiplication involves variables and the resulting product is often the sum of two or more monomials, while numerical multiplication results in a single number.

49. The Distributive Property can be used with a vertical or horizontal format by distributing, multiplying, and combining like terms. The FOIL method is used with a horizontal format. You multiply the first, outer, inner, and last terms of the binomials and then combine like terms. A rectangular method can also be used by writing the terms of the polynomials along the top and left side of a rectangle and then multiplying the terms and combining like terms.

Explore 7-5

5.



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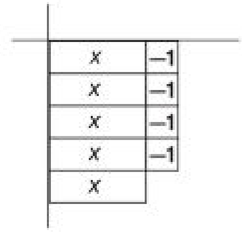

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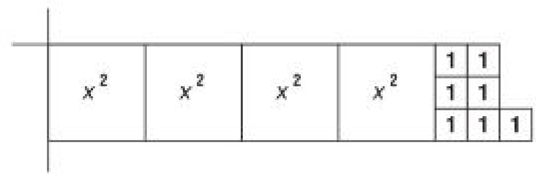

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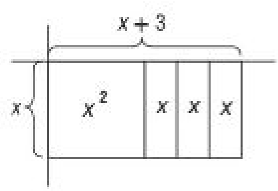
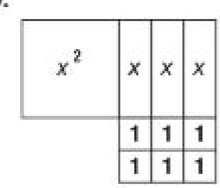
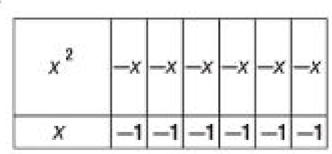


Explore 7-6

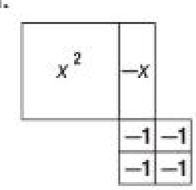
9.



10.



11.



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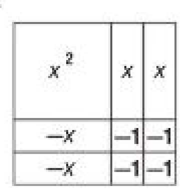
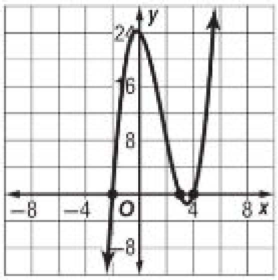


Lesson 7-10

46.



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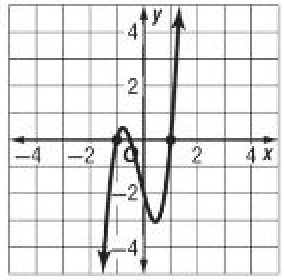

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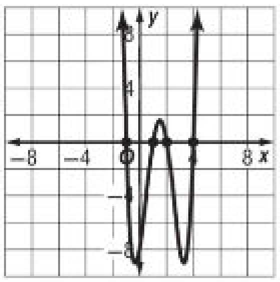
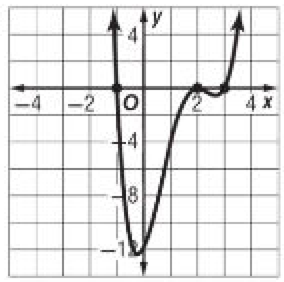
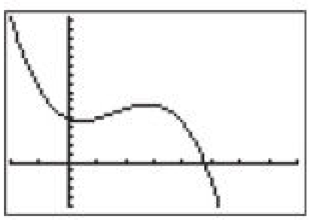


49.



53b.



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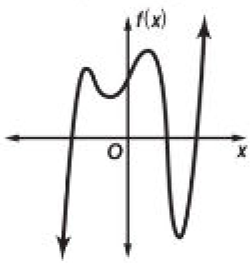
56a.



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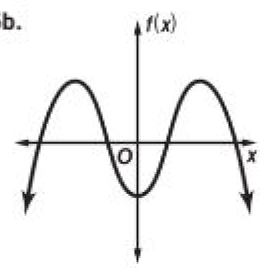

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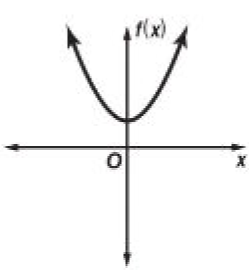


60. Sample answer: To determine the number of positive real roots, determine how many time the signs change in the polynomial as you move from left to right. In this function there are 3 changes in sign. Therefore, there may be 3 or 1 positive real roots. To determine the number of negative real roots, I would first evaluate the polynomial for $-x$. All of the terms with an odd-degree variable would change signs. Then I would again count the number of sign changes as I move from left to right. There would be only one change. Therefore there may be 1 negative root.

Chapter Planner

Diagnostic Assessment
Quick Check

	LESSON 8-1 (45 min: 1 day; 90 min: 0.5 day)	LESSON 8-2 (45 min: 1 day; 90 min: 0.5 day)	LESSON 8-3 (45 min: 1 day; 90 min: 0.5 day)	LESSON 8-4 (45 min: 1 day; 90 min: 0.5 day)
Title	**Multiplication Properties of Exponents**	**Division Properties of Exponents**	**Rational Exponents**	**Scientific Notation**
Objectives	▪ Multiply monomials using the properties of exponents. ▪ Simplify expressions using the multiplication properties of exponents.	▪ Divide monomials using the properties of exponents. ▪ Simplify expressions containing negative and zero exponents.	▪ Evaluate and rewrite expressions involving rational exponents. ▪ Solve equations involving expressions with rational exponents.	▪ Express numbers in scientific notation. ▪ Find products and quotients of numbers expressed in scientific notation.
Key Vocabulary	monomial constant	zero exponents negative exponent order of magnitude	rational exponent cube root *n*th root exponential equation	scientific notation

Formative Assessment
Mid-Chapter Quiz

EXPLORE 8-5 (45 min: 0.5 day; 90 min: 0.5 day)	LESSON 8-5 (45 min: 1 day; 90 min: 0.5 day)	EXTEND 8-5 (45 min: 0.5 day; 90 min: 0.5 day)	LESSON 8-6 (45 min: 1 day; 90 min: 0.5 day)
Graphing Technology Lab: Family of Exponential Functions	**Exponential Functions**	**Graphing Technology Lab: Solving Exponential Equations and Inequalities**	**Growth and Decay**
■ Use a graphing calculator to investigate families of exponential functions.	■ Graph exponential functions. ■ Identify data that display exponential behavior.	■ Use a graphing calculator to solve exponential equations and inequalities.	■ Solve problems involving exponential growth. ■ Solve problems involving exponential decay.
	exponential function exponential growth function exponential decay function		compound interest

Chapter Planner

	EXTEND 8-6 (45 min: 0.5 day; 90 min: 0.5 day)	LESSON 8-7 (45 min: 1 day; 90 min: 0.5 day)	EXTEND 8-7 (45 min: 0.5 day; 90 min: 0.5 day)	LESSON 8-8 (45 min: 1 day; 90 min: 0.5 day)
Title	**Algebra Lab: Transforming Exponential Expressions**	**Geometric Sequences as Exponential Functions**	**Algebra Lab: Average Rate of Change of Exponential Functions**	**Recursive Formulas**
Objectives	▪ Use properties of rational exponents to transform expressions for exponential functions into equivalent forms to solve problems.	▪ Identify and generate geometric sequences. ▪ Relate geometric sequences to exponential functions.	▪ Calculate and interpret the average rate of change of an exponential function.	▪ Use a recursive formula to list the terms in a sequence. ▪ Write recursive formulas for arithmetic and geometric sequences.
Key Vocabulary		geometric sequence common ratio		recursive formula

Summative Assessment
Study Guide and Review
Practice Test

Project CRISS

STUDY SKILL

Power notes can help students organize and outline a lesson or chapter. Students often benefit from making power notes as a cooperative activity.

In the outline at the right, Power 1 is the main idea, Power 2 provides details about the main idea, Power 3 provides details about Power 2, and so on.

You may have students copy the power notes at the right and complete them using the information in Chapter 8. Point out that more than one detail can be placed under each power.

1. Multiplication Properties of Exponents
    2. Multiply Monomials
        3. monomials are products, no addition, subtraction, or division of variables
        3. multiply powers; add the exponents
        3. raise a power to a power; multiply the exponents
        3. raise a product to a power, raise each factor to the power
    2. Simplify Expressions
        3. variables appear only once, no powers of powers, all fractions are simplified

Creating Independence through Student-owned Strategies

NOTES

Assessment

SE = Student Edition, TE = Teacher Edition

	Diagnosis	Prescription
DIAGNOSTIC ASSESSMENT	**Beginning Chapter 8**	
	Get Ready for Chapter 8 **SE**	Response to Intervention **TE**
	Beginning Every Lesson	
	Then, Now, Why? **SE**	Chapter 0 **SE**
FORMATIVE ASSESSMENT	**During/After Every Lesson**	
	Guided Practice **SE**, every example Check Your Understanding **SE** H.O.T. Problems **SE** Spiral Review **SE** Additional Examples **TE** Watch Out! **TE** Step 4, Assess **TE**	Differentiated Instruction **TE**
	Mid-Chapter	
	Mid-Chapter Quiz **SE**	
SUMMATIVE ASSESSMENT	**Before Chapter Test**	
	Chapter Study Guide and Review **SE** Practice Test **SE** Standardized Test Practice **SE**	

Differentiated Instruction

Option 1 Reaching All Learners AL OL BL

Auditory/Musical Music can be a powerful memory tool. Suggest that groups of students make up a song or poem to explain how to use the multiplication and division properties of exponents. Have the groups perform their songs or read their poems for the class.

Visual To help students remember the patterns in the multiplication and division properties of exponents, have students highlight the variables using two different colors. Students can make their own study cards and/or posters to hang in the classroom.

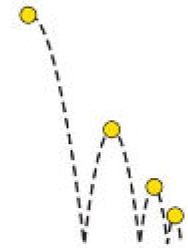
$$a^m \cdot a^p = a^{m+p}$$
$$(a^m)^p = a^{m \cdot p}$$
$$(ab)^m = a^m b^m$$
$$\frac{a^m}{a^p} = a^{m-p}$$
$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$
$$a^0 = 1$$
$$a^{-n} = \frac{1}{a^n}$$

Option 2 Approaching Level AL

Use a bent pipe cleaner and a coordinate grid on an overhead transparency to model the graph of an exponential function. Ask students to identify the *y*-intercept and to describe what happens to the *y*-values as the *x*-values increase. Repeat for the graphs of other exponential functions.

Option 3 Beyond Level BL

Drop a tennis ball while standing in the front of the classroom. Ask students to describe how the bouncing ball can be modeled by a geometric sequence. Have students write an example of a sequence that might model the bouncing ball. Then, ask students to design their own physical models of a geometric sequence.



Focus on Mathematical Content

Vertical Alignment

Before Chapter 8

Related Topics

- make conjectures from patterns or sets of examples
- compare and order rational numbers including integers
- select appropriate operations to solve problems involving rational numbers

Previous Topics

- use symbols to represent unknowns and variables

Chapter 8

- simplify polynomial expressions and apply the laws of exponents in problem-solving situations
- graph and analyze exponential functions
- analyze data and represent situations involving exponential growth and decay using tables, graphs, or algebraic methods
- relate geometric sequences to exponential functions, and write recursive formulas to represent sequences

After Chapter 8

Preparation

- use the necessary algebraic skills required to simplify algebraic expressions and inequalities in problem-solving situations
- use properties of exponents to simplify expressions and to transform and solve exponential and logarithmic equations

Lesson-by-Lesson Preview

8-1 Multiplication Properties of Exponents

A *monomial* is a number, a variable, or a product of a number and one or more variables. An expression involving negative integer exponents is not a monomial. Monomials that are real numbers are called *constants*. When multiplying monomials, use the Commutative and Associative Properties to group constants together and to group powers with the same base together.

- To multiply two powers that have the same base, add their exponents.
- To find the power of a power, multiply the exponents.
- To find the power of a product, find the power of each factor and multiply.

A monomial expression is simplified when

- each base appears exactly once,
- there are no powers of powers, and
- all fractions are in simplest form.

8-2 Division Properties of Exponents

Monomials may also be divided.

- To divide two powers that have the same base, subtract the exponents.
- To find the power of a quotient, find the power of the numerator and the power of the denominator.
- Any nonzero number raised to the zero power is equal to 1.

Expressions can also have negative exponents.

- A nonzero number raised to a negative integer power is the reciprocal of the same number with the opposite, or positive power.
- A fraction that has a negative exponent can be rewritten as its reciprocal with a positive power.

The order of magnitude of a quantity is the number rounded to the nearest power of 10.

8-3 Rational Exponents

Students have evaluated and simplified expressions involving exponents that are integers. Exponents can also be fractions.

- For any real numbers a and b, and any positive integer n, if $a^n = b$, then a is an nth root of b. If $a^n = b$, then $\sqrt[n]{b} = a$.
- For any positive real number b, and any integers m and n with $n > 1$, $b^{\frac{m}{n}} = \sqrt[n]{b^m}$.

8-4 Scientific Notation

scientific notation

$a \times$ ^{n} $\leq a <$ n

-
-
-
-
-
-

8-5 Exponential Functions

$= y\, ab^{x}$ $a \neq$

$b >$ $b \neq$

- $a >$ $b >$ y x x
- $a >$ $< b <$ y x x

8-6 Growth and Decay

- $y = a$ $+ r^{t}$ y a r $r >$ t
- $y = a$ $- r^{t}$

8-7 Geometric Sequences as Exponential Functions

r

- r
- r
- n a_n a $a \neq$ $a_n = a \cdot r^{n-}$ n

8-8 Recursive Formulas

explicit formulas

n

recursive formula

$a =$ $a_n = a_{n-}$ $+$

$a \neq$ $a_n =$ $\cdot a_{n-}$

Chapter Project

INTERE$T-ing Thing About Credit Cards...

Students use what they have learned about exponential functions to complete a project.

This chapter project addresses financial literacy, as well as several specific skills identified as being essential to student success by the Framework for 21st Century Learning.

KeyVocabulary Introduce the key vocabulary in the chapter using the routine below.

Define: A monomial is a number, a variable, or a product of a number and one or more variables.

Example: $\frac{1}{20}n^2$

Ask: Can you name another monomial?

Possible answers: $32ab$, 175, x, y

CHAPTER 8 Exponents and Exponential Functions

Then

You evaluated expressions involving exponents.

Now

In this chapter, you will:

- Simplify and perform operations on expressions involving exponents.
- Extend the properties of integer exponents to rational exponents.
- Use scientific notation.
- Graph and use exponential functions.

Why? ▲

SPACE The Very Large Array is an arrangement of 27 radio antennas in a Y pattern. The data the antennas collect is used by astronomers around the world to study the planets and stars. Astrophysicists use and apply properties of exponents to model the distance and orbit of celestial bodies.

Get Ready for the Chapter

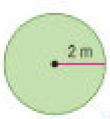
1 Textbook Option Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

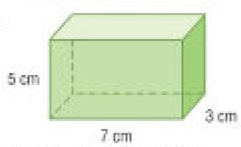
Write each expression using exponents.

1. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ 4^5
2. $y \cdot y \cdot y$ y^3
3. $6 \cdot 6$ 6^2
4. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ 2^9
5. $b \cdot b \cdot b \cdot b \cdot b \cdot b$ b^6
6. $m \cdot m \cdot m \cdot p \cdot p \cdot p \cdot p \cdot p \cdot p$ m^3p^6
7. $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$ $\left(\frac{1}{3}\right)^8$ or $\frac{1}{3^8}$
8. $\frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{w}{z} \cdot \frac{w}{z}$ $\frac{x^4w^2}{y^4z^2}$

Find the area or volume of each figure.

9. 

4π m^2

10. 

105 cm^3

11. **PHOTOGRAPHY** A photo is 10 centimeters by 15 centimeters. What is the area of the photo? 150 cm^2

Evaluate each expression.

12. 2^3 8
13. $(-5)^2$ 25
14. 3^3 27
15. $(-4)^3$ −64
16. $\left(\frac{2}{3}\right)^2$ $\frac{4}{9}$
17. $\left(\frac{1}{2}\right)^4$ $\frac{1}{16}$
18. **SCHOOL** The probability of guessing correctly on 5 true-false questions is $\left(\frac{1}{2}\right)^5$. Express this probability as a fraction without exponents. $\frac{1}{32}$

QuickReview

Example 1 (Used in Lessons 8-1 and 8-2)

Write $5 \cdot 5 \cdot 5 \cdot 5 + x \cdot x \cdot x$ using exponents.

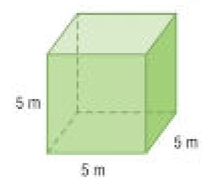
4 factors of 5 is 5^4.

3 factors of x is x^3.

So, $5 \cdot 5 \cdot 5 \cdot 5 + x \cdot x \cdot x = 5^4 + x^3$.

Example 2 (Used in Lessons 8-1 and 8-2)

Find the volume of the figure.



$V = \ell wh$ Volume of a rectangular prism

$= 5 \cdot 5 \cdot 5$ or 125 $\ell = 5$, $w = 5$, and $h = 5$

The volume is 125 cubic meters.

Example 3 (Used in Lessons 8-1 through 8-7)

Evaluate $\left(\frac{5}{7}\right)^2$.

$\left(\frac{5}{7}\right)^2 = \frac{5^2}{7^2}$ Power of a Quotient

$= \frac{25}{49}$ Simplify.

Essential Questions

- How can you make good decisions?
  Sample answer: Determine the available options, compare the advantages and disadvantages of each option, analyze the consequences, and choose the best option.
- What factors can affect good decision making?
  Sample answers: the amount of time that is available, the process used, the environment, the people that are involved, the available options

FOLDABLES Study Organizer

Dinah Zike's Foldables®

Focus Students create a tabbed book on which they organize information about polynomials.

Teach Have students make and label their Foldables as illustrated. Before beginning each lesson, ask students to think of one question that comes to mind as they skim through the lesson. Have them write the questions on the tabbed page of the corresponding lesson. As they read and work through the lesson, ask them to record the answers to their questions under the tabs.

When to Use It Encourage students to add to their Foldables as they work through the chapter and to use them to review for the chapter test.

Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 8. To get ready, identify important terms and organize your resources. You may wish to refer to Chapter 0 to review prerequisite skills.

FOLDABLES Study Organizer

Exponents and Exponential Functions Make this Foldable to help you organize your Chapter 8 notes about exponents and exponential functions. Begin with nine sheets of notebook paper.

1 **Arrange** the paper into a stack.

2 **Staple** along the left side. Starting with the second sheet of paper, cut along the right side to form tabs.

3 **Label** the cover sheet "Exponents and Exponential Functions" and label each tab with a lesson number.

New Vocabulary

- monomial
- constant
- zero exponent
- negative exponent
- order of magnitude
- rational exponent
- cube root
- *n*th root
- exponential equation
- scientific notation
- exponential function
- exponential growth
- exponential decay
- compound interest
- geometric sequence
- common ratio
- recursive formula

Review Vocabulary

base In an expression of the form x^n, the base is x.

Distributive Property For any numbers a, b, and c, $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.

exponent In an expression of the form x^n, the exponent is n. It indicates the number of times x is used as a factor.

$$x^n = \underbrace{x \cdot x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}$$

exponent; base

Additional Answers (Guided Practice)

1A. No; the expression has addition and more than one term.

1B. Yes; this is a product of a number and variables.

1C. Yes; this is a product of variables, with a constant in the denominator.

1D. No; this expression has a variable in the denominator.

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LESSON 8-1 Multiplication Properties of Exponents

Then

- You evaluated expressions with exponents.

Now

1. Multiply monomials using the properties of exponents.
2. Simplify expressions using the multiplication properties of exponents.

Why?

- Many formulas contain *monomials*. For example, the formula for the horsepower of a car is $H = w\left(\frac{v}{234}\right)^3$. *H* represents the horsepower produced by the engine, *w* equals the weight of the car with passengers, and *v* is the velocity of the car at the end of a four tenths of a kilometer. As the velocity increases, the horsepower increases.

NewVocabulary
monomial
constant

Mathematical Practices
Look for and express regularity in repeated reasoning.

1 Multiply Monomials

A **monomial** is a number, a variable, or the product of a number and one or more variables with nonnegative integer exponents. It has only one term. In the formula to calculate the horsepower of a car, the term $w\left(\frac{v}{234}\right)^3$ is a monomial.

An expression that involves division by a variable, like $\frac{ab}{c}$, is not a monomial.

A **constant** is a monomial that is a real number. The monomial $3x$ is an example of a *linear expression* since the exponent of x is 1. The monomial $2x^2$ is a *nonlinear expression* since the exponent is a positive number other than 1.

Example 1 Identify Monomials

Determine whether each expression is a monomial. Write *yes* or *no*. Explain your reasoning.

a. 10 — Yes; this is a constant, so it is a monomial.

b. $f + 24$ — No; this expression has addition, so it has more than one term.

c. h^2 — Yes; this expression is a product of variables.

d. j — Yes; single variables are monomials.

GuidedPractice

1A. $-x + 5$ **1A–1D. See margin.**

1B. $23abcd^2$

1C. $\frac{xyz^2}{2}$

1D. $\frac{mp}{n}$

Recall that an expression of the form x^n is called a *power* and represents the result of multiplying x by itself n times. x is the *base*, and n is the *exponent*. The word *power* is also used sometimes to refer to the exponent.

exponent — 4 factors

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

base

1 Focus

VerticalAlignment

Before Lesson 8-1 Evaluated expressions with exponents.

Lesson 8-1 Multiply monomials using the multiplication properties of exponents. Simplify expressions using the properties of exponents.

After Lesson 8-1 Find quotients of two monomials and simplify expressions containing negative exponents.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What two values do you need to know to be able to use the formula to find the horsepower of a car? weight of the car, *w*, with passengers; velocity of the car, *v*, at the end of four tenths of a kilometer
- Which values in the formula are raised to the third power? *v*, 234
- When would the horsepower of a car be equal to or greater than the weight of a car with passengers? when $v \geq 234$

1 Multiplying Monomials

Example 1 shows how to determine whether an expression is a monomial. **Example 2** shows how to find the product of powers. **Example 3** shows how to find the power of a power. **Example 4** shows how to find the power of a product.

Formative Assessment

Use the Guided Practice exercises after each Example to determine students' understanding of concepts.

Additional Examples

1 Determine whether each expression is a monomial. Write *yes* or *no*. Explain your reasoning.

a. $17 - c$ No; this expression involves subtraction, so it involves more than one term.

b. $8f^2g$ Yes; this expression is the product of a number and two variables.

c. $\frac{3}{4}$ Yes; The expression is a constant.

d. $\frac{5}{t}$ No; this expression has a variable in the denominator.

2 Simplify each expression.

a. $(r^4)(-12r^7)$ $-12r^{11}$

b. $(6cd^5)(5c^5d^2)$ $30c^6d^7$

Teach with Tech

Video Recording Have students record themselves multiplying monomials and explaining their work. Share the videos with the class.

By applying the definition of a power, you can find the product of powers. Look for a pattern in the exponents.

$$2^2 \cdot 2^4 = \overbrace{2 \cdot 2}^{2\text{ factors}} \cdot \overbrace{2 \cdot 2 \cdot 2 \cdot 2}^{4\text{ factors}} \quad (2 + 4 = 6\text{ factors})$$

$$4^3 \cdot 4^2 = \overbrace{4 \cdot 4 \cdot 4}^{3\text{ factors}} \cdot \overbrace{4 \cdot 4}^{2\text{ factors}} \quad (3 + 2 = 5\text{ factors})$$

These examples demonstrate the property for the product of powers.

KeyConcept Product of Powers

Words	To multiply two powers that have the same base, add their exponents.
Symbols	For any real number a and any integers m and p, $a^m \cdot a^p = a^{m+p}$.
Examples	$b^3 \cdot b^5 = b^{3+5}$ or b^8 $g^4 \cdot g^6 = g^{4+6}$ or g^{10}

Example 2 Product of Powers

Simplify each expression.

a. $(6n^3)(2n^7)$

$(6n^3)(2n^7) = (6 \cdot 2)(n^3 \cdot n^7)$ — Group the coefficients and the variables.

$= (6 \cdot 2)(n^{3+7})$ — Product of Powers

$= 12n^{10}$ — Simplify.

b. $(3pt^3)(p^3t^4)$

$(3pt^3)(p^3t^4) = (3 \cdot 1)(p \cdot p^3)(t^3 \cdot t^4)$ — Group the coefficients and the variables.

$= (3 \cdot 1)(p^{1+3})(t^{3+4})$ — Product of Powers

$= 3p^4t^7$ — Simplify.

StudyTip

Coefficients and Powers of 1 A variable with no exponent or coefficient shown can be assumed to have an exponent and coefficient of 1. For example, $x = 1x^1$.

GuidedPractice

2A. $(3y^4)(7y^5)$ $21y^9$

2B. $(-4rx^2t^3)(-6r^5x^2t)$ $24r^6x^4t^4$

We can use the Product of Powers Property to find the power of a power. In the following examples, look for a pattern in the exponents.

$$(3^2)^4 = \overbrace{(3^2)(3^2)(3^2)(3^2)}^{4\text{ factors}} = 3^{2+2+2+2} = 3^8$$

$$(r^4)^3 = \overbrace{(r^4)(r^4)(r^4)}^{3\text{ factors}} = r^{4+4+4} = r^{12}$$

These examples demonstrate the property for the power of a power.

KeyConcept Power of a Power

Words	To find the power of a power, multiply the exponents.
Symbols	For any real number a and any integers m and p, $(a^m)^p = a^{m \cdot p}$.
Examples	$(b^3)^5 = b^{3 \cdot 5}$ or b^{15} $(g^6)^7 = g^{6 \cdot 7}$ or g^{42}

Focus on Mathematical Content

Variables with No Exponents A variable without an exponent can be rewritten with an exponent of 1. For example, x can be written as x^1, and ab can be written as a^1b^1. In order for students to find the products of powers correctly, suggest that they rewrite variables without an exponent with an exponent of 1.

StudyTip

Regularity The power rules are general methods. If you are unsure about when to multiply the exponents and when to add the exponents, write the expression in expanded form.

Standardized Test Example 3 Power of a Power

Simplify $[(2^3)^2]^4$.

A 2^{24} B 2^{12} C 2^{10} D 2^9

Read the Test Item

You need to apply the power of a power rule.

Solve the Test Item

$[(2^3)^2]^4 = (2^{3 \cdot 2})^4$ Power of a Power

$= (2^6)^4$ Simplify.

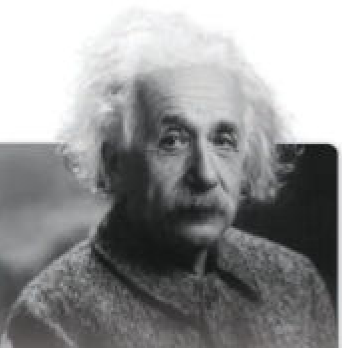
$= 2^{6 \cdot 4}$ or 2^{24} Power of a Power

The correct choice is A.

GuidedPractice

3. Simplify $[(2^2)^2]^4$. **H**

F 2^8 G 2^{10} H 2^{16} J 2^{24}



Math-HistoryLink

Albert Einstein **(1879–1955)** Albert Einstein is perhaps the most well-known scientist of the 20th century. His formula $E = mc^2$, where E represents the energy, m is the mass of the material, and c is the speed of light, shows that if mass is accelerated enough, it could be converted into usable energy.

We can use the Product of Powers Property and the Power of a Power Property to find the power of a product. Look for a pattern in the exponents below.

$$(tw)^3 = \underbrace{(tw)(tw)(tw)}_{\text{3 factors}} = (t \cdot t \cdot t)(w \cdot w \cdot w) = t^3w^3$$

$$(2yz^2)^3 = \underbrace{(2yz^2)(2yz^2)(2yz^2)}_{\text{3 factors}} = (2 \cdot 2 \cdot 2)(y \cdot y \cdot y)(z^2 \cdot z^2 \cdot z^2) = 2^3y^3z^6 \text{ or } 8y^3z^6$$

These examples demonstrate the property for the power of a product.

KeyConcept Power of a Product

Words	To find the power of a product, find the power of each factor and multiply.
Symbols	For any real numbers a and b and any integer m, $(ab)^m = a^mb^m$.
Example	$(-2xy^3)^5 = (-2)^5x^5y^{15}$ or $-32x^5y^{15}$

Example 4 Power of a Product

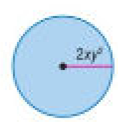
GEOMETRY **Express the area of the circle as a monomial.**

Area $= \pi r^2$ Formula for the area of a circle

$= \pi(2xy^2)^2$ Replace r with $2xy^2$.

$= \pi(2^2x^2y^4)$ Power of a Product

$= 4x^2y^4\pi$ Simplify.



The area of the circle is $4x^2y^4\pi$ square units.

GuidedPractice

4A. Express the area of a square with sides of length $3xy^2$ as a monomial. **$9x^2y^4$**

4B. Express the area of a triangle with height $4a$ and base $5ab^2$ as a monomial. **$10a^2b^2$**

Additional Examples

3 **STANDARDIZED TEST PRACTICE** Simplify $[(2^3)^3]^2$. D

A 8^2

B 8^4

C 2^{11}

D 2^{18}

4 **GEOMETRY** Express the volume of a cube with side length $5xyz$ as a monomial. $(5xyz)^3 = 125x^3y^3z^3$

WatchOut!

Student Misconceptions Students may simplify an expression such as $\frac{4}{6}(x^2y^5)^3[2(xy)^7]$ into $\frac{8}{6}x^{13}y^{22}$, not realizing that the simplification is incomplete because the fraction is not in the simplest form.

Teaching the Mathematical Practices

Regularity Mathematically proficient notice if calculations are repeated, and look both for general methods and for shortcuts. Work through examples to demonstrate each property in this lesson to help students see the repeated reasoning in the calculations.

DifferentiatedInstruction OL BL

Logical Learners Give students an expression such as $240x^{12}y^8$ and challenge them to write ten different monomial expressions that, when simplified, equal the given expression.

2 Simplify Expressions

Example 5 shows how to simplify expressions involving monomials by using the power and product rules.

Tips for New Teachers

Reasoning Remind students that there is often more than one strategy that can be used to simplify an expression. For example, in Example 5, the first step could be to simplify $(-2y)^2$ first and then raise the product to the third power.

Additional Example

5 Simplify $[(8g^3h^4)^2]^2(2gh^5)^4$. $65{,}536g^{16}h^{36}$

3 Practice

Formative Assessment

Use Exercises 1–20 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Additional Answers

1. Yes; constants are monomials.
2. No; there is subtraction and more than one term.
3. No; there is a variable in the denominator.
4. Yes; this is a product of a number and variables.
5. Yes; this is a product of a number and variables.
6. No; there is addition and more than one term.

2 Simplify Expressions

We can combine and use these properties to simplify expressions involving monomials.

KeyConcept Simplify Expressions

To simplify a monomial expression, write an equivalent expression in which:

- each variable base appears exactly once,
- there are no powers of powers, and
- all fractions are in simplest form.

StudyTip

Simplify When simplifying expressions with multiple grouping symbols, begin at the innermost expression and work outward.

Example 5 Simplify Expressions

Simplify $(3xy^4)^2[(-2y)^2]^3$.

$(3xy^4)^2[(-2y)^2]^3 = (3xy^4)^2(-2y)^6$ — Power of a Power

$= (3)^2x^2(y^4)^2(-2)^6y^6$ — Power of a Product

$= 9x^2y^8(64)y^6$ — Power of a Power

$= 9(64)x^2 \cdot y^8 \cdot y^6$ — Commutative

$= 576x^2y^{14}$ — Product of Powers

GuidedPractice

5. Simplify $\left(\frac{1}{2}a^2b^2\right)^3[(-4b)^2]^2$. $32a^6b^{10}$

Check Your Understanding

Example 1 Determine whether each expression is a monomial. Write *yes* or *no*. Explain your reasoning. 1–6. See margin.

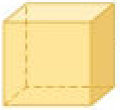
1. 15
2. $2 - 3a$
3. $\frac{5c}{d}$
4. $-15g^2$
5. $\frac{r}{2}$
6. $7b + 9$

Examples 2–3 Simplify each expression.

7. $k(k^3)$ k^4
8. $m^4(m^2)$ m^6
9. $2q^2(9q^4)$ $18q^6$
10. $(5u^4v)(7u^4v^3)$ $35u^8v^4$
11. $[(3^2)^2]^2$ 3^8 or 6561
12. $(xy^4)^6$ x^6y^{24}
13. $(4a^4b^9c)^2$ $16a^8b^{18}c^2$
14. $(-2f^2g^3h^2)^3$ $-8f^6g^9h^6$
15. $(-3p^5t^6)^4$ $81p^{20}t^{24}$

Example 4

16. **GEOMETRY** The formula for the surface area of a cube is $SA = 6s^2$, where SA is the surface area and s is the length of any side.



a. Express the surface area of the cube as a monomial. $6a^6b^2$

b. What is the surface area of the cube if $a = 3$ and $b = 4$? 69,984 units^2

Example 5 Simplify each expression.

17. $(5x^2y)^2(2xy^3z)^3(4xyz)$ $800x^8y^{12}z^4$
18. $(-3d^2f^3g)^2[(-3d^2f)^3]^2$ $6561d^{16}f^{12}g^2$
19. $(-2g^3h)(-3gj^4)^2(-ghj)^2$ $-18g^7h^3j^{10}$
20. $(-7ab^4c)^3[(2a^2c)^2]^3$ $-21{,}952a^{15}b^{12}c^9$

Practice and Problem Solving

Example 1 Determine whether each expression is a monomial. Write *yes* or *no*. Explain your reasoning. **21–26. See margin.**

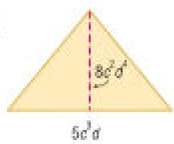
21. 122
22. $3a^4$
23. $2c + 2$
24. $\frac{-2g}{4h}$
25. $\frac{5k}{10}$
26. $6m + 3n$

Examples 2–3 Simplify each expression.

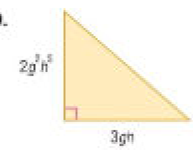
27. $(q^2)(2q^4)$ **$2q^6$**
28. $(-2u^2)(6u^6)$ **$-12u^8$**
29. $(9w^2x^8)(w^6x^4)$ **$9w^8x^{12}$**
30. $(y^6z^9)(6y^4z^2)$ **$6y^{10}z^{11}$**
31. $(b^8c^6d^5)(7b^6c^2d)$ **$7b^{14}c^8d^6$**
32. $(14fg^2h^2)(-3f^4g^2h^2)$ **$-42f^5g^4h^4$**
33. $(j^5k^7)^4$ **$j^{20}k^{28}$**
34. $(n^3p)^4$ **$n^{12}p^4$**
35. $[(2^2)^2]^2$ **2^8 or 256**
36. $[(3^2)^2]^4$ **3^{16} or 43,046,721**
37. $[(4r^2t)^3]^2$ **$4096r^{12}t^6$**
38. $[(-2xy^2)^3]^2$ **$64x^6y^{12}$**

Example 4 **GEOMETRY** Express the area of each triangle as a monomial.

39. **$20c^5d^5$**



40. **$3g^3h^6$**



Example 5 Simplify each expression.

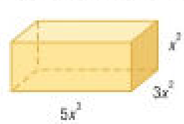
41. $(2a^3)^4(a^3)^3$ **$16a^{21}$**
42. $(c^3)^2(-3c^5)^2$ **$9c^{16}$**
43. $(2gh^4)^3[(-2g^4h)^3]^2$ **$512g^{27}h^{18}$**
44. $(5k^2m)^3[(4km^4)^2]^2$ **$32{,}000k^{10}m^{19}$**
45. $(p^5r^2)^4(-7p^3r^4)^2(6pr^3)$ **$294p^{27}r^{19}$**
46. $(5x^2y)^2(2xy^3z)^3(4xyz)$ **$800x^8y^{12}z^4$**
47. $(5a^2b^3c^4)(6a^3b^4c^2)$ **$30a^5b^7c^6$**
48. $(10xy^5z^3)(3x^4y^6z^3)$ **$30x^5y^{11}z^6$**
49. $(0.5x^3)^2$ **$0.25x^6$**
50. $(0.4h^5)^3$ **$0.064h^{15}$**
51. $\left(-\frac{3}{4}c\right)^3$ **$-\frac{27}{64}c^3$**
52. $\left(\frac{4}{5}a^2\right)^2$ **$\frac{16}{25}a^4$**
53. $(8y^3)(-3x^2y^2)\left(\frac{3}{8}xy^4\right)$ **$-9x^3y^9$**
54. $\left(\frac{4}{7}m\right)^2(49m)(17p)\left(\frac{1}{34}p^5\right)$ **$8m^3p^6$**
55. $(-3r^3w^4)^3(2rw)^2(-3r^2)^3(4rw^2)^3(2r^2w^3)^4$ **$2{,}985{,}984r^{28}w^{32}$**
56. $(3ab^2c)^2(-2a^2b^4)^2(a^4c^2)^3(a^2b^4c^5)^2(2a^3b^2c^4)^3$ **$288a^{31}b^{26}c^{30}$**

57. **FINANCIAL LITERACY** Omar has money in an account that earns 3% simple interest. The formula for computing simple interest is $I = Prt$, where I is the interest earned, P represents the principal that he put into the account, r is the interest rate (in decimal form), and t represents time in years.
   a. Omar makes a deposit of AED $2c$ and leaves it for 2 years. Write a monomial that represents the interest earned. **$0.12c$**
   b. If c represents a graduation gift of AED 250, how much will Omar have in this account after 2 years? **AED 280**

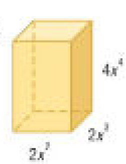
TOOLS Express the volume of each solid as a monomial.

58. **$12x^4\pi$**

59. **$15x^7$**



60. **$16x^9$**



Exercise Alert

Formulas For Exercises 39, 40, and 58–60, students will need to know the formulas for area of a triangle and for volume of solids.

Teaching the Mathematical Practices

Tools Mathematically proficient students are able to identify relevant external mathematical resources and use them to solve problems. In Exercises 58–60, point out that the volume formulas are listed inside the back cover of their book and can be found online.

Additional Answers

21. Yes; constants are monomials.
22. Yes; this is a product of a number and variables.
23. No; there is addition and more than one term.
24. No; there is a variable in the denominator.
25. Yes; this can be written as the product of a number and a variable.
26. No; there is addition and more than one term.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	21–57, 65–84	21–57 odd, 68–71	22–56 even, 65–67, 72–84
OL Core	21–57 odd, 59, 61–63, 65–84	21–57, 68–71	58–63, 65–67, 72–84
BL Advanced	58–78, (optional: 79–84)		

Multiple Representations

In Exercise 63, students use a calculator, a table of values, and verbal analysis to relate the values of expressions involving positive and negative exponents.

61. **PACKAGING** For a commercial art class, Rana must design a new container for individually wrapped pieces of candy. The shape that she chose is a cylinder. The formula for the volume of a cylinder is $V = \pi r^2 h$.

   a. The radius that Rana would like to use is $2p^3$, and the height is $4p^3$. Write a monomial that represents the volume of her container. **$16\pi p^9$**

   b. Make a table for five possible measures for the radius and height of a cylinder having the same volume.

   c. What is the volume of Rana's container if the height is doubled? **$32\pi p^9$**

62. **ENERGY** Albert Einstein's formula $E = mc^2$ shows that if mass is accelerated enough, it could be converted into usable energy. Energy E is measured in joules, mass m in kilograms, and the speed c of light is about 300 million meters per second.

   a. Complete the calculations to convert 3 kilograms of gasoline completely into energy. **270,000,000,000,000,000 joules**

   b. What happens to the energy if the amount of gasoline is doubled? **The energy is also doubled.**

63. **MULTIPLE REPRESENTATIONS** In this problem, you will explore exponents.

   a. **Tabular** Copy and use a calculator to complete the table.

Power	3^4	3^3	3^2	3^1	3^0	3^{-1}	3^{-2}	3^{-3}	3^{-4}
Value	81	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$

   b. **Analytical** What do you think the values of 5^0 and 5^{-1} are? Verify your conjecture using a calculator. **1 and $\frac{1}{5}$**

   c. **Analytical** Complete: For any nonzero number a and any integer n, $a^{-n} =$ ______. **$\frac{1}{a^n}$**

   d. **Verbal** Describe the value of a nonzero number raised to the zero power. **Any nonzero number raised to the zero power is 1.**

61b. Sample answer:

Radius	Height
$4p$	p^7
$4p^2$	p^5
$2p^3$	$4p^3$
$2p^4$	$4p$
$2p$	$4p^7$

H.O.T. Problems Use Higher-Order Thinking Skills

64. **PERSEVERANCE** For any nonzero real numbers a and b and any integers m and t, simplify the expression $\left(-\frac{a^m}{b^t}\right)^{2t}$ and describe each step.

64. Sample answer: First use the power of a power rule to simplify the expression to $\frac{a^{2tm}}{b^{2}t^{2}}$.

65. **REASONING** Copy the table below.

Equation	Related Expression	Power of x	Linear or Nonlinear
$y = x$	x	1	linear
$y = x^2$	x^2	2	nonlinear
$y = x^3$	x^3	3	nonlinear

   a. For each equation, write the related expression and record the power of x.

   b. Graph each equation using a graphing calculator. **See Ch. 8 Answer Appendix.**

   c. Classify each graph as *linear* or *nonlinear*. **See chart above.**

   d. Explain how to determine whether an equation, or its related expression, is linear or nonlinear without graphing. **See Ch. 8 Answer Appendix.**

66. **OPEN ENDED** Write three different expressions that can be simplified to x^6. **Sample answer: $x^4 \cdot x^2$; $x^5 \cdot x$; $(x^3)^2$**

67. **WRITING IN MATH** Write two formulas that have monomial expressions in them. Explain how each is used in a real-world situation.

67. Sample answer: The area of a circle or $A = \pi r^2$, where the radius r can be used to find the area of any circle. The area of a rectangle or $A = w \cdot \ell$, where w is the width and ℓ is the length, can be used to find the area of any rectangle.

484 | Lesson 8-1 | Multiplication Properties of Exponents

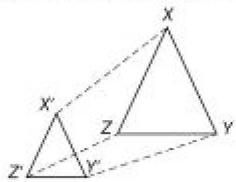
Standardized Test Practice

68. Which of the following is not a monomial? **C**

A $-6xy$ C $-\frac{1}{2b^3}$

B $\frac{1}{2}a^2$ D $5gh^4$

69. **GEOMETRY** The accompanying diagram shows the transformation of $\triangle XYZ$ to $\triangle X'Y'Z'$. **F**



This transformation is an example of a

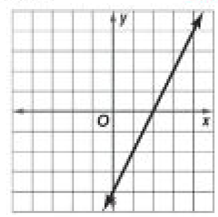
F dilation
G line reflection
H rotation
J translation

70. **CARS** In 2002, the average price of a new domestic car was AED 19,126. In 2008, the average price was AED 28,715. Based on a linear model, what is the predicted average price for 2014? **B**

A AED 45,495 C AED 35,906

B AED 38,304 D AED 26,317

71. **SHORT RESPONSE** If a line has a positive slope and a negative y-intercept, what happens to the x-intercept if the slope and the y-intercept are both doubled? **The x-intercept does not change.**



Spiral Review

Solve each system of inequalities by graphing. **72–75. See margin.**

72. $y < 4x$
$2x + 3y \geq -21$

73. $y \geq 2$
$2y + 2x \leq 4$

74. $y > -2x - 1$
$2y \leq 3x + 2$

75. $3x + 2y < 10$
$2x + 12y < -6$

76. **SPORTS** In the 2006 Winter Olympic Games, the total number of gold and silver medals won by the U.S. was 18. The total points scored for gold and silver medals was 45. Write and solve a system of equations to find how many gold and silver medals were won by the U.S. **$3g + 2v = 45$; $g + v = 18$; 9 gold, 9 silver**

77. **DRIVING** Tires should be kept within 2 kilograms per square centimeter (psi) of the manufacturer's recommended tire pressure. If the recommendation for a tire is 30 psi, what is the range of acceptable pressures? **$\{p \mid 28 \leq p \leq 32\}$**

78. **CHILD CARE** Suhaila charges AED 10 plus AED 4 per hour for child care. Suhaila needs at least AED 40 more to buy a television for which she is saving. Write an inequality for this situation. Will she be able to get her television if she provides child care for 5 hours? **$10 + 4h \geq 40$; no**

Skills Review

Find each quotient.

79. $-64 \div (-8)$ **8**
80. $-78 \div 1.3$ **-60**
81. $42.3 \div (-6)$ **-7.05**
82. $-23.94 \div 10.5$ **-2.28**
83. $-32.5 \div (-2.5)$ **13**
84. $-98.44 \div 4.6$ **-21.4**

Differentiated Instruction OL BL

Extension Tell students that a sports car on a drag-racing strip can reach 160 kilometers per hour in 0.4 kilometer. If s represents the speed in kilometers per hour, then the approximate number of meters that the driver must apply the brakes before stopping is $\frac{1}{153}s^2$. Calculate how far the car would travel on the drag strip, from start to stop, if the driver started braking when the car reached 160 kilometers per hour. 567 m; The initial 0.4 kilometer (400 m) plus braking distance 167 m.

Teaching the Mathematical Practices

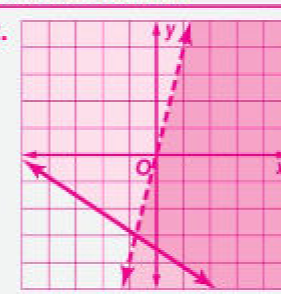
Perseverance Mathematically proficient students try simpler forms of the original problem in order to gain insight into its solution. In Exercise 64, advise students to substitute integers for a and b and simplify the expressions before generalizing.

4 Assess

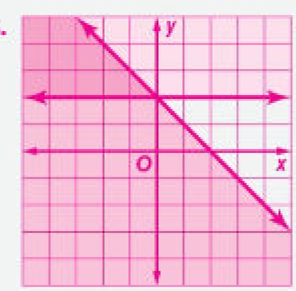
Ticket Out the Door Make several copies each of five monomial expressions that need to be simplified. Give one expression to each student. As students leave the room, ask them to tell you the simplified versions of the expressions they possess.

Additional Answers

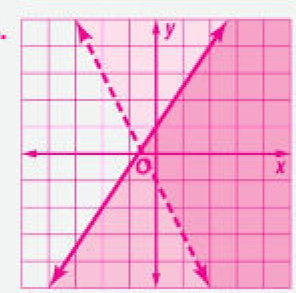
72.



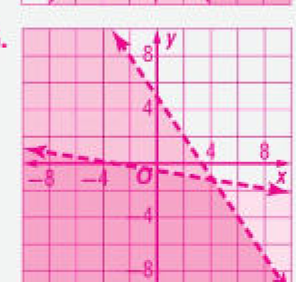
73.



74.



75.



1 Focus

Vertical Alignment

Before Lesson 8-2 Multiply monomials using the properties of exponents.

Lesson 8-2 Divide monomials using the properties of exponents. Simplify expressions containing negative and zero exponents.

After Lesson 8-2 Express numbers in scientific notation. Find products and quotients of numbers expressed in scientific notation.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What is 10^2 when simplified? 100
- What is 10^1 when simplified? 10
- How could you use exponents to write the ratio of the height of a skyscraper that is 900 meters tall to the height of the tallest tree? $\frac{10^3}{10^2}$ or 10^1

LESSON 8-2 Division Properties of Exponents

Then

- You multiplied monomials using the properties of exponents.

Now

1. Divide monomials using the properties of exponents.
2. Simplify expressions containing negative and zero exponents.

Why?

- The tallest redwood tree is 112 meters or about 10^2 meters tall. The average height of a redwood tree is 15 meters. The closest power of ten to 15 is 10^1, so an average redwood is about 10^1 meters tall. The ratio of the tallest tree's height to the average tree's height is $\frac{10^2}{10^1}$ or 10^1. This means the tallest redwood tree is approximately 10 times as tall as the average redwood tree.

New Vocabulary
zero exponent
negative exponent
order of magnitude

Mathematical Practices
Reason abstractly and quantitatively.

1 Divide Monomials We can use the principles for reducing fractions to find quotients of monomials like $\frac{10^2}{10^1}$. In the following examples, look for a pattern in the exponents.

$$\frac{2^7}{2^4} = \frac{\overbrace{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 2 \cdot 2 \cdot 2}^{7\text{ factors}}}{\underbrace{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2}}_{4\text{ factors}}} = 2 \cdot 2 \cdot 2 \text{ or } 2^3 \qquad \frac{t^4}{t^3} = \frac{\overbrace{\cancel{t} \cdot \cancel{t} \cdot \cancel{t} \cdot t}^{4\text{ factors}}}{\underbrace{\cancel{t} \cdot \cancel{t} \cdot \cancel{t}}_{3\text{ factors}}} = t$$

These examples demonstrate the Quotient of Powers Rule.

Key Concept Quotient of Powers

Words	To divide two powers with the same base, subtract the exponents.
Symbols	For any nonzero number a, and any integers m and p, $\frac{a^m}{a^p} = a^{m-p}$.
Examples	$\frac{c^{11}}{c^8} = c^{11-8}$ or c^3 $\qquad \frac{r^5}{r^2} = r^{5-2} = r^3$

Example 1 Quotient of Powers

Simplify $\frac{g^3h^5}{gh^2}$. Assume that no denominator equals zero.

$\frac{g^3h^5}{gh^2} = \left(\frac{g^3}{g}\right)\left(\frac{h^5}{h^2}\right)$ Group powers with the same base.

$= (g^{3-1})(h^{5-2})$ Quotient of Powers

$= g^2h^3$ Simplify.

Guided Practice

Simplify each expression. Assume that no denominator equals zero.

1A. $\frac{x^3y^4}{x^2y}$ xy^3

1B. $\frac{k^7m^{10}p}{k^5m^3p}$ k^2m^7

We can use the Product of Powers Rule to find the powers of quotients for monomials. In the following example, look for a pattern in the exponents.

$$\left(\frac{3}{4}\right)^3 = \underbrace{\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}_{\text{3 factors}} = \frac{\overbrace{3 \cdot 3 \cdot 3}^{\text{3 factors}}}{\underbrace{4 \cdot 4 \cdot 4}_{\text{3 factors}}} = \frac{3^3}{4^3}$$

$$\left(\frac{c}{d}\right)^2 = \underbrace{\left(\frac{c}{d}\right)\left(\frac{c}{d}\right)}_{\text{2 factors}} = \frac{\overbrace{c \cdot c}^{\text{2 factors}}}{\underbrace{d \cdot d}_{\text{2 factors}}} = \frac{c^2}{d^2}$$

StudyTip

Power Rules with Variables The power rules apply to variables as well as numbers. For example, $\left(\frac{3a}{4b}\right)^3 = \frac{(3a)^3}{(4b)^3}$ or $\frac{27a^3}{64b^3}$.

KeyConcept Power of a Quotient

Words To find the power of a quotient, find the power of the numerator and the power of the denominator.

Symbols For any real numbers a and $b \neq 0$, and any integer m, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

Examples $\left(\frac{3}{5}\right)^4 = \frac{3^4}{5^4}$ $\left(\frac{r}{t}\right)^5 = \frac{r^5}{t^5}$

Example 2 Power of a Quotient

Simplify $\left(\frac{3p^3}{7}\right)^2$.

$\left(\frac{3p^3}{7}\right)^2 = \frac{(3p^3)^2}{7^2}$ Power of a Quotient

$= \frac{3^2(p^3)^2}{7^2}$ Power of a Product

$= \frac{9p^6}{49}$ Power of a Power

GuidedPractice

Simplify each expression.

2A. $\left(\frac{3x^4}{4}\right)^3$ $\frac{27x^{12}}{64}$

2B. $\left(\frac{5x^5y}{6}\right)^2$ $\frac{25x^{10}y^2}{36}$

2C. $\left(\frac{2y^2}{3z^3}\right)^2$ $\frac{4y^4}{9z^6}$

2D. $\left(\frac{4x^3}{5y^4}\right)^3$ $\frac{64x^9}{125y^{12}}$

Real-WorldCareer

Astronomer An astronomer studies the universe and analyzes space travel and satellite communications. To be a technician or research assistant, a bachelor's degree is required.

A calculator can be used to explore expressions with 0 as the exponent. There are two methods to explain why a calculator gives a value of 1 for 3^0.

Method 1

$\frac{3^5}{3^5} = 3^{5-5}$ Quotient of Powers

$= 3^0$ Simplify.

Method 2

$\frac{3^5}{3^5} = \frac{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3}}$ Definition of powers

$= 1$ Simplify.

Since $\frac{3^5}{3^5}$ can only have one value, we can conclude that $3^0 = 1$. A **zero exponent** is any nonzero number raised to the zero power.

Teach with Tech

Blog On your secure classroom blog have students create a blog entry explaining how to find the quotient of powers and the power of a quotient.

1 Divide Monomials

Example 1 shows how to find the quotient of powers. **Example 2** shows how to find the power of a quotient. **Example 3** shows how to simplify expressions involving zero exponents.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Simplify $\frac{x^7y^{12}}{x^6y^3}$. Assume that no denominator equals zero. xy^9

2 Simplify $\left(\frac{4c^3d^2}{5}\right)^3$. $\frac{64c^9d^6}{125}$

WatchOut!

Preventing Errors Remind students to also find the powers of the constant terms of the monomials.

Tips for New Teachers

Properties Point out to students that the definition of the Quotient of Powers restricts a to being nonzero. Ask why must a be nonzero? If $a = 0$, we would be dividing by 0, which is undefined.

Focus on Mathematical Content

Powers of Negative Numbers Students may assume that the expression -6^3 means $(-6)(-6)(-6)$. Explain that -6^3 means $-(6^3)$. To express -6 to the third power, they must use parentheses, $(-6)^3$.

Key Concept: Zero Exponent Property

Words Any nonzero number raised to the zero power is equal to 1.

Symbols For any nonzero number a, $a^0 = 1$.

Examples $15^0 = 1$ $\left(\frac{b}{c}\right)^0 = 1$ $\left(\frac{2}{7}\right)^0 = 1$

Example 3 Zero Exponent

Simplify each expression. Assume that no denominator equals zero.

a. $\left(-\frac{4n^2q^5r^2}{9n^3q^2r}\right)^0$

$\left(-\frac{4n^2q^5r^2}{9n^3q^2r}\right)^0 = 1$ $a^0 = 1$

b. $\frac{x^5y^0}{x^3}$

$\frac{x^5y^0}{x^3} = \frac{x^5(1)}{x^3}$ $a^0 = 1$

$= x^2$ Quotient of Powers

Guided Practice

3A. $\frac{b^4c^2d^0}{b^2c}$ b^2c

3B. $\left(\frac{2f^4g^7h^3}{15f^3g^9h^6}\right)^0$ 1

Study Tip

Zero Exponent Be careful of parentheses. The expression $(5x)^0$ is 1 but $5x^0 = 5$.

2 Negative Exponents

Any nonzero real number raised to a negative power is a **negative exponent**. To investigate the meaning of a negative exponent, we can simplify expressions like $\frac{c^2}{c^5}$ using two methods.

Method 1

$\frac{c^2}{c^5} = c^{2-5}$ Quotient of Powers

$= c^{-3}$ Simplify.

Method 2

$\frac{c^2}{c^5} = \frac{\cancel{c} \cdot \cancel{c}}{\cancel{c} \cdot \cancel{c} \cdot c \cdot c \cdot c}$ Definition of powers

$= \frac{1}{c^3}$ Simplify.

Since $\frac{c^2}{c^5}$ can only have one value, we can conclude that $c^{-3} = \frac{1}{c^3}$.

Key Concept: Negative Exponent Property

Words For any nonzero number a and any integer n, a^{-n} is the reciprocal of a^n. Also, the reciprocal of a^{-n} is a^n.

Symbols For any nonzero number a and any integer n, $a^{-n} = \frac{1}{a^n}$.

Examples $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$ $\frac{1}{j^{-4}} = j^4$

WatchOut!

Preventing Errors Ask students why the negative sign in Example 3 does not affect the outcome. Students should explain that any nonzero number raised to the zero power is 1, and a negative number is a nonzero number.

Additional Example

3 Simplify each expression. Assume that no denominator equals zero.

a. $\left(\frac{12m^8n^7}{8m^5n^{10}}\right)^0$ 1

b. $\frac{m^0n^3}{n^2}$ n

Tips for New Teachers

Zero to the Zero Power Ask students to write an expression involving division that will be equivalent to 0^0, for example, $\frac{0^6}{0^6}$. Show students that $\frac{0^6}{0^6} = \frac{0}{0}$ and remind them that division by zero is undefined.

2 Negative Exponents

Example 4 shows how to simplify expressions involving negative exponents. **Example 5** shows how to apply the properties of exponents in real-world situations.

Differentiated Instruction (AL) (OL)

If students have difficulty relating the Key Concepts in this lesson to expressions,

Then have students make flash cards to illustrate each Key Concept. Write an expression that is an example of a Key Concept on the board. Tell students to show their card that correlates to the example. Then ask a student to describe the process of simplifying the expression.

An expression is considered simplified when it contains only positive exponents, each base appears exactly once, there are no powers of powers, and all fractions are in simplest form.

Example 4 Negative Exponents

Simplify each expression. Assume that no denominator equals zero.

a. $\frac{n^{-5}p^4}{r^{-2}}$

$\frac{n^{-5}p^4}{r^{-2}} = \left(\frac{n^{-5}}{1}\right)\left(\frac{p^4}{1}\right)\left(\frac{1}{r^{-2}}\right)$ Write as a product of fractions.

$= \left(\frac{1}{n^5}\right)\left(\frac{p^4}{1}\right)\left(\frac{r^2}{1}\right)$ $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$

$= \frac{p^4r^2}{n^5}$ Multiply.

b. $\frac{5r^{-3}t^4}{-20r^2t^7u^{-5}}$

$\frac{5r^{-3}t^4}{-20r^2t^7u^{-5}} = \left(\frac{5}{-20}\right)\left(\frac{r^{-3}}{r^2}\right)\left(\frac{t^4}{t^7}\right)\left(\frac{1}{u^{-5}}\right)$ Group powers with the same base.

$= \left(-\frac{1}{4}\right)\left(r^{-3-2}\right)\left(t^{4-7}\right)\left(u^5\right)$ Quotient of Powers and Negative Exponents Property

$= -\frac{1}{4}r^{-5}t^{-3}u^5$ Simplify.

$= -\frac{1}{4}\left(\frac{1}{r^5}\right)\left(\frac{1}{t^3}\right)\left(u^5\right)$ Negative Exponent Property

$= -\frac{u^5}{4r^5t^3}$ Multiply.

c. $\frac{2a^2b^3c^{-5}}{10a^{-3}b^{-1}c^{-4}}$

$\frac{2a^2b^3c^{-5}}{10a^{-3}b^{-1}c^{-4}} = \left(\frac{2}{10}\right)\left(\frac{a^2}{a^{-3}}\right)\left(\frac{b^3}{b^{-1}}\right)\left(\frac{c^{-5}}{c^{-4}}\right)$ Group powers with the same base.

$= \left(\frac{1}{5}\right)\left(a^{2-(-3)}\right)\left(b^{3-(-1)}\right)\left(c^{-5-(-4)}\right)$ Quotient of Powers and Negative Exponents Property

$= \frac{1}{5}a^5b^4c^{-1}$ Simplify.

$= \frac{1}{5}\left(a^5\right)\left(b^4\right)\left(\frac{1}{c}\right)$ Negative Exponent Property

$= \frac{a^5b^4}{5c}$ Multiply.

Guided Practice

Simplify each expression. Assume that no denominator equals zero.

4A. $\frac{v^{-3}wx^2}{wy^{-6}}$ $\frac{x^2y^6}{v^3}$ **4B.** $\frac{32a^{-8}b^3c^{-4}}{4a^3b^5c^{-2}}$ $\frac{8}{a^{11}b^2c^2}$ **4C.** $\frac{5j^{-3}k^2m^{-6}}{25k^{-4}m^{-2}}$ $\frac{k^6}{5j^3m^4}$

Order of magnitude is used to compare measures and to estimate and perform rough calculations. The **order of magnitude** of a quantity is the number rounded to the nearest power of 10. For example, the power of 10 closest to 95,000,000,000 is 10^{11}, or 100,000,000,000. So the order of magnitude of 95,000,000,000 is 10^{11}.

Study Tip

Negative Signs Be aware of where a negative sign is placed. $5^{-1} = \frac{1}{5}$, while $-5^1 \neq \frac{1}{5}$.

Real-World Link

An adult human weighs about 70 kilograms and an adult dairy cow weighs about 700 kilograms. Their weights differ by 1 order of magnitude.

Watch Out!

Preventing Errors Have students look at the first step of the solution of Example 4. Point out that rewriting an expression as a product of fractions makes applying the Negative Exponent Property easier. Fractions that have negative exponents can be rewritten as their reciprocals.

Additional Example

4 Simplify each expression. Assume that no denominator equals zero.

a. $\frac{x^{-4}y^9}{z^{-6}}$ $\frac{y^9z^6}{x^4}$

b. $\frac{75p^3m^{-5}}{15p^5m^{-4}r^{-8}}$ $\frac{5r^8}{p^2m}$

Tips for New Teachers

Monomials Point out to students that a variable with a negative exponent such as x^{-2} is not a monomial. A monomial does not involve division by variables and $x^{-2} = \frac{1}{x^2}$.

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Additional Example

5 **SAVINGS** Usama has AED123,456 in his savings account. Ismail has AED156 in his savings account. Determine the order of magnitude of Usama's account and Ismail's account. How many orders of magnitude as great is Usama's account as Ismail's account? Usama: 10^5, Ismail: 10^2; Usama's account is 3 orders of magnitude as great as Ismail's account.

3 Practice

Formative Assessment

Use Exercises 1–18 to check for understanding.

Use the chart at the bottom of the next page to customize assignments for your students.

Real-World Link

There are over 14,000 species of ants living all over the world. Some ants can carry objects that are 50 times their own weight.

Source: Maine Animal Coalition

Real-World Example 5 Apply Properties of Exponents

HEIGHT **Suppose the average height of a man is about 1.7 meters, and the average height of an ant is 0.0008 meter. How many orders of magnitude as tall as an ant is a man?**

Understand We must find the order of magnitude of the heights of the man and ant. Then find the ratio of the orders of magnitude of the man's height to that of the ant's height.

Plan Round each height to the nearest power of ten. Then find the ratio of the height of the man to the height of the ant.

Solve The average height of a man is close to 1 meter. So, the order of magnitude is 10^0 meter. The average height of an ant is about 0.001 meter. So, the order of magnitude is 10^{-3} meters.

The ratio of the height of a man to the height of an ant is about $\frac{10^0}{10^{-3}}$.

$$\frac{10^0}{10^{-3}} = 10^{0-(-3)} \quad \text{Quotient of Powers}$$
$$= 10^3 \quad 0-(-3) = 0+3 \text{ or } 3$$
$$= 1000 \quad \text{Simplify.}$$

So, a man is approximately 1000 times as tall as an ant, or a man is 3 orders of magnitude as tall as an ant.

Check The ratio of the man's height to the ant's height is $\frac{1.7}{0.0008} = 2125$. The order of magnitude of 2125 is 10^3. ✓

Guided Practice

5. **ASTRONOMY** The order of magnitude of the mass of Earth is about 10^{27}. The order of magnitude of the Milky Way galaxy is about 10^{44}. How many orders of magnitude as big is the Milky Way galaxy as Earth? 17

Check Your Understanding

Examples 1–4 **Simplify each expression. Assume that no denominator equals zero.**

1. $\frac{t^5u^4}{t^2u}$ t^3u^3
2. $\frac{a^6b^4c^{10}}{a^3b^2c}$ $a^3b^2c^9$
3. $\frac{m^6r^5p^3}{m^5r^2p^3}$ mr^3
4. $\frac{b^4c^6f^8}{b^4c^3f^5}$ c^3f^3
5. $\frac{g^8h^2m}{hg^7}$ ghm
6. $\frac{r^4t^7v^2}{t^7v^2}$ r^4
7. $\frac{x^3y^2z^6}{z^5x^2y}$ xyz
8. $\frac{n^4q^4w^6}{q^2n^3w}$ nq^2w^5
9. $\left(\frac{2a^3b^5}{3}\right)^2$ $\frac{4a^6b^{10}}{9}$
10. $\frac{r^3v^{-2}}{t^{-7}}$ $\frac{r^3t^7}{v^2}$
11. $\left(\frac{2c^3d^5}{5g^2}\right)^5$ $\frac{32c^{15}d^{25}}{3125g^{10}}$
12. $\left(-\frac{3xy^4z^2}{x^3yz^4}\right)^0$ 1
13. $\left(\frac{3f^4gh^4}{32f^3g^4h}\right)^0$ 1
14. $\frac{4r^2v^0t^5}{2rt^3}$ $2rt^2$
15. $\frac{f^{-3}g^2}{h^{-4}}$ $\frac{g^2h^4}{f^3}$
16. $\frac{-8x^2y^8z^{-5}}{12x^4y^{-7}z^7}$ $\frac{-2y^{15}}{3x^2z^{12}}$
17. $\frac{2a^2b^{-7}c^{10}}{6a^{-3}b^2c^{-3}}$ $\frac{a^5c^{13}}{3b^9}$

Example 5

18. **FINANCIAL LITERACY** The gross domestic product (GDP) for the United States in 2008 was AED 52.199 trillion, and the GDP per person was AED 174,850. Use order of magnitude to approximate the population of the United States in 2008. 10^8 or 100,000,000

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Practice and Problem Solving

Examples 1–4 **Simplify each expression. Assume that no denominator equals zero.**

19. $\frac{m^4p^2}{m^2p}$ m^2p

20. $\frac{p^{12}t^3r}{p^2tr}$ $p^{10}t^2$

21. $\frac{3m^{-3}r^4p^2}{12t^4}$ $\frac{r^4p^2}{4m^3t^4}$

22. $\frac{c^4d^4f^3}{c^2d^4f^3}$ c^2

23. $\left(\frac{3xy^4}{5z^2}\right)^2$ $\frac{9x^2y^8}{25z^4}$

24. $\left(\frac{3t^6u^2v^5}{9tuv^{21}}\right)^0$ 1

25. $\left(\frac{p^2t^7}{10}\right)^3$ $\frac{p^6t^{21}}{1000}$

26. $\frac{x^{-4}y^9}{z^{-2}}$ $\frac{y^9z^2}{x^4}$

27. $\frac{a^7b^8c^8}{a^5bc^7}$ a^2b^7c

28. $\left(\frac{3np^3}{7q^2}\right)^2$ $\frac{9n^2p^6}{49q^4}$

29. $\left(\frac{2r^3t^6}{5u^9}\right)^4$ $\frac{16r^{12}t^{24}}{625u^{36}}$

30. $\left(\frac{3m^5r^3}{4p^8}\right)^4$ $\frac{81m^{20}r^{12}}{256p^{32}}$

31. $\left(-\frac{5f^9g^4h^2}{fg^2h^3}\right)^0$ 1

32. $\frac{p^{12}t^7r^2}{p^2t^7r}$ $p^{10}r$

33. $\frac{p^4t^{-3}}{r^{-2}}$ $\frac{p^4r^2}{t^3}$

34. $-\frac{5c^2d^5}{8cd^5f^0}$ $-\frac{5c}{8}$

35. $\frac{-2f^3g^2h^0}{8f^2g^2}$ $\frac{-f}{4}$

36. $\frac{12m^{-4}p^2}{-15m^3p^{-9}}$ $\frac{4p^{11}}{-5m^7}$

37. $\frac{k^4m^3p^2}{k^2m^2}$ k^2mp^2

38. $\frac{14f^{-3}g^2h^{-7}}{21k^3}$ $\frac{2g^2}{3f^3h^7k^3}$

39. $\frac{39t^4uv^{-2}}{13t^{-3}u^7}$ $\frac{3t^7}{u^6v^2}$

40. $\left(\frac{a^{-2}b^4c^5}{a^{-4}b^{-4}c^3}\right)^2$ $a^4b^{16}c^4$

41. $\frac{r^3t^{-1}x^{-5}}{tx^5}$ $\frac{r^3}{t^2x^{10}}$

42. $\frac{g^0h^7j^{-2}}{g^{-5}h^0j^{-2}}$ g^5h^7

43. 10^6; 10^8; about 10^2 or 100 times as many users as hosts

Example 5

43. **INTERNET** In a recent year, there were approximately 3.95 million Internet hosts. Suppose there were 208 million Internet users. Determine the order of magnitude for the Internet hosts and Internet users. Using the orders of magnitude, how many Internet users were there compared to Internet hosts?

B 44. **PROBABILITY** The probability of rolling a die and getting an even number is $\frac{1}{2}$. If you roll the die twice, the probability of getting an even number both times is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ or $\left(\frac{1}{2}\right)^2$.

a. What does $\left(\frac{1}{2}\right)^4$ represent? probability of all evens on 4 rolls

b. Write an expression to represent the probability of rolling a die d times and getting an even number every time. Write the expression as a power of 2. $\left(\frac{1}{2}\right)^d$; 2^{-d}

Simplify each expression. Assume that no denominator equals zero.

45. $\frac{-4w^{12}}{12w^3}$ $-\frac{w^9}{3}$

46. $\frac{13r^7}{39r^4}$ $\frac{r^3}{3}$

47. $\frac{(4k^3m^2)^3}{(5k^2m^{-3})^{-2}}$ $1600k^{13}$

48. $\frac{3wy^{-2}}{(w^{-1}y)^3}$ $\frac{3w^4}{y^5}$

49. $\frac{20qr^{-2}t^{-5}}{4q^0r^4t^{-2}}$ $\frac{5q}{r^6t^3}$

50. $\frac{-12c^3d^0f^{-2}}{6c^5d^{-3}f^4}$ $\frac{-2d^3}{c^2f^6}$

51. $\frac{(2g^3h^{-2})^2}{(g^2h^0)^{-3}}$ $\frac{4g^{12}}{h^4}$

52. $\frac{(5pr^{-2})^{-2}}{(3p^{-1}r)^3}$ $\frac{pr}{675}$

53. $\left(\frac{-3x^{-6}y^{-1}z^{-2}}{6x^{-2}yz^{-5}}\right)^{-2}$ $\frac{4x^8y^4}{z^6}$

54. $\left(\frac{2a^{-2}b^4c^2}{-4a^{-2}b^{-5}c^{-7}}\right)^{-1}$ $\frac{-2}{b^9c^9}$

55. $\frac{(16x^2y^{-1})^0}{(4x^0y^{-4}z)^{-2}}$ $\frac{16z^2}{y^8}$

56. $\left(\frac{4^0c^2d^3f}{2c^{-4}d^{-5}}\right)^{-3}$ $\frac{8}{c^{18}d^{24}f^3}$

57. **SENSE-MAKING** The processing speed of an older desktop computer is about 10^8 instructions per second. A new computer can process about 10^{10} instructions per second. The newer computer is how many times as fast as the older one? 100

Tips for New Teachers

Checking Answers When simplifying expressions, students may wish to check their answers by choosing nonzero values for the variables and evaluating the original and simplified expressions. If the results are the same, the expressions are likely to be equivalent.

Teaching the Mathematical Practices

Sense-Making Mathematically proficient students try simpler forms of the original problem in order to gain insight into its solution. In Exercise 57, encourage students to start by finding the number of instructions per second in a computer ten times as fast.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	19–43, 61, 62, 64–89	19–43 odd, 66–69	20–42 even, 61, 62, 64, 65, 70–89
OL Core	19–43 odd, 44, 45–55 odd, 57–62, 64–89	19–43, 66–69	44–62, 64, 65, 70–89
BL Advanced	44–81, (optional: 82–89)		

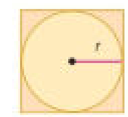
Multiple Representations

In Exercise 60, students use a geometric sketch, algebraic formulas, and a table of values to analyze the relation between a square and an inscribed circle.

58. **ASTRONOMY** The brightness of a star is measured in magnitudes. The lower the magnitude, the brighter the star. A magnitude 9 star is 2.51 times as bright as a magnitude 10 star. A magnitude 8 star is $2.51 \cdot 2.51$ or 2.51^2 times as bright as a magnitude 10 star. **2.51^7 or 627.647857**
   a. How many times as bright is a magnitude 3 star as a magnitude 10 star?
   b. Write an expression to compare a magnitude m star to a magnitude 10 star. **2.51^{10-m}**
   c. A full moon is considered to be magnitude -13, approximately. Does your expression make sense for this magnitude? Explain.

59. **PROBABILITY** The probability of rolling a die and getting a 3 is $\frac{1}{6}$. If you roll the die twice, the probability of getting a 3 both times is $\frac{1}{6} \cdot \frac{1}{6}$ or $\left(\frac{1}{6}\right)^2$.
   a. Write an expression to represent the probability of rolling a die d times and getting a 3 each time. **$\left(\frac{1}{6}\right)^d$**
   b. Write the expression as a power of 6. **6^{-d}**

60. **MULTIPLE REPRESENTATIONS** To find the area of a circle, use $A = \pi r^2$. The formula for the area of a square is $A = s^2$.
   a. **Algebraic** Find the ratio of the area of the circle to the area of the square. **$\frac{\pi}{4}$**
   b. **Algebraic** If the radius of the circle and the length of each side of the square are doubled, find the ratio of the area of the circle to the square. **$\frac{\pi}{4}$**
   c. **Tabular** Copy and complete the table.



58c. Sample answer: Yes; according to the expression, a full Moon would be $2.51^{10-m} = 2.51^{10-(-13)} = 2.51^{23}$ or 1,557,742,231 times as bright as a magnitude 10 star. Since we know that the lower the magnitude the brighter the object, it follows that a magnitude -13 object is significantly brighter than a magnitude 10 object.

61. Sometimes; sample answer: The equation is true when $x = 0$, $y = 2$, and $z = 3$, but it is false when $x = 1$, $y = 2$, and $z = 3$.

Radius	Area of Circle	Area of Square	Ratio
r	πr^2	$4r^2$	$\frac{\pi}{4}$
$2r$	$\pi 4r^2$	$16r^2$	$\frac{\pi}{4}$
$3r$	$\pi 9r^2$	$36r^2$	$\frac{\pi}{4}$
$4r$	$\pi 16r^2$	$64r^2$	$\frac{\pi}{4}$
$5r$	$\pi 25r^2$	$100r^2$	$\frac{\pi}{4}$
$6r$	$\pi 36r^2$	$144r^2$	$\frac{\pi}{4}$

   d. **Analytical** What conclusion can be drawn from this?
   The ratio of the area of the circle to the area of the square will always be $\frac{\pi}{4}$.

H.O.T. Problems Use Higher-Order Thinking Skills

61. **REASONING** Is $x^y \cdot x^z = x^{yz}$ *sometimes*, *always*, or *never* true? Explain.

62. **OPEN ENDED** Name two monomials with a quotient of $24a^2b^3$. **Sample answer: $24a^4b^6$ and a^2b^3**

63. **CHALLENGE** Use the Quotient of Powers Property to explain why $x^{-n} = \frac{1}{x^n}$. **$\frac{1}{x^n} = \frac{x^0}{x^n} = x^{0-n} = x^{-n}$**

64. **REGULARITY** Write a convincing argument to show why $3^0 = 1$. **See margin.**

65. **WRITING IN MATH** Explain how to use the Quotient of Powers property and the Power of a Quotient property. **See margin.**

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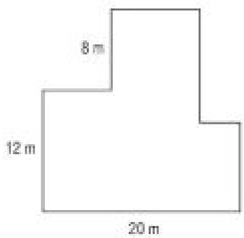
Standardized Test Practice

66. What is the perimeter of the figure in meters? **B**

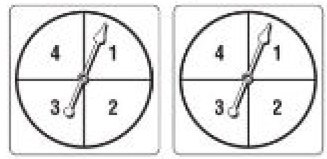


A 40 meters
B 80 meters
C 160 meters
D 400 meters

67. In researching her science project, Wafa learned that light travels at a constant rate and that it takes 500 seconds for light to travel the 150 million km from the Sun to Earth. Mars is 228 million km from the Sun. About how many seconds will it take for light to travel from the Sun to Mars? **J**

F 235 seconds
G 327 seconds
H 642 seconds
J 763 seconds

68. EXTENDED RESPONSE Hiyam and Hana are playing a game using the spinners below. Each spinner is equally likely to stop on any of the four numbers. In the game, a player spins both spinners and calculates the product of the two numbers on which the spinners have stopped.



a. What product has the greatest probability of occurring? **4**

b. What is the probability of that product occurring? $\frac{3}{16}$

69. Simplify $(4^{-2} \cdot 5^0 \cdot 64)^3$. **B**

A $\frac{1}{64}$
B 64
C 320
D 1024

Spiral Review

70. GEOMETRY A rectangular prism has a width of $7x^3$ units, a length of $4x^2$ units, and a height of $3x$ units. What is the volume of the prism? (Lesson 8-1) **$84x^6$ units^3**

Solve each system of inequalities by graphing. **71–74. See Ch. 8 Answer Appendix.**

71. $y \geq 1$, $x < -1$ **72.** $y \geq -3$, $y - x < 1$ **73.** $y < 3x + 2$, $y \geq -2x + 4$ **74.** $y - 2x < 2$, $y - 2x > 4$

Solve each inequality. Check your solution.

75. $5(2h - 6) > 4h$ **$h > 5$**
76. $22 \geq 4(b - 8) + 10$ **$b \leq 11$**
77. $5(u - 8) \leq 3(u + 10)$ **$u \leq 35$**
78. $8 + t \leq 3(t + 4) + 2$ **$t \geq -3$**
79. $9n + 3(1 - 6n) \leq 21$ **$n \geq -2$**
80. $-6(b + 5) > 3(b - 5)$ **$b < -\frac{5}{3}$**

81. GRADES In a high school science class, a test is worth three times as much as a quiz. What is the student's average grade? **87**

Science Grades

Tests	Quizzes
85	82
92	75
	95

Skills Review

Evaluate each expression.

82. 9^2 **81** **83.** 11^2 **121** **84.** 10^6 **1,000,000** **85.** 10^4 **10,000**

86. 3^5 **243** **87.** 5^3 **125** **88.** 12^3 **1728** **89.** 4^6 **4096**

Exercise Alert

Grid Paper For Exercises 71–74, students will need grid paper.

Teaching the Mathematical Practices

Regularity Mathematically proficient students notice if calculations are repeated. In Exercise 64, encourage students to examine the sequences $3^5, 3^4, 3^3, 3^2, \ldots$ and 243, 81, 27, 9, ...

4 Assess

Yesterday's News Ask students to write two ways in which the concepts of multiplying monomials helped them to understand dividing monomials.

Additional Answers

64. Since each number is obtained by dividing the previous number by 3, $3^1 = 3$ and $3^0 = 1$.

65. The Quotient of Powers Property is used when dividing two powers with the same base. The exponents are subtracted. The Power of a Quotient Property is used to find the power of a quotient. You find the power of the numerator and the power of the denominator.

Differentiated Instruction OL BL

Extension Tell students that if you roll a color cube, the probability of getting red is $\frac{1}{2}$. If you throw the cube n times, the probability of getting red each time is $\left(\frac{1}{2}\right)^n$. Ask them to determine how many times the cube is rolled if the probability of getting red each time is $\frac{1}{512}$. $\frac{1}{512} = \left(\frac{1}{2}\right)^9$, $n = 9$, so 9 times

1 Focus

Vertical Alignment

Before Lesson 8-3 Used the laws of exponents to find products and quotients of monomials.

Lesson 8-3 Evaluate and rewrite expressions involving rational exponents. Solve equations involving expressions with rational exponents.

After Lesson 8-3 Express numbers in scientific notation. Find products and quotients of numbers expressed in scientific notation.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- How does $50f^{0.2}$ differ from other exponential expressions students have seen before? Sample answer: The exponent is not an integer.
- Does a higher SPF offer more or less protection from sun damage? more
- What SPF numbers have you seen on sunscreens in stores? Sample answer: 2, 4, 15, 30

LESSON 8-3 Rational Exponents

Then

- You used the laws of exponents to find products and quotients of monomials.

Now

1. Evaluate and rewrite expressions involving rational exponents.
2. Solve equations involving expressions with rational exponents.

Why?

- It's important to protect your skin with sunscreen to prevent damage. The sun protection factor (SPF) of a sunscreen indicates how well it protects you. Sunscreen with an SPF of f absorbs about p percent of the UV-B rays, where $p = 50f^{0.2}$.

New Vocabulary
rational exponent
cube root
nth root
exponential equation

Mathematical Practices
Use appropriate tools strategically.

1 Rational Exponents

You know that an exponent represents the number of times that the base is used as a factor. But how do you evaluate an expression with an exponent that is not an integer like the one above? Let's investigate **rational exponents** by assuming that they behave like integer exponents.

$$\left(b^{\frac{1}{2}}\right)^2 = b^{\frac{1}{2}} \cdot b^{\frac{1}{2}} \quad \text{Write as a multiplication expression.}$$
$$= b^{\frac{1}{2}+\frac{1}{2}} \quad \text{Product of Powers}$$
$$= b^1 \text{ or } b \quad \text{Simplify.}$$

Thus, $b^{\frac{1}{2}}$ is a number with a square equal to b. So $b^{\frac{1}{2}} = \sqrt{b}$.

Key Concept $b^{\frac{1}{2}}$

Words	For any nonnegative real number b, $b^{\frac{1}{2}} = \sqrt{b}$.	
Examples	$16^{\frac{1}{2}} = \sqrt{16}$ or 4	$38^{\frac{1}{2}} = \sqrt{38}$

Example 1 Radical and Exponential Forms

Write each expression in radical form, or write each radical in exponential form.

a. $25^{\frac{1}{2}}$

$25^{\frac{1}{2}} = \sqrt{25}$ Definition of $b^{\frac{1}{2}}$

$= 5$ Simplify.

b. $\sqrt{18}$

$\sqrt{18} = 18^{\frac{1}{2}}$ Definition of $b^{\frac{1}{2}}$

c. $5x^{\frac{1}{2}}$

$5x^{\frac{1}{2}} = 5\sqrt{x}$ Definition of $b^{\frac{1}{2}}$

d. $\sqrt{8p}$

$\sqrt{8p} = (8p)^{\frac{1}{2}}$ Definition of $b^{\frac{1}{2}}$

Guided Practice

1A. $a^{\frac{1}{2}}$ $\sqrt{a}$ **1B.** $\sqrt{22}$ $22^{\frac{1}{2}}$ **1C.** $(7w)^{\frac{1}{2}}$ $\sqrt{7w}$ **1D.** $2\sqrt{x}$ $2x^{\frac{1}{2}}$

You know that to find the square root of a number a you find a number with a square of a. In the same way, you can find other roots of numbers. If $a^3 = b$, then a is the **cube root** of b, and if $a^n = b$ for a positive integer n, then a is an **nth root** of b.

StudyTip

Tools You can use a graphing calculator to find nth roots. Enter n, then press MATH and choose $\sqrt[x]{\ }$.

KeyConcept nth Root

Words	For any real numbers a and b and any positive integer n, if $a^n = b$, then a is an nth root of b.
Symbols	If $a^n = b$, then $\sqrt[n]{b} = a$.
Example	Because $2^4 = 16$, 2 is a fourth root of 16; $\sqrt[4]{16} = 2$.

Since $3^2 = 9$ and $(-3)^2 = 9$, both 3 and -3 are square roots of 9. Similarly, since $2^4 = 16$ and $(-2)^4 = 16$, both 2 and -2 are fourth roots of 16. The positive roots are called *principal roots*. Radical symbols indicate principal roots, so $\sqrt[4]{16} = 2$.

Example 2 nth roots

Simplify.

a. $\sqrt[3]{27}$

$\sqrt[3]{27} = \sqrt[3]{3 \cdot 3 \cdot 3}$

$= 3$

b. $\sqrt[5]{32}$

$\sqrt[5]{32} = \sqrt[5]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$

$= 2$

GuidedPractice

2A. $\sqrt[3]{64}$ 4

2B. $\sqrt[4]{10{,}000}$ 10

Like square roots, nth roots can be represented by rational exponents.

$\left(b^{\frac{1}{n}}\right)^n = \underbrace{b^{\frac{1}{n}} \cdot b^{\frac{1}{n}} \cdot \ldots \cdot b^{\frac{1}{n}}}_{n \text{ factors}}$ Write as a multiplication expression.

$= b^{\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}}$ Product of Powers

$= b^1$ or b Simplify.

Thus, $b^{\frac{1}{n}}$ is a number with an nth power equal to b. So $b^{\frac{1}{n}} = \sqrt[n]{b}$.

KeyConcept $b^{\frac{1}{n}}$

Words	For any positive real number b and any integer $n > 1$, $b^{\frac{1}{n}} = \sqrt[n]{b}$.
Example	$8^{\frac{1}{3}} = \sqrt[3]{8} = \sqrt[3]{2 \cdot 2 \cdot 2}$ or 2

1 Rational Exponents

Example 1 shows how to write an expression with a rational exponent in radical form and how to write an expression in radical form as an expression with a rational exponent. **Example 2** shows how to evaluate nth roots. **Example 3** shows how to evaluate expressions of the form $b^{\frac{1}{n}}$. **Example 4** shows how to evaluate expressions of the form $b^{\frac{m}{n}}$.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Write each expression in radical form, or write each radical in exponential form.

a. $81^{\frac{1}{2}}$ 9

b. $\sqrt{38}$ $38^{\frac{1}{2}}$

c. $12m^{\frac{1}{2}}$ $12\sqrt{m}$

d. $\sqrt{32w}$ $(32w)^{\frac{1}{2}}$

2 Simplify.

a. $\sqrt[4]{256}$ 4

b. $\sqrt[6]{15{,}625}$ 5

Teaching the Mathematical Practices

Tools Mathematically proficient students are familiar with the tools appropriate for their course and make sound decisions about when each might be helpful. Encourage students to use mental math to verify answers they find with a calculator.

Tips for New Teachers

Order of Operations Remind students to observe the order of operations with radical and exponential expressions. Emphasize that expressions like $5x^{\frac{1}{2}}$ and $(5x)^{\frac{1}{2}}$ are different.

Notation Emphasize that if there is no index on a radical symbol, it denotes a square root.

Additional Examples

3 Simplify.

a. $1331^{\frac{1}{3}}$ 11

b. $2401^{\frac{1}{4}}$ 7

4 Simplify.

a. $32^{\frac{2}{5}}$ 4

b. $81^{\frac{5}{2}}$ 59,049

Focus on Mathematical Content

Rational Exponents The meaning of a rational exponent follows from extending the properties of integer exponents to those values. This allows for a notation for radicals in terms of rational exponents. For example, we define $2^{\frac{1}{2}}$ to be the square root of 2 because we want $\left(2^{\frac{1}{2}}\right)^2 = 2^{\frac{1}{2}\cdot 2} = 2^1$ to hold, so $\left(2^{\frac{1}{2}}\right)^2$ must equal 2.

StudyTip

Rational Exponents on a Calculator Use parentheses to evaluate expressions involving rational exponents on a graphing calculator. For example to find $125^{\frac{1}{3}}$, press 125 [^] [(] 1 [÷] 3 [)] [ENTER].

Example 3 Evaluate $b^{\frac{1}{n}}$ Expressions

Simplify.

a. $125^{\frac{1}{3}}$

$125^{\frac{1}{3}} = \sqrt[3]{125}$ $b^{\frac{1}{n}} = \sqrt[n]{b}$

$= \sqrt[3]{5 \cdot 5 \cdot 5}$ $125 = 5^3$

$= 5$ Simplify.

b. $1296^{\frac{1}{4}}$

$1296^{\frac{1}{4}} = \sqrt[4]{1296}$ $b^{\frac{1}{n}} = \sqrt[n]{b}$

$= \sqrt[4]{6 \cdot 6 \cdot 6 \cdot 6}$ $1296 = 6^4$

$= 6$ Simplify.

GuidedPractice

3A. $27^{\frac{1}{3}}$ 3

3B. $256^{\frac{1}{4}}$ 4

The Power of a Power property allows us to extend the definition of $b^{\frac{1}{n}}$ to $b^{\frac{m}{n}}$.

$b^{\frac{m}{n}} = \left(b^{\frac{1}{n}}\right)^m$ Power of a Power

$= \left(\sqrt[n]{b}\right)^m$ or $\sqrt[n]{b^m}$ $b^{\frac{1}{n}} = \sqrt[n]{b}$

KeyConcept $b^{\frac{m}{n}}$

Words For any positive real number b and any integers m and $n > 1$,

$$b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m \text{ or } \sqrt[n]{b^m}.$$

Example $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2$ or 4

Example 4 Evaluate $b^{\frac{m}{n}}$ Expressions

Simplify.

a. $64^{\frac{2}{3}}$

$64^{\frac{2}{3}} = \left(\sqrt[3]{64}\right)^2$ $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$

$= \left(\sqrt[3]{4 \cdot 4 \cdot 4}\right)^2$ $64 = 4^3$

$= 4^2$ or 16 Simplify.

b. $36^{\frac{3}{2}}$

$36^{\frac{3}{2}} = \left(\sqrt[2]{36}\right)^3$ $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$

$= 6^3$ $\sqrt{36} = 6$

$= 216$ Simplify.

GuidedPractice

4A. $27^{\frac{2}{3}}$ 9

4B. $256^{\frac{5}{4}}$ 1024

DifferentiatedInstruction AL OL BL

Interpersonal Learners Divide the class into groups of two or three students. Have students discuss what they knew about exponents before starting the lesson and how it relates to rational exponents. For example, if an exponent tells you how many times to use the base as a factor, what does an exponent of $\frac{3}{2}$ represent?

496 | Lesson 8-3 | Rational Exponents

2 Solve Exponential Equations

In an **exponential equation**, variables occur as exponents. The Power Property of Equality and the other properties of exponents can be used to solve exponential equations.

KeyConcept Power Property of Equality

Words	For any real number $b > 0$ and $b \neq 1$, $b^x = b^y$ if and only if $x = y$.
Examples	If $5^x = 5^3$, then $x = 3$. If $n = \frac{1}{2}$, then $4^n = 4^{\frac{1}{2}}$.

Example 5 Solve Exponential Equations

Solve each equation.

a. $6^x = 216$

$6^x = 216$ Original equation

$6^x = 6^3$ Rewrite 216 as 6^3.

$x = 3$ Property of Equality

CHECK $6^x = 216$

$6^3 \stackrel{?}{=} 216$

$216 = 216$ ✓

b. $25^{x-1} = 5$

$25^{x-1} = 5$ Original equation

$(5^2)^{x-1} = 5$ Rewrite 25 as 5^2.

$5^{2x-2} = 5^1$ Power of a Power, Distributive Property

$2x - 2 = 1$ Power Property of Equality

$2x = 3$ Add 2 to each side.

$x = \frac{3}{2}$ Divide each side by 2.

CHECK $25^{x-1} = 5$

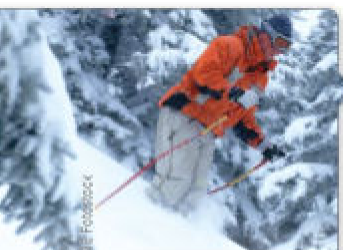
$25^{\frac{3}{2}-1} \stackrel{?}{=} 5$

$25^{\frac{1}{2}} = 5$ ✓

GuidedPractice

5A. $5^x = 125$ **3**

5B. $12^{2x+3} = 144$ $-\frac{1}{2}$



Real-WorldLink

Use extra caution near snow, water, and sand because they reflect the damaging rays of the Sun, which can increase your chance of sunburn.

Source: American Academy of Dermatology

Real-World Example 6 Solve Exponential Equations

SUNSCREEN **Refer to the beginning of the lesson. Find the SPF that absorbs 100% of UV-B rays.**

$p = 50f^{0.2}$ Original equation

$100 = 50f^{0.2}$ $p = 100$

$2 = f^{0.2}$ Divide each side by 50.

$2 = f^{\frac{1}{5}}$ $0.2 = \frac{1}{5}$

$(2^5)^{\frac{1}{5}} = f^{\frac{1}{5}}$ $2 = 2^1 = (2^5)^{\frac{1}{5}}$

$2^5 = f$ Power Property of Equality

$32 = f$ Simplify.

GuidedPractice

6. CHEMISTRY The radius r of the nucleus of an atom of mass number A is $r = 1.2A^{\frac{1}{3}}$ femtometers. Find A if $r = 3.6$ femtometers. **27**

2 Exponential Equations

Example 5 shows how to solve an exponential equation by applying the Power Property of Equality. **Example 6** shows how to solve a real-world exponential equation.

Additional Examples

5 Solve each equation.

a. $9^x = 729$ 3

b. $16^{2x-1} = 8$ $\frac{7}{8}$

6 **BIOLOGY** The population p of a culture that begins with 40 bacteria and doubles every 8 hours can be modeled by $p = 40(2)^{\frac{t}{8}}$, where t is time in hours. Find t if $p = 20{,}480$. 72 hours

3 Practice

Formative Assessment

Use Exercises 1–16 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Teaching the Mathematical Practices

Tools Mathematically proficient students consider the available tools when solving a mathematical problem. In Exercise 16, discuss how a calculator could be used to solve and check the solution.

Modeling Mathematically proficient students can analyze relationships mathematically to draw conclusions. In Exercise 85, encourage students to use a calculator and substitute many whole number values of n to observe the relationship between n and f. Then they can use the results to check the reasonableness of their answers.

Check Your Understanding

Example 1 Write each expression in radical form, or write each radical in exponential form.

1. $12^{\frac{1}{2}}$ $\sqrt{12}$
2. $3x^{\frac{1}{2}}$ $3\sqrt{x}$
3. $\sqrt{33}$ $33^{\frac{1}{2}}$
4. $\sqrt{8n}$ $(8n)^{\frac{1}{2}}$

Examples 2–4 Simplify.

5. $\sqrt[3]{512}$ 8
6. $\sqrt[5]{243}$ 3
7. $343^{\frac{1}{3}}$ 7
8. $\left(\frac{1}{16}\right)^{\frac{1}{4}}$ $\frac{1}{2}$
9. $343^{\frac{2}{3}}$ 49
10. $81^{\frac{3}{4}}$ 27
11. $216^{\frac{4}{3}}$ 1296
12. $\left(\frac{1}{49}\right)^{\frac{3}{2}}$ $\frac{1}{343}$

Example 5 Solve each equation.

13. $8^x = 4096$ 4
14. $3^{3x+1} = 81$ 1
15. $4^{x-3} = 32$ 5.5

Example 6

16. **TOOLS** A weir is used to measure water flow in a channel. For a rectangular broad crested weir, the flow Q in cubic meters per second is related to the weir length L in meters and height H of the water by $Q = 1.6LH^{\frac{3}{2}}$. Find the water height for a weir that is 3 meters long and has flow of 38.4 cubic meters per second. 4 m

Practice and Problem Solving

Example 1 Write each expression in radical form, or write each radical in exponential form.

17. $15^{\frac{1}{2}}$ $\sqrt{15}$
18. $24^{\frac{1}{2}}$ $\sqrt{24}$
19. $4k^{\frac{1}{2}}$ $4\sqrt{k}$
20. $(12y)^{\frac{1}{2}}$ $\sqrt{12y}$
21. $\sqrt{26}$ $26^{\frac{1}{2}}$
22. $\sqrt{44}$ $44^{\frac{1}{2}}$
23. $2\sqrt{ab}$ $2(ab)^{\frac{1}{2}}$
24. $\sqrt{3xyz}$ $(3xyz)^{\frac{1}{2}}$

Examples 2–4 Simplify.

25. $\sqrt[3]{8}$ 2
26. $\sqrt[5]{1024}$ 4
27. $\sqrt[3]{216}$ 6
28. $\sqrt[4]{10{,}000}$ 10
29. $\sqrt[3]{0.001}$ 0.1
30. $\sqrt[4]{\frac{16}{81}}$ $\frac{2}{3}$
31. $1331^{\frac{1}{3}}$ 11
32. $64^{\frac{1}{6}}$ 2
33. $3375^{\frac{1}{3}}$ 15
34. $512^{\frac{1}{9}}$ 2
35. $\left(\frac{1}{81}\right)^{\frac{1}{4}}$ $\frac{1}{3}$
36. $\left(\frac{3125}{32}\right)^{\frac{1}{5}}$ $\frac{5}{2}$
37. $8^{\frac{2}{3}}$ 4
38. $625^{\frac{3}{4}}$ 125
39. $729^{\frac{5}{6}}$ 243
40. $256^{\frac{3}{8}}$ 8
41. $125^{\frac{4}{3}}$ 625
42. $49^{\frac{5}{2}}$ 16,807
43. $\left(\frac{9}{100}\right)^{\frac{3}{2}}$ $\frac{27}{1000}$
44. $\left(\frac{8}{125}\right)^{\frac{4}{3}}$ $\frac{16}{625}$

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	17–58, 89, 90, 92–114	18–58 even, 94–97	17–57 odd, 89, 90, 92, 93, 98–114
OL Core	17–83 odd, 85–90, 92–114	17–58, 94–97	59–90, 92, 93, 98–114
BL Advanced	59–97 (Optional: 98–114)		

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Example 5 **Solve each equation.**

45. $3^x = 243$ **5**
46. $12^x = 144$ **2**
47. $16^x = 4$ $\frac{1}{2}$

48. $27^x = 3$ $\frac{1}{3}$
49. $9^x = 27$ $\frac{3}{2}$
50. $32^x = 4$ $\frac{2}{5}$

51. $2^{x-1} = 128$ **8**
52. $4^{2x+1} = 1024$ **2**
53. $6^{x-4} = 1296$ **8**

54. $9^{2x+3} = 2187$ $\frac{1}{4}$
55. $4^{3x} = 512$ $\frac{3}{2}$
56. $128^{3x} = 8$ $\frac{1}{7}$

Example 6

57. CONSERVATION Water collected in a rain barrel can be used to water plants and reduce city water use. Water flowing from an open rain barrel has velocity $v = 8h^{\frac{1}{2}}$, where v is in meters per second and h is the height of the water in meters. Find the height of the water if it is flowing at 8 meters per second. **1 m**

58. ELECTRICITY The radius r in millimeters of a platinum wire L centimeters long with resistance 0.1 ohm is $r = 0.059L^{\frac{1}{2}}$. How long is a wire with radius 0.236 millimeter? **16 cm**

B **Write each expression in radical form, or write each radical in exponential form.**

59. $17^{\frac{1}{3}}$ $\sqrt[3]{17}$
60. $q^{\frac{1}{4}}$ $\sqrt[4]{q}$
61. $7b^{\frac{1}{3}}$ $7\sqrt[3]{b}$
62. $m^{\frac{2}{3}}$ $\sqrt[3]{m^2}$

63. $\sqrt[3]{29}$ $29^{\frac{1}{3}}$
64. $\sqrt[5]{h}$ $h^{\frac{1}{5}}$
65. $2\sqrt[3]{a}$ $2a^{\frac{1}{3}}$
66. $\sqrt[3]{xy^2}$ $x^{\frac{1}{3}}y^{\frac{2}{3}}$

Simplify.

67. $\sqrt[3]{0.027}$ **0.3**
68. $\sqrt[4]{\frac{n^4}{16}}$ $\frac{n}{2}$
69. $a^{\frac{1}{3}} \cdot a^{\frac{2}{3}}$ a
70. $c^{\frac{1}{2}} \cdot c^{\frac{3}{2}}$ c^2

71. $(8^2)^{\frac{2}{3}}$ **16**
72. $\left(y^{\frac{3}{4}}\right)^{\frac{1}{2}}$ $y^{\frac{3}{8}}$
73. $9^{-\frac{1}{2}}$ $\frac{1}{3}$
74. $16^{-\frac{3}{2}}$ $\frac{1}{64}$

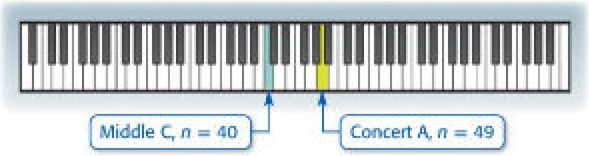
75. $(3^2)^{-\frac{3}{2}}$ $\frac{1}{27}$
76. $\left(81^{\frac{1}{4}}\right)^{-2}$ $\frac{1}{9}$
77. $k^{-\frac{1}{2}}$ $\frac{1}{\sqrt{k}}$
78. $\left(d^{\frac{4}{3}}\right)^0$ **1**

Solve each equation.

79. $2^{5x} = 8^{2x-4}$ **12**
80. $81^{2x-3} = 9^{x+3}$ **3**
81. $2^{4x} = 32^{x+1}$ **−5**

82. $16^x = \frac{1}{2}$ $-\frac{1}{4}$
83. $25^x = \frac{1}{125}$ $-\frac{3}{2}$
84. $6^{8-x} = \frac{1}{216}$ **11**

85. MODELING The frequency f in hertz of the nth key on a piano is $f = 440\left(2^{\frac{1}{12}}\right)^{n-49}$.



a. What is the frequency of Concert A? **440 Hz**

b. Which note has a frequency of 220 Hz? **the A below middle C, the 37th note**

499

86. RANDOM WALKS Suppose you go on a walk where you choose the direction of each step at random. The path of a molecule in a liquid or a gas, the path of a foraging animal, and a fluctuating stock price are all modeled as random walks. The number of possible random walks w of n steps where you choose one of d directions at each step is $w = d^n$.

a. How many steps have been taken in a 2-direction random walk if there are 4096 possible walks? **12**

b. How many steps have been taken in a 4-direction random walk if there are 65,536 possible walks? **8**

c. If a walk of 7 steps has 2187 possible walks, how many directions could be taken at each step? **3**

87. SOCCER The radius r of a ball that holds V cubic units of air is modeled by $r = 0.62V^{\frac{1}{3}}$. What are the possible volumes of each size soccer ball? **Size 3, 3320.8 to 3889 cm^3; Size 4, 4518.5 to 4857.3 cm^3; Size 5, 5584.7 to 6381.4 cm^3**

Soccer Ball Dimensions	
Size	**Diameter (cm.)**
3	18.5–19.5
4	20.5–21
5	22–23

88. MULTIPLE REPRESENTATIONS In this problem, you will explore the graph of an exponential function.

a. **TABULAR** Copy and complete the table below.

x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$f(x) = 4^x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

b. **GRAPHICAL** Graph $f(x)$ by plotting the points and connecting them with a smooth curve. **b–c. See margin.**

c. **VERBAL** Describe the shape of the graph of $f(x)$. What are its key features? Is it linear?

H.O.T. Problems Use Higher-Order Thinking Skills

89. OPEN ENDED Write two different expressions with rational exponents equal to $\sqrt{2}$. **See margin.**

90. ARGUMENTS Determine whether each statement is *always*, *sometimes*, or *never* true. Assume that x is a nonnegative real number. Explain your reasoning. **a–f. See margin.**

a. $x^2 = x^{\frac{1}{2}}$ b. $x^{-2} = x^{\frac{1}{2}}$ c. $x^{\frac{1}{3}} = x^{\frac{1}{2}}$

d. $\sqrt{x} = x^{\frac{1}{2}}$ e. $\left(x^{\frac{1}{2}}\right)^2 = x$ f. $x^{\frac{1}{2}} \cdot x^2 = x$

91. CHALLENGE For what values of x is $x = x^{\frac{1}{3}}$? **−1, 0, 1**

92. ERROR ANALYSIS Jassim and Jamal are solving $128^x = 4$. Is either of them correct? Explain your reasoning. **Jassim; Jamal did not write the expressions with equal bases before applying the Power Property of Equality.**

Jassim	Jamal
$128^x = 4$	$128^x = 4$
$(2^7)^x = 2^2$	$(2^7)^x = 4$
$2^{7x} = 2^2$	$2^{7x} = 4^1$
$7x = 2$	$7x = 1$
$x = \frac{2}{7}$	$x = \frac{1}{7}$

93. WRITING IN MATH Explain why 2 is the principal fourth root of 16.
Sample answer: 2 is the principal fourth root of 16 because 2 is positive and $2^4 = 16$.

Differentiated Instruction BL

Extension For Exercise 86, students found the numbers of random walks of a given length of steps. Have students investigate the applications of random walks.

Multiple Representations

In Exercise 88, students use a table and a graph to explore the graph of an exponential function.

Teaching the Mathematical Practices

Arguments Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. In Exercise 90, point out that x is a nonnegative real number. Tell students to consider values of x that are whole numbers and fractions that are greater than and less than 1. Point out that 1 is often a special case.

WatchOut!

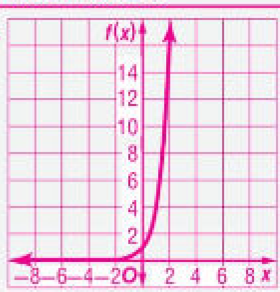
Error Analysis In Exercise 92, students should recognize that the bases of the expressions must have equal bases to apply the Power Property of Equality.

4 Assess

Ticket Out the Door Have each student an exponential expression and an equivalent radical expression.

Additional Answers

88b.



88c. The graph of $f(x) = 4^x$ is a curve. It has no x-intercept, a y-intercept of 1, the domain is all reals, the range is all positive reals, it is increasing over the entire domain, as x approaches infinity $f(x)$ approaches infinity, as x approaches negative infinity $f(x)$ approaches 0.

89. Sample answer: $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$

Standardized Test Practice

94. What is the value of $16^{\frac{3}{4}} + 9^{\frac{3}{2}}$? **D**

A 5   C 25

B 11   D 35

95. At a movie theater, the costs for various numbers of popcorn and sandwiches are shown. **G**

Sandwiches	Boxes of Popcorn	Total Cost
1	1	AED 8.50
2	4	AED 21.60

Which pair of equations can be used to find p, the cost of a box of popcorn, and h, the cost of a sandwiches?

F $p + h = 8.5$
$p + 2h = 10.8$

H $p + h = 8.5$
$2p + 4h = 21.6$

G $p + h = 8.5$
$2h + 4p = 21.6$

J $p + h = 8.5$
$2p + 2h = 21.6$

96. SHORT RESPONSE Find the dimensions of the rectangle if its perimeter is 52 centimeters. **17.5 cm. × 8.5 cm.**

97. If $3^4 = 9^x$, then $x =$ **B**

A 1

B 2

C 4

D 5

Spiral Review

Simplify each expression. Assume that no denominator equals zero. (Lesson 8-2)

98. $\frac{a^3b^5}{ab^3}$ a^2b^2

99. $\frac{c^8d^{11}}{c^4d^5}$ c^4d^6

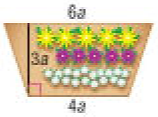
100. $\frac{4x^3y^3z^6}{xyz^5}$ $4x^2y^2z$

101. $\frac{a^5b^3c}{a^5bc}$ b^2

102. $\left(\frac{3m^4}{4p^2}\right)^2$ $\frac{9m^8}{16p^4}$

103. $\left(\frac{3df^2}{9d^2f}\right)^0$ 1

104. GARDENING Salem is planting a flower garden that is shaped like a trapezoid as shown at the right. Use the formula $A = \frac{1}{2}h(b_1 + b_2)$ to find the area of the garden. (Lesson 8-1) $15a^2$



Write each equation in slope-intercept form.

105. $y - 2 = 3(x - 1)$ $y = 3x - 1$

106. $y - 5 = 6(x + 1)$ $y = 6x + 11$

107. $y + 2 = -2(x + 5)$ $y = -2x - 12$

108. $y + 3 = \frac{1}{2}(x + 4)$ $y = \frac{1}{2}x - 1$

109. $y - 1 = \frac{2}{3}(x + 9)$ $y = \frac{2}{3}x + 7$

110. $y + 3 = -\frac{1}{4}(x + 2)$ $y = -\frac{1}{4}x - \frac{7}{2}$

Skills Check

Find each power.

111. 10^3 1000

112. 10^5 100,000

113. 10^{-1} 0.1

114. 10^{-4} 0.0001

Additional Answers

90a. sometimes; true only when $x = 1$

90b. sometimes; true only when $x = 1$

90c sometimes; true only when $x = 1$

90d. always; by definition of $x^{\frac{1}{2}}$

90e. always; $\left(x^{\frac{1}{2}}\right)^2 = x^{\frac{1}{2} \cdot 2} = x^1$ or x

90f. sometimes; true only when $x = 1$

1 Focus

VerticalAlignment

Before Lesson 8-4 Use the laws of exponents to find products and quotients of monomials.

Lesson 8-4 Express numbers in scientific notation. Find products and quotients of numbers expressed in scientific notation.

After Lesson 8-4 Graph exponential functions.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What would AED20 million look like when written out? AED20,000,000
- What would you have to multiply AED2 by to get AED20 million? 10,000,000
- How can you write 10,000,000 as a power of 10? 10^7
- What would you have to multiply 1.4 by to get 0.0000000014? 0.000000001
- How can you write 0.000000001 as a power of 10? 10^{-9}
- What is AED2 $\times 10^7$? 20,000,000 What is 1.4×10^{-9}? 0.0000000014

LESSON 8-4 Scientific Notation

Then

- You used the laws of exponents to find products and quotients of monomials.

Now

1. Express numbers in scientific notation.
2. Find products and quotients of numbers expressed in scientific notation.

Why?

- Space tourism is a multibillion dollar industry. For a price of AED 73 million, a civilian can travel on a rocket or shuttle and visit the International Space Station (ISS) for a week.

NewVocabulary
scientific notation

Mathematical Practices
Construct viable arguments and critique the reasoning of others.
Attend to precision.

1 Scientific Notation

Very large and very small numbers such as AED 73 million can be cumbersome to use in calculations. For this reason, numbers are often expressed in scientific notation. A number written in **scientific notation** is of the form $a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

KeyConcept Standard Form to Scientific Notation

Step 1 Move the decimal point until it is to the right of the first nonzero digit. The result is a real number a.

Step 2 Note the number of places n and the direction that you moved the decimal point.

Step 3 If the decimal point is moved left, write the number as $a \times 10^n$. If the decimal point is moved right, write the number as $a \times 10^{-n}$.

Step 4 Remove the unnecessary zeros.

Example 1 Standard Form to Scientific Notation

Express each number in scientific notation.

a. 201,000,000

Step 1 201,000,000 → 2.01000000 $a = 2.01000000$

Step 2 The decimal point moved 8 places to the left, so $n = 8$.

Step 3 $201{,}000{,}000 = 2.01000000 \times 10^8$

Step 4 2.01×10^8

b. 0.000051

Step 1 0.000051 → 00005.1 $a = 00005.1$

Step 2 The decimal point moved 5 places to the right, so $n = 5$.

Step 3 $0.000051 = 00005.1 \times 10^{-5}$

Step 4 5.1×10^{-5}

GuidedPractice

1A. 68,700,000,000 6.87×10^{10} **1B.** 0.0000725 7.25×10^{-5}

You can also rewrite numbers in scientific notation in standard form.

WatchOut!

Negative Signs Be careful about the placement of negative signs. A negative sign in the exponent means that the number is between 0 and 1. A negative sign before the number means that it is less than 0.

KeyConcept Scientific Notation to Standard Form

Step 1 In $a \times 10^n$, note whether $n > 0$ or $n < 0$.

Step 2 If $n > 0$, move the decimal point n places right.
If $n < 0$, move the decimal point $-n$ places left.

Step 3 Insert zeros, decimal point, and commas as needed for place value.

Example 2 Scientific Notation to Standard Form

Express each number in standard form.

a. 6.32×10^9

Step 1 The exponent is 9, so $n = 9$.

Step 2 Since $n > 0$, move the decimal point 9 places to the right.
$6.32 \times 10^9 \longrightarrow 6320000000$

Step 3 $6.32 \times 10^9 = 6{,}320{,}000{,}000$ — Rewrite; insert commas.

b. 4×10^{-7}

Step 1 The exponent is -7, so $n = -7$.

Step 2 Since $n < 0$, move the decimal point 7 places to the left.
$4 \times 10^{-7} \longrightarrow 0000004$

Step 3 $4 \times 10^{-7} = 0.0000004$ — Rewrite; insert a 0 before the decimal point.

GuidedPractice

2A. 3.201×10^6 3,201,000 **2B.** 9.03×10^{-5} 0.0000903

2 Product and Quotients in Scientific Notation

You can use scientific notation to simplify multiplying and dividing very large and very small numbers.

Problem-SolvingTip

Tools Estimating an answer before computing the solution can help you determine if your answer is reasonable.

Example 3 Multiply with Scientific Notation

Evaluate $(3.5 \times 10^{-3})(7 \times 10^5)$. Express the result in both scientific notation and standard form.

$(3.5 \times 10^{-3})(7 \times 10^5)$	Original expression
$= (3.5 \times 7)(10^{-3} \times 10^5)$	Commutative and Associative Properties
$= 24.5 \times 10^2$	Product of Powers
$= (2.45 \times 10^1) \times 10^2$	$24.5 = 2.45 \times 10$
$= 2.45 \times 10^3$ or 2450	Product of Powers

GuidedPractice

Evaluate each product. Express the results in both scientific notation and standard form.

3A. $(6.5 \times 10^{12})(8.7 \times 10^{-15})$ **3B.** $(7.8 \times 10^{-4})^2$

3A. 5.655×10^{-2}; 0.05655
3B. 6.084×10^{-7}; 0.0000006084

DifferentiatedInstruction (AL)

If students have trouble keeping track of a moving decimal point,

Then have students write each digit in the number 201,000,000, for example, on an index card. Using an object such as a paper clip or coin as the decimal point, students can actually move the decimal point and count the number of places it moved. Repeat using other numbers from the examples, such as 0.000051.

1 Scientific Notation

Example 1 shows how to express a very large and a very small number in scientific notation. **Example 2** shows how to express a number written in scientific notation in standard form.

Formative Assessment

Use the Guided Practice exercises after each Example to determine students' understanding of concepts.

Additional Examples

1 Express each number in scientific notation.

a. 4,062,000,000,000 4.062×10^{12}

b. 0.000000823 8.23×10^{-7}

2 Express each number in standard form.

a. 6.49×10^5 649,000

b. 1.8×10^{-3} 0.0018

2 Products and Quotients in Scientific Notation

Example 3 shows how to multiply with scientific notation. **Example 4** shows how to divide with scientific notation. **Example 5** shows how to solve a real-world problem using scientific notation.

Additional Example

3 Evaluate $(5 \times 10^{-6})(2.3 \times 10^{12})$. Express the result in both scientific notation and standard form. 1.15×10^7; 11,500,000

Teaching the Mathematical Practices

Tools Mathematically proficient detect possible errors by strategically using estimation and other mathematical knowledge. Model this behavior by verbalizing estimation as you solve example problems.

Additional Examples

4 Evaluate $\frac{4.5 \times 10^8}{1.5 \times 10^{10}}$. Express the result in both scientific notation and standard form. 3×10^{-2}; 0.03

5 **WATERCRAFT** Last year Amna's state registered over 400 thousand watercraft. Boat sales in her state generated more than AED15.4 million in state sales taxes that same year.

a. Express the number of watercraft registered and the state sales tax generated from boat sales last year in Amna's state in standard notation. Watercraft registered: 400,000; state sales tax generated: AED15,400,000

b. Write each number in scientific notation. 4×10^5; 1.54×10^7

c. How many watercraft have been registered in Amna's state if 12 times the number registered last year have been registered in all? Write your answer in scientific notation and standard form. 4,800,000; 4.8×10^6

Tips for New Teachers

E Notation On some calculators, the powers of 10 are written using E notation. For example, 3×10^{-6}, would be displayed as **3** E **−6**. The E means *times 10 to the given power*.

Teach with Tech

Interactive Whiteboard Write a number not in scientific notation on the board. Grab the decimal point and drag it to the left or right as you count the number of places you have moved it.

Example 4 Divide with Scientific Notation

Evaluate $\frac{3.066 \times 10^8}{7.3 \times 10^3}$. Express the result in both scientific notation and standard form.

$\frac{3.066 \times 10^8}{7.3 \times 10^3} = \left(\frac{3.066}{7.3}\right)\left(\frac{10^8}{10^3}\right)$	Product rule for fractions
$= 0.42 \times 10^5$	Quotient of Powers
$= 4.2 \times 10^{-1} \times 10^5$	$0.42 = 4.2 \times 10^{-1}$
$= 4.2 \times 10^4$	Product of Powers
$= 42{,}000$	Standard form

StudyTip

Quotient of Powers Recall that the Quotient of Powers Property is only valid for powers that have the same base. Since 10^8 and 10^3 have the same base, the property applies.

GuidedPractice

Evaluate each quotient. Express the results in both scientific notation and standard form.

4A. $\frac{2.3958 \times 10^3}{1.98 \times 10^8}$ 1.21×10^{-5}; 0.0000121 **4B.** $\frac{1.305 \times 10^3}{1.45 \times 10^{-4}}$ 9×10^6; 9,000,000

Real-World Example 5 Use Scientific Notation

MUSIC In the United States, a CD reaches gold status once 500 thousand copies are sold. A CD reaches platinum status once 1 million or more copies are sold.

a. Express the number of copies of CDs that need to be sold to reach each status in standard notation.

gold status: 500 thousand = 500,000; platinum status: 1 million = 1,000,000

b. Write each number in scientific notation.

gold status: $500{,}000 = 5 \times 10^5$; platinum status: $1{,}000{,}000 = 1 \times 10^6$

c. How many copies of a CD have sold if it has gone platinum 13 times? Write your answer in scientific notation and standard form.

A CD reaches platinum status once it sells 1 million records. Since the CD has gone platinum 13 times, we need to multiply by 13.

$(13)(1 \times 10^6)$	Original expression
$= (13 \times 1)(10^6)$	Associative Property
$= 13 \times 10^6$	$13 \times 1 = 13$
$= (1.3 \times 10^1) \times 10^6$	$13 = 1.3 \times 10$
$= 1.3 \times 10^7$	Product of Powers
$= 13{,}000{,}000$	Standard form

GuidedPractice

5. SATELLITE RADIO Suppose a satellite radio company earned AED 125.4 million in one year.

A. Write this number in standard form. 125,400,000

B. Write this number in scientific notation. 1.254×10^8

C. If the following year the company earned 2.5 times the amount earned the previous year, determine the amount earned. Write your answer in scientific notation and standard form. 3.135×10^8; 313,500,000

Focus on Mathematical Content

Multiplying and Dividing with Scientific Notation If students think of numbers in scientific notation as monomials, then the procedures for multiplying and dividing are the same. For example, think of $(3 \times 10^{-2})(1.2 \times 10^5)$ as $(3x^{-2})(1.2x^5)$. In this case, you first multiply the constants and then the powers with the same bases. Similarly, with scientific notation, you multiply the constants, then the powers of 10.

Check Your Understanding

Example 1 Express each number in scientific notation.

1. 185,000,000 1.85×10^8
2. 1,902,500,000 1.9025×10^9
3. 0.000564 5.64×10^{-4}
4. 0.00000804 8.04×10^{-6}

MONEY Express each number in scientific notation.

5. Teens spend AED 13 billion annually on clothing. 1.3×10^{10}
6. Teens have an influence on their families' spending habit. They control about AED 1.5 billion of discretionary income. 1.5×10^9

Example 2 Express each number in standard form.

7. 1.98×10^7 19,800,000
8. 4.052×10^6 4,052,000
9. 3.405×10^{-8} 0.00000003405
10. 6.8×10^{-5} 0.000068

Example 3 Evaluate each product. Express the results in both scientific notation and standard form. 11–14. See margin.

11. $(1.2 \times 10^3)(1.45 \times 10^{12})$
12. $(7.08 \times 10^{14})(5 \times 10^{-9})$
13. $(5.18 \times 10^2)(9.1 \times 10^{-5})$
14. $(2.18 \times 10^{-2})^2$

Example 4 Evaluate each quotient. Express the results in both scientific notation and standard form. 15–18. See margin.

15. $\frac{1.035 \times 10^8}{2.3 \times 10^4}$
16. $\frac{2.542 \times 10^5}{4.1 \times 10^{-10}}$
17. $\frac{1.445 \times 10^{-7}}{1.7 \times 10^5}$
18. $\frac{2.05 \times 10^{-8}}{4 \times 10^{-2}}$

Example 5

19. **PRECISION** Zayed bought an air purifier to help him deal with his allergies. The filter in the purifier will stop particles as small as one hundredth of a micron. A micron is one millionth of a millimeter.
   a. Write one hundredth and one micron in standard form. 0.01, 0.000001
   b. Write one hundredth and one micron in scientific notation. 1×10^{-2}, 1×10^{-6}
   c. What is the smallest size particle in meters that the filter will stop? Write the result in both standard form and scientific notation. 0.00000000001; 1×10^{-11}

Practice and Problem Solving

Example 1 Express each number in scientific notation.

20. 1,220,000 1.22×10^6
21. 58,600,000 5.86×10^7
22. 1,405,000,000,000 1.405×10^{12}
23. 0.0000013 1.3×10^{-6}
24. 0.000056 5.6×10^{-5}
25. 0.000000000709 7.09×10^{-10}

EMAIL Express each number in scientific notation.

26. Approximately 100 million emails sent to the President are put into the National Archives. 1×10^8
27. By 2015, the email security market will generate AED 6.5 billion. 6.5×10^9

Example 2 Express each number in standard form. 28. 1,000,000,000,000

28. 1×10^{12}
29. 9.4×10^7 94,000,000
30. 8.1×10^{-3} 0.0081
31. 5×10^{-4} 0.0005
32. 8.73×10^{11} 873,000,000,000
33. 6.22×10^{-6} 0.00000622

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	20–54, 70, 71, 73–93	21–53 odd, 76–79	20–54 even, 70, 71, 73–75, 80–93
OL Core	21–61 odd, 63–71, 73–93	20–54, 76–79	55–71, 73–75, 80–93
BL Advanced	55–87, (optional: 88–93)		

Tips for New Teachers

Sense-Making Students may have encountered the use of scientific notation in their science classes, where they rounded numbers to two decimal places when converting from standard to scientific notation. Make sure that students do not automatically round numbers to two decimal places when they convert to scientific notation in this lesson.

3 Practice

Formative Assessment

Use Exercises 1–19 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Teaching the Mathematical Practices

Precision Mathematically proficient students express numerical answers with a degree of precision appropriate for the problem context. In Exercise 19, point out that in some scientific fields units of measure as small as microns or even smaller are commonly used.

Additional Answers

11. 1.74×10^{15}; 1,740,000,000,000,000
12. 3.54×10^6; 3,540,000
13. 4.7138×10^{-2}; 0.047138
14. 4.7524×10^{-4}; 0.00047524
15. 4.5×10^3; 4500
16. 6.2×10^{14}; 620,000,000,000,000
17. 8.5×10^{-13}; 0.00000000000085
18. 5.125×10^{-7}; 0.0000005215

Exercise Alert

Change in Units For Exercises 63–65, students are given the speed of light in meters per second but are asked to find results in kilometers per second.

Teaching the Mathematical Practices

Modeling Mathematically proficient students are able to identify important quantities in a practical situation and map their relationships. In Exercise 66, the relationship may be clearer to students if they write the fraction using words for each quantity like $\frac{\text{Company B's data processed per second}}{\text{Company A's data processed per second}}$.

WatchOut!

Error Analysis For Exercise 71, help students see that everything is the same in the two methods except the exponent in the power of 10 in the last line. Ask students if the decimal point was moved to the left or right, and how the direction affects the power of 10.

Additional Answers

65.

Time	Kilometers Traveled
1 day	2.592×10^{10}
1 week	1.8144×10^{11}
1 month	7.776×10^{11}
1 year	9.4608×10^{12}

66. $\frac{1.41 \times 10^6}{7.95 \times 10^5} \approx 1.774$; The phone from company B is about 1.774 times as fast as the phone from company A.

75. Sample answer: Divide the numbers to the left of the × symbols. Then divide the powers of 10. If necessary, rewrite the results in scientific notation. To convert that to standard form, check to see if the exponent is positive or negative. If positive, move the decimal point to the right, and if negative, to the left. The number of places to move the decimal point is the absolute value of the exponent. Fill in with zeros as needed.

Example 2 **INTERNET** **Express each number in standard form.**

34. About 2.1×10^7 people, aged 12 to 17, use the Internet. 21,000,000

35. Approximately 1.1×10^7 teens go online daily. 11,000,000

Examples 3–4 **Evaluate each product or quotient. Express the results in both scientific notation and standard form.**

36. $(3.807 \times 10^3)(5 \times 10^2)$ 1.9035×10^6; 1,903,500

37. $\frac{9.6 \times 10^3}{1.2 \times 10^{-4}}$ 8×10^7; 80,000,000

38. $\frac{2.88 \times 10^3}{1.2 \times 10^{-5}}$ 2.4×10^8; 240,000,000

39. $(6.5 \times 10^7)(7.2 \times 10^{-2})$ 4.68×10^6; 4,680,000

40. $(9.5 \times 10^{-18})(9 \times 10^9)$ 8.55×10^{-8}; 0.0000000855

41. $\frac{8.8 \times 10^3}{4 \times 10^{-4}}$ 2.2×10^7; 22,000,000

42. $\frac{9.15 \times 10^{-3}}{6.1 \times 10}$ 1.5×10^{-4}; 0.00015

43. $(1.4 \times 10^6)^2$ 1.96×10^{12}; 1,960,000,000,000

44. $(2.58 \times 10^2)(3.6 \times 10^6)$ 9.288×10^8; 928,800,000

45. $\frac{5.6498 \times 10^{10}}{8.2 \times 10^4}$ 6.89×10^5; 689,000

46. $\frac{1.363 \times 10^{16}}{2.9 \times 10^6}$ 4.7×10^9; 4,700,000,000

47. $(5 \times 10^3)(1.8 \times 10^{-7})$ 9×10^{-4}; 0.0009

48. $(2.3 \times 10^{-3})^2$ 5.29×10^{-6}; 0.00000529

49. $\frac{6.25 \times 10^{-4}}{1.25 \times 10^2}$ 5×10^{-6}; 0.000005

50. $\frac{3.75 \times 10^{-9}}{1.5 \times 10^{-4}}$ 2.5×10^{-5}; 0.000025

51. $(7.2 \times 10^7)^2$ 5.184×10^{15}; 5,184,000,000,000,000

52. $\frac{8.6 \times 10^4}{2 \times 10^{-6}}$ 4.3×10^{10}; 43,000,000,000

53. $(6.3 \times 10^{-5})^2$ 3.969×10^{-9}; 0.000000003969

Example 5

54. **ASTRONOMY** The distance between Earth and the Sun varies throughout the year. Earth is closest to the Sun in January when the distance is 146 million kilometers. In July, the distance is greatest at 152 million kilometers.

a. Write 146 million in both standard form and in scientific notation. 146,000,000, 1.46×10^8

b. Write 152 million in both standard form and in scientific notation. 152,000,000, 1.52×10^8

c. What is the percent increase in distance from January to July? Round to the nearest tenth of a percent. 3.3%

B **Evaluate each product or quotient. Express the results in both scientific notation and standard form.**

55. $(4.65 \times 10^{-2})(5.91 \times 10^6)$ 2.74185×10^5; 274,815

56. $\frac{2.548 \times 10^5}{2.8 \times 10^{-2}}$ 9.1×10^6; 9,100,000

57. $\frac{2.135 \times 10^5}{3.5 \times 10^{12}}$ 6.1×10^{-8}; 0.000000061

58. $(3.16 \times 10^{-2})^2$ 9.9856×10^{-4}; 0.00099856

59. $(2.01 \times 10^{-4})(8.9 \times 10^{-3})$ 1.7889×10^{-6}; 0.0000017889

60. $\frac{5.184 \times 10^{-5}}{7.2 \times 10^3}$ 7.2×10^{-9}; 0.0000000072

61. $(9.04 \times 10^6)(5.2 \times 10^{-4})$ 4.7008×10^3; 4700.8

62. $\frac{1.032 \times 10^{-4}}{8.6 \times 10^{-5}}$ 1.2×10^0; 1.2

LIGHT **The speed of light is approximately 3×10^8 meters per second.**

63. Write an expression to represent the speed of light in kilometers per second. 3×10^5

64. Write an expression to represent the speed of light in kilometers per hour. 1.08×10^9

65. Make a table to show how many kilometers light travels in a day, a week, a 30-day month, and a 365-day year. Express your results in scientific notation. See margin.

66. **MODELING** A recent cell phone study showed that company A's phone processes up to 7.95×10^5 bits of data every second. Company B's phone processes up to 1.41×10^6 bits of data every second. Evaluate and interpret $\frac{1.41 \times 10^6}{7.95 \times 10^5}$. See margin.

Differentiated Instruction OL BL

Extension Write the expression 0.00042 × 316,000,000 on the board. Ask students to write this expression using scientific notation. Then have students evaluate the product.
$4.2 \times 10^{-4} \times 3.16 \times 10^8$; 132,720

67 **EARTH** The population of Earth is about 6.623×10^9. The land surface of Earth is 1.483×10^8 square kilometers. What is the population density for the land surface area of Earth? **about 44.7 persons/km²**

68. **RIVERS** A drainage basin separated from adjacent basins by a ridge, hill, or mountain is known as a watershed. The watershed of the Amazon River is 5,900,000 square kilometers. The watershed of the Mississippi River is 3,100,000 square kilometers.

a. Write each of these numbers in scientific notation. **5.9×10^6; 3.1×10^6**

b. How many times as large is the Amazon River watershed as the Mississippi River watershed? **about 1.9 times as large**

69. **AGRICULTURE** In a recent year, farmers planted approximately 37.6 hectares. of corn. They also planted 25.9 hectares of soybeans and 4.5 hectares of cotton. **a. corn: 3.76×10^7, 37,600,000 soybeans: 2.59×10^7, 25,900,000 cotton: 4.5×10^7, 4,500,000**

a. Write each of these numbers in scientific notation and in standard form.

b. How many times as much corn was planted as soybeans? Write your results in standard form and in scientific notation. Round your answer to four decimal places. **about 1.4517×10^0; 1.4517**

c. How many times as much corn was planted as cotton? Write your results in standard form and in scientific notation. Round your answer to four decimal places. **about 8.3555×10^0; 8.3555**

70. 10^{100}; $100^{10} = (10^2)^{10}$ or 10^{20} and $10^{100} > 10^{20}$, so $10^{100} > 100^{10}$
71. Pete; Syreeta moved the decimal point in the wrong direction.

H.O.T. Problems Use Higher-Order Thinking Skills

70. **REASONING** Which is greater, 100^{10} or 10^{100}? Explain your reasoning.

71. **ERROR ANALYSIS** Rana and Reham are solving a division problem with scientific notation. Is either of them correct? Explain your reasoning.

73. Sample answer: Always; if the numbers are $a \times 10^m$ and $b \times 10^n$ in scientific notation, then $1 \le a < 10$ and $1 \le b < 10$. So $1 \le ab < 100$.

Rana	Reham
$\frac{3.65 \times 10^{12}}{5 \times 10^5} = 0.73 \times 10^{17}$ $= 7.3 \times 10^{16}$	$\frac{3.65 \times 10^{12}}{5 \times 10^5} = 0.73 \times 10^{17}$ $= 7.3 \times 10^{18}$

74. Sample answer: product: 2.5×10^5, 5.2×10^{-9}; quotient: 2.6×10^3, 2×10^6

72. **CHALLENGE** Order these numbers from least to greatest without converting them to standard form. **-4.65×10^5, -5.64×10^4, 4.56×10^{-4}, 5.46×10^{-3}, 6.54×10^3**

5.46×10^{-3}, 6.54×10^3, 4.56×10^{-4}, -5.64×10^4, -4.65×10^5

73. **ARGUMENTS** Determine whether the statement is *always*, *sometimes*, or *never* true. Give examples or a counterexample to verify your reasoning.

When multiplying two numbers written in scientific notation, the resulting number can have no more than two digits to the left of the decimal point.

74. **OPEN ENDED** Write two numbers in scientific notation with a product of 1.3×10^{-3}. Then name two numbers in scientific notation with a quotient of 1.3×10^{-3}.

75. **WRITING IN MATH** Write the steps that you would use to divide two numbers written in scientific notation. Then describe how you would write the results in standard form. Demonstrate by finding $\frac{a}{b}$ for $a = 2 \times 10^3$ and $b = 4 \times 10^5$. **See margin.**

Teaching the Mathematical Practices

Arguments Mathematically proficient students can recognize and use counterexamples. In Exercise 73, stress that examples alone do not prove a conjecture but one counterexample disproves one.

4 Assess

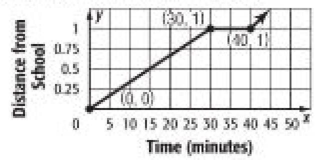
Name the Math Ask students to describe how they would evaluate $(3.5 \times 10^{-5})(4 \times 10^{2})$.

Standardized Test Practice

76. Which number represents 0.05604×10^{8} written in standard form? **C**

A 0.0000000005604
B 560,400
C 5,604,000
D 50,604,000

77. Ayesha left school and rode her bike home. The graph below shows the relationship between her distance from the school and time.



Which explanation could account for the section of the graph from $x = 30$ to $x = 40$? **H**

F Ayesha rode her bike down a hill.
G Ayesha ran all the way home.
H Ayesha stopped at a friend's house on her way home.
J Ayesha returned to school to get her mathematics book.

78. SHORT RESPONSE In his first four years of coaching soccer, Coach Tarek's team won 5 games the first year, 10 games the second year, 8 games the third year, and 7 games the fourth year. How many games does the team need to win during the fifth year to have an average of 8 wins per year? **10 games**

79. The table shows the relationship between Calories and grams of fat contained in an order of fried chicken from various restaurants.

Calories	305	410	320	500	510	440
Fat (g)	28	34	28	41	42	38

Assuming that the data can best be described by a linear model, about how many grams of fat would you expect to be in a 275-Calorie order of fried chicken? **B**

A 22
B 25
C 28
D 30

Spiral Review

80. HEALTH A ponderal index p is a measure of a person's body based on height h in centimeters and mass m in kilograms. One such formula is $p = 100m^{\frac{1}{3}}h^{-1}$. If a person who is 182 centimeters tall has a ponderal index of about 2.2, how much does the person weigh in kilograms? (Lesson 8-3) **64.2 kg**

Simplify. Assume that no denominator is equal to zero. (Lesson 8-2)

81. $\frac{8^9}{8^6}$ **8^3 or 512**

82. $\frac{6^5}{6^3}$ **6^2 or 36**

83. $\frac{r^8t^{12}}{r^2t^7}$ **r^6t^5**

84. $\left(\frac{3a^4b^4}{8c^2}\right)^4$ **$\frac{81a^{16}b^{16}}{4096c^8}$**

85. $\left(\frac{5d^3g^2}{3h^4}\right)^2$ **$\frac{25d^6g^4}{9h^8}$**

86. $\left(\frac{4n^2p^4}{8p^3}\right)^3$ **$\frac{n^6p^3}{8}$**

87. CHEMISTRY Lemon juice is 10^2 times as acidic as tomato juice. Tomato juice is 10^3 times as acidic as egg whites. How many times as acidic is lemon juice as egg whites? (Lesson 8-2) **10^5**

Skills Review

Evaluate $a(b^x)$ for each of the given values.

88. $a = 1, b = 2, x = 4$ **16**

89. $a = 4, b = 1, x = 7$ **4**

90. $a = 5, b = 3, x = 0$ **5**

91. $a = 0, b = 6, x = 8$ **0**

92. $a = -2, b = 3, x = 1$ **−6**

93. $a = -3, b = 5, x = 2$ **−75**

508 | Lesson 8-4 | Scientific Notation

CHAPTER 8

Mid-Chapter Quiz

Lessons 8-1 through 8-4

Simplify each expression. (Lesson 8-1)

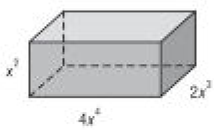
1. $(x^3)(4x^5)$ **$4x^8$**

2. $(m^2p^5)^3$ **m^6p^{15}**

3. $[(2xy^3)^2]^3$ **$64x^6y^{18}$**

4. $(6ab^3c^4)(-3a^2b^3c)$ **$-18a^3b^6c^5$**

5. **MULTIPLE CHOICE** Express the volume of the solid as a monomial. (Lesson 8-1) **B**



A $6x^9$

B $8x^9$

C $8x^{24}$

D $7x^{24}$

Simplify each expression. Assume that no denominator equals 0. (Lesson 8-2)

6. $\left(\frac{2a^4b^3}{c^6}\right)^3$ **$\frac{8a^{12}b^9}{c^{18}}$**

7. $\frac{2xy^0}{6x}$ **$\frac{1}{3}$**

8. $\frac{m^7n^4p}{m^3n^3p}$ **m^4n**

9. $\frac{p^4t^{-2}}{r^{-5}}$ **$\frac{p^4r^5}{t^2}$**

10. **ASTRONOMY** Physicists estimate that the number of stars in the universe has an order of magnitude of 10^{21}. The number of stars in the Milky Way galaxy is around 100 billion. Using orders of magnitude, how many times as many stars are there in the universe as the Milky Way? (Lesson 8-2) **10^{10}**

Write each expression in radical form, or write each radical in exponential form. (Lesson 8-3)

11. $42^{\frac{1}{2}}$ **$\sqrt{42}$**

12. $11x^{\frac{1}{2}}$ **$11\sqrt{x}$**

13. $(11g)^{\frac{1}{2}}$ **$\sqrt{11g}$**

14. $\sqrt{55}$ **$55^{\frac{1}{2}}$**

15. $\sqrt{5k}$ **$(5k)^{\frac{1}{2}}$**

16. $4\sqrt{p}$ **$4p^{\frac{1}{2}}$**

Simplify. (Lesson 8-3)

17. $\sqrt[3]{729}$ **9**

18. $\sqrt[4]{625}$ **5**

19. $1331^{\frac{1}{3}}$ **11**

20. $\left(\frac{16}{81}\right)^{\frac{1}{4}}$ **$\frac{2}{3}$**

21. $8^{\frac{2}{3}}$ **4**

22. $625^{\frac{3}{4}}$ **125**

23. $216^{\frac{5}{3}}$ **7776**

24. $\left(\frac{1}{4}\right)^{\frac{3}{2}}$ **$\frac{1}{8}$**

Solve each equation. (Lesson 8-3)

25. $4^x = 4096$ **6**

26. $5^{2x+1} = 125$ **1**

27. $4^{x-3} = 128$ **6.5**

Express each number in scientific notation. (Lesson 8-4)

28. 0.00000054 **5.4×10^{-7}**

29. 0.0042 **4.2×10^{-3}**

30. 234,000 **2.34×10^5**

31. 418,000,000 **4.18×10^8**

Express each number in standard form. (Lesson 8-4)

32. 4.1×10^{-3} **0.0041**

33. 2.74×10^5 **274,000**

34. 3×10^9 **3,000,000,000**

35. 9.1×10^{-5} **0.000091**

Evaluate each product or quotient. Express the results in scientific notation. (Lesson 8-4)

36. $(2.13 \times 10^2)(3 \times 10^5)$ **6.39×10^7**

37. $(7.5 \times 10^6)(2.5 \times 10^{-2})$ **1.875×10^5**

38. $\frac{7.5 \times 10^8}{2.5 \times 10^4}$ **3×10^4**

39. $\frac{6.6 \times 10^5}{2 \times 10^{-3}}$ **3.3×10^8**

40. **MAMMALS** A blue whale has been caught that was 1.9×10^5 kilograms. The smallest mammal is a bumblebee bat, which is about 0.0019 kilograms. (Lesson 8-4)

a. Write the whale's weight in standard form. **190,000 kilograms**

b. Write the bat's weight in scientific notation. **1.9×10^{-3} kilograms**

c. How many orders of magnitude as big as a blue whale is a bumblebee bat? **10^8**

Formative Assessment

Use the Mid-Chapter Quiz to assess students' progress in the first half of the chapter.

For problems answered incorrectly, have students review the lessons indicated in parentheses.

FOLDABLES StudyOrganizer

Dinah Zike's Foldables®

Before students complete the Mid-Chapter Quiz, encourage them to review the information for Lessons 8-1 through 8-4 in their Foldables.

1 Focus

Objective Use a graphing calculator to investigate families of exponential functions.

Materials

- graphing calculator

Teaching Tips

- Remind students that to set their viewing window to standard, use [ZOOM] 6. Otherwise, use the window settings shown.
- For Activity 2, remind students that to enter functions like $y = \left(\frac{1}{2}\right)^x$, they will need to type [( ] 1 [÷] 2 [ )] [^] [X,T,θ,n] [ENTER] in the Y= list.

2 Teach

Working in Cooperative Groups

Divide the class into pairs. Have one student work through Activity 1 and the other work through Activity 2. Ask students to describe to each other what they see on the calculator screens. Have them switch activities and see if they notice any other similarities or differences.

Then ask students to work with their partners to complete Activities 3 and 4.

Practice Have students complete Exercises 1 and 2.

EXPLORE 8-5 Graphing Technology Lab
Family of Exponential Functions

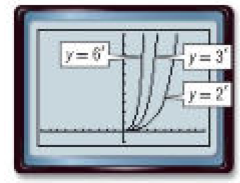
An exponential function is a function of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$. You have studied the effects of changing parameters in linear functions. You can use a graphing calculator to analyze how changing the parameters a and b affects the graphs in the family of exponential functions.

Activity 1 b in $y = b^x$, $b > 1$

Graph the set of equations on the same screen. Describe any similarities and differences among the graphs.

$y = 2^x$, $y = 3^x$, $y = 6^x$

Enter the equations in the [Y=] list and graph.



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There are many similarities in the graphs. The domain for each function is all real numbers, and the range is all positive real numbers. The functions are increasing over the entire domain. The graphs do not display any line symmetry.

Use the [ZOOM] feature to investigate the key features of the graphs.

Zooming in twice on a point near the origin allows closer inspection of the graphs. The y-intercept is 1 for all three graphs.

Tracing along the graphs reveals that there are no x-intercepts, no maxima and no minima.

The graphs are different in that the graphs for the equations in which b is greater are steeper.

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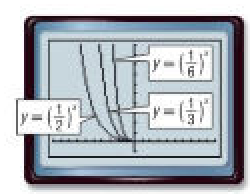
The effect of b on the graph is different when $0 < b < 1$.

Activity 2 b in $y = b^x$, $0 < b < 1$

Graph the set of equations on the same screen. Describe any similarities and differences among the graphs.

$y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{6}\right)^x$

The domain for each function is all real numbers, and the range is all positive real numbers. The function values are all positive and the functions are decreasing over the entire domain. The graphs display no line symmetry. There are no x-intercepts, and the y-intercept is 1 for all three graphs. There are no maxima or minima.



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However, the graphs in which b is lesser are steeper.

WatchOut!

Common Misconceptions Be sure students do not confuse polynomial functions and exponential functions. While $y = x^2$ and $y = 2^x$ each have an exponent, $y = x^2$ is a polynomial function, and $y = 2^x$ is an exponential function.

Activity 3 a in $y = ab^x$, $a > 0$

Graph each set of equations on the same screen. Describe any similarities and differences among the graphs.

$y = 2^x$, $y = 3(2^x)$, $y = \frac{1}{6}(2^x)$

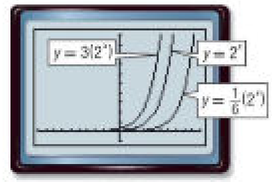
The domain for each function is all real numbers, and the range is all positive real numbers. The functions are increasing over the entire domain. The graphs do not display any line symmetry.

Use the ZOOM feature to investigate the key features of the graphs.

Zooming in twice on a point near the origin allows closer inspection of the graphs.

Tracing along the graphs reveals that there are no x-intercepts, no maxima and no minima.

However, the graphs in which a is greater are steeper. The y-intercept is 1 in the graph of $y = 2^x$, 3 in $y = 3(2^x)$, and $\frac{1}{6}$ in $y = \frac{1}{6}(2^x)$.



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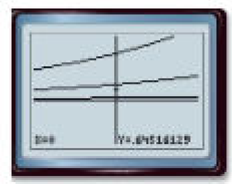

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Activity 4 a in $y = ab^x$, $a < 0$

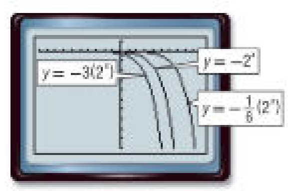
Graph each set of equations on the same screen. Describe any similarities and differences among the graphs.

$y = -2^x$, $y = -3(2^x)$, $y = -\frac{1}{6}(2^x)$

The domain for each function is all real numbers, and the range is all negative real numbers. The functions are decreasing over the entire domain. The graphs do not display any line symmetry.

There are no x-intercepts, no maxima and no minima.

However, the graphs in which the absolute value of a is greater are steeper. The y-intercept is -1 in the graph of $y = -2^x$, -3 in $y = -3(2^x)$, and $-\frac{1}{6}$ in $y = -\frac{1}{6}(2^x)$.



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Model and Analyze 1–3. See margin.

1. How does b affect the graph of $y = ab^x$ when $b > 1$ and when $0 < b < 1$? Give examples.
2. How does a affect the graph of $y = ab^x$ when $a > 0$ and when $a < 0$? Give examples.
3. **REGULARITY** Make a conjecture about the relationship of the graphs of $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$. Verify your conjecture by graphing both functions.

3 Assess

Formative Assessment

Use Exercise 3 to assess each student's knowledge of graphing exponential functions.

From Concrete to Abstract

Ask students to summarize how to use technology to find the solutions to exponential equations and inequalities.

Additional Answers

1. The value of b affects the steepness of the graph. When $b > 1$, the greater the value of b the steeper the graph. When $0 < b < 1$, the lesser the value of b the steeper the graph. Sample answer: The graph of $y = 5^x$ is steeper than the graph of $y = 3^x$. The graph of $y = \left(\frac{1}{5}\right)^x$ is steeper than the graph of $y = \left(\frac{1}{3}\right)^x$.
2. The value of a affects the steepness and direction of opening of the graph. The greater the absolute value of a, the steeper the graph. When $a > 0$ the graph opens up, and when $a < 0$ the graph opens down. Sample answer: The graph of $y = 4(2^x)$ is steeper than the graph of $y = 3(2^x)$. The graph of $y = 4(2^x)$ opens up and the graph of $y = -4(2^x)$ opens down.
3. Sample answer: The graph of $y = \left(\frac{1}{3}\right)^x$ is the graph of $y = 3^x$ reflected over the y-axis.

Teaching the Mathematical Practices

Regularity Mathematically proficient students look for and express regularity in repeated reasoning. In Exercise 3, guide students to look at the equations in Activity 2 to observe a pattern.

1 Focus

VerticalAlignment

Before Lesson 8-5 Evaluate numerical expressions involving exponents.

Lesson 8-5 Graph exponential functions. Identify data that display exponential behavior.

After Lesson 8-5 Solve problems involving exponential growth and decay.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- How is this equation different from a linear equation? The independent variable x is an exponent.
- What is the value of y when $x = 0$? $y = 3$
- Can the value of y ever be 0? no

LESSON 8-5 Exponential Functions

Then

- You evaluated numerical expressions involving exponents.

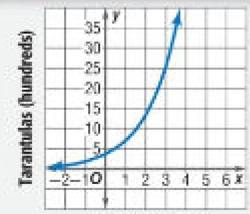
Now

1. Graph exponential functions.
2. Identify data that display exponential behavior.

Why?

- Tarantulas can appear scary with their large hairy bodies and legs, but they are harmless to humans. The graph shows a tarantula spider population that increases over time. Notice that the graph is not linear.

The graph represents the function $y = 3(2)^x$. This is an example of an *exponential* function.



NewVocabulary
exponential function
exponential growth function
exponential decay function

Mathematical Practices
Make sense of problems and persevere in solving them.

1 Graph Exponential Functions

An **exponential function** is a function of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$. Notice that the base is a constant and the exponent is a variable. Exponential functions are nonlinear.

KeyConcept Exponential Function

Words An exponential function is a function that can be described by an equation of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$.

Examples $y = 2(3)^x$ $\quad y = 4^x$ $\quad y = \left(\frac{1}{2}\right)^x$

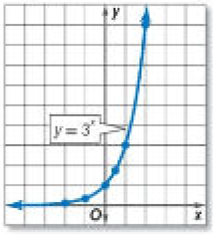
Example 1 Graph with $a > 0$ and $b > 1$

Graph $y = 3^x$. Find the y-intercept, and state the domain and range.

The graph crosses the y-axis at 1, so the y-intercept is 1. The domain is all real numbers, and the range is all positive real numbers.

Notice that the graph approaches the x-axis but there is no x-intercept. The graph is increasing on the entire domain.

x	3^x	y
−2	3^{-2}	$\frac{1}{9}$
−1	3^{-1}	$\frac{1}{3}$
0	3^0	1
$\frac{1}{2}$	$3^{\frac{1}{2}}$	≈1.73
1	3^1	3
2	3^2	9



GuidedPractice

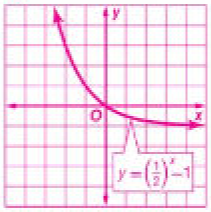
1. Graph $y = 7^x$. Find the y-intercept, and state the domain and range. See Ch. 8 Answer Appendix.

Functions of the form $y = ab^x$, where $a > 0$ and $b > 1$, are called **exponential growth functions** and all have the same shape as the graph in Example 1. Functions of the form $y = ab^x$, where $a > 0$ and $0 < b < 1$ are called **exponential decay functions** and also have the same general shape.

StudyTip

$a < 0$ If the value of a is less than 0, the graph will be reflected across the x-axis.

2.



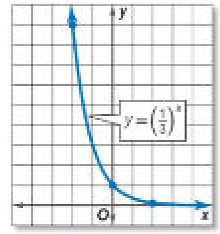
(0, 0); D = {all real numbers}; R = {y | $y > -1$}

Example 2 Graph with $a > 0$ and $0 < b < 1$

Graph $y = \left(\frac{1}{3}\right)^x$. Find the y-intercept, and state the domain and range.

The y-intercept is 1. The domain is all real numbers, and the range is all positive real numbers. Notice that as x increases, the y-values decrease less rapidly.

x	$\left(\frac{1}{3}\right)^x$	y
−2	$\left(\frac{1}{3}\right)^{-2}$	9
0	$\left(\frac{1}{3}\right)^0$	1
2	$\left(\frac{1}{3}\right)^2$	$\frac{1}{9}$



GuidedPractice

2. Graph $y = \left(\frac{1}{2}\right)^x - 1$. Find the y-intercept, and state the domain and range.

The key features of the graphs of exponential functions can be summarized as follows.

KeyConcept Graphs of Exponential Functions

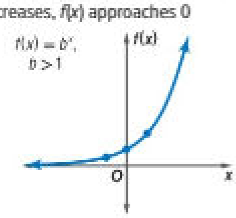
Exponential Growth Functions

Equation: $f(x) = ab^x$, $a > 0$, $b > 1$

Domain, Range: all reals; all positive reals

Intercepts: one y-intercept, no x-intercepts

End behavior: as x increases, $f(x)$ increases; as x decreases, $f(x)$ approaches 0



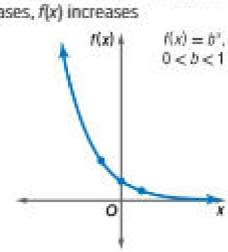
Exponential Decay Functions

Equation: $f(x) = ab^x$, $a > 0$, $0 < b < 1$

Domain, Range: all reals; all negative reals

Intercepts: one y-intercept, no x-intercepts

End behavior: as x increases, $f(x)$ approaches 0; as x decreases, $f(x)$ increases



Exponential functions occur in many real world situations.

Real-WorldLink

The United States is the largest soda consumer in the world. In a recent year, the United States accounted for one third of the world's total soda consumption.

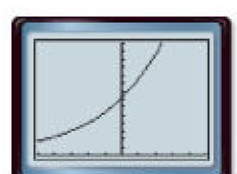
Source: Worldwatch Institute

Real-World Example 3 Use Exponential Functions to Solve Problems

SODA The function $C = 179(1.029)^t$ models the amount of soda consumed in the world, where C is the amount consumed in billions of liters and t is the number of years since 2000.

a. Graph the function. What values of C and t are meaningful in the context of the problem?

Since t represents time, $t > 0$. At $t = 0$, the consumption is 179 billion liters. Therefore, in the context of this problem, $C > 179$ is meaningful.



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1 Graph Exponential Functions

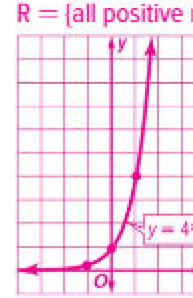
Example 1 shows how to graph an exponential function when $a > 0$ and $b > 1$. **Example 2** shows how to graph an exponential function when $a > 0$ and $0 < b < 1$. **Example 3** shows how to use an exponential function to solve a real-world problem.

Formative Assessment

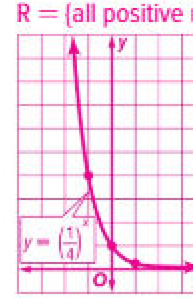
Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Graph $y = 4^x$. Find the y-intercept and state the domain and range. y-intercept: 1, D = {all real numbers}, R = {all positive numbers}



2 Graph $y = \left(\frac{1}{4}\right)^x$. Find the y-intercept and state the domain and range. y-intercept: 1, D = {all real numbers}, R = {all positive numbers}



Teach with Tech

Interactive Whiteboard Create a template for a table of values, and drag a coordinate grid onto the board. Write a function and have students complete the table of values. Then have students draw the graph of the function.

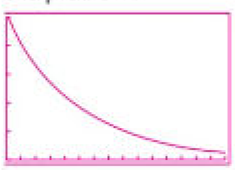
WatchOut!

Student Misconceptions Make sure students understand that the graphs of exponential functions never actually touch the x-axis. It is acceptable for hand-drawn graphs to show the graph just above and about parallel to the x-axis as long as students understand that the graph gets infinitely close to the axis without touching it.

Additional Example

3 DEPRECIATION Some people say that the value of a new car decreases as soon as it is driven off the lot. The function $V = 25{,}000 \cdot 0.82^t$ models the depreciation in the value of a new car that originally cost AED25,000. *V* represents the value of the car and *t* represents the time in years from the time of purchase.

a. Graph the function. What values of *V* and *t* are meaningful in the context of the problem?



[0, 15] scl: 1 by [0, 25,000] scl: 500; Only values of $V \le 25{,}000$ and $t \ge 0$ are meaningful.

b. What is the car's value after five years? about AED9270

2 Identify Exponential Behavior

Example 4 shows how to determine whether a set of data displays exponential behavior.

Additional Example

4 Determine whether the set of data shown below displays exponential behavior. Write *yes* or *no*. Explain why or why not.

x	0	10	20	30
y	10	25	62.5	156.25

The domain values are at regular intervals, and the range values have a common factor of 2.5, so the set is probably exponential. Also, the graph shows rapidly increasing values of *y* as *x* increases.

b. How much soda was consumed in 2005?

$C = 179(1.029)^t$ Original equation

$= 179(1.029)^5$ $t = 5$

≈ 206.5 Use a calculator.

The world soda consumption in 2005 was approximately 206.5 billion liters.

Guided Practice

3. BIOLOGY A certain bacteria population doubles every 20 minutes. Beginning with 10 cells in a culture, the population can be represented by the function $B = 10(2)^t$, where *B* is the number of bacteria cells and *t* is the time in 20 minute increments. How many will there be after 2 hours? **640**

2 Identify Exponential Behavior

Recall from Lesson 3-3 that linear functions have a constant rate of change. Exponential functions do not have constant rates of change, but they do have constant ratios.

Problem-Solving Tip

Make an Organized List Making an organized list of *x*-values and corresponding *y*-values is helpful in graphing the function. It can also help you identify patterns in the data.

Example 4 Identify Exponential Behavior

Determine whether the set of data shown below displays exponential behavior. Write *yes* or *no*. Explain why or why not.

x	0	5	10	15	20	25
y	64	32	16	8	4	2

Method 1 Look for a pattern.

The domain values are at regular intervals of 5. Look for a common factor among the range values.

64 32 16 8 4 2

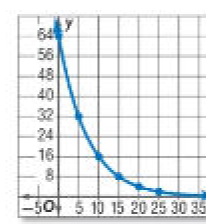
$\times\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$

The range values differ by the common factor of $\frac{1}{2}$.

Since the domain values are at regular intervals and the range values differ by a positive common factor, the data are probably exponential. Its equation may involve $\left(\frac{1}{2}\right)^x$.

Method 2 Graph the data.

Plot the points and connect them with a smooth curve. The graph shows a rapidly decreasing value of *y* as *x* increases. This is a characteristic of exponential behavior in which the base is between 0 and 1.



Guided Practice

4. Determine whether the set of data shown below displays exponential behavior. Write *yes* or *no*. Explain why or why not.

x	0	3	6	9	12	15
y	12	16	20	24	28	32

4. No; the domain values are at regular intervals, but the range values have a common difference of 4.

Differentiated Instruction OL BL

Logical Learners Ask students to write a comparison of an exponential function to a linear function.

Check Your Understanding

Examples 1–2 **Graph each function. Find the y-intercept and state the domain and range.**

1. $y = 2^x$
2. $y = -5^x$
3. $y = -\left(\frac{1}{5}\right)^x$
4. $y = 3\left(\frac{1}{4}\right)^x$
5. $f(x) = 6^x + 3$
6. $f(x) = 2 - 2^x$

1–6. See Ch. 8 Answer Appendix.

Example 3

7. **BIOLOGY** The function $f(t) = 100(1.05)^t$ models the growth of a fruit fly population, where $f(t)$ is the number of flies and t is time in days.
   a. What values for the domain and range are reasonable in the context of this situation? Explain. $D = \{d \mid d \geq 0\}$, the number of days is greater than or equal to 0; $R = \{y \mid y \geq 100\}$, the number of fruit flies is greater than or equal to 100.
   b. After two weeks, approximately how many flies are in this population? about 198 fruit flies

Example 4 **Determine whether the set of data shown below displays exponential behavior. Write *yes* or *no*. Explain why or why not.** 8–9. See margin.

8.

x	1	2	3	4	5	6
y	−4	−2	0	2	4	6

9.

x	2	4	6	8	10	12
y	1	4	16	64	256	1024

Practice and Problem Solving

Examples 1–2 **Graph each function. Find the y-intercept and state the domain and range.** 10–19. See Ch. 8 Answer Appendix.

10. $y = 2 \cdot 8^x$
11. $y = 2 \cdot \left(\frac{1}{6}\right)^x$
12. $y = \left(\frac{1}{12}\right)^x$
13. $y = -3 \cdot 9^x$
14. $y = -4 \cdot 10^x$
15. $y = 3 \cdot 11^x$
16. $y = 4^x + 3$
17. $y = \frac{1}{2}(2^x - 8)$
18. $y = 5(3^x) + 1$
19. $y = -2(3^x) + 5$

Example 3

20. **MODELING** A population of bacteria in a culture increases according to the model $p = 300(2.7)^{0.02t}$, where t is the number of hours and $t = 0$ corresponds to 9:00 A.M.
   a. Use this model to estimate the number of bacteria at 11 A.M. about 312
   b. Graph the function and name the p-intercept. Describe what the p-intercept represents, and describe a reasonable domain and range for this situation. See Ch. 8 Answer Appendix.

Example 4 **Determine whether the set of data shown below displays exponential behavior. Write *yes* or *no*. Explain why or why not.** 21–24. See margin.

21.

x	−4	0	4	8	12
y	2	−4	8	−16	32

22.

x	−6	−3	0	3
y	5	10	15	20

23.

x	−8	−6	−4	−2
y	0.25	0.5	1	2

24.

x	20	30	40	50	60
y	1	0.4	0.16	0.064	0.0256

B 25. **PHOTOGRAPHY** Manal is enlarging a photograph to make a poster for school. She will enlarge the picture repeatedly at 150%. The function $P = 1.5^x$ models the new size of the picture being enlarged, where x is the number of enlargements. How many times as big is the picture after 4 enlargements? about 506% bigger than the original

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	10–24, 42–68	11–23 odd, 46–49	10–24 even, 42–45, 50–68
OL Core	11–39 odd, 26, 40, 42–68	10–24, 46–49	25–40, 42–45, 50–68
BL Advanced	25–62, (optional: 63–68)		

3 Practice

Formative Assessment

Use Exercises 1–9 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Teaching the Mathematical Practices

Modeling Mathematically proficient students can use a function to describe how one quantity of interest depends on another. In Exercise 20, ask students how time will affect the population.

Additional Answers

8. No; the domain values are at regular intervals, but the range values have a common difference of 2.
9. Yes; the domain values are at regular intervals, and the range values have a common factor of 4.
21. No; the domain values are at regular intervals, but the range values do not have a positive common factor.
22. No; the domain values are at regular intervals, but the range values have a common difference of 5.
23. Yes; the domain values are at regular intervals, and the range values have a common factor of 2.
24. Yes; the domain values are at regular intervals, and the range values have a common factor of 0.4.

Follow-up

Students have explored modeling using exponential functions.

Ask:

- How can mathematical models help you make good decisions? Sample answer: Mathematical models can be used to compare different options that are available, as well as to predict the impact of an option if it is chosen.

26. FINANCIAL LITERACY Mohammad deposited AED 500 into a savings account and after 8 years, his investment is worth AED 807.07. The equation $A = d(1.005)^{12t}$ models the value of Mohammad's investment A after t years with an initial deposit d.

a. What would the value of Mohammad's investment be if he had deposited AED 1,000? about AED 403.54

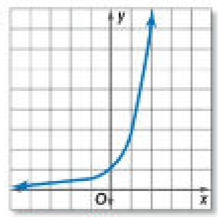
b. What would the value of Mohammad's investment be if he had deposited AED 250? about AED 403.54

c. Interpret $d(1.005)^{12t}$ to explain how the amount of the original deposit affects the value of Mohammad's investment.

26c. Sample answer: Since Mohammad's investment is the product of d and a factor not depending on d, it varies directly as the original deposit.

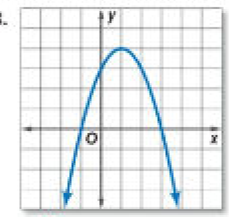
Identify each function as *linear, exponential,* or *neither.*

27.



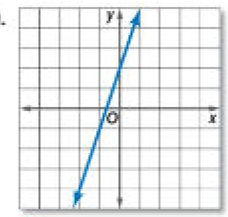
exponential

28.



neither

29.



linear

30. $y = 4^x$ exponential

31. $y = 2x(x - 1)$ neither

32. $5x + y = 8$ linear

33. **GRADUATION** The number of graduates at a high school has increased by a factor of 1.055 every year since 2001. In 2001, 110 students graduated. The function $N = 110(1.055)^t$ models the number of students N expected to graduate t years after 2001. How many students will graduate in 2012? about 198 students

C **Describe the graph of each equation as a transformation of the graph of $y = 2^x$.**

34. $y = 2^x + 6$ a translation 6 units up

35. $y = 3(2)^x$ a vertical stretch by a factor of 3

36. $y = -\frac{1}{4}(2)^x$ 36. a reflection over the x-axis and a vertical compression

37. $y = -3 + 2^x$ a translation down 3 units

38. $y = \left(\frac{1}{2}\right)^x$ a reflection over the y-axis

39. $y = -5(2)^x$ 39. a vertical stretch by a factor of 5 and a reflection over the x-axis.

40. **DEER** The deer population at a national park doubles every year. In 2000, there were 25 deer in the park. The function $N = 25(2)^t$ models the number of deer N in the park t years after 2000. What will the deer population be in 2015? 819,200

42. Never; the graph never crosses the x-axis because the powers of b are always positive and $a \neq 0$. Thus, ab^x is never 0.

H.O.T. Problems Use Higher-Order Thinking Skills

41. **PERSEVERANCE** Write an exponential function for which the graph passes through the points at $(0, 3)$ and $(1, 6)$. $f(x) = 3(2)^x$

42. **REASONING** Determine whether the graph of $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$, *sometimes, always,* or *never* has an x-intercept. Explain your reasoning.

43. **OPEN ENDED** Find an exponential function that represents a real-world situation, and graph the function. Analyze the graph, and explain why the situation is modeled by an exponential function rather than a linear function. See margin.

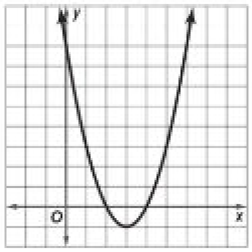
44. **REASONING** Use tables and graphs to compare and contrast an exponential function $f(x) = ab^x + c$, where $a \neq 0$, $b > 0$, and $b \neq 1$, and a linear function $g(x) = ax + c$. Include intercepts, intervals where the functions are increasing, decreasing, positive, or negative, relative maxima and minima, symmetry, and end behavior. See Ch. 8 Answer Appendix.

45. **WRITING IN MATH** Explain how to determine whether a set of data displays exponential behavior. See margin.

516 | Lesson 8-5 | Exponential Functions

Standardized Test Practice

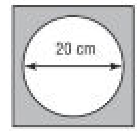
46. SHORT RESPONSE What are the x-intercepts of the function graphed below? **2 and 4**



47. Mazen invested AED 300 into a savings account. The equation $A = 300(1.005)^{12t}$ models the amount in Mazen's account A after t years. How much will be in Mazen's account after 7 years? **B**

A AED 25,326 C AED 382.01
B AED 456.11 D AED 301.52

48. GEOMETRY Noura placed a circular piece of paper on a square picture as shown below. If the picture extends 4 centimeter beyond the circle on each side, what is the perimeter of the square picture? **J**



F 64 cm H 94 cm
G 80 cm J 112 cm

49. The points with coordinates $(0, -3)$ and $(2, 7)$ are on line l. Line p contains $(3, -1)$ and is perpendicular to line l. What is the x-coordinate of the point where l and p intersect? **A**

A $\frac{1}{2}$ B $-\frac{2}{5}$
C $-\frac{1}{2}$ D -3

Spiral Review

Evaluate each product. Express the results in both scientific notation and standard form. (Lesson 8-4)

50. $(1.9 \times 10^2)(4.7 \times 10^6)$ **8.93×10^8, 893,000,000**

51. $(4.5 \times 10^{-3})(5.6 \times 10^4)$ **2.52×10^2, 252**

52. $(3.8 \times 10^{-4})(6.4 \times 10^{-8})$ **2.432×10^{-11}, 0.00000000002432**

Simplify. (Lesson 8-3)

53. $\sqrt[3]{343}$ **7**

54. $\sqrt[6]{729}$ **3**

55. $\left(\frac{1}{32}\right)^{\frac{1}{5}}$ **$\frac{1}{2}$**

56. $729^{\frac{5}{6}}$ **243**

57. $216^{\frac{5}{3}}$ **7776**

58. $\left(\frac{1}{81}\right)^{\frac{3}{2}}$ **$\frac{1}{729}$**

59. DEMOLITION DERBY When a car hits an object, the damage is measured by the collision impact. For a certain car the collision impact I is given by $I = 2v^2$, where v represents the speed in kilometers per minute. What is the collision impact if the speed of the car is 4 kilometers per minute? (Lesson 8-1)
32 km^2/min^2

Use elimination to solve each system of equations. (Lesson 6-3)

60. $x + y = -3$, $x - y = 1$ **(−1, −2)**

61. $3a + b = 5$, $2a + b = 10$ **(−5, 20)**

62. $3x - 5y = 16$, $-3x + 2y = -10$ **(2, −2)**

Skills Review

Find the next three terms of each arithmetic sequence.

63. 1, 3, 5, 7, ... **9, 11, 13**

64. −6, −4, −2, 0, ... **2, 4, 6**

65. 6.5, 9, 11.5, 14, ... **16.5, 19, 21.5**

66. 10, 3, −4, −11, ... **−18, −25, −32**

67. $\frac{1}{2}, \frac{5}{4}, 2, \frac{11}{4}, \ldots$ **$\frac{7}{2}, \frac{17}{4}, 5$**

68. $1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \ldots$ **$0, -\frac{1}{4}, -\frac{1}{2}$**

Differentiated Instruction OL BL

Extension Give students this scenario: a wise man asked his ruler to provide rice for his people. The wise man asked the ruler to give him 2 grains of rice for the first square on a chess board, 4 grains for the second, and so on, doubling the amount of rice with each square of the board.

Ask:

- How many grains of rice will the wise man receive for the sixty-fourth square on the chessboard? 2^{64} or about 1.84×10^{19} grains
- If one pound of rice has approximately 24,000 grains, how many tons of rice will the wise man receive on the last day? (Hint: 1 ton = 2000 kilograms) about 3.84×10^{11} tons

Teaching the Mathematical Practices

Perseverance Mathematically proficient students analyze givens, constraints, relationships, and goals of a problem. In Exercise 41, students may struggle since they are given only two points. Suggest that they start with the general form of an exponential equation $y = ab^x$.

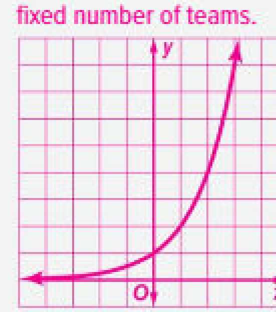
4 Assess

Crystal Ball Ask students to write how they think exponential functions will connect with the next lesson, which involves real-world growth and decay problems.

Additional Answers

43. Sample answer: The number of teams competing in a basketball tournament can be represented by $y = 2^x$, where the number of teams competing is y and the number of rounds is x.

The y-intercept of the graph is 1. The graph increases rapidly for $x > 0$. With an exponential model, each team that joins the tournament will play all of the other teams. If the scenario were modeled with a linear function, each team that joined would play a fixed number of teams.



45. Sample answer: First, look for a pattern by making sure that the domain values are at regular intervals and the range values differ by a common factor.

1 Focus

Objective Use a graphing calculator to solve exponential equations and inequalities.

Materials

- graphing calculator

Teaching Tips

- For Activity 1, remind students that to enter $3^x + 4$, they will need to use the ^ key for the exponent and use the down arrow before entering the + 4.
- When changing the windows settings, use the **tab** key to move from one field to another.
- In Activity 2, student will need to use the **tab** key to move the curser to the entry line to type **f2(*x*)**.

2 Teach

Working in Cooperative Groups

Divide the class into pairs. Work through Activity 1 and Activity 2 as a class. Then ask students to work with their partners to complete Exercises 1–9 and Activities 3 and 4.

Practice Have students complete Exercises 10–12.

Teaching the Mathematical Practices

Tools Mathematically proficient students are sufficiently familiar with tools appropriate to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. Point out that Activities 2, 3, and 4 offer various methods for solving equations and inequalities. Discuss when to use the methods and the technology tools available.

EXTEND 8-5

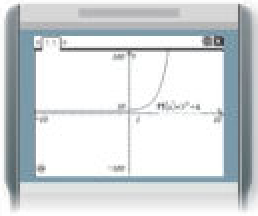
Graphing Technology Lab
Solving Exponential Equations and Inequalities

You can use a graphing calculator to solve exponential equations and inequalities by graphing and by using tables.

Mathematical Practices
Use appropriate tools strategically.

Activity 1 Graph an Exponential Equation

Graph $y = 3^x + 4$ using a graphing calculator.



Step 1 Add a new **Graphs** page.

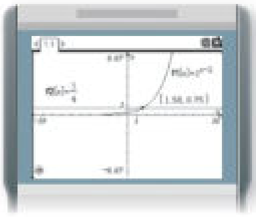
Step 2 Enter $3^x + 4$ as **f1(*x*)**.

Step 3 Use the **Window Settings** option from the **Window/Zoom** menu to adjust the window so that x is from −10 to 10 and y is from −100 to 100. Keep the scales as **Auto**.

To solve an equation by graphing, graph both sides of the equation and locate the point(s) of intersection.

Activity 2 Solve an Exponential Equation by Graphing

Solve $2^{x-2} = \frac{3}{4}$.



Step 1 Add a new **Graphs** page.

Step 2 Enter 2^{x-2} as **f1(*x*)** and $\frac{3}{4}$ as **f2(*x*)**.

Step 3 Use the **Intersection Point(s)** tool from the **Points & Lines** menu to find the intersection of the two graphs. Select the graph of **f1(*x*) enter** and then the graph of **f2(*x*) enter**.

The graphs intersect at about (1.58, 0.75). Therefore, the solution of $2^{x-2} = \frac{3}{4}$ is 1.58.

Exercises

TOOLS Use a graphing calculator to solve each equation.

1. $\left(\frac{1}{3}\right)^{x-1} = \frac{3}{4}$ **≈1.26**
2. $2^{2x-1} = 2x$ **0.5, 1**
3. $\left(\frac{1}{2}\right)^{2x} = 2^{2x}$ **0**
4. $5^{\frac{1}{3}x+2} = -x$ **≈−3.61**
5. $\left(\frac{1}{8}\right)^{2x} = -2x + 1$ **0, ≈0.409**
6. $2^{\frac{1}{4}x-1} = 3^{x+1}$ **≈−1.94**
7. $2^{3x-1} = 4^x$ **1**
8. $4^{2x-3} = 5^{-x+1}$ **≈1.32**
9. $3^{2x-4} = 2^x + 1$ **3**

Activity 3 Solve an Exponential Equation by Using a Table

Solve $2\left(\frac{1}{2}\right)^{x+2} = \frac{1}{4}$ using a table.

Step 1 Add a new **Lists & Spreadsheet** page.

Step 2 Label column A as x. Enter values from -4 to 4 in cells A1 to A9.

Step 3 In column B in the formula row, enter the left side of the rational equation. In column C of the formula row, enter $= \frac{1}{4}$. Specify **Variable Reference** when prompted.

Scroll until you see where the values in Columns B and C are equal. This occurs at $x = 1$. Therefore, the solution of $2\left(\frac{1}{2}\right)^{x+2} = \frac{1}{4}$ is 1.

You can also use a graphing calculator to solve exponential inequalities.

Activity 4 Solve an Exponential Inequality

Solve $4^{x-3} \leq \left(\frac{1}{4}\right)^{2x}$.

Step 1 Add a new **Graphs** page.

Step 2 Enter the left side of the inequality into **f1(x)**. Press **del**, select $\geq$, and enter 4^{x-3}. Enter the right side of the inequality into **f2(x)**. Press **tab del** $\leq$, and enter $\left(\frac{1}{4}\right)^{2x}$.

The x-values of the points in the region where the shading overlap is the solution set of the original inequality. Therefore the solution of $4^{x-3} \leq \left(\frac{1}{4}\right)^{2x}$ is $x \leq 1$.

Exercises

TOOLS Use a graphing calculator to solve each equation or inequality.

10. $\left(\frac{1}{3}\right)^{3x} = 3^x$ **0**

11. $\left(\frac{1}{6}\right)^{2x} = 4^x$ **0**

12. $3^{1-x} \leq 4^x$ **$x \geq 0.442$**

13. $4^{3x} \leq 2x + 1$ **$-0.409 \leq x \leq 0$**

14. $\left(\frac{1}{4}\right)^x > 2^{x+4}$ **$x < -1.33$**

15. $\left(\frac{1}{3}\right)^{x-1} \geq 2^x$ **$x \leq 0.613$**

3 Assess

Formative Assessment

Use Exercises 13–15 to assess each student's knowledge of solving exponential equations and inequalities.

From Concrete to Abstract

Ask students to summarize the use of technology to find the solutions to exponential equations and inequalities.

1 Focus

VerticalAlignment

Before Lesson 8-6 Analyze exponential functions.

Lesson 8-6 Solve problems involving exponential growth. Solve problems involving exponential decay.

After Lesson 8-6 Relate geometric sequences to exponential functions.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- Looking at the equation, how do you know the function is not linear? Time, represented as t, is an exponent so the function is not linear.
- Use the equation to predict the average number of blogs in millions in the 8th month. about 3 million Would you describe the average number of blogs per month as growing or decaying? growing

1 Exponential Growth

Example 1 shows how to solve a real-world problem involving exponential growth. **Example 2** shows how to solve a real-world problem involving compound interest.

LESSON 8-6 Growth and Decay

:: Then

- You analyzed exponential functions.

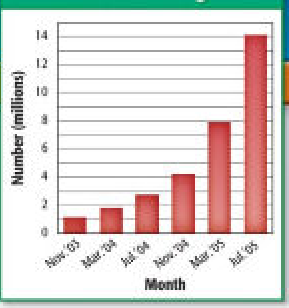
:: Now

1. Solve problems involving exponential growth.
2. Solve problems involving exponential decay.

:: Why?

- The number of Weblogs or blogs increased at a monthly rate of about 13.7% over 21 months. The average number of blogs per month can be modeled by $y = 1.1(1 + 0.137)^t$ or $y = 1.1(1.137)^t$, where y represents the total number of blogs in millions and t is the number of months since November 2003.

Growth of Blogs



NewVocabulary
compound interest

Mathematical Practices
Model with mathematics.

1 Exponential Growth

The equation for the number of blogs is in the form $y = a(1 + r)^t$. This is the general equation for exponential growth.

KeyConcept **Equation for Exponential Growth**

$$y = a(1 + r)^t$$

- a is the initial amount.
- t is time.
- y is the final amount.
- r is the rate of change expressed as a decimal, $r > 0$.

Real-World Example 1 Exponential Growth

CONTEST **The prize for a radio station contest begins with a AED 100 gift card. Once a day, a name is announced. The person has 15 minutes to call or the prize increases by 2.5% for the next day.**

a. Write an equation to represent the amount of the gift card in dirhams after t days with no winners.

$y = a(1 + r)^t$	Equation for exponential growth
$y = 100(1 + 0.025)^t$	$a = 100$ and $r = 2.5\%$ or 0.025
$y = 100(1.025)^t$	Simplify.

In the equation $y = 100(1.025)^t$, y is the amount of the gift card and t is the number of days since the contest began.

b. How much will the gift card be worth if no one wins after 10 days?

$y = 100(1.025)^t$	Equation for amount of gift card
$= 100(1.025)^{10}$	$t = 10$
≈ 128.01	Use a calculator.

In 10 days, the gift card will be worth AED 128.01.

GuidedPractice $y = 10{,}850(1.05)^t$; about AED 22,556.37

1. **TUITION** A college's tuition has risen 5% each year since 2000. If the tuition in 2000 was AED 10,850, write an equation for the amount of the tuition t years after 2000. Predict the cost of tuition for this college in 2015.

Compound interest is interest earned or paid on both the initial investment and previously earned interest. It is an application of exponential growth.

KeyConcept Equation for Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A is the current amount.

n is the number of times the interest is compounded each year, and t is time in years.

P is the principal or initial amount.

r is the annual interest rate expressed as a decimal, $r > 0$.

Real-WorldCareer

Financial Advisor Financial advisors help people plan their financial futures. A good financial advisor has mathematical, problem-solving, and communication skills. A bachelor's degree is strongly preferred but not required.

Real-World Example 2 Compound Interest

FINANCE **Huda's parents invested AED 14,000 at 6% per year compounded monthly. How much money will there be in the account after 10 years?**

$A = P\left(1 + \frac{r}{n}\right)^{nt}$	Compound interest equation
$= 14{,}000\left(1 + \frac{0.06}{12}\right)^{12(10)}$	$P = 14{,}000$, $r = 6\%$ or 0.06, $n = 12$, and $t = 10$
$= 14{,}000(1.005)^{120}$	Simplify.
$\approx 25{,}471.55$	Use a calculator.

There will be about AED 25,471.55 in 10 years.

GuidedPractice

2. **FINANCE** Determine the amount of an investment if AED 300 is invested at an interest rate of 3.5% compounded monthly for 22 years. **about AED 647.20**

2 Exponential Decay

In exponential decay, the original amount decreases by the same percent over a period of time. A variation of the growth equation can be used as the general equation for exponential decay.

StudyTip

Growth and Decay Since r is added to 1, the value inside the parentheses will be greater than 1 for exponential growth functions. For exponential decay functions, this value will be less than 1 since r is subtracted from 1.

KeyConcept Equation for Exponential Decay

$$y = a(1 - r)^t$$

a is the initial amount.

t is time.

y is the final amount.

r is the rate of decay expressed as a decimal, $0 < r < 1$.

Real-World Example 3 Exponential Decay

SWIMMING **A fully inflated child's raft for a pool is losing 6.6% of its air every day. The raft originally contained 74,000 cubic centimeters of air.**

a. Write an equation to represent the loss of air.

$y = a(1 - r)^t$	Equation for exponential decay
$= 74000(1 - 0.066)^t$	$a = 74000$ and $r = 6.6\%$ or 0.066
$= 74000(0.934)^t$	Simplify.

$y = 74000(0.934)^t$, where y is the air in the raft in cubic centimeters after t days.

Focus on Mathematical Content

Compound Interest In contrast to simple interest, compound interest is applied to the original principal and any previously earned interest. There are four ways to increase the amount in a compound-interest account: the investor can increase the initial principal, increase the annual interest rate, increase the number of compoundings per year, or increase the time that the money is in the account.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 **POPULATION** In 2008 the town of Flat Creek had a population of about 280,000 and a growth rate of 0.85% per year.

a. Write an equation to represent the population of Flat Creek since 2008. $y = 280{,}000\ (1.0085)^t$

b. According to the equation, what will be the population of Flat Creek in the year 2018? about 304,731

2 **COLLEGE** When Eiman was born, her grandparents invested AED1000 in a fixed rate savings account at a rate of 7% compounded annually. Eiman will receive the money when she turns 18 to help with her college expenses. What amount of money will Eiman receive from the investment? She will receive about AED3380.

2 Exponential Decay

Example 3 shows how to solve a real-world problem involving exponential decay.

WatchOut!

Student Misconceptions Remind students that in growth and decay equations, the amount inside the parentheses will be greater than 1 for growth and less than 1 for decay.

Additional Example

3 CHARITY During an economic recession, a charitable organization found that its donations dropped by 1.1% per year. Before the recession, its donations were AED390,000.

a. Write an equation to represent the charity's donations since the beginning of the recession. $A = 390{,}000(0.989)^t$

b. Estimate the amount of the donations 5 years after the start of the recession. about AED369,017

3 Practice

Formative Assessment

Use Exercises 1–3 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Teach with Tech

Web Search Have students search the Web to find the half-life of different radioactive elements. Have them choose three elements, and ask them to calculate the amount of a 500 gram sample that remains after 3 years.

Exercise Alert

Grid Paper For Exercises 25–27 and 42–44, students will need grid paper.

Teaching the Mathematical Practices

Precision Mathematically proficient students express answers with a degree of precision appropriate for the problem context. In Exercise 9, point out that an estimate is appropriate to answer the question. In Exercise 17, students can approximate by graphing on a graphing calculator or by guessing and checking on a scientific calculator.

Estimate the amount of air in the raft after 7 days.

$y = 74000(0.934)^t$ — Equation for air loss

$= 74000(0.934)^7$ — $t = 7$

≈ 45880 — Use a calculator.

The amount of air in the raft after 7 days will be about 45880 cubic centmeters.

Guided Practice

3. **POPULATION** The population of Campbell County, Kentucky, has been decreasing at an average rate of about 0.3% per year. In 2000, its population was 88,647. Write an equation to represent the population since 2000. If the trend continues, predict the population in 2010. $y = 88{,}647(1 - 0.003)^t$; about 86,023

Check Your Understanding

Example 1

1. **SALARY** Ms. Hidaya received a job as a teacher with a starting salary of AED 125,000. According to her contract, she will receive a 1.5% increase in her salary every year. How much will Ms. Hidaya earn in 7 years? about AED 138,730.61

Example 2

2. **MONEY** Yousif invested AED 400 into an account with a 5.5% interest rate compounded monthly. How much will Yousif's investment be worth in 8 years? about AED 620.46

Example 3

3. **ENROLLMENT** In 2000, 2200 students attended Polaris High School. The enrollment has been declining 2% annually.
   a. Write an equation for the enrollment of Polaris High School t years after 2000. $y = 2200(0.98)^t$
   b. If this trend continues, how many students will be enrolled in 2015? about 1624

Practice and Problem Solving

Example 1

4. **MEMBERSHIPS** The Work-Out Gym sold 550 memberships in 2001. Since then the number of memberships sold has increased 3% annually.
   a. Write an equation for the number of memberships sold at Work-Out Gym t years after 2001. $y = 550(1.03)^t$
   b. If this trend continues, predict how many memberships the gym will sell in 2020. about 964
5. **COMPUTERS** The number of people who own computers has increased 23.2% annually since 1990. If half a million people owned a computer in 1990, predict how many people will own a computer in 2015. about 92,095,349
6. **COINS** Majed purchased a rare coin from a dealer for AED 300. The value of the coin increases 5% each year. Determine the value of the coin in 5 years. about AED 382.88

Example 2

7. **INVESTMENTS** Mahmoud invested AED 6,600 at an interest rate of 4.5% compounded monthly. Determine the value of his investment in 4 years. about AED 7,898.97
8. **COMPOUND INTEREST** Nisreen invested AED 1,200 at an interest rate of 5.75% compounded quarterly. Determine the value of her investment in 7 years. about AED 1,789.54
9. **PRECISION** Najla is saving money for a trip to the Bahamas that costs AED 1,087.76. She puts AED 550 into a savings account that pays 7.25% interest compounded quarterly. Will she have enough money in the account after 4 years? Explain.

9. Sample answer: No; she will have about AED 733.13 in the account in 4 years.

Example 3

10. **INVESTMENTS** Ali's investment of AED 4,500 has been losing its value at a rate of 2.5% each year. What will his investment be worth in 5 years? about AED 3,964.93

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	4–12, 16, 18–44	5–11 odd, 21–24	4–12 even, 16, 18–20, 25–44
OL Core	5–11 odd, 13–16, 18–44	4–12, 21–24	13–16, 18–20, 25–44
BL Advanced	13–41, (optional: 42–44)		

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11. **POPULATION** In the years from 2010 to 2015, the population of Washington DC is expected decrease about 0.9% annually. In 2010, the population was about 530,000. What is the population of Washington DC expected to be in 2015? **about 506,575**

12. **CARS** Faris purchases a car for AED 18,995. The car depreciates at a rate of 18% annually. After 6 years, Faleh offers to buy the car for AED 4,500. Should Faris sell the car? Explain. **Sample answer: No; the car is worth about AED 5,774.61.**

B 13. **HOUSING** The median house price in the United States increased an average of 1.4% each year between 2005 and 2007. Assume that this pattern continues.

a. Write an equation for the median house price for t years after 2007. **$I = 910{,}000(1.014)^t$**

b. Predict the median house price in 2018. **about AED 1,060,373**

Median House Price	
2005	AED 885,000
2006	AED 905,800
2007	AED 910,000

Source: *Real Estate Journal*

C 14. **ELEMENTS** A radioactive element's half-life is the time it takes for one half of the element's quantity to decay. The half-life of Plutonium-241 is 14.4 years. The number of grams A of Plutonium-241 left after t years can be modeled by $A = p(0.5)^{\frac{t}{14.4}}$, where p is the original amount of the element.

a. How much of a 0.2-gram sample remains after 72 years? **0.00625 g**

b. How much of a 5.4-gram sample remains after 1095 days? **≈4.7 g**

15. **COMBINING FUNCTIONS** A swimming pool holds a maximum of 77, 600 liters of water. It evaporates at a rate of 0.5% per hour. The pool currently contains 71,900 liters of water.

15c. $C(t) = 300t + 19000(0.995)^t$; The function represents the number of milliliters of water in the pool at any time after the hose is turned on.

a. Write an exponential function $w(t)$ to express the amount of water remaining in the pool after time t where t is the number of hours after the pool has reached 71,900 liters. **$w(t) = 71{,}900(0.995)^t$**

b. At this same time, a hose is turned on to refill the pool at a rate of 1,100 liters per hour. Write a function $p(t)$, where t is the time in hours the hose is running, to express the amount of water that is pumped into the pool. **$p(t) = 1{,}100t$**

c. Find $C(t) = p(t) + w(t)$. What does this new function represent? **$C(t) = 300t + 19000(0.995)^t$**

d. Use the graph of $C(t)$ to determine how long the hose must run to fill the pool to its maximum capacity. **about 7.3 h**

H.O.T. Problems Use Higher-Order Thinking Skills

16. **REASONING** Determine the growth rate (as a percent) of a population that quadruples every year. Explain. **300%; Solving $y = a(1 + r)^t$ for $y = 4$, $a = 1$, and $t = 1$ gives $r = 3$ or 300%.**

17. **PRECISION** Mansour invested AED 1,200 into an account with an interest rate of 8% compounded monthly. Use a calculator to approximate how long it will take for Mansour's investment to reach AED 2,500. **about 9.2 yr**

18. **REASONING** The amount of water in a container doubles every minute. After 8 minutes, the container is full. After how many minutes was the container half full? Explain. **7; Sample answer: Since the amount of water doubles every minute, the container would be half full a minute before it was full.**

19. **WRITING IN MATH** What should you consider when using exponential models to make decisions? **See Ch. 8 Answer Appendix.**

20. **WRITING IN MATH** Compare and contrast the exponential growth formula and the exponential decay formula. **See Ch. 8 Answer Appendix.**

Follow-up

Students have explored growth and decay.

Ask:

- How can being financially literate help you to make good decisions? **Sample answer: If you are financially literate, you understand the vocabulary of financial terms and know how to analyze data and trends. Successfully applying these skills when considering your available options can help you to make good decisions in many real-world situations such as opening a bank account, applying for college loans, and buying a car.**

4 Assess

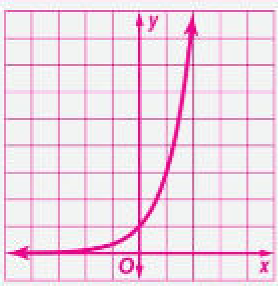
Ticket Out the Door Make several copies each of five equations for exponential growth or decay. Give one equation to each student. As students leave the room, ask them to tell you whether their equations are for growth or decay.

Formative Assessment

Check for student understanding of Lessons 8-5 and 8-6.

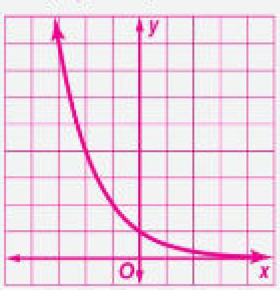
Additional Answers

25.



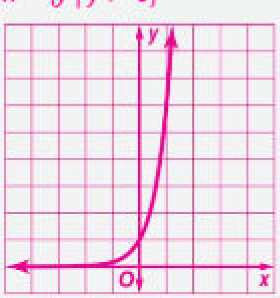
1; D = {all real numbers};
R = {y | y > 0}

26.



1; D = {all real numbers};
R = {y | y > 0}

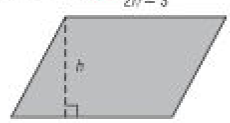
27.



1; D = {all real numbers};
R = {y | y > 0}

Standardized Test Practice

21. GEOMETRY The parallelogram has an area of 35 square centimeters. Find the height h of the parallelogram. **C**



A 3.5 centimeters
B 4 centimeters
C 5 centimeters
D 7 centimeters

22. Which is greater than $64^{\frac{1}{3}}$? **H**

F 2^2
G $64^{\frac{1}{6}}$
H $64^{\frac{1}{2}}$
J 64^{-3}

23. Eissa purchased a car for AED 22,900. The car depreciated at an annual rate of 16%. Which of the following equations models the value of Eissa's car after 5 years? **D**

A $A = 22{,}900(1.16)^5$
B $A = 22{,}900(0.16)^5$
C $A = 16(22{,}900)^5$
D $A = 22{,}900(0.84)^5$

24. GRIDDED RESPONSE A deck measures 4 meters by 6 meters. If a painter charges AED 9.75 per square meter including tax, how much will it cost in dirhams to have the deck painted? **234**

Spiral Review

Graph each function. Find the y-intercept and state the domain and range. (Lesson 8-5) **25–27. See margin.**

25. $y = 3^x$ **26.** $y = \left(\frac{1}{2}\right)^x$ **27.** $y = 6^x$

Evaluate each product. Express the results in both scientific notation and standard form. (Lesson 8-4)

28. $(4.2 \times 10^3)(3.1 \times 10^{10})$ **1.302×10^{14}; 130,200,000,000,000**

29. $(6.02 \times 10^{23})(5 \times 10^{-14})$ **3.01×10^{10}; 30,100,000,000**

30. $(7 \times 10^5)^2$ **4.9×10^{11}; 490,000,000,000**

31. $(1.1 \times 10^{-2})^2$ **1.21×10^{-4}; 0.000121**

32. $(9.1 \times 10^{-2})(4.2 \times 10^{-7})$ **3.822×10^{-8}; 0.00000003822**

33. $(3.14 \times 10^2)(6.1 \times 10^{-3})$ **1.9154×10^0; 1.9154**

34. EVENT PLANNING A hall does not charge a rental fee as long as at least AED 4,000 is spent on food. For the graduation, the hall charges AED 28.95 per person for a buffet. How many people must attend the graduation to avoid a rental fee for the hall? **at least 139 people**

Determine whether the graphs of each pair of equations are *parallel*, *perpendicular*, or *neither*.

35. $y = -2x + 11$, $y + 2x = 23$ **parallel**

36. $3y = 2x + 14$, $-3x - 2y = 2$ **perpendicular**

37. $y = -5x$, $y = 5x - 18$ **neither**

38. AGES The table shows equivalent ages for horses and humans. Write an equation that relates human age to horse age and find the equivalent horse age for a human who is 16 years old. **$y = 3x$; 5 yr 4 mo**

Horse age (x)	0	1	2	3	4	5
Human age (y)	0	3	6	9	12	15

Find the total price of each item.

39. umbrella: AED 14.00, tax: 5.5% **AED 14.77**

40. sandals: AED 29.99, tax: 5.75% **AED 31.71**

41. backpack: AED 35.00, tax: 7% **AED 37.45**

Skills Review

Graph each set of ordered pairs. **42–44. See Ch. 8 Answer Appendix.**

42. (3, 0), (0, 1), (−4, −6) **43.** (0, −2), (−1, −6), (3, 4) **44.** (2, 2), (−2, −3), (−3, −6)

Differentiated Instruction OL BL

If you think students need a challenge in this lesson,

Then ask students to write their own exponential growth or decay problems, using data from periodicals or the Internet. Have students share their problems with the class when they are complete.

EXTEND 8-6 Algebra Lab
Transforming Exponential Expressions

You can use the properties of rational exponents to transform exponential functions into other forms in order to solve real-world problems.

Activity Write Equivalent Exponential Expressions

Nabila is trying to decide between two savings account plans. Plan A offers a monthly compounding interest rate of 0.25%, while Plan B offers 2.5% interest compounded annually. Which is the better plan? Explain.

In order to compare the plans, we must compare rates with the same compounding frequency. One way to do this is to compare the approximate monthly interest rates of each plan, also called the *effective* monthly interest rate. While you can use the compound interest formula to find this rate, you can also use the properties of exponents.

Write a function to represent the amount A Nabila would earn after t years with Plan B. For convenience, let the initial amount of Nabila's investment be AED 1.

$y = a(1 + r)^t$	Equation for exponential growth
$A(t) = 1(1 + 0.025)^t$	$y = A(t)$, $a = 1$, $r = 2.5\%$ or 0.025
$= 1.025^t$	Simplify.

Now write a function equivalent to $A(t)$ that represents 12 compoundings per year, with a power of $12t$, instead of 1 per year, with a power of $1t$.

$A(t) = 1.025^{1t}$	Original function
$= 1.025^{\left(\frac{1}{12} \cdot 12\right)t}$	$1 = \frac{1}{12} \cdot 12$
$= \left(1.025^{\frac{1}{12}}\right)^{12t}$	Power of a Power
$\approx 1.0021^{12t}$	$(1.025)^{\frac{1}{12}} = \sqrt[12]{1.025}$ or about 1.0021

From this equivalent function, we can determine that the effective monthly interest by Plan B is about 0.0021 or about 0.21% per month. This rate is less than the monthly interest rate of 0.25% per month offered by Plan A, so Plan A is the better plan.

Model and Analyze

1. Use the compound interest formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to determine the effective monthly interest rate for Plan B. How does this rate compare to the rate calculated using the method in the Activity above? **About 0.21% per month; they are approximately the same.**
2. Write a function to represent the amount A Nabila would earn after t months by Plan A. Then use the properties of exponents to write a function equivalent to $A(t)$ that represents the amount earned after t years. **$A(t) = (1.0025)^t$; $A(t) \approx (1.030)^{\frac{1}{12}t}$**
3. From the expression you wrote in Exercise 2, identify the effective annual interest rate by Plan A. Use this rate to explain why Plan A is the better plan. **See margin.**
4. Suppose Plan A offered a quarterly compounded interest rate of 1.5%. Use the properties of exponents to explain which is the better plan. **See margin.**

1 Focus

Objective Use properties of rational exponents to transform expressions for exponential functions into equivalent forms to solve problems.

2 Teach

Working in Cooperative Groups

Organize students into groups of 2, mixing abilities. Then have groups complete the Activity and Exercises 1–3.

Teaching Tip

Point out to students that the annual interest formula is approximated as a monthly interest rate using $\frac{1}{12} \cdot 12$ because there are 12 months in a year.

Practice Have students complete Exercise 4.

3 Assess

Formative Assessment

Use Exercise 4 to assess whether students understand how to use the properties of exponents to write equivalent expressions in order to compare interest rates.

From Concrete to Abstract

Ask students to justify that $A = P\left(1 + \frac{r}{n}\right)^{nt}$ is approximately equivalent to $A = P\left[(1 + r)^{\frac{1}{n}}\right]^{nt}$ graphically by fixing the values for P, r and n. Depending on the values chosen, students should see that the graphs of the two functions nearly coincide for a large interval of their domain.

Additional Answers

3. About 3.0% per year; this rate is greater than the 2.5% per year offered by Plan B.

4. The function $A(t) = (1.025)^t$ gives the amount Plan B earned after t years.

$A(t) = (1.025)^{1t}$	Original function
$= (1.025)^{\left(\frac{1}{4} \cdot 4\right)t}$	$1 = \frac{1}{4} \cdot 4$
$= \left[(1.025)^{\frac{1}{4}}\right]^{4t}$	Power of a Power
$\approx (1.0062)^{4t}$	$(1.025)^{\frac{1}{4}} = \sqrt[4]{1.025}$ or about 1.0062

The effective quarterly interest rate is about 0.0062 or 0.62%, which is less than the quarterly interest rate of 1.5% offered by Plan A, so Plan A is the better plan.

1 Focus

Vertical Alignment

Before Lesson 8-7 Relate arithmetic sequences to linear functions.

Lesson 8-7 Identify and generate geometric sequences. Relate geometric sequences to exponential functions.

After Lesson 8-7 Write recursive formulas for arithmetic and geometric sequences.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- If one e-mail was sent in the first round, how many were sent in the second round? the third round? the fourth round? 5; 25; 125
- How do you determine the number of e-mails sent in each subsequent round of e-mails? Multiply the previous number by 5.
- What is the equation to find the number of e-mails y after x rounds? $y = 5^x$

LESSON 8-7 Geometric Sequences as Exponential Functions

Then

- You related arithmetic sequences to linear functions.

Now

- 1 Identify and generate geometric sequences.
- 2 Relate geometric sequences to exponential functions.

Why?

- You send a chain email to a friend who forwards the email to five more people. Each of these five people forwards the email to five more people. The number of new email generated forms a geometric sequence.

New Vocabulary
geometric sequence
common ratio

Mathematical Practices
Look for and make use of structure.

1 Recognize Geometric Sequences

The first person generates 5 emails. If each of these people sends the email to 5 more people, 25 emails are generated. If each of the 25 people sends 5 emails, 125 emails are generated. The sequence of emails generated, 1, 5, 25, 125, ... is an example of a **geometric sequence**.

In a geometric sequence, the first term is nonzero and each term after the first is found by multiplying the previous term by a nonzero constant r called the **common ratio**. The common ratio can be found by dividing any term by its previous term.

Example 1 Identify Geometric Sequences

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain.

a. 256, 128, 64, 32, ...

Find the ratios of consecutive terms.

256 128 64 32

$\frac{128}{256} = \frac{1}{2}$ $\frac{64}{128} = \frac{1}{2}$ $\frac{32}{64} = \frac{1}{2}$

Since the ratios are constant, the sequence is geometric. The common ratio is $\frac{1}{2}$.

b. 4, 9, 12, 18, ...

Find the ratios of consecutive terms.

4 9 12 18

$\frac{9}{4} = 2\frac{1}{4}$ $\frac{12}{9} = 1\frac{1}{3}$ $\frac{18}{12} = 1\frac{1}{2}$

The ratios are not constant, so the sequence is not geometric.

Find the differences of consecutive terms.

4 9 12 18

$9 - 4 = 5$ $12 - 9 = 3$ $18 - 12 = 6$

There is no common difference, so the sequence is not arithmetic.
Thus, the sequence is neither geometric nor arithmetic.

Guided Practice 1A–1C. See margin.

1A. 1, 3, 9, 27, ... **1B.** −20, −15, −10, −5, ... **1C.** 2, 8, 14, 22, ...

Once the common ratio is known, more terms of a sequence can be generated. The formula can be rewritten as $a_n = a_1 r^{n-1}$, where n is a counting number and r is the common ratio.

Example 2 Find Terms of Geometric Sequences

Find the next three terms in each geometric sequence.

a. $1, -4, 16, -64, \ldots$

Step 1 Find the common ratio.

$1 \quad -4 \quad 16 \quad -64$

$\frac{-4}{1} = -4 \qquad \frac{16}{-4} = -4 \qquad \frac{-64}{16} = -4$

Step 2 Multiply each term by the common ratio to find the next three terms.

$-64 \quad 256 \quad -1024 \quad 4096$

$\times(-4) \qquad \times(-4) \qquad \times(-4)$

The next three terms are 256, −1024, and 4096.

b. $9, 3, 1, \frac{1}{3} \ldots$

Step 1 Find the common ratio.

$9 \quad 3 \quad 1 \quad \frac{1}{3}$

$\frac{3}{9} = \frac{1}{3} \qquad \frac{1}{3} = \frac{1}{3} \qquad \frac{\frac{1}{3}}{1} = \frac{1}{3}$

The value of r is $\frac{1}{3}$.

Step 2 Multiply each term by the common ratio to find the next three terms.

$\frac{1}{3} \quad \frac{1}{9} \quad \frac{1}{27} \quad \frac{1}{81}$

$\times\frac{1}{3} \qquad \times\frac{1}{3} \qquad \times\frac{1}{3}$

The next three terms are $\frac{1}{9}$, $\frac{1}{27}$, and $\frac{1}{81}$.

Guided Practice 2A. −1875, 9375, −46,875 2B. 121.5, 182.25, 273.375

2A. $-3, 15, -75, 375, \ldots$ **2B.** $24, 36, 54, 81, \ldots$

Study Tip

Structure If the terms of a geometric sequence alternate between positive and negative terms or vice versa, the common ratio is negative.

Math History Link

Thomas Robert Malthus (1766–1834) Malthus studied populations and had pessimistic views about the future population of the world. In his work, he stated: "Population increases in a geometric ratio, while the means of subsistence increases in an arithmetic ratio."

2 Geometric Sequences and Functions

Finding the nth term of a geometric sequence would be tedious if we used the above method. The table below shows a rule for finding the nth term of a geometric sequence.

Position, n	1	2	3	4	...	n
Term, a_n	a_1	a_1r	a_1r^2	a_1r^3	...	a_1r^{n-1}

Notice that the common ratio between the terms is r. The table shows that to get the nth term, you multiply the first term by the common ratio r raised to the power $n - 1$. A geometric sequence can be defined by an exponential function in which n is the independent variable, a_n is the dependent variable, and r is the base. The domain is the counting numbers.

1 Recognize Geometric Sequences

Example 1 shows how to determine whether a sequence is arithmetic, geometric, or neither. **Example 2** shows how to find additional terms of a geometric sequence.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain.

a 0, 8, 16, 24, 32, ... Arithmetic; the common difference is 8.

b. 64, 48, 36, 27, ... Geometric; the common ratio is $\frac{3}{4}$.

2 Find the next three terms in each geometric sequence.

a. 1, −8, 64, −512, ... 4096; −32,768; 262,144

b. 40, 20, 10, 5, ... $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}$

2 Geometric Sequences and Functions

Example 3 shows how to find the nth term of a geometric sequence. **Example 4** shows how to use real-world data to draw a graph of a geometric sequence.

Teaching the Mathematical Practices

Structure Mathematically proficient students look closely to discern a pattern or structure. Ask students to explain why the common difference is negative when the signs of the terms alternate.

Teach with Tech

Blog On your secure classroom blog have students write a blog entry describing how geometric sequences and exponential functions are related.

Additional Answers (Guided Practice)

1A. Geometric; the common ratio is 3.

1B. Arithmetic; the common difference is 5.

1C. Neither; there is no common ratio or common difference.

Focus on Mathematical Content

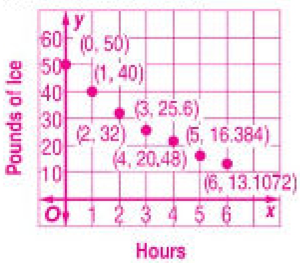
Common Ratio When finding the common ratio, it is important to set up the ratio in the correct order. For example, in Example 1, had the ratio been set up as $\frac{256}{128}$ instead of $\frac{128}{256}$, an incorrect ratio of 2 would have been found. It should be noted that when a geometric sequence is decreasing, the common ratio must be between 0 and 1.

Additional Examples

3 a. Write an equation for the *n*th term of the geometric sequence 1, −2, 4, −8, ... $a_n = 1 \cdot (-2)^{n-1}$

b. Find the 12th term of this sequence. −2048

4 ART A 50-kilogram ice sculpture is melting at a rate in which 80% of its weight remains each hour. Draw a graph to represent how many kilograms of the sculpture is left at each hour.



Tips for New Teachers

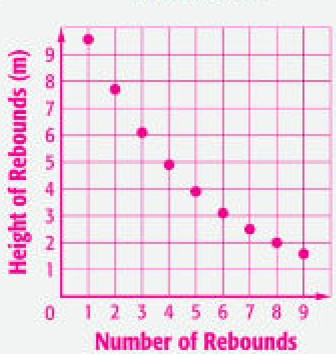
The Power of *r* Make sure students raise *r* to the power $(n - 1)$, in their equations for the *n*th term of a geometric sequence, instead of *n*.

Teaching the Mathematical Practices

Reasoning Mathematically proficient students attend to the meaning of quantities, not just how to compute them. In Exercise 32, ask students why the common ratio is not 0.2. How does the common ratio relate to the meaning of the sequence?

Additional Answer (Guided Practice)

4. **Tennis Ball**



KeyConcept *n*th term of a Geometric Sequence

The *n*th term a_n of a geometric sequence with first term a_1 and common ratio r is given by the following formula, where n is any positive integer and $a_1, r \neq 0$.

$$a_n = a_1 r^{n-1}$$

Example 3 Find the *n*th Term of a Geometric Sequence

a. Write an equation for the *n*th term of the sequence −6, 12, −24, 48, ...

The first term of the sequence is −6. So, $a_1 = -6$. Now find the common ratio.

−6	12	−24	48
	$\frac{12}{-6} = -2$	$\frac{-24}{12} = -2$	$\frac{48}{-24} = -2$

The common ratio is −2.

$a_n = a_1 r^{n-1}$ Formula for *n*th term

$a_n = -6(-2)^{n-1}$ $a_1 = -6$ and $r = 2$

b. Find the ninth term of this sequence.

$a_n = a_1 r^{n-1}$ Formula for *n*th term

$a_9 = -6(-2)^{9-1}$ For the *n*th term, $n = 9$.

$= -6(-2)^8$ Simplify.

$= -6(256)$ $(-2)^8 = 256$

$= -1536$

WatchOut!

Negative Common Ratio If the common ratio is negative, as in Example 3, make sure to enclose the common ratio in parentheses. $(-2)^8 \neq -2^8$

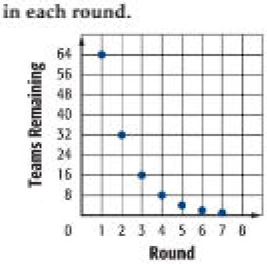
GuidedPractice 3. $a_n = 96 \cdot \left(\frac{1}{2}\right)^{n-1}$; $\frac{3}{16}$

3. Write an equation for the *n*th term of the geometric sequence 96, 48, 24, 12, ... Then find the tenth term of the sequence.

Real-World Example 4 Graph a Geometric Sequence

BASKETBALL The NCAA women's basketball tournament begins with 64 teams. In each round, one half of the teams are left to compete, until only one team remains. Draw a graph to represent how many teams are left in each round.

Compared to the previous rounds, one half of the teams remain. So, $r = \frac{1}{2}$. Therefore, the geometric sequence that models this situation is 64, 32, 16, 8, 4, 2, 1. So in round two, 32 teams compete, in round three 16 teams compete and so forth. Use this information to draw a graph.



Real-WorldLink

The first NCAA Division I women's basketball tournament was held in 1982. The University of Tennessee has won the most national titles with 8 titles as of 2010.

Source: NCAA Sports

GuidedPractice

4. **TENNIS** A tennis ball is dropped from a height of 12 meters. Each time the ball bounces back to 80% of the height from which it fell. Draw a graph to represent the height of the ball after each bounce. See margin.

DifferentiatedInstruction

Interpersonal Learners Put students into groups of mixed math abilities. Have groups discuss the differences between arithmetic and geometric sequences. Suggest they help each other organize clear, concise, and accurate notes about these and other concepts taught in this lesson.

Check Your Understanding

Example 1 Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain.

1. 200, 40, 8, ... 2. 2, 4, 16, ... 3. −6, −3, 0, 3, ... 4. 1, −1, 1, −1, ...

Example 2 Find the next three terms in each geometric sequence. 5–8. See margin.

1. Geometric; the common ratio is $\frac{1}{5}$.

5. 10, 20, 40, 80, ... 6. 100, 50, 25, ... 7. 4, −1, $\frac{1}{4}$, ... 8. −7, 21, −63, ...

Example 3 Write an equation for the *n*th term of each geometric sequence, and find the indicated term.

9. the fifth term of −6, −24, −96, ... $a_n = -6 \cdot (4)^{n-1}$; −1536
10. the seventh term of −1, 5, −25, ... $a_n = -1 \cdot (-5)^{n-1}$; −15,625
11. the tenth term of 72, 48, 32, ... $a_n = 72 \cdot \left(\frac{2}{3}\right)^{n-1}$; $\frac{4096}{2187}$
12. the ninth term of 112, 84, 63, ... $a_n = 112 \cdot \left(\frac{3}{4}\right)^{n-1}$; $\frac{45,927}{4096}$

2. Neither; there is no common ratio or difference.
3. Arithmetic; the common difference is 3.
4. Geometric; the common ratio is −1.

Example 4 13. **EXPERIMENT** In a physics class experiment, Lamis drops a ball from a height of 16 meters. Each bounce has 70% the height of the previous bounce. Draw a graph to represent the height of the ball after each bounce. See margin.

Practice and Problem Solving

Example 1 Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain. 14–19. See margin.

14. 4, 1, 2, ... 15. 10, 20, 30, 40, ... 16. 4, 20, 100, ...
17. 212, 106, 53, ... 18. −10, −8, −6, −4, ... 19. 5, −10, 20, 40, ...

Example 2 Find the next three terms in each geometric sequence. 20–25. See margin.

20. 2, −10, 50, ... 21. 36, 12, 4, ... 22. 4, 12, 36, ...
23. 400, 100, 25, ... 24. −6, −42, −294, ... 25. 1024, −128, 16, ...

Example 3

26. The first term of a geometric series is 1 and the common ratio is 9. What is the 8th term of the sequence? 4,782,969
27. The first term of a geometric series is 2 and the common ratio is 4. What is the 14th term of the sequence? 134,217,728
28. What is the 15th term of the geometric sequence −9, 27, −81, ...? −43,046,721
29. What is the 10th term of the geometric sequence 6, −24, 96, ...? −1,572,864

Example 4

30. **PENDULUM** The first swing of a pendulum is shown. On each swing after that, the arc length is 60% of the length of the previous swing. Draw a graph that represents the arc length after each swing. See Ch. 8 Answer Appendix.

31. Find the eighth term of a geometric sequence for which $a_3 = 81$ and $r = 3$. 19,683

B 32. **REASONING** At an online mapping site, Mr. Adnan notices that when he clicks a spot on the map, the map zooms in on that spot. The magnification increases by 20% each time. a. $a_n = 1.2^n$ 32b. 2.0736; The map will be magnified at approximately 207% of the original size after the fourth click.
   a. Write a formula for the *n*th term of the geometric sequence that represents the magnification of each zoom level. (*Hint:* The common ratio is not just 0.2.)
   b. What is the fourth term of this sequence? What does it represent?

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Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	14–31, 39–69	15–31 odd, 43–46	14–30 even, 39–42, 47–69
OL Core	15–31 odd, 32–37, 39–69	14–31, 43–46	32–37, 39–42, 47–69
BL Advanced	32–63, (optional: 64–69)		

3 Practice

Formative Assessment

Use Exercises 1–13 to check for understanding.

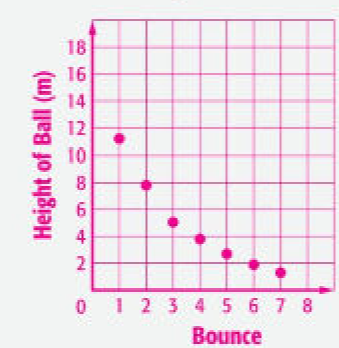
Use the chart at the bottom of this page to customize assignments for your students.

Exercise Alert

Grid Paper For Exercises 13, 30, 37, and 53–55, students will need grid paper.

Additional Answers

5. 160, 320, 640
6. 12.5, 6.25, 3.125
7. $-\frac{1}{16}, \frac{1}{64}, -\frac{1}{256}$
8. 189, −567, 1701
13.



14. Neither; there is no common ratio or difference.
15. Arithmetic; the common difference is 10.
16. Geometric; the common ratio is 5.
17. Geometric; the common ratio is $\frac{1}{2}$.
18. Arithmetic; the common difference is 2.
19. Neither; there is no common ratio or difference.
20. −250, 1250, −6250
21. $\frac{4}{3}, \frac{4}{9}, \frac{4}{27}$
22. 108, 324, 972
23. $\frac{25}{4}, \frac{25}{16}, \frac{25}{64}$
24. −2058; −14,406; −100,842
25. $-2, \frac{1}{4}, -\frac{1}{32}$

Teaching the Mathematical Practices

Critique Mathematically proficient students can read the arguments of others and decide whether they make sense. In Exercise 39, advise students to start by comparing the arguments line by line to find the differences.

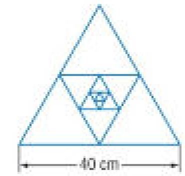
33. ALLOWANCE Laila's parents have offered her two different options to earn her allowance for a 9-week period over the summer. She can either get paid AED 30 each week or AED 1 the first week, AED 2 for the second week, AED 4 for the third week, and so on. **a. Yes; the common ratio is 2.**

a. Does the second option form a geometric sequence? Explain.

b. Which option should Laila choose? Explain.

33b. The second option; she would earn AED 511, which is much more than she would earn with the first option.

34. SIERPINSKI'S TRIANGLE Consider the inscribed equilateral triangles at the right. The perimeter of each triangle is one half of the perimeter of the next larger triangle. What is the perimeter of the smallest triangle? **7.5 cm**



35. If the second term of a geometric sequence is 3 and the third term is 1, find the first and fourth terms of the sequence. **$9; \frac{1}{3}$**

36. If the third term of a geometric sequence is −12 and the fourth term is 24, find the first and fifth terms of the sequence. **−3; −48**

37. EARTHQUAKES The Richter scale is used to measure the force of an earthquake. The table shows the increase in magnitude for the values on the Richter scale.

Richter Number (x)	Increase in Magnitude (y)	Rate of Change (slope)
1	1	–
2	10	9
3	100	**90**
4	1000	**900**
5	10,000	**9000**

a. Copy and complete the table. Remember that the rate of change is the change in y divided by the change in x. **b–c. See Ch. 8 Answer Appendix.**

b. Plot the ordered pairs (Richter number, increase in magnitude).

c. Describe the graph that you made of the Richter scale data. Is the rate of change between any two points the same?

d. Write an exponential equation that represents the Richter scale. **$y = 1 \cdot (10)^{x-1}$**

H.O.T. Problems Use Higher-Order Thinking Skills

38. CHALLENGE Write a sequence that is both geometric and arithmetic. Explain your answer.

38. 1, 1, 1, 1, ...; The common ratio is 1 making it a geometric sequence, but the common difference is 0 making it an arithmetic sequence as well.

39. See Ch. 8 Answer Appendix.

39. CRITIQUE Ibrahim and Ahmed are finding the ninth term of the geometric sequence −5, 10, −20, ... Is either of them correct? Explain your reasoning.

Ibrahim	Ahmed
$r = \frac{10}{-5}$ or -2	$r = \frac{10}{-5}$ or -2
$a_9 = -5(-2)^{9-1}$	$a_9 = -5 \cdot (-2)^{9-1}$
$= -5(512)$	$= -5 \cdot -256$
$= -2560$	$= 1280$

40. REASONING Write a sequence of numbers that form a pattern but are neither arithmetic nor geometric. Explain the pattern.

40. Sample answer: 1, 4, 9, 16, 25, 36, ...; This is the sequence of squares of counting numbers.

41. WRITING IN MATH How are graphs of geometric sequences and exponential functions similar? different? **See Ch. 8 Answer Appendix.**

42. WRITING IN MATH Summarize how to find a specific term of a geometric sequence. **See Ch. 8 Answer Appendix.**

530 | Lesson 8-7 | Geometric Sequences as Exponential Functions

Standardized Test Practice

43. Find the eleventh term of the sequence 3, −6, 12, −24, ... **B**

A 6144 C 33

B 3072 D −6144

44. What is the total amount of the investment shown in the table below if interest is compounded monthly? **H**

Principal	AED 500
Length of Investment	4 years
Annual Interest Rate	5.25%

F AED 613.56 H AED 616.56

G AED 616.00 J AED 718.75

45. SHORT RESPONSE Asma has AED 6.50 in 25 fils and 10 fils. If she has 35 coins in total, how many of each coin does she have? **fifteen 10 fils coins and twenty 25 fils coins**

46. What are the domain and range of the function $y = 4(3^x) - 2$? **A**

A D = {all real numbers}, R = $\{y \mid y > -2\}$

B D = {all real numbers}, R = $\{y \mid y > 0\}$

C D = {all integers}, R = $\{y \mid y > -2\}$

D D = {all integers}, R = $\{y \mid y > 0\}$

Spiral Review

Find the next three terms in each geometric sequence. (Lesson 8-6) **47–52. See margin.**

47. 2, 6, 18, 54, ... **48.** −5, −10, −20, −40, ... **49.** $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

50. −3, 1.5, −0.75, 0.375, ... **51.** 1, 0.6, 0.36, 0.216, ... **52.** 4, 6, 9, 13.5, ...

Graph each function. Find the *y*-intercept and state the domain and range. (Lesson 8-5) **53–55. See margin.**

53. $y = \left(\frac{1}{4}\right)^x - 5$ **54.** $y = 2(4)^x$ **55.** $y = \frac{1}{2}(3^x)$

56. LANDSCAPING A blue spruce grows an average of 15 centimeters per year. A hemlock grows an average of 10 centimeters per year. If a blue spruce is 120 centimeters tall and a hemlock is 180 centimeters tall, write a system of equations to represent their growth. Find and interpret the solution in the context of the situation. **See Ch. 8 Answer Appendix.**

57. MONEY City Bank requires a minimum balance of AED 1,500 to maintain free checking services. If Mr. Ismail is going to write checks for the amounts listed in the table, how much money should he start with in order to have free checking? **at least AED 3,747**

Check	Amount
750	AED 1,300
751	AED 947

Write an equation in slope-intercept form of the line with the given slope and *y*-intercept. **58–63. See Ch. 8 Answer Appendix.**

58. slope: 4, *y*-intercept: 2 **59.** slope: −3, *y*-intercept: $-\frac{2}{3}$

60. slope: $-\frac{1}{4}$, *y*-intercept: −5 **61.** slope: $\frac{1}{2}$, *y*-intercept: −9

62. slope: $-\frac{2}{5}$, *y*-intercept: $\frac{3}{4}$ **63.** slope: −6, *y*-intercept: −7

Skills Review

Simplify each expression. If not possible, write *simplified*.

64. $3u + 10u$ **$13u$** **65.** $5a - 2 + 6a$ **$11a - 2$** **66.** $6m^2 - 8m$ **simplified**

67. $4w^2 + w + 15w^2$ **$19w^2 + w$** **68.** $13(5 + 4a)$ **$65 + 52a$** **69.** $(4t - 6)16$ **$64t - 96$**

DifferentiatedInstruction OL BL

Extension Often sequences of numbers do not appear, on first calculations, to have a pattern. Sometimes the differences between terms themselves create a sequence that can be used to determine the next term in the original sequence. Ask students to determine the 6th term in the sequence 4, 7, 14, 25, 40, ... Have them also explain how they found the term.

4 7 14 25 40 59

+3 +7 +11 +15 +19

+4 +4 +4 +4

4 Assess

Name the Math Give each student one of five different geometric sequences. Ask students to explain how to find the common ratio for their sequence.

Additional Answers

47. 162, 486, 1458

48. −80, −160, −320

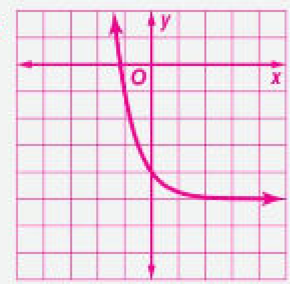
49. $\frac{1}{16}, -\frac{1}{32}, \frac{1}{64}$

50. −0.1875, 0.09375, −0.046875

51. 0.1296, 0.07776, 0.046656

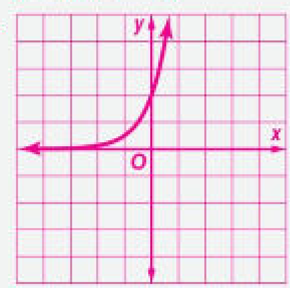
52. 20.25, 30.375, 45.5625

53.



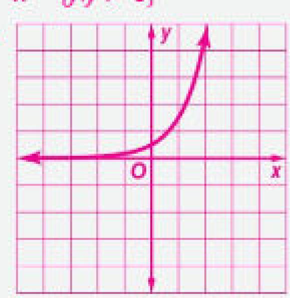
−4; D = {all real numbers}; R = $\{y \mid y > -5\}$

54.



2; D = {all real numbers}; R = $\{y \mid y > 0\}$

55.



$\frac{1}{2}$; D = {all real numbers}; R = $\{y \mid y > 0\}$

1 Focus

Objective Calculate and interpret the average rate of change of an exponential function.

Materials for Each Student

- grid paper

Teaching Tip

Have students use increments of 25 on the vertical axis. This will give a clearer illustration of the difference in the two graphs.

2 Teach

Working in Cooperative Groups

Put students in groups of two or three, mixing abilities. Have groups complete the activity.

- Discuss how the length of time of the investment affects the comparison of the plans.
- Have students describe the shape of the graph for each plan and discuss how the shape is related to the average rates of change.

Practice Have students complete Exercises 1–4.

3 Assess

Formative Assessment

Use Exercises 1–3 to assess whether students can calculate and interpret an average rate of change.

EXTEND 8-7

Algebra Lab: Average Rate of Change of Exponential Functions

You know that the rate of change of a linear function is the same for any two points on the graph. The rate of change of an exponential function is not constant.

Activity Evaluating Investment Plans

Ali has AED 2,000 to invest in one of two plans. Plan 1 offers to increase his principal by AED 75 each year, while Plan 2 offers to pay 3.6% interest compounded monthly. The dirham value of each investment after t years is given by $A_1 = 2000 + 75t$ and $A_2 = 2000(1.003)^{12t}$, respectively. Use the function values, the average rate of change, and the graphs of the equations to interpret and compare the plans.

Step 1 Copy and complete the table below by finding the missing values for A_1 and A_2.

t	0	1	2	3	4	5
A_1	2000	2075	2150	2225	2300	2375
A_2	2000	2073.2	2149.08	2227.74	2309.27	2393.79

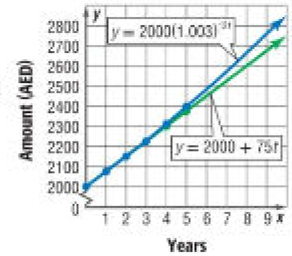
Step 2 Find the average rate of change for each plan from $t = 0$ to 1, $t = 3$ to 4, and $t = 0$ to 5.

Plan 1: $\frac{2075 - 2000}{1 - 0}$ or 75 $\frac{2300 - 2225}{4 - 3}$ or 75 $\frac{2375 - 2000}{5 - 0}$ or 75

Plan 2: $\frac{2073.2 - 2000}{1 - 0}$ or 73.2 $\frac{2309.27 - 2227.74}{4 - 3}$ or about 82 $\frac{2393.79 - 2000}{5 - 0}$ or about 79

Step 3 Graph the ordered pairs for each function. Connect each set of points with a smooth curve.

Step 4 Use the graph and the rates of change to compare the plans. Both graphs have a rate of change for the first year of about AED 75 per year. From year 3 to 4, Plan 1 continues to increase at AED 75 per year, but Plan 2 grows at a rate of more than AED 81 per year. The average rate of change over the first five years for Plan 1 is AED 75 per year and for Plan 2 is over AED 78 per year. This indicates that as the number of years increases, the investment in Plan 2 grows at an increasingly faster pace. This is supported by the widening gap between their graphs.



3. Sample answer: The value of the equipment decreases at a slower rate as the number of years increases.

Exercises

The value of a company's piece of equipment decreases over time due to depreciation. The function $y = 16{,}000(0.985)^{2t}$ represents the value after t years.

1. What is the average rate of change over the first five years? **−AED 448.86 per year**
2. What is the average rate of change of the value from year 5 to year 10? **−AED 386 per year**
3. What conclusion about the value can we make based on these average rates of change?
4. **REGULARITY** Copy and complete the table for $y = x^4$.

x	−3	−2	−1	0	1	2	3
y	81	16	1	0	1	16	81

Compare and interpret the average rate of change for $x = -3$ to 0 and for $x = 0$ to 3.

4. Sample answer: The average rate of change for $x = -3$ to 0 is −27 while the average rate of change for $x = 0$ to 3 is 27. This would indicate that the graph of the function was going down and then changed to going up.

From Concrete to Abstract

After students have completed Exercise 4, have them discuss what characteristics of a graph they can determine by examining the rate of change for a function.

Teaching the Mathematical Practices

Regularity Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. In Exercise 4, advise students to look for and use regularity in their calculations.

532 | Extend 8-7 | Algebra Lab: Average Rate of Change of Exponential Functions

LESSON 8-8 Recursive Formulas

Then

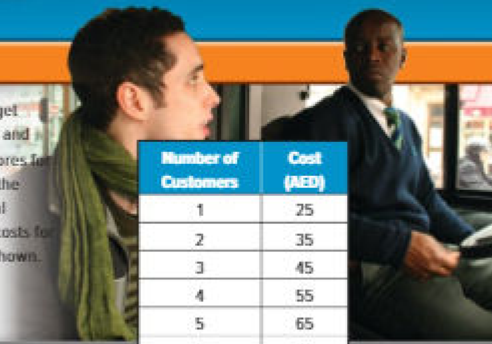
- You wrote explicit formulas to represent arithmetic and geometric sequences.

Now

1. Use a recursive formula to list terms in a sequence.
2. Write recursive formulas for arithmetic and geometric sequences.

Why?

- Clients of a shuttle service get picked up from their homes and driven to premium outlet stores for shopping. The total cost of the service depends on the total number of customers. The costs for the first six customers are shown.



Number of Customers	Cost (AED)
1	25
2	35
3	45
4	55
5	65
6	75

NewVocabulary
recursive formula

Mathematical Practices
Construct viable arguments and critique the reasoning of others.

1 Using Recursive Formulas

An explicit formula allows you to find any term a_n of a sequence by using a formula written in terms of n. For example, $a_n = 2n$ can be used to find the fifth term of the sequence 2, 4, 6, 8, ...: $a_5 = 2(5)$ or 10.

A **recursive formula** allows you to find the nth term of a sequence by performing operations to one or more of the preceding terms. Since each term in the sequence above is 2 greater than the term that preceded it, we can add 2 to the fourth term to find that the fifth term is 8 + 2 or 10. We can then write a recursive formula for a_n.

$$\begin{aligned} a_1 &= & &= 2 \\ a_2 &= a_1 + 2 \text{ or } 2 + 2 & &= 4 \\ a_3 &= a_2 + 2 \text{ or } 4 + 2 & &= 6 \\ a_4 &= a_3 + 2 \text{ or } 6 + 2 & &= 8 \\ &\vdots \quad \vdots \\ a_n &= a_{n-1} + 2 \end{aligned}$$

A recursive formula for the sequence above is $a_1 = 2$, $a_n = a_{n-1} + 2$, for $n \geq 2$ where n is an integer. The term denoted a_{n-1} represents the term immediately before a_n. Notice that the first term a_1 is given, along with the domain for n.

Example 1 Use a Recursive Formula

Find the first five terms of the sequence in which $a_1 = 7$ and $a_n = 3a_{n-1} - 12$, if $n \geq 2$.

Use $a_1 = 7$ and the recursive formula to find the next four terms.

$a_2 = 3a_{2-1} - 12$ — $n = 2$
$= 3a_1 - 12$ — Simplify.
$= 3(7) - 12$ or 9 — $a_1 = 7$

$a_4 = 3a_{4-1} - 12$ — $n = 4$
$= 3a_3 - 12$ — Simplify.
$= 3(15) - 12$ or 33 — $a_3 = 15$

$a_3 = 3a_{3-1} - 12$ — $n = 3$
$= 3a_2 - 12$ — Simplify.
$= 3(9) - 12$ or 15 — $a_2 = 9$

$a_5 = 3a_{5-1} - 12$ — $n = 5$
$= 3a_4 - 12$ — Simplify.
$= 3(33) - 12$ or 87 — $a_4 = 33$

The first five terms are 7, 9, 15, 33, and 87.

GuidedPractice

1. Find the first five terms of the sequence in which $a_1 = -2$ and $a_n = (-3)a_{n-1} + 4$, if $n \geq 2$. **−2, 10, −26, 82, −242**

1 Focus

VerticalAlignment

Before Lesson 8-8 Wrote explicit formulas to represent arithmetic and geometric sequences.

Lesson 8-8 Use a recursive formula to list terms in a sequence. Write recursive formulas for arithmetic and geometric sequences.

After Lesson 8-8 Identify linear, quadratic, and exponential functions from given data.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- How does the total cost of the shuttle service change as a customer is added? Sample answer: The total cost increases by AED10.
- Is this sequence *arithmetic*, *geometric*, or neither? arithmetic
- How much would it cost for 9 customers? AED105

1 Using Recursive Formulas

Example 1 shows how to find the first five terms of a sequence using a recursive formula.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Example

1 Find the first five terms of the sequence in which $a_1 = -8$ and $a_n = -2a_{n-1} + 5$, if $n \geq 2$. $-8, 21, -37, 79, -153$

2 Writing Recursive Formulas

Example 2 shows how to write a recursive formula for a sequence. **Example 3** shows how to write recursive and explicit formulas for a sequence. **Example 4** shows how to translate between recursive and explicit formulas.

Additional Example

2 Write a recursive formula for each sequence.

a. 23, 29, 35, 41, ... $a_1 = 23$, $a_n = a_{n-1} + 6, n \geq 2$

b. 7, −21, 63, −189, ... $a_1 = 7, a_n = -3a_{n-1}, n \geq 2$

2 Writing Recursive Formulas

To write a recursive formula for an arithmetic or geometric sequence, complete the following steps.

StudyTip

Defining n For the nth term of a sequence, the value of n must be a positive integer. Although we must still state the domain of n, from this point forward, we will assume that n is an integer.

KeyConcept Writing Recursive Formulas

Step 1 Determine if the sequence is arithmetic or geometric by finding a common difference or a common ratio.

Step 2 Write a recursive formula.

Arithmetic Sequences $a_n = a_{n-1} + d$, where d is the common difference

Geometric Sequences $a_n = r \cdot a_{n-1}$, where r is the common ratio

Step 3 State the first term and domain for n.

StudyTip

Domain The domain for n is decided by the given terms. Since the first term is already given, it makes sense that the first term to which the formula would apply is the 2nd term of the sequence, or when $n = 2$.

Example 2 Write Recursive Formulas

Write a recursive formula for each sequence.

a. 17, 13, 9, 5, ...

Step 1 First subtract each term from the term that follows it.

$13 - 17 = -4 \quad 9 - 13 = -4 \quad 5 - 9 = -4$

There is a common difference of −4. The sequence is arithmetic.

Step 2 Use the formula for an arithmetic sequence.

$a_n = a_{n-1} + d$ Recursive formula for arithmetic sequence

$a_n = a_{n-1} + (-4)$ $d = -4$

Step 3 The first term a_1 is 17, and $n \geq 2$.

A recursive formula for the sequence is $a_1 = 17, a_n = a_{n-1} - 4, n \geq 2$.

b. 6, 24, 96, 384, ...

Step 1 First subtract each term from the term that follows it.

$24 - 6 = 18 \quad 96 - 24 = 72 \quad 384 - 96 = 288$

There is no common difference. Check for a common ratio by dividing each term by the term that precedes it.

$\frac{24}{6} = 4 \quad \frac{96}{24} = 4 \quad \frac{384}{96} = 4$

There is a common ratio of 4. The sequence is geometric.

Step 2 Use the formula for a geometric sequence.

$a_n = r \cdot a_{n-1}$ Recursive formula for geometric sequence

$a_n = 4a_{n-1}$ $r = 4$

Step 3 The first term a_1 is 6, and $n \geq 2$.

A recursive formula for the sequence is $a_1 = 6, a_n = 4a_{n-1}, n \geq 2$.

GuidedPractice

2A. 4, 10, 25, 62.5, ... $a_1 = 4, a_n = 2.5a_{n-1}, n \geq 2.$

2B. 9, 36, 63, 90, ... $a_1 = 9, a_n = a_{n-1} + 27, n \geq 2.$

Tips for New Teachers

Notation Recursive formulas are occasionally defined for a_{n+1} and written in terms of a_n. For part **a** of Example 2, the recursive formula can be written as $a_1 = 17, a_{n+1} = a_n - 4, n \geq 1$.

Real-World Career

Transportation The number of jobs in the transportation industry is expected to grow by an estimated 1.1 million between 2004 and 2014. The specific fields dictate the educational requirements, which include a high school diploma and some form of specialized training.

Source: United States Department of Labor

A sequence can be represented by both an explicit formula and a recursive formula.

Example 3 Write Recursive and Explicit Formulas

COST Refer to the beginning of the lesson. Let n be the number of customers.

a. Write a recursive formula for the sequence.

Steps 1 and 2 First subtract each term from the term that follows it.

$35 - 25 = 10 \qquad 45 - 35 = 10 \qquad 55 - 45 = 10$

There is a common difference of 10. The sequence is arithmetic.

Step 3 Use the formula for an arithmetic sequence.

$a_n = a_{n-1} + d$ — Recursive formula for arithmetic sequence

$a_n = a_{n-1} + 10$ — $d = 10$

Step 4 The first term a_1 is 25, and $n \geq 2$.

A recursive formula for the sequence is $a_1 = 25$, $a_n = a_{n-1} + 10$, $n \geq 2$.

b. Write an explicit formula for the sequence.

Step 1 The common difference is 10.

Step 2 Use the formula for the nth term of an arithmetic sequence.

$a_n = a_1 + (n-1)d$ — Formula for the nth term

$= 25 + (n-1)10$ — $a_1 = 25$ and $d = 10$

$= 25 + 10n - 10$ — Distributive Property

$= 10n + 15$ — Simplify.

An explicit formula for the sequence is $a_n = 10n + 15$.

Guided Practice

3. **SAVINGS** The money that Badr has in his savings account earns interest each year. He does not make any withdrawals or additional deposits. The account balance at the beginning of each year is AED 10,000, AED 10,300, AED 10,609, AED 10,927.27, and so on. Write a recursive formula and an explicit formula for the sequence. $a_1 = 10{,}000$, $a_n = 1.03a_{n-1}$, $n \geq 2$; $a_n = 10{,}000(1.03)^{n-1}$

If several successive terms of a sequence are needed, a recursive formula may be useful, whereas if just the nth term of a sequence is needed, an explicit formula may be useful. Thus, it is sometimes beneficial to translate between the two forms.

Example 4 Translate between Recursive and Explicit Formulas

a. Write a recursive formula for $a_n = 6n + 3$.

$a_n = 6n + 3$ is an explicit formula for an arithmetic sequence with $d = 6$ and $a_1 = 6(1) + 3$ or 9. Therefore, a recursive formula for a_n is $a_1 = 9$, $a_n = a_{n-1} + 6$, $n \geq 2$.

b. Write an explicit formula for $a_1 = 120$, $a_n = 0.8a_{n-1}$, $n \geq 2$.

$a_n = 0.8a_{n-1}$ is a recursive formula for a geometric sequence with $a_1 = 120$ and $r = 0.8$. Therefore, an explicit formula for a_n is $a_n = 120(0.8)^{n-1}$.

Study Tip

Geometric Sequence Recall that the formula for the nth term of a geometric sequence is $a_n = a_1r^{n-1}$.

Guided Practice

4A. Write a recursive formula for $a_n = 4(3)^{n-1}$. $a_1 = 4$, $a_n = 3a_{n-1}$, $n \geq 2$.

4B. Write an explicit formula for $a_1 = -16$, $a_n = a_{n-1} - 7$, $n \geq 2$. $a_n = -7n - 9$.

Additional Examples

3 CARS The price of a car depreciates at the end of each year.

Year	Price (AED)
1	12,000
2	7200
3	4320
4	2592

a. Write a recursive formula for the sequence. $a_1 = 12{,}000$, $a_n = 0.6a_{n-1}$

b. Write an explicit formula for the sequence. $a_n = 12{,}000(0.6)^{n-1}$

4 a. Write a recursive formula for $a_n = 2n - 4$. $a_1 = -2$, $a_n = a_{n-1} + 2$, $n \geq 2$

b. Write an explicit formula for $a_1 = 84$, $a_n = 1.5a_{n-1}$, $n \geq 2$. $a_n = 84(1.5)^{n-1}$

Tips for New Teachers

Terms The first term of a sequence is occasionally denoted as a_0.

Differentiated Instruction AL OL

Interpersonal Learners Divide the class into groups of two or three students. Have each student write a sequence on one note card and the recursive formula for the sequence on another note card. Repeat the process for 10 sequences. Then, have the students lay the cards face down. Each student should take turns flipping over two cards, attempting to find a match between a sequence and its recursive formula.

3 Practice

Formative Assessment

Use Exercises 1–9 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Teaching the Mathematical Practices

Modeling Mathematically proficient students use tools such as diagrams to map the relationships of important quantities in a practical situation. In Exercise 22, tell students that they can sketch the patio to help them see the pattern.

Arguments Mathematically proficient students can recognize and use counterexamples. In Exercise 33, tell students that they can start with a sequence for which they know a recursive formula and see if they can write another recursive formula that fits the sequence to see if they can find a counterexample to the statement.

WatchOut!

Error Analysis In Exercise 31, students should recognize that the sequence is geometric with a common ratio of −1. Therefore, the sequence can be represented as both an explicit formula and a recursive formula.

Additional Answer

31. Both; sample answer: The sequence can be written as the recursive formula $a_1 = 2$, $a_n = (-1)a_{n-1}$, $n \geq 2$. The sequence can also be written as the explicit formula $a_n = 2(-1)^{n-1}$.

Check Your Understanding

Example 1 Find the first five terms of each sequence.

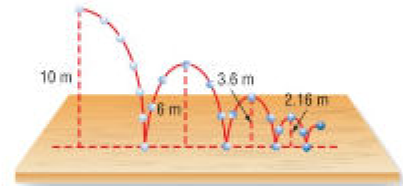
1. $a_1 = 16$, $a_n = a_{n-1} - 3$, $n \geq 2$
   16, 13, 10, 7, 4
2. $a_1 = -5$, $a_n = 4a_{n-1} + 10$, $n \geq 2$
   −5, −10, −30, −110, −430

Example 2 Write a recursive formula for each sequence.

3. 1, 6, 11, 16, ...
   $a_1 = 1$, $a_n = a_{n-1} + 5$, $n \geq 2$
4. 4, 12, 36, 108, ...
   $a_1 = 4$, $a_n = 3a_{n-1}$, $n \geq 2$

Example 3

5. **BALL** A ball is dropped from an initial height of 10 meters. The maximum heights the ball reaches on the first three bounces are shown.



   a. Write a recursive formula for the sequence.
   b. Write an explicit formula for the sequence.
   5a. $a_1 = 10$, $a_n = 0.6a_{n-1}$, $n \geq 2$
   5b. $a_n = 10(0.6)^{n-1}$

Example 4 For each recursive formula, write an explicit formula. For each explicit formula, write a recursive formula.

6. $a_1 = 4$, $a_n = a_{n-1} + 16$, $n \geq 2$
   $a_n = 16n - 12$
7. $a_n = 5n + 8$
   $a_1 = 13$, $a_n = a_{n-1} + 5$, $n \geq 2$
8. $a_n = 15(2)^{n-1}$
   $a_1 = 15$, $a_n = 2a_{n-1}$, $n \geq 2$
9. $a_1 = 22$, $a_n = 4a_{n-1}$, $n \geq 2$
   $a_n = 22(4)^{n-1}$

Practice and Problem Solving

Example 1 Find the first five terms of each sequence.

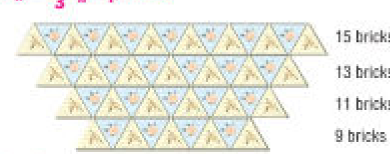
10. $a_1 = 23$, $a_n = a_{n-1} + 7$, $n \geq 2$
    23, 30, 37, 44, 51
11. $a_1 = 48$, $a_n = -0.5a_{n-1} + 8$, $n \geq 2$
    48, −16, 16, 0, 8
12. $a_1 = 8$, $a_n = 2.5a_{n-1}$, $n \geq 2$
    8, 20, 50, 125, 312.5
13. $a_1 = 12$, $a_n = 3a_{n-1} - 21$, $n \geq 2$
    12, 15, 24, 51, 132
14. $a_1 = 13$, $a_n = -2a_{n-1} - 3$, $n \geq 2$
    13, −29, 55, −113, 223
15. $a_1 = \frac{1}{2}$, $a_n = a_{n-1} + \frac{3}{2}$, $n \geq 2$
    $\frac{1}{2}, 2, \frac{7}{2}, 5, \frac{13}{2}$

Example 2 Write a recursive formula for each sequence.

16. 12, −1, −14, −27, ...
    $a_1 = 12$, $a_n = a_{n-1} - 13$, $n \geq 2$
17. 27, 41, 55, 69, ...
    $a_1 = 27$, $a_n = a_{n-1} + 14$, $n \geq 2$
18. 2, 11, 20, 29, ...
    $a_1 = 2$, $a_n = a_{n-1} + 9$, $n \geq 2$
19. 100, 80, 64, 51.2, ...
    $a_1 = 100$, $a_n = 0.8a_{n-1}$, $n \geq 2$
20. 40, −60, 90, −135, ...
    $a_1 = 40$, $a_n = -1.5a_{n-1}$, $n \geq 2$
21. 81, 27, 9, 3, ...
    $a_1 = 81$, $a_n = \frac{1}{3}a_{n-1}$, $n \geq 2$

Example 3

22. **MODELING** A landscaper is building a brick patio. Part of the patio includes a pattern constructed from triangles. The first four rows of the pattern are shown. 22a. $a_1 = 15$, $a_n = a_{n-1} - 2$, $n \geq 2$



    a. Write a recursive formula for the sequence.
    b. Write an explicit formula for the sequence. $a_n = 17 - 2n$

Example 4 For each recursive formula, write an explicit formula. For each explicit formula, write a recursive formula.

23. $a_n = 3(4)^{n-1}$
    $a_1 = 3$, $a_n = 4a_{n-1}$, $n \geq 2$
24. $a_1 = -2$, $a_n = a_{n-1} - 12$, $n \geq 2$ $a_n = -12n + 10$
25. $a_1 = 38$, $a_n = \frac{1}{2}a_{n-1}$, $n \geq 2$
    $a_n = 38\left(\frac{1}{2}\right)^{n-1}$
26. $a_n = -7n + 52$ $a_1 = 45$, $a_n = a_{n-1} - 7$, $n \geq 2$

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	10–26, 31, 33–59	11–25 odd, 36–39	10–26 even, 31, 33–35, 40–59
OL Core	11–25 odd, 27–31, 33–59	10–26, 36–39	27–31, 33–35, 40–59
BL Advanced	27–53, (optional: 54–59)		

536 | Lesson 8-8 | Recursive Formulas

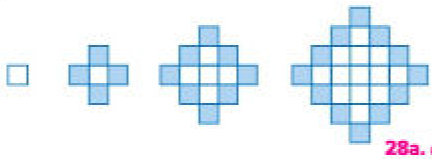
B **27.** **TEXTING** Amani received a chain text that she forwarded to five of her friends. Each of her friends forwarded the text to five more friends, and so on.

a. Find the first five terms of the sequence representing the number of people who receive the text in the nth round. **1, 5, 25, 125, 625**

b. Write a recursive formula for the sequence. $a_1 = 1, a_n = 5a_{n-1}, n \geq 2$

c. If Amani represents a_1, find a_8. **78,125**

28. **GEOMETRY** Consider the pattern below. The number of blue boxes increases according to a specific pattern.

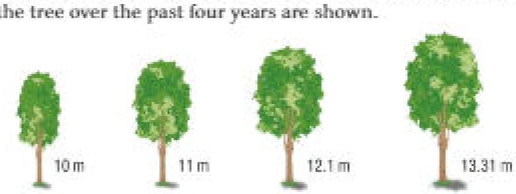


28a. $a_1 = 0, a_n = a_{n-1} + 4, n \geq 2$

a. Write a recursive formula for the sequence of the number of blue boxes in each figure.

b. If the first box represents a_1, find the number of blue boxes in a_8. **28**

29. **TREE** The growth of a certain type of tree slows as the tree continues to age. The heights of the tree over the past four years are shown.



a. Write a recursive formula for the height of the tree. $a_1 = 10, a_n = 1.1a_{n-1}, n \geq 2$

b. If the pattern continues, how tall will the tree be in two more years? Round your answer to the nearest tenth of a meter. **16.1 m**

C **30.** **MULTIPLE REPRESENTATIONS** The Fibonacci sequence is neither arithmetic nor geometric and can be defined by a recursive formula. The first terms are 1, 1, 2, 3, 5, 8, ...

a. **Logical** Determine the relationship between the terms of the sequence. What are the next five terms in the sequence? 30b. $a_1 = 1, a_2 = 1, a_n = a_{n-2} + a_{n-1}, n \geq 3$

b. **Algebraic** Write a formula for the nth term if $a_1 = 1$, $a_2 = 1$, and $n \geq 3$.

c. **Algebraic** Find the 15th term. **610**

d. **Analytical** Explain why the Fibonacci sequence is not an arithmetic sequence. **Sample answer: There is no common difference.**

30a. Sample answer: The first two terms are 1. Starting with the third term, the two previous terms are added together to get the next term; 13, 21, 34, 55, 89.

H.O.T. Problems Use Higher-Order Thinking Skills

31. **ERROR ANALYSIS** Bilal and Jassim are working on a math problem that involves the sequence 2, −2, 2, −2, 2, ... Bilal thinks that the sequence can be written as a recursive formula. Jassim believes that the sequence can be written as an explicit formula. Is either of them correct? Explain. **See margin.**

32. **CHALLENGE** Find a_1 for the sequence in which $a_4 = 1104$ and $a_n = 4a_{n-1} + 16$. **12**

33. **ARGUMENTS** Determine whether the following statement is *true* or *false*. Justify your reasoning.

There is only one recursive formula for every sequence.

34. **CHALLENGE** Find a recursive formula for 4, 9, 19, 39, 79, ... $a_1 = 4, a_n = 2a_{n-1} + 1, n \geq 2$

35. **WRITING IN MATH** Explain the difference between an explicit formula and a recursive formula.

33. False; sample answer: A recursive formula for the sequence 1, 2, 3, ... can be written as $a_1 = 1, a_n = a_{n-1} + 1, n \geq 2$ or as $a_1 = 1, a_2 = 2, a_n = a_{n-2} + 2, n \geq 3$.

35. Sample answer: In an explicit formula, the nth term a_n is given as a function of n. In a recursive formula, the nth term a_n is found by performing operations to one or more of the terms that precede it.

Multiple Representations

In Exercise 30, students use logic, analysis, and algebra to explore the Fibonacci sequence and find terms as needed.

4 Assess

Ticket Out the Door Have each student create a sequence by writing the first five terms. Then have them write an explicit formula and a recursive formula for the sequence.

Standardized Test Practice

36. Find a recursive formula for the sequence 12, 24, 36, 48, ... **C**

A $a_1 = 12, a_n = 2a_{n-1}, n \geq 2.$

B $a_1 = 12, a_n = 4a_{n-1} - 24, n \geq 2.$

C $a_1 = 12, a_n = a_{n-1} + 12, n \geq 2.$

D $a_1 = 12, a_n = 12a_{n-1} + 12, n \geq 2.$

37. GEOMETRY The area of a rectangle is $36m^4n^6$ square meters. The length of the rectangle is $6m^3n^3$ meters. What is the width of the rectangle? **G**

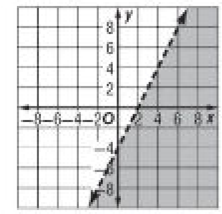
F $216m^7n^9$ m

G $6mn^3$ m

H $42m^7n^3$ m

J $30mn^3$ m

38. Find an inequality for the graph shown. **C**



A $y > 2x - 4$ C $y < 2x - 4$

B $y \geq 2x - 4$ D $y \leq 2x - 4$

39. Write an equation of the line that passes through $(-2, -20)$ and $(4, 58)$. **F**

F $y = 13x + 6$ H $y = 19x + 18$

G $y = 19x - 18$ J $y = 13x - 6$

Spiral Review

Find the next three terms in each geometric sequence. (Lesson 8-7)

40. 675, 225, 75, ... **40.** $25, \frac{25}{3}, \frac{25}{9}$

41. 16, −24, 36, ... **−54, 81, −121.5**

42. 6, 18, 54, ... **162, 486, 1458**

43. 512, −256, 128, ... **−64, 32, −16**

44. 125, 25, 5, ... **$1, \frac{1}{5}, \frac{1}{25}$**

45. 12, 60, 300, ... **1500; 7500; 37,500**

46. INVESTMENT Ahmed invested AED 2,000 with a 5.75% interest rate compounded monthly. How much money will Ahmed have after 5 years? (Lesson 8-6) **AED 2,664.35**

47. TOURS Rashid's family and Saleh's family are traveling together on a trip to visit a candy factory. The number of people in each family and the total cost are shown in the table below. Find the adult and children's admission prices. (Lesson 6-3) **adults: AED 14; children: AED 10**

Family	Number of Adults	Number of Children	Total Cost
Rashid	2	3	AED 58
Saleh	2	1	AED 38

Write each equation in standard form. (Lesson 4-3)

48. $y + 6 = -3(x + 2)$ **$3x + y = -12$**

49. $y - 12 = 4(x - 7)$ **$4x - y = 16$**

50. $y + 9 = 5(x - 3)$ **$5x - y = 24$**

51. $y - 1 = \frac{1}{3}(x + 15)$ **$x - 3y = -18$**

52. $y + 10 = \frac{2}{5}(x - 6)$ **$2x - 5y = 62$**

53. $y - 4 = -\frac{2}{7}(x + 1)$ **$2x + 7y = 26$**

Skills Review

Simplify each expression. If not possible, write *simplified*.

54. $8x + 3y^2 + 7x - 2y$ **$3y^2 + 15x - 2y$**

55. $4(x - 16) + 6x$ **$10x - 64$**

56. $4n - 3m + 9m - n$ **$3n + 6m$**

57. $6r^2 + 7r$ **simplified**

58. $-2(4g - 5h) - 6g$ **$-14g + 10h$**

59. $9x^2 - 7x + 16y^2$ **simplified**

Differentiated Instruction BL

Extension For Exercise 30, students wrote a recursive formula for the Fibonacci sequence, which is neither arithmetic nor geometric. Have students write a recursive formula for another sequence that is neither arithmetic nor geometric.

CHAPTER 8 Study Guide and Review

Study Guide

KeyConcepts

Multiplication and Division Properties of Exponents (Lessons 8-1 and 8-2)

For any nonzero real numbers a and b and any integers m, n, and p, the following are true.

- Product of Powers: $a^m \cdot a^n = a^{m+n}$
- Power of a Power: $(a^m)^n = a^{m \cdot n}$
- Power of a Product: $(ab)^m = a^m b^m$
- Quotient of Powers: $\frac{a^m}{a^p} = a^{m-p}$
- Power of a Quotient: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
- Zero Exponent: $a^0 = 1$
- Negative Exponent: $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$

Rational Exponents (Lesson 8-3)

For any positive real number b and any integers m and $n > 1$, the following are true.

$b^{\frac{1}{2}} = \sqrt{b}$ $\quad$ $b^{\frac{1}{n}} = \sqrt[n]{b}$ $\quad$ $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$ or $\sqrt[n]{b^m}$

Scientific Notation (Lesson 8-4)

- A number is in scientific notation if it is in the form $a \times 10^n$, where $1 \leq a < 10$.
- To write in standard form:
  - If $n > 0$, move the decimal n places right.
  - If $n < 0$, move the decimal n places left.

Exponential Functions (Lessons 8-5 and 8-6)

- The equation for exponential growth is $y = a(1 + r)^t$, where $r > 0$. The equation for exponential decay is $y = a(1 - r)^t$, where $0 < r < 1$. y is the final amount, a is the initial amount, r is the rate of change, and t is the time in years.

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.

KeyVocabulary

common ratio	monomial
compound interest	negative exponent
constant	nth root
cube root	order of magnitude
exponential decay	rational exponent
exponential equation	recursive formula
exponential function	scientific notation
exponential growth	zero exponent
geometric sequence	

VocabularyCheck

Choose the word or term that best completes each sentence.

1. $7xy^4$ is an example of a(n) ________. **monomial**
2. The ________ of 95,234 is 10^5. **order of magnitude**
3. 2 is a(n) ________ of 8. **cube root**
4. The rules for operations with exponents can be extended to apply to expressions with a(n) ________ such as $7^{\frac{2}{3}}$. **rational exponent**
5. A number written in ________ is of the form $a \times 10^n$, where $1 \leq a < 10$ and n is an integer. **scientific notation**
6. $f(x) = 3^x$ is an example of a(n) ________. **exponential function**
7. $a_1 = 4$ and $a_n = 3a_{n-1} - 12$, if $n \geq 2$, is a(n) ________ for the sequence 4, −8, −20, −32, ... **recursive formula**
8. $2^{3x-1} = 16$ is an example of a(n) ________. **exponential equation**
9. The equation for ________ is $y = C(1 - r)^t$. **exponential decay**
10. If $a^n = b$ for a positive integer n, then a is a(n) ________ of b. **nth root**

Formative Assessment

KeyVocabulary If students have difficulty answering questions 1–10, remind them that they can review the lessons to refresh their memories about the vocabulary terms.

FOLDABLES StudyOrganizer

Dinah Zike's Foldables®

Have students look through the chapter to make sure they have included examples in their Foldables for each lesson of the chapter. Suggest that students keep their Foldables handy while completing the Study Guide and Review pages. Point out that their Foldables can serve as a quick review when studying for the chapter test.

Lesson-by-Lesson Review

Intervention If the given examples are not sufficient to review the topics covered by the questions, remind students that the lesson references tell them where to review that topic in their textbook.

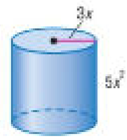
Study Guide and Review *Continued*

Lesson-by-Lesson Review

8-1 Multiplication Properties of Exponents

Simplify each expression.

11. $x \cdot x^3 \cdot x^5$ **x^9**
12. $(2xy)(-3x^2y^5)$ **$-6x^3y^6$**
13. $(-4ab^4)(-5a^5b^2)$ **$20a^6b^6$**
14. $(6x^3y^2)^2$ **$36x^6y^4$**
15. $[(2r^3t)^3]^2$ **$64r^{18}t^6$**
16. $(-2u^3)(5u)$ **$-10u^4$**
17. $(2x^2)^3(x^3)^3$ **$8x^{15}$**
18. $\frac{1}{2}(2x^3)^3$ **$4x^9$**
19. **GEOMETRY** Use the formula $V = \pi r^2 h$ to find the volume of the cylinder. **$45\pi x^4$**



Example 1

Simplify $(5x^2y^3)(2x^4y)$.

$(5x^2y^3)(2x^4y)$

$= (5 \cdot 2)(x^2 \cdot x^4)(y^3 \cdot y)$ Commutative Property

$= 10x^6y^4$ Product of Powers

Example 2

Simplify $(3a^2b^4)^3$.

$(3a^2b^4)^3 = 3^3(a^2)^3(b^4)^3$ Power of a Product

$= 27a^6b^{12}$ Simplify.

8-2 Division Properties of Exponents

Simplify each expression. Assume that no denominator equals zero.

20. $\frac{(3x)^0}{2a}$ **$\frac{1}{2a}$**
21. $\left(\frac{3xy^3}{2z}\right)^3$ **$\frac{27x^3y^9}{8z^3}$**
22. $\frac{12y^{-4}}{3y^{-5}}$ **$4y$**
23. $a^{-3}b^0c^6$ **$\frac{c^6}{a^3}$**
24. $\frac{-15x^7y^8z^4}{-45x^3y^5z^3}$ **$\frac{x^4y^3z}{3}$**
25. $\frac{(3x^{-1})^{-2}}{(3x^2)^{-2}}$ **x^6**
26. $\left(\frac{6xy^{11}z^9}{48x^6yz^{-7}}\right)^0$ **1**
27. $\left(\frac{12}{2}\right)\left(\frac{x}{y^5}\right)\left(\frac{y^4}{x^4}\right)$ **$\frac{6}{yx^3}$**
28. **GEOMETRY** The area of a rectangle is $25x^2y^4$ square meters. The width of the rectangle is $5xy$ meters. What is the length of the rectangle? **$5xy^3$ m**

Example 3

Simplify $\frac{2k^4m^3}{4k^2m}$. Assume that no denominator equals zero.

$\frac{2k^4m^3}{4k^2m} = \left(\frac{2}{4}\right)\left(\frac{k^4}{k^2}\right)\left(\frac{m^3}{m}\right)$ Group powers with the same base.

$= \left(\frac{1}{2}\right)k^{4-2}m^{3-1}$ Quotient of Powers

$= \frac{k^2m^2}{2}$ Simplify.

Example 4

Simplify $\frac{t^4uv^{-2}}{t^{-3}u^7}$. Assume that no denominator equals zero.

$\frac{t^4uv^{-2}}{t^{-3}u^7} = \left(\frac{t^4}{t^{-3}}\right)\left(\frac{u}{u^7}\right)(v^{-2})$ Group the powers with the same base.

$= (t^{4+3})(u^{1-7})(v^{-2})$ Quotient of Powers

$= t^7u^{-6}v^{-2}$ Simplify.

$= \frac{t^7}{u^6v^2}$ Simplify.

8-3 Rational Exponents

Simplify.

29. $\sqrt[3]{343}$ **7** **30.** $\sqrt[6]{729}$ **3**

31. $625^{\frac{1}{4}}$ **5** **32.** $\left(\frac{8}{27}\right)^{\frac{1}{3}}$ $\frac{2}{3}$

33. $256^{\frac{3}{4}}$ **64** **34.** $32^{\frac{2}{5}}$ **4**

35. $343^{\frac{4}{3}}$ **2401** **36.** $\left(\frac{4}{49}\right)^{\frac{3}{2}}$ $\frac{8}{343}$

Solve each equation.

37. $6^x = 7776$ **5** **38.** $4^{4x-1} = 32$ $\frac{7}{8}$

Example 5

Simplify $125^{\frac{2}{3}}$.

$125^{\frac{2}{3}} = (\sqrt[3]{125})^2$ $b^{\frac{m}{n}} = (\sqrt[n]{b})^m$

$= (\sqrt[3]{5 \cdot 5 \cdot 5})^2$ $64 = 4^3$

$= 5^2$ or 25 Simplify.

Example 6

Solve $9^{x-1} = 729$.

$9^{x-1} = 729$ Original equation

$9^{x-1} = 9^3$ Rewrite 729 as 9^3.

$x - 1 = 3$ Power Property of Equality

$x = 4$ Add 1 to each side.

8-4 Scientific Notation

Express each number in scientific notation.

39. 2,300,000 2.3×10^6 **40.** 0.0000543 5.43×10^{-5}

41. **ASTRONOMY** Earth has a diameter of about 12,700 kilometers. Jupiter has a diameter of about 140,000 kilometers. Write in scientific notation the ratio of Earth's diameter to Jupiter's diameter. **about 9.1×10^{-2}**

Example 7

Express 300,000,000 in scientific notation.

Step 1 300,000,000 ⟶ 3.00000000

Step 2 The decimal point moved 8 places to the left, so $n = 8$.

Step 3 $300{,}000{,}000 = 3 \times 10^8$

8-5 Exponential Functions

Graph each function. Find the y-intercept, and state the domain and range. **42–45. See margin.**

42. $y = 2^x$

43. $y = 3^x + 1$

44. $y = 4^x + 2$

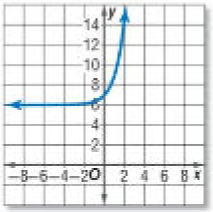
45. $y = 2^x - 3$

46. **BIOLOGY** The population of bacteria in a petri dish increases according to the model $p = 550(2.7)^{0.008t}$, where t is the number of hours and $t = 0$ corresponds to 1:00 P.M. Use this model to estimate the number of bacteria in the dish at 5:00 P.M. **about 568**

Example 8

Graph $y = 3^x + 6$. Find the y-intercept, and state the domain and range.

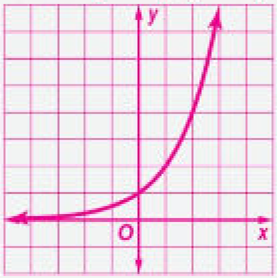
x	$3^x + 6$	y
−3	$3^{-3} + 6$	6.04
−2	$3^{-2} + 6$	6.11
−1	$3^{-1} + 6$	6.33
0	$3^0 + 6$	7
1	$3^1 + 6$	9



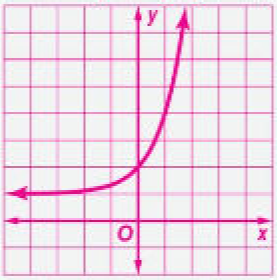
The y-intercept is (0, 7). The domain is all real numbers, and the range is all real numbers greater than 6.

Additional Answers

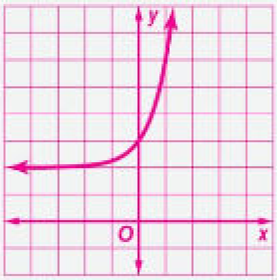
42. y-intercept 1; D = {all real numbers}; R = $\{y \mid y > 0\}$



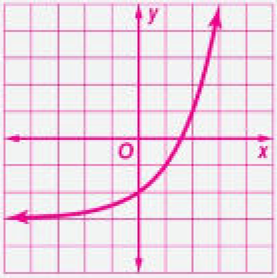
43. y-intercept 2; D = {all real numbers}; R = $\{y \mid y > 1\}$



44. y-intercept 3; D = {all real numbers}; R = $\{y \mid y > 2\}$



45. y-intercept −2; D = {all real numbers}; R = $\{y \mid y > -3\}$

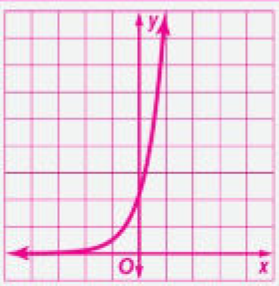


Additional Answer

55. **Basketball Rebound**

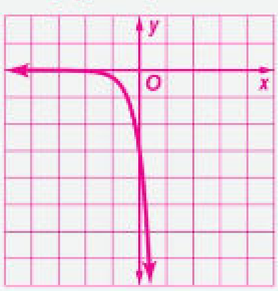
Additional Answers (Practice Test)

25.



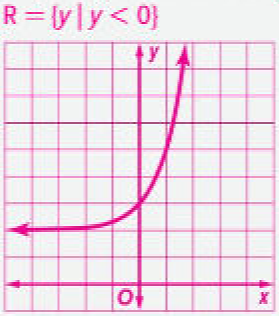
2; D = {all real numbers}, R = $\{y \mid y > 0\}$

26.



−3; D = {all real numbers}, R = $\{y \mid y < 0\}$

27.



3; D = {all real numbers}, R = $\{y \mid y > 2\}$

CHAPTER 8 Study Guide and Review *Continued*

8-6 Growth and Decay

47. Find the final value of AED 2,500 invested at an interest rate of 2% compounded monthly for 10 years. **AED 3,053.00**

48. COMPUTERS Alia's computer is depreciating at a rate of 3% per year. She bought the computer for AED 1,200.

a. Write an equation to represent this situation.

b. What will the computer's value be after 5 years? **AED 1,030.48**

48a. $y = 1200(1 - 0.03)^t$

Example 9

Find the final value of AED 2,000 invested at an interest rate of 3% compounded quarterly for 8 years.

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ — Compound interest equation

$= 2000\left(1 + \frac{0.03}{4}\right)^{4(8)}$ — $P = 2000$, $r = 0.03$, $n = 4$, and $t = 8$

$\approx$ AED 2,540.22 — Use a calculator.

8-7 Geometric Sequences as Exponential Functions

Find the next three terms in each geometric sequence.

49. −1, 1, −1, 1, ... **−1, 1, −1**

50. 3, 9, 27, ... **81, 243, 729**

51. 256, 128, 64, ... **32, 16, 8**

Write the equation for the *n*th term of each geometric sequence.

52. −1, 1, −1, 1, ... $a_n = -1(-1)^{n-1}$

53. 3, 9, 27, ... $a_n = 3(3)^{n-1}$

54. 256, 128, 64, ... $a_n = 256\left(\frac{1}{2}\right)^{n-1}$

55. SPORTS A basketball is dropped from a height of 20 meters. It bounces to $\frac{1}{2}$ its height after each bounce. Draw a graph to represent the situation. **See margin.**

Example 10

Find the next three terms in the geometric sequence 2, 6, 18, ...

Step 1 Find the common ratio. Each number is 3 times the previous number, so $r = 3$.

Step 2 Multiply each term by the common ratio to find the next three terms.

$18 \times 3 = 54$, $54 \times 3 = 162$, $162 \times 3 = 486$

The next three terms are 54, 162, and 486.

Example 11

Write the equation for the *n*th term of the geometric sequence −3, 12, −48, ...

The common ratio is −4. So $r = -4$.

$a_n = a_1 r^{n-1}$ — Formula for the *n*th term

$a_n = -3(-4)^{n-1}$ — $a_1 = -3$ and $r = -4$

8-8 Recursive Formulas

Find the first five terms of each sequence.

56. $a_1 = 11$, $a_n = a_{n-1} - 4$, $n \geq 2$ **11, 7, 3, −1, −5**

57. $a_1 = 3$, $a_n = 2a_{n-1} + 6$, $n \geq 2$ **3, 12, 30, 66, 138**

Write a recursive formula for each sequence.

58. 2, 7, 12, 17, ... $a_1 = 2, a_n = a_{n-1} + 5, n \geq 2$

59. 32, 16, 8, 4, ... $a_1 = 32, a_n = 0.5a_{n-1}, n \geq 2$

60. 2, 5, 11, 23, ... $a_1 = 2, a_n = 2a_{n-1} + 1, n \geq 2$

Example 12

Write a recursive formula for 3, 1, −1, −3, ...

Step 1 First subtract each term from the term that follows it.

$1 - 3 = -2$, $-1 - 1 = -2$, $-3 - (-1) = -2$

There is a common difference of −2. The sequence is arithmetic.

Step 2 Use the formula for an arithmetic sequence.

$a_n = a_{n-1} + d$ — Recursive formula

$a_n = a_{n-1} + (-2)$ — $d = -2$

Step 3 The first term a_1 is 3, and $n \geq 2$.

A recursive formula is $a_1 = 3$, $a_n = a_{n-1} - 2$, $n \geq 2$.

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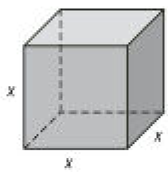
CHAPTER 8 Practice Test

Simplify each expression.

1. $(x^2)(7x^8)$ $7x^{10}$

2. $(5a^7bc^2)(-6a^2bc^5)$ $-30a^9b^2c^7$

3. **MULTIPLE CHOICE** Express the volume of the solid as a monomial. **A**



A x^3 C $6x^3$

B $6x$ D x^6

Simplify each expression. Assume that no denominator equals 0.

4. $\frac{x^6y^8}{x^2}$ x^4y^8

5. $\left(\frac{2a^4b^3}{c^6}\right)^0$ 1

6. $\frac{2xy^{-7}}{8x}$ $\frac{1}{4y^7}$

Simplify.

7. $\sqrt[3]{1000}$ 10

8. $\sqrt[5]{3125}$ 5

9. $1728^{\frac{1}{3}}$ 12

10. $\left(\frac{16}{81}\right)^{\frac{1}{2}}$ $\frac{4}{9}$

11. $27^{\frac{2}{3}}$ 9

12. $10{,}000^{\frac{3}{4}}$ 1000

13. $27^{\frac{5}{3}}$ 243

14. $\left(\frac{1}{121}\right)^{\frac{3}{2}}$ $\frac{1}{1331}$

Solve each equation.

15. $12^x = 1728$ 3

16. $7^{x-1} = 2401$ 5

17. $9^{x-3} = 729$ 6

Express each number in scientific notation.

18. 0.00021 2.1×10^{-4}

19. 58,000 5.8×10^4

Express each number in standard form.

20. 2.9×10^{-5} 0.000029

21. 9.1×10^6 9,100,000

Evaluate each product or quotient. Express the results in scientific notation.

22. $(2.5 \times 10^3)(3 \times 10^4)$ 7.5×10^7

23. $\frac{8.8 \times 10^2}{4 \times 10^{-4}}$ 2.2×10^6

24. **ASTRONOMY** The average distance from Mercury to the Sun is 57,910,000 km. Express this distance in scientific notation. 5.791×10^7

Graph each function. Find the *y*-intercept, and state the domain and range.

25. $y = 2(5)^x$ 25–27. See margin.

26. $y = -3(11)^x$

27. $y = 3^x + 2$

Find the next three terms in each geometric sequence.

28. 2, −6, 18, ... −54, 162, −486

29. 1000, 500, 250, ... 125, 62.5, 31.25

30. 32, 8, 2, ... $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}$

31. **MULTIPLE CHOICE** Shaima invested AED 500 into an account with a 6.5% interest rate compounded monthly. How much will Shaima's investment be worth in 10 years? **H**

F AED 600.00

G AED 938.57

H AED 956.09

J AED 957.02

32. **INVESTMENTS** Suha's investment of AED 3,000 has been losing value at a rate of 3% each year. What will her investment be worth in 6 years? AED 2,498.92

Find the first five terms of each sequence.

33. $a_1 = 18$, $a_n = a_{n-1} - 4$, $n \geq 2$ 18, 14, 10, 6, 2

34. $a_1 = -2$, $a_n = 4a_{n-1} + 5$, $n \geq 2$ −2, −3, −7, −23, −87

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1 Focus

Objective Use the strategy of using a scientific calculator to solve standardized test problems.

2 Teach

Scaffolding Questions

Ask:

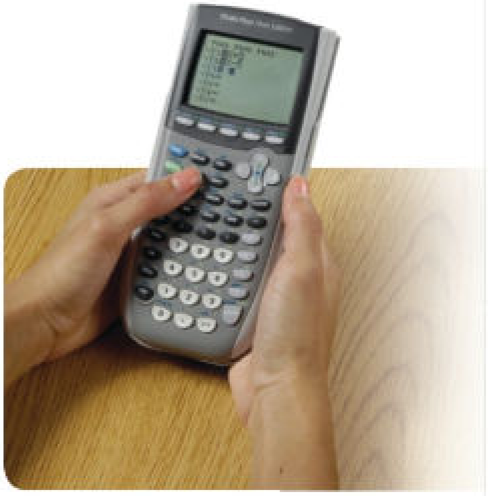
- For which types of math problems have you used a scientific calculator? Answers will vary.
- Are there any types of math problems in which you find that it is faster or easier to *not* use a scientific calculator? Answers will vary.
- Are you more likely or less likely to make calculation errors when using a scientific calculator? Explain. Sample answer: You are less likely to make a calculation error, but you could still make errors if you enter the numbers incorrectly, or use formulas incorrectly, or round incorrectly.

CHAPTER 8

Preparing for Standardized Tests

Using a Scientific or Graphing Calculator

Scientific and graphing calculators are powerful problem-solving tools. There are times when a calculator can be used to make computations faster and easier, such as computations with very large numbers. However, there are times when using a calculator is necessary, like the estimation of irrational numbers.



Strategies for Using a Scientific or Graphing Calculator

Step 1

Familiarize yourself with the various functions of a scientific or graphing calculator as well as when they should be used:

- **Exponents** scientific notation, calculating with large or small numbers
- **Pi** solving circle problems, like circumference and area
- **Square roots** distance on a coordinate plane, Pythagorean theorem
- **Graphs** analyzing paired data in a scatter plot, graphing functions, finding roots of equations

Step 2

Use your scientific or graphing calculator to solve the problem.

- Remember to work as efficiently as possible. Some steps may be done mentally or by hand, while others should be completed using your calculator.
- If time permits, check your answer.

Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

The distance from the Sun to Jupiter is approximately 7.786×10^{11} meters. If the speed of light is about 3×10^{8} meters per second, how long does it take for light from the Sun to reach Jupiter? Round to the nearest minute.

A about 43 minutes
B about 51 minutes
C about 1876 minutes
D about 2595 minutes

Read the problem carefully. You are given the approximate distance from the Sun to Jupiter as well as the speed of light. Both quantities are given in scientific notation. You are asked to find how many minutes it takes for light from the Sun to reach Jupiter. Use the relationship distance = rate × time to find the amount of time.

$d = r \times t$

$\frac{d}{r} = t$

To find the amount of time, divide the distance by the rate. Notice, however, that the units for time will be seconds.

$\frac{7.786 \times 10^{11} \text{ m}}{3 \times 10^{8} \text{ m/s}} = t$ seconds

Use a scientific calculator to quickly find the quotient. On most scientific calculators, the EE key is used to enter numbers in scientific notation.

KEYSTROKES: (7.786 2nd [EE] 11) ÷ (3 2nd [EE] 8) ENTER

The result is 2595.33333333 seconds. To convert this number to minutes, use your calculator to divide the result by 60. This gives an answer of about 43.2555 minutes. The answer is A.

Exercises

Read each problem. Identify what you need to know. Then use the information in the problem to solve.

1. Since its creation 5 years ago, approximately 2.504×10^7 items have been sold or traded on a popular online website. What is the average daily number of items sold or traded over the 5-year period? **B**

   A about 9640 items per day

   B about 13,720 items per day

   C about 1,025,000 items per day

   D about 5,008,000 items per day

2. Evaluate $\sqrt{ab}$ if $a = 121$ and $b = 23$. **J**

   F about 5.26

   G about 9.90

   H about 12

   J about 52.75

3. The population of the United States is about 3.034×10^8 people. The land area of the country is about 3.54 to 9.17 square kilometer. What is the average *population density* (number of people per square kilometer) of the United States? **D**

   A about 136.3 people per square kilometer

   B about 30.2 people per square kilometer

   C about 94.3 people per square kilometer

   D about 33.1 people per square kilometer

4. Ghaya is making a cover for the marching band's drum. The drum has a diameter of 20 centimeters. Estimate the area of the face of the bass drum. **J**

   F 31.41 square centimeters

   G 62.83 square centimeters

   H 78.54 square centimeters

   J 314.16 square centimeters

Additional Example

Light travels at about 9.46×10^{12} kilometers per year. The star Altair is about 1.5136×10^{14} kilometers from Earth. About many months would it take light to travel from Earth to Altair? **B**

A about 1920 months

B about 192 months

C about 160 months

D about 16 months

3 Assess

Use Exercise 1–4 to assess students' understanding.

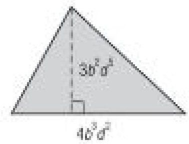
CHAPTER 8 Standardized Test Practice

Cumulative, Chapters 1 through 8

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Express the area of the triangle below as a monomial. **D**



A $12b^5d^7$

B $12b^6d^{10}$

C $6b^6d^{10}$

D $6b^5d^7$

2. Simplify the following expression. **G**

$$\left(\frac{2w^2z^5}{3y^4}\right)^3$$

F $\frac{2w^5z^8}{3y^7}$

G $\frac{8w^6z^{15}}{27y^{12}}$

H $\frac{8w^5z^8}{27y^7}$

J $\frac{2w^6z^{15}}{3y^{12}}$

3. Which equation of a line is perpendicular to $y = \frac{3}{5}x - 3$? **A**

A $y = -\frac{5}{3}x + 2$

B $y = -\frac{3}{5}x + 2$

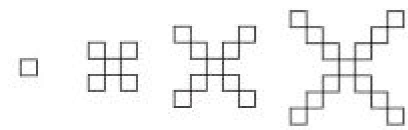
C $y = \frac{5}{3}x - 2$

D $y = \frac{3}{5}x - 2$

Test-Taking Tip

Question 2 Use the laws of exponents to simplify the expression. Remember, to find the power of a power, multiply the exponents.

4. Write a recursive formula for the sequence of the number of squares in each figure. **H**



F $a_1 = 1, a_n = 4a_{n-1} - 3, n \geq 1$

G $a_1 = 1, a_n = 4a_{n-1}, n \geq 2$

H $a_1 = 1, a_n = a_{n-1} + 4, n \geq 2$

J $a_1 = 1, a_n = 4a_{n-1} + 4, n \geq 2$

5. Evaluate $(4.2 \times 10^6)(5.7 \times 10^8)$. **A**

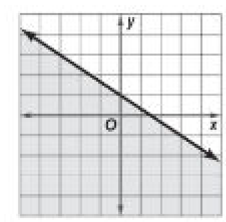
A 2.394×10^{15}

B 23.94×10^{14}

C 9.9×10^{14}

D 2.394×10^{48}

6. Which inequality is shown in the graph? **H**



F $y \leq -\frac{2}{3}x - 1$

G $y \leq -\frac{3}{4}x - 1$

H $y \leq -\frac{2}{3}x + 1$

J $y \leq -\frac{3}{4}x + 1$

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Short Response/Gridded Response

7. Saeed created a Web site for the Science Olympiad team. The total number of hits the site has received is shown.

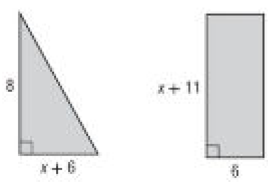
Day	Total Hits	Day	Total Hits
3	5	17	27
6	7	21	33
10	12	26	40
13	17	34	55

$y = 1.67x - 2.64$

a. Find an equation for the regression line.

b. Predict the number of total hits that the Web site will have received on day 46. about 74

8. Find the value of x so that the figures have the same area. 4



9. What is the solution to the following system of equations? Show your work. no solution

$$\begin{cases} y = 6x - 1 \\ y = 6x + 4 \end{cases}$$

10. **GRIDDED RESPONSE** At a family fun center, Amer's and Abdalla's families each bought video game tokens and batting cage tokens as shown in the table.

Family	Amer	Abdalla
Number of Video Game Tokens	25	30
Number of Batting Cage Tokens	8	6
Total Cost	AED 26.50	AED 25.50

What is the cost in dirhams of a batting cage token at the family fun center? 1.75

Extended Response

Record your answers on a sheet of paper. Show your work.

11. The table below shows the distances from the Sun to Mercury, Earth, Mars, and Saturn. Use the data to answer each question.

Planet	Distance from Sun (km)
Mercury	5.79×10^7
Earth	1.50×10^8
Mars	2.28×10^8
Saturn	1.43×10^9

a. Of the planets listed, which one is the closest to the Sun? Mercury

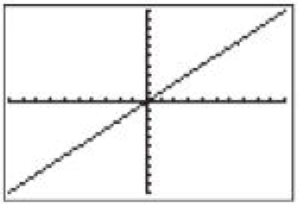
b. About how many times as far from the Sun is Mars as Earth? about 1.52 times farther

Homework Option

Get Ready for Chapter 9 Assign students the exercises on p. 549 as homework to assess whether they possess the prerequisite skills needed for the next chapter.

Lesson 8-1

65b.



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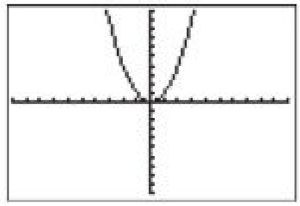

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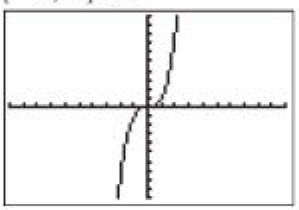

[−10, 10] scl: 1 by
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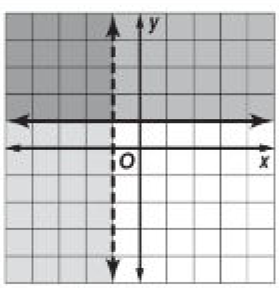
65d. If the power of *x* is 1, the equation or its related expression is linear. Otherwise, it is nonlinear.

Lesson 8-2

71.



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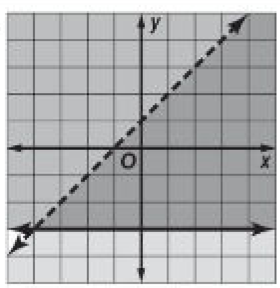

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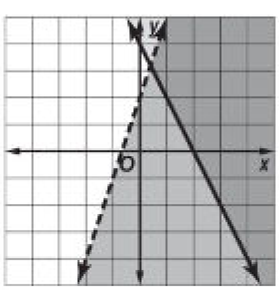

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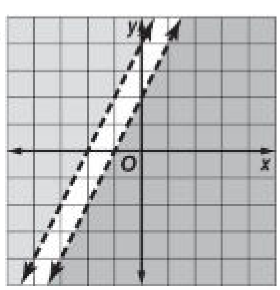
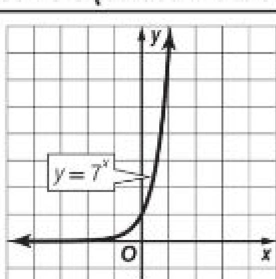


no solution

Lesson 8-5 (Guided Practice)

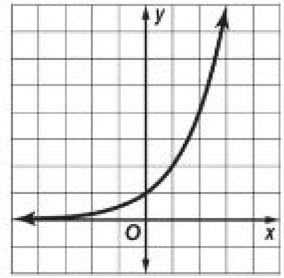
1A.



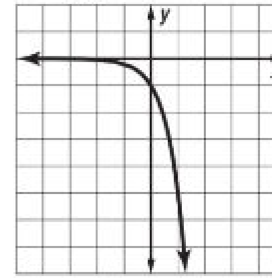
(0, 1); D = {all real numbers}
R = {y | $y > 0$}

Lesson 8-5

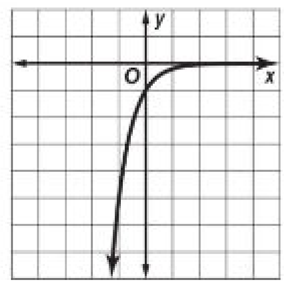
1. 1; D = {all real numbers};
R = {y | $y > 0$}



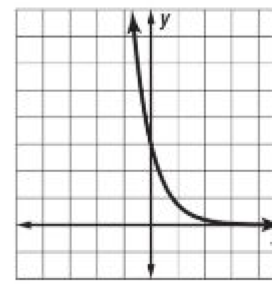
2. −1; D = {all real numbers};
R = {y | $y < 0$}



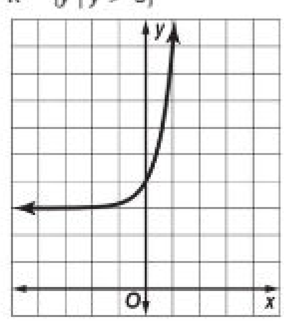
3. −1; D = {all real numbers};
R = {y | $y < 0$}



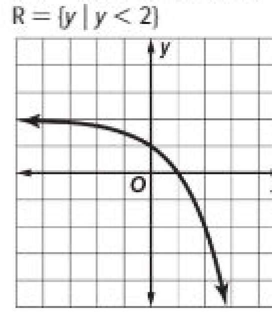
4. 3; D = {all real numbers};
R = {y | $y > 0$}



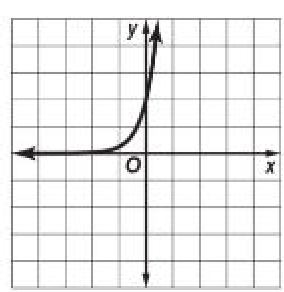
5. 4; D = {all real numbers};
R = {y | $y > 3$}



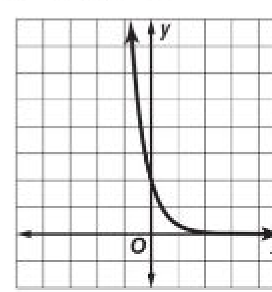
6. 1; D = {all real numbers};
R = {y | $y < 2$}



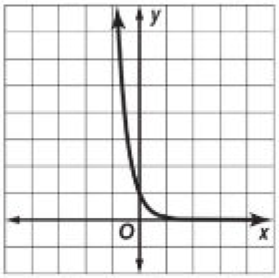
10. 2; D = {all real numbers};
R = {y | $y > 0$}



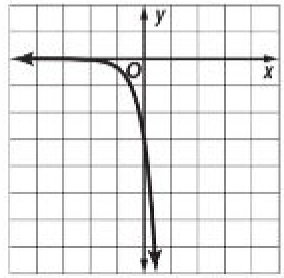
11. 2; D = {all real numbers};
R = {y | $y > 0$}



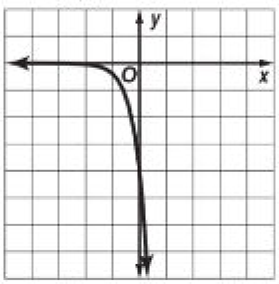
12. 1; D = {all real numbers}; R = {y | $y > 0$}



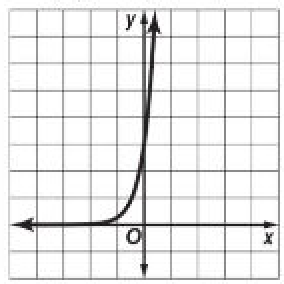
13. −3; D = {all real numbers}; R = {y | $y < 0$}



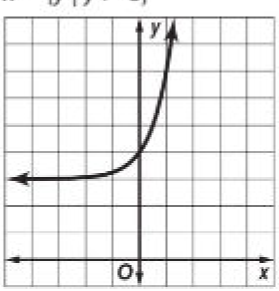
14. −4; D = {all real numbers}; R = {y | $y < 0$}



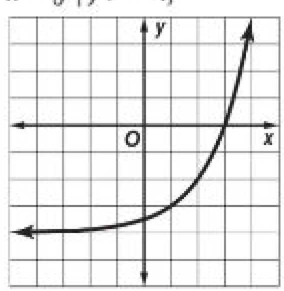
15. 3; D = {all real numbers}; R = {y | $y > 0$}



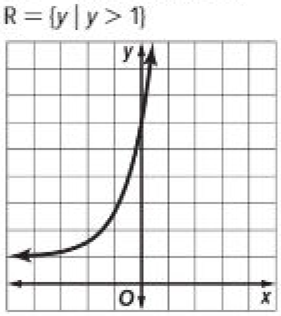
16. 4; D = {all real numbers} R = {y | $y > 3$}



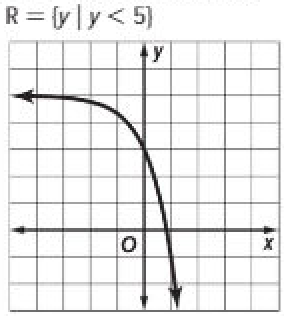
17. −3.5; D = {all real numbers} R = {y | $y > -4$}



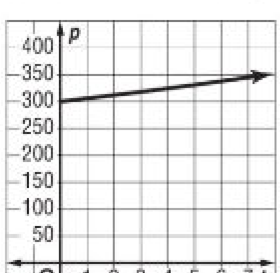
18. 6; D = {all real numbers} R = {y | $y > 1$}



19. 3; D = {all real numbers}; R = {y | $y < 5$}

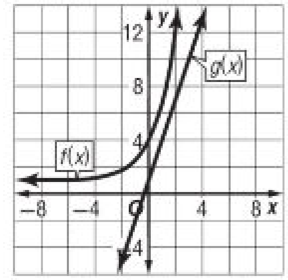


20b.



The p-intercept is 300; there are 300 bacteria at 9:00 A.M.; D = {t | $t \geq 0$}; R = {p | $p \geq 300$}.

44. Sample answer: Let $a = 3$, $b = 2$ and $c = 1$.



x	$f(x) = 3 \times 2^x + 1$	$g(x) = 3x + 1$
−5	1.09375	−14
−4	1.1875	−11
−3	1.375	−8
−2	1.75	−5
−1	2.5	−2
0	4	1
1	7	4
2	13	7
3	25	10
4	49	13
5	97	16

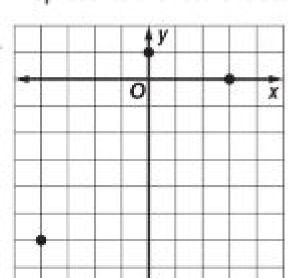
The y-intercept of $f(x)$ is 4 and the y-intercept of $g(x)$ is 1. Both $f(x)$ and $g(x)$ increase as x increases. All function values for $f(x)$ are positive, while $g(x)$ has both positive and negative values. Neither $f(x)$ nor $g(x)$ have maximum or minimum points, and neither has symmetry.

Lesson 8-6

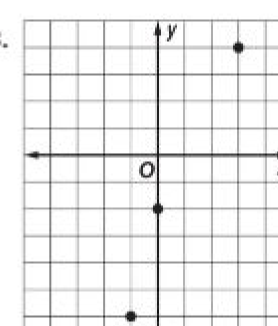
19. Sample answer: Exponential models can grow without bound, which is usually not the case of the situation that is being modeled. For instance, a population cannot grow without bound due to space and food constraints. Therefore, when using a model, the situation that is being modeled should be carefully considered when used to make decisions.

20. The exponential growth formula is $y = a(1 + r)^t$, where a is the initial amount, t is time, y is the final amount, and r is the rate of change expressed as a decimal. The exponential decay formula is basically the same except the rate is subtracted from 1 and r represents the rate of decay.

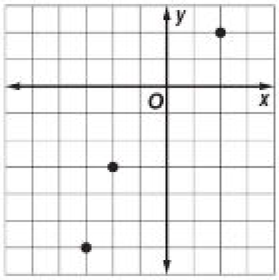
42.



43.



44.



58. $y = 4x + 2$

59. $y = -3x - \frac{2}{3}$

60. $y = -\frac{1}{4}x - 5$

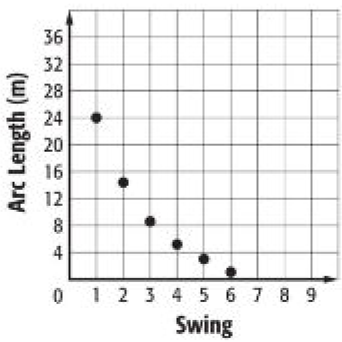
61. $y = \frac{1}{2}x - 9$

62. $y = -\frac{2}{5}x + \frac{3}{4}$

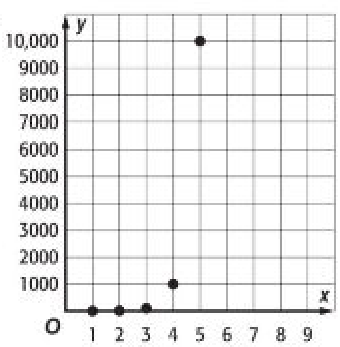
63. $y = -6x - 7$

Lesson 8-7

30. **Pendulum**



37b.



37c. The graph appears to be exponential. The rate of change between any two points does not match any others.

39. Neither; Ibrahim calculated the exponent incorrectly. Ahmed did not calculate $(-2)^8$ correctly.

41. Sample answer: When graphed, the terms of a geometric sequence lie on a curve that can be represented by an exponential function. They are different in that the domain of a geometric sequence is the set of natural numbers, while the domain of an exponential function is all real numbers. Thus, geometric sequences are discrete, while exponential functions are continuous.

42. Sample answer: First find the common ratio. Then use the formula $a_n = a_1 \cdot r^{n-1}$. Substitute the first term for a_1 and the common ratio for r. Let n be equal to the number of the term you are finding. Then solve the equation.

56. $y = 120 + 15x$, $y = 180 + 10x$; (12, 300) means that in 12 years the trees will be the same height, 300 centimeters or 3 meters.

NOTES

Chapter Planner

	Diagnostic Assessment Quick Check		
	LESSON 9-1 45 min: 1 day 90 min: 0.5 day	EXTEND 9-1 45 min: 0.5 day 90 min: 0.5 day	LESSON 9-2 45 min: 1 day 90 min: 0.5 day
Title	**Square Root Functions**	**Graphing Technology Lab: Graphing Square Root Functions**	**Simplifying Radical Expressions**
Objectives	▪ Graph and analyze dilations of radical functions. ▪ Graph and analyze reflections and translations of radical functions.	▪ Use a graphing calculator to investigate the graphs of square root functions.	▪ Simplify radical expressions by using the Product Property of Square Roots. ▪ Simplify radical expressions by using the Quotient Property of Square Roots.
Key Vocabulary	square root function radical function radicand		radical expression rationalizing the denominator conjugate

EXTEND 9-2 (45 min: 0.5 day; 90 min: 0.5 day)	LESSON 9-3 (45 min: 1 day; 90 min: 0.5 day)	LESSON 9-4 (45 min: 1 day; 90 min: 0.5 day)
Algebra Lab: Rational and Irrational Numbers	**Operations with Radical Expressions**	**Radical Equations**
■ Investigate the products and sums of two rational numbers, two irrational numbers, and a rational and irrational number.	■ Add and subtract radical expressions. ■ Multiply radical expressions.	■ Solve radical equations. ■ Solve radical equations with extraneous solutions.
closed		radical equations extraneous solutions
		Formative Assessment Mid-Chapter Quiz

Chapter Planner

	LESSON 9-5 (45 min: 1 day; 90 min: 0.5 day)	EXPLORE 9-6 (45 min: 0.5 day; 90 min: 0.5 day)	LESSON 9-6 (45 min: 1 day; 90 min: 0.5 day)
Title	**Inverse Variation**	**Graphing Technology Lab: Family of Rational Functions**	**Rational Functions**
Objectives	▪ Identify and use inverse variations. ▪ Graph inverse variations.	▪ Use graphing technology to investigate families of rational functions.	▪ Identify excluded values. ▪ Identify and use asymptotes to graph rational functions.
Key Vocabulary	inverse variation product rule		rational function excluded value asymptote

LESSON 9-7 (45 min: 0.5 day; 90 min: 0.5 day)	EXTEND 9-7 (45 min: 0.5 day; 90 min: 0.25 day)
Rational Equations	**Graphing Technology Lab: Solving Rational Equations**
▪ Solve rational equations. ▪ Use rational equations to solve problems.	▪ Use graphing technology to solve rational equations.
rational equation extraneous solution work and rate problems	

Summative Assessment
Study Guide and Review
Practice Test

Assessment

SE = Student Edition, TE = Teacher Edition

	Diagnosis	Prescription
DIAGNOSTIC ASSESSMENT	**Beginning Chapter 9**	
	Get Ready for Chapter 9 **SE**	Response to Intervention **TE**
	Beginning Every Lesson	
	Then, Now, Why? **SE**	Chapter 0 **SE**
FORMATIVE ASSESSMENT	**During/After Every Lesson**	
	Guided Practice **SE**, every example Check Your Understanding **SE** H.O.T. Problems **SE** Spiral Review **SE** Additional Examples **TE** Watch Out! **TE** Step 4, Assess **TE**	Differentiated Instruction **TE** Differentiated Homework Options **TE**
	Mid-Chapter	
	Mid-Chapter Quiz **SE**	
SUMMATIVE ASSESSMENT	**Before Chapter Test**	
	Chapter Study Guide and Review **SE** Practice Test **SE** Standardized Test Practice **SE**	

Differentiated Instruction

Option 1 Reaching All Learners AL OL BL

Social Have students work in groups. Assign each student a polynomial division problem. Tell the students to study their problems for a few minutes to decide on a method for finding the quotient. Then, ask each student to “teach” the group how to find the quotient. Have groups discuss whether the method “taught” is correct.

Auditory In pairs or small groups, students work on Exercise 18 and talk about each step of the solution as they show their work. Have them check their solutions and discuss why one of the solutions is extraneous.

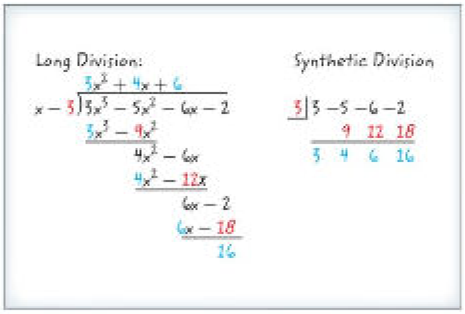
Kinesthetic Average speed r is given by the equation $r = \frac{d}{t}$, where d is the distance traveled and t is the time it takes to travel the distance. Ask students to choose a distance like 50 meters that people are willing to run. Once they have chosen the distance, ask them to create a graph, perform an experiment with the given distance, and then determine whether the graph accurately depicts the performance of their participants. Analyses should be performed to show how the data points relate to the graph.

Option 2 Approaching Level AL

Have students write division problems involving rational expressions on note cards, pieces of scrap paper, or any other item that they can manipulate. Then have students physically “flip” the fractions to multiply by the reciprocal. The act of “flipping” the fractions will help cement the concept in students’ minds.

Option 3 Beyond Level BL

Ask students to research a method of dividing polynomials known as synthetic division. Then ask students to compare this method to using the long division method. An example below is shown for $(3x^3 - 5x^2 - 6x - 2) \div (x - 3)$.



Focus on Mathematical Content

Vertical Alignment

Before Chapter 9

Related Topics

- represent squares and square roots using geometric models
- approximate the values of irrational numbers as they arise from problem situations

Previous Topics

- use the Distributive Property to simplify algebraic expressions
- solve quadratic equations using algebraic methods
- solve problems involving proportional change
- graph and analyze linear, exponential, and quadratic functions
- rewrite expressions involving radicals and rational exponents using the properties of exponents

Chapter 9

Related Topics

- add, subtract, multiply, and simplify radical expressions
- solve radical equations

After Chapter 9

Preparation

- use the Pythagorean Theorem
- identify and apply patterns from right triangles to solve meaningful problems
- use formulas for length, slope, and midpoint
- use ratios to solve problems involving similar figures
- formulate equations and inequalities based on square root functions, use a variety of methods to solve them, and analyze the solutions in terms of the situation

Lesson-by-Lesson Preview

9-1 Square Root Functions

A square root function is so named because the function contains a variable inside a square root symbol. The parent function of a square root function is $f(x) = \sqrt{x}$. In order for a square root to be a real number, the radicand cannot be negative. When graphing a square root function, numbers that make the radicand negative must be excluded from the domain of the function. The graph of $y = a\sqrt{x}$ starts at the origin and passes through the point $(1, a)$. If a is positive, the graph is in the first quadrant. If a is negative, the graph is a reflection of the graph of $y = |a|\sqrt{x}$ and lies in the fourth quadrant.

9-2 Simplifying Radical Expressions

When an expression contains a square root, it is called a radical expression. When the radicand, the expression under the square root sign, contains no perfect square factors other than 1, it is said to be in simplest form. Properties of square roots can be used to simplify radical expressions.

- Product Property of Square Roots: For any numbers greater than or equal to 0, the square root of their product is equal to the product of each number's square root. For example, $\sqrt{18} = \sqrt{2 \cdot 3 \cdot 3} = \sqrt{2} \cdot \sqrt{3^2} = 3\sqrt{2}$. Principle square roots are never negative, so absolute value symbols must be used to signify that some results are not negative, for example $\sqrt{x^2} = |x|$.
- Quotient Property of Square Roots: For any number greater than or equal to 0 divided by a number greater than 0, the square root of their quotient is equal to the quotient of each number's square root, for example, $\sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2}$.
- When there is a radical expression in the denominator, the expression is not in simplest form. A process called *rationalizing the denominator* eliminates radicals from a denominator. Since squaring and taking a square root are inverse functions, multiply the numerator and the denominator by the same radical expression so that the radical in the denominator contains a perfect square. For example, $\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{15}}{5}$.
- If the denominator is an expression containing a radical, multiply by its *conjugate* to rationalize the denominator. For example, if the denominator is in the form $a + \sqrt{b}$, multiply both the numerator and the denominator by $a - \sqrt{b}$.

9-3 Operations with Radical Expressions

- Adding or subtracting expressions with radicals uses the process of combining like terms. For terms to be combined, their radicands must be the same.
- Multiplying two radical expressions with two terms is similar to multiplying binomials. The radicands do not have to be like radicands when multiplying.

9-4 Radical Equations

Equations that contain radicals with variables in the radicands are called *radical equations*. To solve radical equations, the radical must first be isolated on one side of the equation. Then both sides are squared. This will eliminate the radical. Squaring each side sometimes produces *extraneous solutions*, results that are not solutions of the original equation. Be sure to substitute all solutions back into the original equation to check their validity.

9-5 Inverse Variation

Some situations in which y increases as x increases are known as *direct variations* (Lesson 3-4). Some situations in which y decreases as x increases, or vice versa, are known as *inverse variations*. Inverse variations can be represented by equations of the form $y = \frac{k}{x}$ or $xy = k$, where $x \neq 0$ and $y \neq 0$. The value of an inverse variation is undefined when $x = 0$.

The *product* rule of inverse variation states that if (x_1, y_1) and (x_2, y_2) are solutions of an inverse variation, then $x_1y_1 = x_2y_2$ because both $x_1y_1 = k$ and $x_2y_2 = k$. The equation $x_1y_1 = x_2y_2$ can be used to solve for missing values of x and y.

9-6 Rational Functions

A rational function is a function of the form $y = \frac{p}{q}$, where the numerator, p, and denominator, q, are both polynomials. Any value that makes the value of q equal to 0 is called an *excluded value* of the rational function and must be excluded from the domain of the function. Depending on the situation represented by a rational function, there may be additional values excluded as well. The graph of a rational function of the form $y = \frac{a}{x - b}$ (where $a \neq 0$), has two asympotes, the line $x = b$ and the line $y = 0$. The graph of the function *appears* to approach these two lines.

9-7 Rational Equations

Rational equations are equations that contain rational expressions. If both sides of a rational equation are single fractions, cross products can be used to solve the equation. Or, you can multiply each side of the equation by the LCD of the fractions to eliminate the fractions, and then solve the resulting equation.

When solving a rational equation, more than one solution may result. Always check all solutions in the original equation, as some solutions may be *extraneous solutions*.

Chapter Project

The More You Know

Students use what they have learned about rational functions to complete a project.

This chapter project addresses civic literacy, as well as several specific skills identified as being essential to student success by the Framework for 21st Century Learning.

KeyVocabulary Introduce the key vocabulary in the chapter using the routine below.

Define: A radicand is the expression that is under the radical sign.

Example: $\frac{2ab}{c}$ is the radicand in the expression $\sqrt{\frac{2ab}{c}}$.

Ask: What does the radical sign indicate in regards to the radicand? the square root of the radicand

CHAPTER 9

Radical Functions, Rational Functions, and Geometry

Then

- You solved quadratic and exponential equations.

Now

- In this chapter, you will:
  - Graph and transform radical functions.
  - Simplify, add, subtract, and multiply radical expressions.
  - Solve radical equations.
  - Use the Pythagorean Theorem.
  - Find trigonometric ratios.

Why?

- **OCEANS** Tsunamis, or large waves, are generated by undersea earthquakes. A radical equation can be used to find the speed of a tsunami in meters per second or the depth of the ocean in meters.

Get Ready for the Chapter

1 Textbook Option Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Find each square root. If necessary, round to the nearest hundredth.

1. $\sqrt{82}$ **9.06**

2. $\sqrt{26}$ **5.10**

3. $\sqrt{15}$ **3.87**

4. $\sqrt{99}$ **9.95**

5. SANDBOX Eissa is making a square sandbox with an area of 100 square meters. How long is a side of the sandbox? **10 m**

Simplify each expression.

6. $(21x + 15y) - (9x - 4y)$ **$12x + 19y$**

7. $13x - 5y + 2y$ **$13x - 3y$**

8. $(10a - 5b) + (6a + 5b)$ **$16a$**

9. $6m + 5n + 4 - 3m - 2n + 6$ **$3m + 3n + 10$**

10. $x + y - 3x - 4y + 2x - 8y$ **$-11y$**

Solve each equation. **11–14. See margin.**

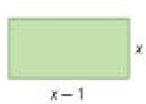
11. $2x^2 - 4x = 0$

12. $6x^2 - 5x - 4 = 0$

13. $x^2 - 7x + 10 = 0$

14. $2x^2 + 7x - 5 = -1$

15. GEOMETRY The area of the rectangle is 90 square meters. Find x. **10**



QuickReview

Example 1 **(Used in Lesson 9-2)**

Find the square root of $\sqrt{50}$. If necessary, round to the nearest hundredth.

$\sqrt{50} = 7.071067812....$ Use a calculator.

To the nearest hundredth, $\sqrt{50} = 7.07$.

Example 2 **(Used in Lesson 9-3)**

Simplify $3x + 7y - 4x - 8y$.

$3x + 7y - 4x - 8y$

$= (3x - 4x) + (7y - 8y)$ Combine like terms.

$= -x - y$ Simplify.

Example 3 **(Used in Lesson 9-4)**

Solve $x^2 - 5x + 6 = 0$.

$x^2 - 5x + 6 = 0$ Original equation

$(x - 3)(x - 2) = 0$ Factor.

$x - 3 = 0$ or $x - 2 = 0$ Zero Product Property

$x = 3$ $\quad x = 2$ Solve each equation.

Additional Answers

11. 0, 2

12. $-\frac{1}{2}, \frac{4}{3}$

13. 2, 5

14. $-4, \frac{1}{2}$

Essential Question

- How can you choose a model to represent a real-world situation? Sample answer: To model a set of data, you could create a scatter plot and choose a function with a graph that fits the data. To model a physical situation, you could create a diagram to represent a situation using a figure that has known properties.

FOLDABLES StudyOrganizer

Dinah Zike's Foldables®

Focus As students read and study this chapter, they show examples and write notes about radical functions and geometry in their booklet.

Teach Have students make and label their Foldables as illustrated. Before beginning, have them skim the chapter and write down any problems from the chapter that they think are challenging on the right-hand page of the problems' corresponding lessons. As students work through the chapter, ascertain that students are able to find the answers to their problems.

When to Use It Encourage students to add to their Foldables as they work through the chapter and to use them to review for the chapter test.

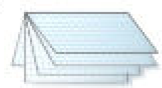
Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 9. To get ready, identify important terms and organize your resources. You may wish to refer to Chapter 0 to review prerequisite skills.

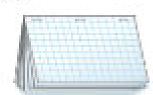
FOLDABLES StudyOrganizer

Radical Functions and Geometry Make this Foldable to help you organize your Chapter 9 notes about radical functions and geometry. Begin with four sheets of grid paper.

1 **Fold** in half along the width.



2 **Staple** along the fold.



3 **Turn** the fold to the left and write the title of the chapter on the front. On each left-hand page of the booklet, write the title of a lesson from the chapter.

NewVocabulary

square root function
radicand
radical function
radical expression
conjugate
rationalize the denominator
closed
radical equations
extraneous solutions
inverse variation
product rule
rational function
excluded values
asymptote
rational equation
work problem
rate problem

ReviewVocabulary

FOIL method to multiply two binomials, find the sum of the products of the First terms, Outer terms, Inner terms, and Last terms

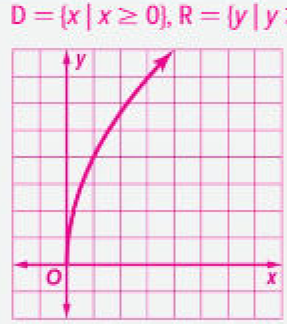
perfect square a number with a square root that is a rational number

proportion an equation of the form $\frac{a}{b} = \frac{c}{d}$, $b \neq 0$, $d \neq 0$ stating that two ratios are equivalent

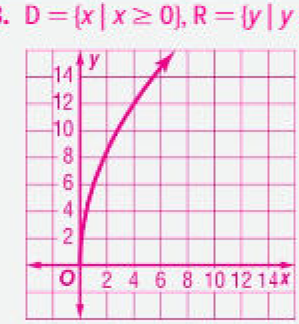
$$\frac{a}{b} \times \frac{c}{d}$$
$$ad = bc$$

Additional Answers (Lesson 10-1, Guided Practice)

1A. $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq 0\}$



1B. $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq 0\}$



LESSON 9-1 Square Root Functions

Then

- You graphed and analyzed linear, exponential, and quadratic functions.

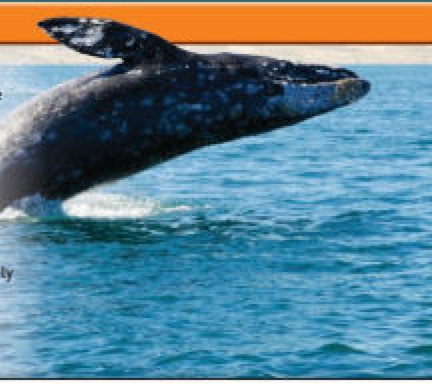
Now

- 1 Graph and analyze dilations of radical functions.
- 2 Graph and analyze reflections and translations of radical functions.

Why?

- Scientists use sounds of whales to track their movements. The distance to a whale can be found by relating time to the speed of sound in water.

The speed of sound in water can be described by the *square root function* $c = \sqrt{\frac{E}{d}}$, where E represents the bulk modulus elasticity of the water and d represents the density of the water.



NewVocabulary
square root function
radical function
radicand

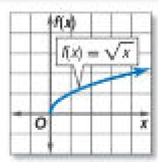
Mathematical Practices
Attend to precision.

1 Dilations of Radical Functions

A **square root function** contains the square root of a variable. Square root functions are a type of **radical function**. The expression under the radical sign is called the **radicand**. For a square root to be a real number, the radicand cannot be negative. Values that make the radicand negative are not included in the domain.

KeyConcept Square Root Function

Parent Function:	$f(x) = \sqrt{x}$
Type of Graph:	curve
Domain:	$\{x \mid x \geq 0\}$
Range:	$\{y \mid y \geq 0\}$



Example 1 Dilation of the Square Root Function

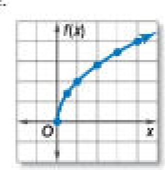
Graph $f(x) = 2\sqrt{x}$. State the domain and range.

Step 1 Make a table.

x	0	0.5	1	2	3	4
$f(x)$	0	≈1.4	2	≈2.8	≈3.5	4

The domain is $\{x \mid x \geq 0\}$, and the range is $\{y \mid y \geq 0\}$. Notice that the graph is increasing on the entire domain, the minimum value is 0, and there is no symmetry.

Step 2 Plot points. Draw a smooth curve.



GuidedPractice **1A–1B. See margin.**

1A. $g(x) = 4\sqrt{x}$ **1B.** $h(x) = 6\sqrt{x}$

1 Focus

VerticalAlignment

Before Lesson 9-1 Graph and analyze linear, exponential, and quadratic functions.

Lesson 9-1 Graph and analyze dilations of radical functions.Graph and analyze reflections and translations of radical functions.

After Lesson 9-1 Solve radical equations.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What part of the equation indicates that it is a square root function? the radical symbol, which represents a square root
- If the number under the square root symbol cannot be negative, what must be true about $\frac{E}{d}$? The quotient must be greater than or equal to zero.
- If d is constant and E increases, what happens to c? It increases.

1 Dilations of Radical Functions

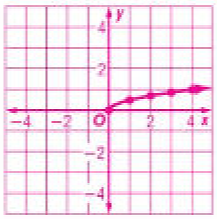
Example 1 shows how to graph a dilation of a radical function.

Formative Assessment

Use the Guided Practice exercises after each Example to determine students' understanding of concepts.

Additional Example

1 Graph $y = \frac{1}{2}\sqrt{x}$. State the domain and range.
$D = \{x \mid x \geq 0\}$; $R = \{y \mid y \geq 0\}$

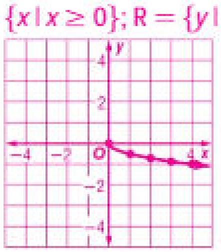


2 Reflections and Translations of Radical Functions

Examples 2 and 3 show how to graph reflections and translations of radical functions. **Example 4** shows how to solve a real-world problem involving a radical function. **Example 5** shows how to graph a radical function that has more than one transformation.

Additional Example

2 Graph $y = -\frac{1}{2}\sqrt{x}$. Compare to the parent graph. State the domain and range. vertical compression of $y = \sqrt{x}$ and reflection over the x-axis; $D = \{x \mid x \geq 0\}$; $R = \{y \mid y \leq 0\}$



2 Reflections and Translations of Radical Functions

Recall that when the value of a is negative in the quadratic function $f(x) = ax^2$, the graph of the parent function is reflected across the x-axis.

StudyTip

Graphing Radical Functions Choose perfect squares for x-values that will result in coordinates that are easy to plot.

KeyConcept Graphing $y = a\sqrt{x+h} + k$

Step 1 Draw the graph of $y = a\sqrt{x}$. The graph starts at the origin and passes through $(1, a)$. If $a > 0$, the graph is in quadrant I. If $a < 0$, the graph is reflected across the x-axis and is in quadrant IV.

Step 2 Translate the graph k units up if $k > 0$ and $|k|$ units down if $k < 0$.

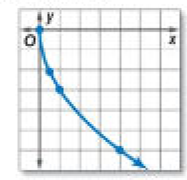
Step 3 Translate the graph h units left if $h > 0$ and $|h|$ units right if $h < 0$.

Example 2 Reflection of the Square Root Function

Graph $y = -3\sqrt{x}$. Compare to the parent graph. State the domain and range.

Make a table of values. Then plot the points on a coordinate system and draw a smooth curve that connects them.

x	0	0.5	1	4
y	0	≈ -2.1	-3	-6



Notice that the graph is in the 4th quadrant. It is obtained by stretching the graph of $y = \sqrt{x}$ vertically and then reflecting across the x-axis. The domain is $\{x \mid x \geq 0\}$, and the range is $\{y \mid y \leq 0\}$.

GuidedPractice 2A–2B. See Ch. 9 Answer Appendix.

2A. $y = -2\sqrt{x}$

2B. $y = -4\sqrt{x}$

StudyTip

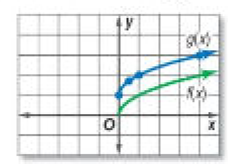
Translating Radical Functions If $h > 0$, a radical function $f(x) = \sqrt{x-h}$ is a horizontal translation h units to the right. $f(x) = \sqrt{x+h}$ is a horizontal translation h units to the left.

Example 3 Translation of the Square Root Function

Graph each function. Compare to the parent graph. State the domain and range.

a. $g(x) = \sqrt{x} + 1$

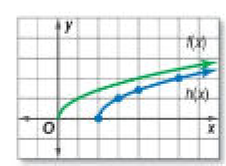
x	0	0.5	1	4	9
y	0	≈ 1.7	2	3	4



Notice that the values of $g(x)$ are 1 greater than those of $f(x) = \sqrt{x}$. This is a vertical translation 1 unit up from the parent function. The domain is $\{x \mid x \geq 0\}$, and the range is $\{y \mid y \geq 1\}$.

b. $h(x) = \sqrt{x-2}$

x	2	3	4	6
y	0	1	≈ 1.4	2



This is a horizontal translation 2 units to the right of the parent function. The domain is $\{x \mid x \geq 2\}$, and the range is $\{y \mid y \geq 0\}$.

Focus on Mathematical Content

Square Root Functions Square root functions are a type of radical function. The square root function $y = \sqrt{x}$ is the inverse function of $y = x^2$ for $x \geq 0$. It can also be represented as $y = x^{\frac{1}{2}}$. To graph a square root function, exclude any values from the domain that result in a negative radicand.

Tips for New Teachers

Sense-Making Show students that the limits on the domain and range of a square root function are also representative of the inital point of the graph of the function, where x is the minimum value in the domain and y is the minimum value in the range.

Guided Practice 3A–3B. See Ch. 9 Answer Appendix.

3A. $g(x) = \sqrt{x} - 4$ **3B.** $h(x) = \sqrt{x + 3}$

Physical phenomena such as motion can be modeled by radical functions. Often these functions are transformations of the parent square root function.

Real-World Link

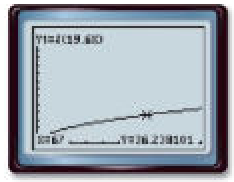
Approximately 39 million cars cross the Golden Gate Bridge in San Francisco each year.

Source: San Francisco Convention and Visitors Bureau

Real-World Example 4 Analyze a Radical Function

BRIDGES The Golden Gate Bridge is about 67 meters above the water. The velocity v of a freely falling object that has fallen h meters is given by $v = \sqrt{2gh}$, where g is the constant 9.8 meters per second squared. Graph the function. If an object is dropped from the bridge, what is its velocity when it hits the water?

Use a graphing calculator to graph the function. To find the velocity of the object, substitute 67 meters for h.



$v = \sqrt{2gh}$ Original function

$= \sqrt{2(9.8)(67)}$ $g = 9.8$ and $h = 67$

$= \sqrt{1313.2}$ Simplify.

≈ 36.2 m/s Use a calculator.

The velocity of the object is about 36.2 meters per second after dropping 67 meters.

Guided Practice

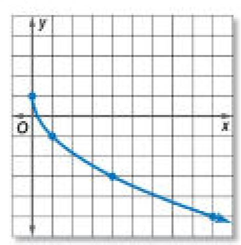
4. Use the graph above to estimate the initial height of an object if it is moving at 20 meters per second when it hits the water. ≈20 m

Transformations such as reflections, translations, and dilations can be combined in one equation.

Example 5 Transformations of the Square Root Function

Graph $y = -2\sqrt{x} + 1$, and compare to the parent graph. State the domain and range.

x	0	1	4	9
y	1	−1	−3	−5

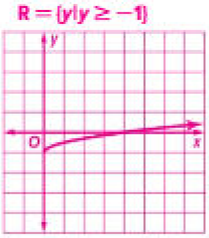


This graph is the result of a vertical stretch of the graph of $y = \sqrt{x}$ followed by a reflection across the x-axis, and then a translation 1 unit up. The domain is $\{x \mid x \geq 0\}$, and the range is $\{y \mid y \leq 1\}$.

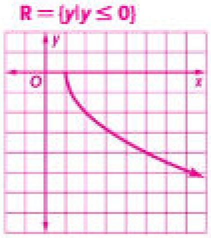
Guided Practice

5A. $y = \frac{1}{2}\sqrt{x} - 1$ **5B.** $y = -2\sqrt{x - 1}$

5A. compressed vertically and translated down 1; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq -1\}$



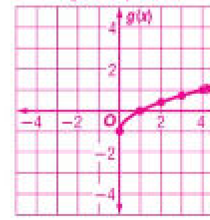
5B. stretched vertically and reflected across the x-axis, and translated right 1; $D = \{x \mid x \geq 1\}$, $R = \{y \mid y \leq 0\}$



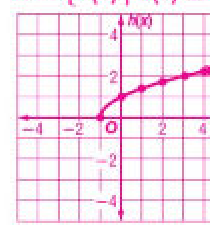
Additional Examples

3 Graph each function. Compare to the parent graph. State the domain and range.

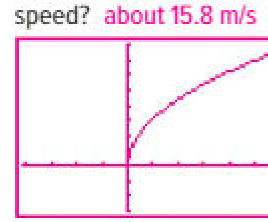
a. $g(x) = \sqrt{x} - 1$ translated 1 unit down; $D = \{x \mid x \geq 0\}$; $R = \{g(x) \mid g(x) \geq -1\}$



b. $h(x) = \sqrt{x + 1}$ translated 1 unit left; $D = \{x \mid x \geq -1\}$; $R = \{h(x) \mid h(x) \geq 0\}$



4 **TSUNAMIS** The speed s of a tsunami, in meters per second, is given by $s = 3.1\sqrt{d}$, where d is the depth of the ocean water in meters. Graph the function. If a tsunami is traveling in water 26 meters deep, what is its speed? about 15.8 m/s



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Differentiated Instruction OL BL

If students demonstrate an understanding of the translations of the graphs of square root functions,

Then have students work in pairs or groups to graph square root functions such as $y = \sqrt{x^2 + 2}$, $y = \sqrt{9 - x^2}$, and $y = \sqrt{x^2 - 2x}$. Have them state the domain and range of the functions and describe the graphs. Ask how they might use a graph of the quadratic in the radicand to find the domain and range of the function.

Follow-up

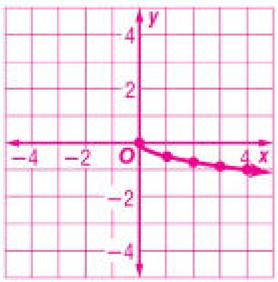
Students have explored square root functions.

Ask:

- Why would you choose a square root function to model a set of data instead of a polynomial function? Sample answer: The end behavior of a square root function might fit the data better. Also, the domain of the square root function is $x \geq 0$, which may be applicable to many real-world situations that involve quantities such as time and distance.

Additional Example

5 Graph $y = 3\sqrt{x - 2}$ and compare to the parent graph. State the domain and range. vertical stretch of $y = \sqrt{x}$ translation 2 units right; $D = \{x \mid x \geq 2\}$; $R = \{y \mid y \geq 0\}$



3 Practice

Formative Assessment

Use Exercises 1–13 to check for understanding.

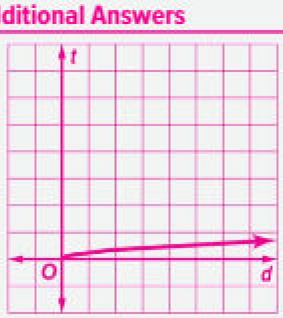
Use the chart at the bottom of this page to customize assignments for your students.

Multiple Representations

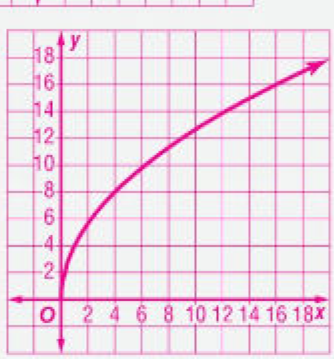
In Exercise 44, students use graphs and a piecewise-defined function to compare quadratic and square root functions.

Additional Answers

9.



34a.



Check Your Understanding

Examples 1–3 Graph each function. Compare to the parent graph. State the domain and range. **1–8. See Ch. 9 Answer Appendix.**

1. $y = 3\sqrt{x}$
2. $y = -5\sqrt{x}$
3. $y = \frac{1}{3}\sqrt{x}$
4. $y = -\frac{1}{2}\sqrt{x}$
5. $y = \sqrt{x} + 3$
6. $y = \sqrt{x} - 2$
7. $y = \sqrt{x + 2}$
8. $y = \sqrt{x - 3}$

Example 4

9. **FREE FALL** The time t, in seconds, that it takes an object to fall a distance d, in meters, is given by the function $t = \frac{5}{11}\sqrt{d}$ (assuming zero air resistance). Graph the function, and state the domain and range. **See margin for graph; $D = \{d \mid d \geq 0\}$, $R = \{t \mid t \geq 0\}$.**

Example 5 Graph each function, and compare to the parent graph. State the domain and range. **10–13. See Ch. 9 Answer Appendix.**

10. $y = \frac{1}{2}\sqrt{x} + 2$
11. $y = -\frac{1}{4}\sqrt{x} - 1$
12. $y = -2\sqrt{x + 1}$
13. $y = 3\sqrt{x - 2}$

Practice and Problem Solving

Examples 1–3 Graph each function. Compare to the parent graph. State the domain and range. **14–33. See Ch. 9 Answer Appendix.**

14. $y = 5\sqrt{x}$
15. $y = \frac{1}{2}\sqrt{x}$
16. $y = -\frac{1}{3}\sqrt{x}$
17. $y = 7\sqrt{x}$
18. $y = -\frac{1}{4}\sqrt{x}$
19. $y = -\sqrt{x}$
20. $y = -\frac{1}{5}\sqrt{x}$
21. $y = -7\sqrt{x}$
22. $y = \sqrt{x} + 2$
23. $y = \sqrt{x} + 4$
24. $y = \sqrt{x} - 1$
25. $y = \sqrt{x} - 3$
26. $y = \sqrt{x} + 1.5$
27. $y = \sqrt{x} - 2.5$
28. $y = \sqrt{x + 4}$
29. $y = \sqrt{x - 4}$
30. $y = \sqrt{x + 1}$
31. $y = \sqrt{x - 0.5}$
32. $y = \sqrt{x + 5}$
33. $y = \sqrt{x - 1.5}$

Example 4

34. **GEOMETRY** The perimeter of a square is given by the function $P = 4\sqrt{A}$, where A is the area of the square.
    a. Graph the function. **See margin.**
    b. Determine the perimeter of a square with an area of 225 m^2. **60 m**
    c. When will the perimeter and the area be the same value? **When the sides of the square have length 4 m, the perimeter and area are both 16 m.**

Example 5 Graph each function, and compare to the parent graph. State the domain and range. **35–40. See Ch. 9 Answer Appendix.**

35. $y = -2\sqrt{x} + 2$
36. $y = -3\sqrt{x} - 3$
37. $y = \frac{1}{2}\sqrt{x + 2}$
38. $y = -\sqrt{x - 1}$
39. $y = \frac{1}{4}\sqrt{x - 1} + 2$
40. $y = \frac{1}{2}\sqrt{x - 2} + 1$

B 41. **ENERGY** An object has kinetic energy when it is in motion. The velocity in meters per second of an object of mass m kilograms with an energy of E joules is given by the function $v = \sqrt{\frac{2E}{m}}$. Use a graphing calculator to graph the function that represents the velocity of a basketball with a mass of 0.6 kilogram. **See Ch. 9 Answer Appendix.**

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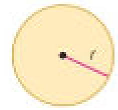
Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	14–40, 47, 49–64	15–39 odd, 53–56	14–40 even, 47, 49–52, 57–64
OL Core	15–39 odd, 41–44, 47, 49–64	14–40, 53–56	41–44, 47, 49–52, 57–64
BL Advanced	41–64		

42. **GEOMETRY** The radius of a circle is given by $r = \sqrt{\frac{A}{\pi}}$, where A is the area of the circle.



a. Graph the function. **See Ch. 9 Answer Appendix.**

b. Use a graphing calculator to determine the radius of a circle that has an area of 27 cm^2. **about 2.9 cm.**

43. **SPEED OF SOUND** The speed of sound in air is determined by the temperature of the air. The speed c in meters per second is given by $c = 331.5\sqrt{1 + \frac{t}{273.15}}$, where t is the temperature of the air in degrees Celsius.

a. Use a graphing calculator to graph the function. **See Ch. 9 Answer Appendix.**

b. How fast does sound travel when the temperature is 55°C? **about 363.3 m/s**

c. How is the speed of sound affected when the temperature increases to 65°C?

44. **MULTIPLE REPRESENTATIONS** In this problem, you will explore the relationship between the graphs of square root functions and parabolas. **a–e. See Ch. 9 Answer Appendix.**

a. **Graphical** Graph $y = x^2$ on a coordinate system.

b. **Algebraic** Write a piecewise-defined function to describe the graph of $y^2 = x$ in each quadrant.

c. **Graphical** On the same coordinate system, graph $y = \sqrt{x}$ and $y = -\sqrt{x}$.

d. **Graphical** On the same coordinate system, graph $y = x$. Plot the points (2, 4), (4, 2), and (1, 1).

e. **Analytical** Compare the graph of the parabola to the graphs of the square root functions.

43c. When t is 65°C, c is about 368.8 m/s, so this 10-degree increase results in an increase in speed of about 5.5 m/s.

H.O.T. Problems Use Higher-Order Thinking Skills

CHALLENGE Determine whether each statement is *true* or *false*. Provide an example or counterexample to support your answer. **45–46. See Ch. 9 Answer Appendix.**

45. Numbers in the domain of a radical function will always be nonnegative.

46. Numbers in the range of a radical function will always be nonnegative.

47. **WRITING IN MATH** Why are there limitations on the domain and range of square root functions? **See Ch. 9 Answer Appendix.**

48. **TOOLS** Write a radical function with a domain of all real numbers greater than or equal to 2 and a range of all real numbers less than or equal to 5.
Sample answer: $y = -\sqrt{x-2} + 5$

49. **WHICH DOES NOT BELONG?** Identify the equation that does not belong. Explain.

$y = 3\sqrt{x}$	$y = 0.7\sqrt{x}$	$y = \sqrt{x} + 3$	$y = \frac{\sqrt{x}}{6}$

See Ch. 9 Answer Appendix.

50. **OPEN ENDED** Write a function that is a reflection, translation, and a dilation of the parent graph $y = \sqrt{x}$. **Sample answer: $y = -3\sqrt{x-1}$**

51. **REASONING** If the range of the function $y = a\sqrt{x}$ is $\{y \mid y \le 0\}$, what can you conclude about the value of a? Explain your reasoning. **See Ch. 9 Answer Appendix.**

52. **WRITING IN MATH** Compare and contrast the graphs of $f(x) = \sqrt{x} + 2$ and $g(x) = \sqrt{x+2}$. **See Ch. 9 Answer Appendix.**

Teaching the Mathematical Practices

Tools Mathematically proficient students consider the available tools when solving a mathematical problem. In Exercise 48, encourage students to consider using graph paper or technology to help as they write their function.

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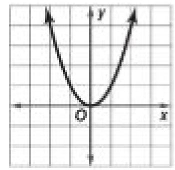
4 Assess

Name the Math Give students a square root function such as $y = \sqrt{x - 5}$ and have them explain how to find the domain and range of the function.

Standardized Test Practice

53. 

Which function *best* represents the graph? **A**

A $y = x^2$ C $y = \sqrt{x}$

B $y = 2^x$ D $y = x$

54. The statement "$x < 10$ and $3x - 2 \geq 7$" is true when x is equal to what? **H**

F 0 H 8

G 2 J 12

55. Which of the following is the equation of a line parallel to $y = -\frac{1}{2}x + 3$ and passing through $(-2, -1)$? **D**

A $y = \frac{1}{2}x$ C $y = -\frac{1}{2}x + 2$

B $y = 2x + 3$ D $y = -\frac{1}{2}x - 2$

56. **SHORT RESPONSE** A landscaper needs to mulch 6 rectangular flower beds that are 8 meter by 4 meter and 4 circular flower beds each with a radius of 3 meter. One bag of mulch covers 25 square meter. How many bags of mulch are needed to cover the flower beds? **13 bags of mulch**

Spiral Review

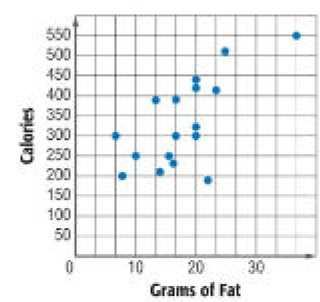
57. **HEALTH** Khawla exercises every day by walking and jogging at least 3 kilometer. Khawla walks at a rate of 4 kilometer per hour and jogs at a rate of 8 kilometer per hour. Suppose she has at most one half-hour to exercise today. **a–b. See Ch. 9 Answer Appendix.**

a. Draw a graph showing the possible amounts of time she can spend walking and jogging today.

b. List three possible solutions.

58. **NUTRITION** Determine whether the graph shows a *positive*, *negative*, or *no* correlation. If there is a positive or negative correlation, describe its meaning in the situation. **Positive; as the number of grams of fat increases, the amount of calories increases.**

Fast-Food Choices



Skills Review

Factor each monomial completely.

59. $28n^3$ **$2 \cdot 2 \cdot 7 \cdot n \cdot n \cdot n$**

60. $-33a^2b$ **$-1 \cdot 3 \cdot 11 \cdot a \cdot a \cdot b$**

61. $150rt$ **$2 \cdot 3 \cdot 5 \cdot 5 \cdot r \cdot t$**

62. $-378nq^2r^2$ **$-1 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7 \cdot n \cdot q \cdot q \cdot r \cdot r$**

63. $225a^3b^2c$ **63. $3 \cdot 3 \cdot 5 \cdot 5 \cdot a \cdot a \cdot a \cdot b \cdot b \cdot c$**

64. $-160x^2y^4$ **$-1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$**

Differentiated Instruction OL BL

Extension Write several square root functions on the board and have students state the domain and range of each. Examples may include $y = \sqrt{3x + 2}$, $y = -5\sqrt{\frac{x+2}{8}}$, or $y = \sqrt{\frac{1}{x+2}}$. Encourage students to graph the functions to check the domain and range. $D = \left\{x \mid x \geq -\frac{2}{3}\right\}$; $R = \{y \mid y \geq 0\}$; $D = \{x \mid x \geq -2\}$; $R = \{y \mid y \leq 0\}$; $D = \{x \mid x > -2\}$; $R = \{y \mid y > 0\}$

556 | Lesson 9-1 | Square Root Functions

EXTEND 9-1

Graphing Technology Lab
Graphing Square Root Functions

For a square root to be a real number, the radicand cannot be negative. When graphing a radical function, determine when the radicand would be negative and exclude those values from the domain.

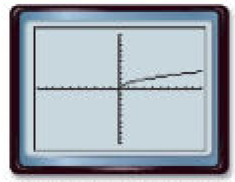
Mathematical Practices
Use appropriate tools strategically.

Activity 1 Parent Function

Graph $y = \sqrt{x}$.

Enter the equation in the Y= list, and graph in the standard viewing window.

KEYSTROKES: [Y=] [2nd] [√] [X,T,θ,n] [ )] [ZOOM] 6



[−10, 10] scl: 1 by [−10, 10] scl: 1

1A. Examine the graph. What is the domain of the function? $\{x|x \geq 0\}$

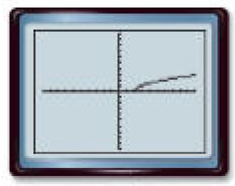
1B. What is the range of the function? $\{y|y \geq 0\}$

Activity 2 Translation of Parent Function

Graph $y = \sqrt{x-2}$.

Enter the equation in the Y= list, and graph in the standard viewing window.

KEYSTROKES: [Y=] [2nd] [√] [X,T,θ,n] [−] 2 [ )] [ZOOM] 6



[−10, 10] scl: 1 by [−10, 10] scl: 1

2A. What are the domain and range of the function? $\{x|x \geq 2\}$; $\{y|y \geq 0\}$

2B. How does the graph of $y = \sqrt{x-2}$ compare to the graph of the parent function $y = \sqrt{x}$? It is translated 2 units to the right.

Exercises

Graph each equation, and sketch the graph on your paper. State the domain and range. Describe how the graph differs from that of the parent function $y = \sqrt{x}$. **1–8. See Ch. 9 Answer Appendix.**

1. $y = \sqrt{x-1}$
2. $y = \sqrt{x+3}$
3. $y = \sqrt{x} - 2$
4. $y = \sqrt{-x}$
5. $y = -\sqrt{x}$
6. $y = \sqrt{2x}$
7. $y = \sqrt{2-x}$
8. $y = \sqrt{x-3} + 2$

Solve each equation for y. Does the equation represent a function? Explain your reasoning.

9. $x = y^2$
10. $x^2 + y^2 = 4$
11. $x^2 + y^2 = 2$

9. $y = \pm\sqrt{x}$; No; you must consider the graph of $y = \sqrt{x}$ and the graph of $y = -\sqrt{x}$. This graph fails the vertical line test.

10. $y = \pm\sqrt{4 - x^2}$; No; there are both positive and negative values of y for most values of x.

11. $y = \pm\sqrt{2 - x^2}$; No; there are both positive and negative values of y for most values of x.

Write a function with a graph that translates $y = \sqrt{x}$ in each way.

12. shifted 4 units to the left $y = \sqrt{x+4}$
13. shifted up 7 units $y = \sqrt{x} + 7$
14. shifted down 6 units $y = \sqrt{x} - 6$
15. shifted 5 units to the right and up 3 units $y = \sqrt{x-5} + 3$

1 Focus

Objective Use a graphing calculator to investigate the graphs of square root functions.

Materials for Each Student

- graphing calculator
- grid paper

Teaching Tip

Before starting this lab, familiarize students with the **ZoomFit** option in the [ZOOM] menu. This option lets the calculator automatically zoom the viewing window to fit the graph. Suggest that students use this option to get a better view of the shape of the graph of a square root function.

2 Teach

Working in Cooperative Groups

Have students work in groups of two or three, mixing abilities, to complete Activities 1 and 2.

- Have students use the value operation from the CALC menu to find the value of the function at different x-values. Press [2nd] [CALC] 1 and then enter an x-value. Students should see that the value of the function is the square root of the radicand of the function.

Practice Have students complete Exercises 1–5.

3 Assess

Formative Assessment

Use Exercise 8 to assess whether students understand how to graph square root functions and can describe how the graph differs from the parent function.

From Concrete to Abstract

Use Exercise 15 to assess whether students can write a function given a description of the graph of the function.

1 Focus

VerticalAlignment

Before Lesson 9-2 Simplify radicals.

Lesson 9-2 Simplify radical expressions by using the Product Property of Square Roots. Simplify radical expressions by using the Quotient Property of Square Roots.

After Lesson 9-2 Add, subtract, and multiply radical expressions.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- Which is the radical expression in the equation for the diameter of a steel cable? $\sqrt{\frac{w}{8}}$
- What does the radical sign in the equation mean? You must find the square root of the value under the radical sign.
- Based on what you know about the order of operations, when do you think you should simplify the radical expression? Simplify the expression under the radical sign before finding the square root.

LESSON 9-2 Simplifying Radical Expressions

Then

- You simplified radicals.

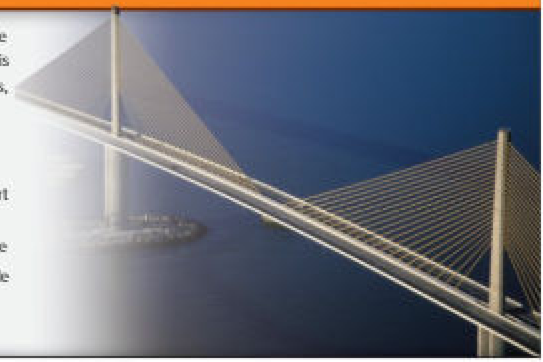
Now

1. Simplify radical expressions by using the Product Property of Square Roots.
2. Simplify radical expressions by using the Quotient Property of Square Roots.

Why?

- The Sunshine Skyway Bridge across Florida's Tampa Bay is supported by 21 steel cables, each 22 centimeters in diameter.

To find the diameter a steel cable should have to support a given weight, you can use the equation $d = \sqrt{\frac{w}{8}}$, where d is the diameter of the cable in centimeter and w is the weight in tons.



NewVocabulary
radical expression
rationalizing the denominator
conjugate

Mathematical Practices
Look for and make use of structure.
Look for and express regularity in repeated reasoning.

1 Product Property of Square Roots A **radical expression** contains a radical, such as a square root. Recall the expression under the radical sign is called the radicand. A radicand is in simplest form if the following three conditions are true.

- No radicands have perfect square factors other than 1.
- No radicands contain fractions.
- No radicals appear in the denominator of a fraction.

The following property can be used to simplify square roots.

KeyConcept Product Property of Square Roots

Words	For any nonnegative real numbers a and b, the square root of ab is equal to the square root of a times the square root of b.
Symbols	$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, if $a \geq 0$ and $b \geq 0$
Examples	$\sqrt{4 \cdot 9} = \sqrt{36}$ or 6 $\quad \sqrt{4 \cdot 9} = \sqrt{4} \cdot \sqrt{9} = 2 \cdot 3$ or 6

Example 1 Simplify Square Roots

Simplify $\sqrt{80}$.

$\sqrt{80} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}$ — Prime factorization of 80

$= \sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{5}$ — Product Property of Square Roots

$= 2 \cdot 2 \cdot \sqrt{5}$ or $4\sqrt{5}$ — Simplify.

GuidedPractice

1A. $\sqrt{54}$ $3\sqrt{6}$ **1B.** $\sqrt{180}$ $6\sqrt{5}$

Example 2 Multiply Square Roots

Simplify $\sqrt{2} \cdot \sqrt{14}$.

$\sqrt{2} \cdot \sqrt{14} = \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{7}$ — Product Property of Square Roots

$= \sqrt{2^2} \cdot \sqrt{7}$ or $2\sqrt{7}$ — Product Property of Square Roots

GuidedPractice

2A. $\sqrt{5} \cdot \sqrt{10}$ **$5\sqrt{2}$**

2B. $\sqrt{6} \cdot \sqrt{8}$ **$4\sqrt{3}$**

Consider the expression $\sqrt{x^2}$. It may seem that $x = \sqrt{x^2}$, but when finding the principal square root of an expression containing variables, you have to be sure that the result is not negative. Consider $x = -3$.

$\sqrt{x^2} \stackrel{?}{=} x$

$\sqrt{(-3)^2} \stackrel{?}{=} -3$ — Replace x with -3.

$\sqrt{9} \stackrel{?}{=} -3$ — $(-3)^2 = 9$

$3 \neq -3$ — $\sqrt{9} = 3$

Notice in this case, if the right hand side of the equation were $|x|$, the equation would be true. For expressions where the exponent of the variable inside a radical is even and the simplified exponent is odd, you must use absolute value.

$\sqrt{x^2} = |x| \qquad \sqrt{x^3} = x\sqrt{x} \qquad \sqrt{x^4} = x^2 \qquad \sqrt{x^6} = |x^3|$

Example 3 Simplify a Square Root with Variables

Simplify $\sqrt{90x^3y^4z^5}$.

$\sqrt{90x^3y^4z^5} = \sqrt{2 \cdot 3^2 \cdot 5 \cdot x^3 \cdot y^4 \cdot z^5}$ — Prime factorization

$= \sqrt{2} \cdot \sqrt{3^2} \cdot \sqrt{5} \cdot \sqrt{x^2} \cdot \sqrt{x} \cdot \sqrt{y^4} \cdot \sqrt{z^4} \cdot \sqrt{z}$ — Product Property

$= \sqrt{2} \cdot 3 \cdot \sqrt{5} \cdot x \cdot \sqrt{x} \cdot y^2 \cdot z^2 \cdot \sqrt{z}$ — Simplify.

$= 3y^2z^2x\sqrt{10xz}$ — Simplify.

GuidedPractice

3A. $\sqrt{32r^2k^4t^5}$ **$4|r|k^2t^2\sqrt{2t}$**

3B. $\sqrt{56xy^{10}z^5}$ **$2|y^5|z^2\sqrt{14xz}$**

2 Quotient Property of Square Roots

To divide square roots and simplify radical expressions, you can use the Quotient Property of Square Roots.

ReadingMath

Fractions in the Radicand The expression $\sqrt{\frac{a}{b}}$ is read *the square root of a over b, or the square root of the quantity of a over b.*

KeyConcept Quotient Property of Square Roots

Words	For any real numbers a and b, where $a \geq 0$ and $b > 0$, the square root of $\frac{a}{b}$ is equal to the square root of a divided by the square root of b.
Symbols	$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

1 Product Property of Square Roots

Example 1 shows how to simplify a radical expression in which the radicand is not a perfect square. **Example 2** shows how to multiply square roots. **Example 3** shows how to simplify a square root that contains variables.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1. Simplify $\sqrt{52}$. **$2\sqrt{13}$**
2. Simplify $\sqrt{2} \cdot \sqrt{24}$. **$4\sqrt{3}$**
3. Simplify $\sqrt{45a^4b^5c^6}$. **$3a^2b^2|c^3|\sqrt{5b}$**

Focus on Mathematical Content

Product Property of Square Roots The Product Property can be used to simplify radical expressions. For all nonnegative numbers, the product of each square root equals the square root of the products.

Teach with Tech

Blog Have half of the students show and explain how they simplified a radical expression. Have different students show how to simplify products and quotients of radicals. When all agree, post on the secure classroom website for additional reference.

Tips for New Teachers

Intervention In order to simplify square roots with the Product Property of Square Roots, students need to be able to find the prime factorization of the radicand. Take a few minutes to review finding prime factorizations so that students can focus on learning the new concept rather than trying to recall earlier material.

2 Quotient Property of Square Roots

Example 4 is a test item that shows how to rationalize the denominator of a radical expression to eliminate radicals from the denominator. **Example 5** shows how to use conjugates to rationalize the denominator of a radical expression.

Focus on Mathematical Content

Quotient Property of Square Roots The Quotient Property can be used to simplify radical expressions. For nonnegative dividends and positive divisors, the square root of a quotient is equal to the quotient of the principal square roots of the numbers.

Additional Examples

4 STANDARDIZED TEST PRACTICE Which expression is equivalent to $\frac{\sqrt{3n}}{\sqrt{8}}$? D

A $\frac{\sqrt{3n}}{8}$ C $\frac{\sqrt{6n}}{2}$

B $\frac{\sqrt{3n}}{4}$ D $\frac{\sqrt{6n}}{4}$

5 Simplify $\frac{2}{4-\sqrt{5}}$. $\frac{8+2\sqrt{5}}{11}$

Teaching the Mathematical Practices

Structure Mathematically proficient students can see complicated things as single objects or as being composed of several objects. Point out that simplifying the radicand first doesn't change the final answer, but it does simplify the calculations.

You can use the properties of square roots to **rationalize the denominator** of a fraction with a radical. This involves multiplying the numerator and denominator by a factor that eliminates radicals in the denominator.

Standardized Test Example 4 Rationalize a Denominator

Which expression is equivalent to $\sqrt{\frac{35}{15}}$?

A $\frac{5\sqrt{21}}{15}$ B $\frac{\sqrt{21}}{3}$ C $\frac{\sqrt{525}}{15}$ D $\frac{\sqrt{35}}{15}$

Test-Taking Tip
Structure Look at the radicand to see if it can be simplified first. This may make your computations simpler.

Read the Test Item The radical expression needs to be simplified.

Solve the Test Item

$\sqrt{\frac{35}{15}} = \sqrt{\frac{7}{3}}$ Reduce $\frac{35}{15}$ to $\frac{7}{3}$.

$= \frac{\sqrt{7}}{\sqrt{3}}$ Quotient Property of Square Roots

$= \frac{\sqrt{7}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$ Multiply by $\frac{\sqrt{3}}{\sqrt{3}}$.

$= \frac{\sqrt{21}}{3}$ Product Property of Square Roots

The correct choice is B.

Guided Practice

4. Simplify $\frac{\sqrt{6y}}{\sqrt{12}}$. H

F $\frac{\sqrt{y}}{2}$ G $\frac{\sqrt{y}}{4}$ H $\frac{\sqrt{2y}}{2}$ J $\frac{\sqrt{2y}}{4}$

Binomials of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where a, b, c, and d are rational numbers, are called **conjugates**. For example, $2 + \sqrt{7}$ and $2 - \sqrt{7}$ are conjugates. The product of two conjugates is a rational number and can be found using the pattern for the difference of squares.

Example 5 Use Conjugates to Rationalize a Denominator

Simplify $\frac{3}{5+\sqrt{2}}$.

$\frac{3}{5+\sqrt{2}} = \frac{3}{5+\sqrt{2}} \cdot \frac{5-\sqrt{2}}{5-\sqrt{2}}$ The conjugate of $5+\sqrt{2}$ is $5-\sqrt{2}$.

$= \frac{3(5-\sqrt{2})}{5^2-(\sqrt{2})^2}$ $(a-b)(a+b) = a^2 - b^2$

$= \frac{15-3\sqrt{2}}{25-2}$ or $\frac{15-3\sqrt{2}}{23}$ $(\sqrt{2})^2 = 2$

Guided Practice Simplify each expression.

5A. $\frac{3}{2+\sqrt{2}}$ $\frac{6-3\sqrt{2}}{2}$

5B. $\frac{7}{3-\sqrt{7}}$ $\frac{21+7\sqrt{7}}{2}$

Differentiated Instruction AL OL

If students need further practice with conjugates,

Then have students use their calculators to show that using conjugates produces equivalent expressions in Example 5. Have students show that $\frac{3}{5+\sqrt{2}}$, $\frac{3(5-\sqrt{2})}{(5+\sqrt{2})(5-\sqrt{2})}$, and $\frac{15-3\sqrt{2}}{23}$ are equivalent. They should find that each expression is about 0.4677.

Check Your Understanding

Examples 1–3 Simplify each expression.

1. $\sqrt{24}$ $2\sqrt{6}$
2. $3\sqrt{16}$ 12
3. $2\sqrt{25}$ 10
4. $\sqrt{10} \cdot \sqrt{14}$ $2\sqrt{35}$
5. $\sqrt{3} \cdot \sqrt{18}$ $3\sqrt{6}$
6. $3\sqrt{10} \cdot 4\sqrt{10}$ 120
7. $\sqrt{60x^4y^7}$ $2x^2y^3\sqrt{15y}$
8. $\sqrt{88m^3p^2r^5}$ $2m|p|r^2\sqrt{22mr}$
9. $\sqrt{99ab^5c^2}$ $3b^2|c|\sqrt{11ab}$

Example 4

10. **MULTIPLE CHOICE** Which expression is equivalent to $\sqrt{\frac{45}{10}}$? D

A $\frac{5\sqrt{2}}{10}$ B $\frac{\sqrt{45}}{10}$ C $\frac{\sqrt{50}}{10}$ D $\frac{3\sqrt{2}}{2}$

Example 5 Simplify each expression.

11. $\frac{3}{3+\sqrt{5}}$ $\frac{9-3\sqrt{5}}{4}$
12. $\frac{5}{2-\sqrt{6}}$ $\frac{10+5\sqrt{6}}{-2}$
13. $\frac{2}{1-\sqrt{10}}$ $\frac{2+2\sqrt{10}}{-9}$
14. $\frac{1}{4+\sqrt{12}}$ $\frac{2-\sqrt{3}}{2}$
15. $\frac{4}{6-\sqrt{7}}$ $\frac{24+4\sqrt{7}}{29}$
16. $\frac{6}{5+\sqrt{11}}$ $\frac{15-3\sqrt{11}}{7}$

Practice and Problem Solving

Examples 1–3 Simplify each expression.

17. $\sqrt{52}$ $2\sqrt{13}$
18. $\sqrt{56}$ $2\sqrt{14}$
19. $\sqrt{72}$ $6\sqrt{2}$
20. $3\sqrt{18}$ $9\sqrt{2}$
21. $\sqrt{243}$ $9\sqrt{3}$
22. $\sqrt{245}$ $7\sqrt{5}$
23. $\sqrt{5} \cdot \sqrt{10}$ $5\sqrt{2}$
24. $\sqrt{10} \cdot \sqrt{20}$ $10\sqrt{2}$
25. $3\sqrt{8} \cdot 2\sqrt{7}$ $12\sqrt{14}$
26. $4\sqrt{2} \cdot 5\sqrt{8}$ 80
27. $3\sqrt{25t^2}$ $15|t|$
28. $5\sqrt{81q^5}$ $45q^2\sqrt{q}$
29. $\sqrt{28a^2b^3}$ $2|a|b\sqrt{7b}$
30. $\sqrt{75qr^3}$ $5r\sqrt{3qr}$
31. $7\sqrt{63m^3p}$ $21m\sqrt{7mp}$
32. $4\sqrt{66g^2h^4}$ $4|g|h^2\sqrt{66}$
33. $\sqrt{2ab^2} \cdot \sqrt{10a^5b}$ $2a^3b\sqrt{5b}$
34. $\sqrt{4c^3d^3} \cdot \sqrt{8c^3d}$ $4c^3d^2\sqrt{2}$

35. **ROLLER COASTER** Starting from a stationary position, the velocity v of a roller coaster in meter per second at the bottom of a hill can be approximated by $v = \sqrt{19.6h}$, where h is the height of the hill in meter.

   a. Determine the velocity of a roller coaster at the bottom of a 41 meters hill. about 28.34 m/s

36. **PRECISION** When fighting a fire, the velocity v of water being pumped into the air is modeled by the function $v = \sqrt{2hg}$, where h represents the maximum height of the water and g represents the acceleration due to gravity (9.8 m/s²).

   a. Solve the function for h. $h = \frac{v^2}{2g}$

   b. The Hollowville Fire Department needs a pump that will propel water 24 meters into the air. Will a pump advertised to project water with a velocity of 21 m per second meet their needs? Explain. No; sample answer: The advertised pump will pump water only to a maximum height of about 20.28 meters.

   c. The Fire Department must purchase a pump that will propel water 27.5 m into the air. Will a pump that is advertised to project water with a velocity of 23 m per second meet the fire department's need? Explain. Yes; sample answer: The advertised pump will pump water only to a maximum height of about 23.05 meters.

WatchOut!

Preventing Errors Students may not think the simplified form of a fraction containing radicals looks any simpler than the original fraction. Explain that meeting the three conditions for simplifying a radical expression, not the look of the fraction, dictates whether a fraction containing radicals is simplified.

3 Practice

Formative Assessment

Use Exercises 1–16 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Teaching the Mathematical Practices

Precision Mathematically proficient students express numerical answers with a degree of precision appropriate for the problem context. In Exercise 36, point out that the calculated values only have to be accurate enough to determine if the needs are met.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	17–48, 52, 54, 55, 57–75	17–47 odd, 58–61	18–48 even, 52, 54, 55, 57, 62–75
OL Core	17–47 odd, 49–52, 54, 55, 57–75	17–48, 58–61	49–52, 54, 55, 57, 62–75
BL Advanced	49–75		

Teaching the Mathematical Practices

Structure Mathematically proficient students can step back for an overview and shift perspective. In Exercise 52, tell students one way to start is by thinking about when two squared numbers are equal.

Examples 4–5 Simplify each expression.

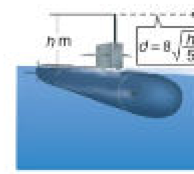
37. $\sqrt{\frac{32}{t^4}}$ $\frac{4\sqrt{2}}{t^2}$
38. $\sqrt{\frac{27}{m^5}}$ $\frac{3\sqrt{3m}}{m^3}$
39. $\frac{\sqrt{68ac^3}}{\sqrt{27a^2}}$ $\frac{2c\sqrt{51ac}}{9|a|}$
40. $\frac{\sqrt{h^3}}{\sqrt{8}}$ $\frac{h\sqrt{2h}}{4}$
41. $\sqrt{\frac{3}{16}} \cdot \sqrt{\frac{9}{5}}$ $\frac{3\sqrt{15}}{20}$
42. $\sqrt{\frac{7}{2}} \cdot \sqrt{\frac{5}{3}}$ $\frac{\sqrt{210}}{6}$
43. $\frac{7}{5 + \sqrt{3}}$ $\frac{35 - 7\sqrt{3}}{22}$
44. $\frac{9}{6 - \sqrt{8}}$ $\frac{27 + 9\sqrt{2}}{14}$
45. $\frac{3\sqrt{3}}{-2 + \sqrt{6}}$ $\frac{6\sqrt{3} + 9\sqrt{2}}{2}$
46. $\frac{3}{\sqrt{7} - \sqrt{2}}$ $\frac{3\sqrt{7} + 3\sqrt{2}}{5}$
47. $\frac{5}{\sqrt{6} + \sqrt{3}}$ $\frac{5\sqrt{6} - 5\sqrt{3}}{3}$
48. $\frac{2\sqrt{5}}{2\sqrt{7} + 3\sqrt{3}}$ $4\sqrt{35} - 6\sqrt{15}$

B

49. **ELECTRICITY** The amount of current in amperes I that an appliance uses can be calculated using the formula $I = \sqrt{\frac{P}{R}}$, where P is the power in watts and R is the resistance in ohms.
   a. Simplify the formula. $I = \frac{\sqrt{PR}}{R}$
   b. How much current does an appliance use if the power used is 75 watts and the resistance is 5 ohms? **about 3.9 amps**

50. **KINETIC ENERGY** The speed v of a ball can be determined by the equation $v = \sqrt{\frac{2k}{m}}$, where k is the kinetic energy and m is the mass of the ball.
   a. Simplify the formula if the mass of the ball is 3 kilograms. $v = \frac{\sqrt{6k}}{3}$
   b. If the ball is traveling 7 meters per second, what is the kinetic energy of the ball in Joules? **73.5 Joules**

51. **SUBMARINES** The greatest distance d in kilometers that a lookout can see on a clear day is modeled by the formula shown. Determine how high the submarine would have to raise its periscope to see a ship, if the submarine is the given distances away from the ship. Round your answers to one decimal place.



Distance	3	6	9	12	15
Height	0.7	2.8	6.3	11.3	17.6

H.O.T. Problems Use Higher-Order Thinking Skills

C

52. **STRUCTURE** Explain how to solve $\frac{\sqrt{3+2}}{x} = \frac{\sqrt{3-1}}{\sqrt{3}}$. **See margin.**

53. **CHALLENGE** Simplify each expression.
   a. $\sqrt[3]{27}$ 3
   b. $\sqrt[3]{40}$ $2\sqrt[3]{5}$
   c. $\sqrt[3]{750}$ $5\sqrt[3]{6}$

54. **REASONING** Amna takes a number, subtracts 4, multiplies by 4, takes the square root, and takes the reciprocal to get $\frac{1}{2}$. What number did she start with? Write a formula to describe the process.

   54. $5; \frac{1}{\sqrt{4(x-4)}} = \frac{1}{2}$

55. **OPEN ENDED** Write two binomials of the form $a\sqrt{b} + c\sqrt{f}$ and $a\sqrt{b} - c\sqrt{f}$. Then find their product.
   Sample answer: $1 + \sqrt{2}$ and $1 - \sqrt{2}$; $(1 + \sqrt{2}) \bullet (1 - \sqrt{2}) = 1 - 2 = -1$

56. **CHALLENGE** Use the Quotient Property of Square Roots to derive the Quadratic Formula by solving the quadratic equation $ax^2 + bx + c = 0$. (*Hint*: Begin by completing the square.) **See Ch. 9 Answer Appendix.**

57. **WRITING IN MATH** Summarize how to write a radical expression in simplest form. **See margin.**

562 | Lesson 9-2 | Simplifying Radical Expressions

Standardized Test Practice

58. Salem's electric bill is AED 23 less than his natural gas bill. The two bills are a total of AED 109. Which of the following equations can be used to find the amount of his natural gas bill? **D**

A $g + g = 109$ C $g - 23 = 109$

B $23 + 2g = 109$ D $2g - 23 = 109$

59. Solve $a^2 - 2a + 1 = 25$. **H**

F $-4, -6$ H $-4, 6$

G $4, -6$ J $4, 6$

60. The expression $\sqrt{160x^2y^5}$ is equivalent to which of the following? **C**

A $16|x|y^2\sqrt{10y}$ C $4|x|y^2\sqrt{10y}$

B $|x|y^2\sqrt{160y}$ D $10|x|y^2\sqrt{4y}$

61. GRIDDED RESPONSE Faris earns AED 10 an hour and 10% commission on sales. If Faris worked 38 hours and had a total sales of AED 1275 last week, how much did she make? **507.50**

Spiral Review

Graph each function. Compare to the parent graph. State the domain and range. (Lesson 9-1) **62–67. See Ch. 9 Answer Appendix.**

62. $y = 2\sqrt{x} - 1$ **63.** $y = \frac{1}{2}\sqrt{x}$ **64.** $y = 2\sqrt{x + 2}$

65. $y = -\sqrt{x + 1}$ **66.** $y = -3\sqrt{x - 3}$ **67.** $y = -2\sqrt{x} + 1$

68. POPULATION The country of Latvia has been experiencing a 1.1% annual decrease in population. In 2009, its population was 2,261,294. If the trend continues, predict Latvia's population in 2019. **about 2,024,510**

69. TOMATOES There are more than 10,000 varieties of tomatoes. One seed company produces seed packages for 200 varieties of tomatoes. For how many varieties do they not provide seeds? **Sample answer: Let t = the number of tomato varieties for which they do not produce seeds, $t + 200 > 10{,}000$; $\{t \mid t > 9800\}$.**

Skills Review

Write the prime factorization of each number.

70. 24 $2^3 \cdot 3$ **71.** 88 $2^3 \cdot 11$ **72.** 180 $2^2 \cdot 3^2 \cdot 5$

73. 31 **31** **74.** 60 $2^2 \cdot 3 \cdot 5$ **75.** 90 $2 \cdot 3^2 \cdot 5$

4 Assess

Ticket Out the Door Ask students to write radical expressions in which they need to use a conjugate to rationalize the denominators. Have students simplify the expressions.

Additional Answers

52. Sample answer: Cross multiply then divide. Rationalize the denominator to find that $x = \frac{5\sqrt{3} + 9}{2}$

57. No radicals can appear in the denominator of a fraction. So, rationalize the denominator to get rid of the radicand in the denominator. Then check if any of the radicands have perfect square factors other than 1. If so, simplify.

Differentiated Instruction BL

Extension Find a number other than 1 that has a whole number square root, cube root, and fourth root. Sample answer: 4096

1 Focus

Objective Investigate the products and sums of two rational numbers, two irrational numbers, and a rational and irrational number.

Teaching Tip

Emphasize that using examples with actual numbers cannot actually prove the conclusions made in the activities. This would need to be done algebraically. To increase the reliability of each conclusion, examples should be done using all types of numbers in the set(s).

2 Teach

Working in Cooperative Groups

Have students work in pairs, mixing abilities. Have them first make a list of subsets for rational and irrational numbers. After reading Activities 1 and 2, have them test more pairs of numbers from the subsets they listed.

Ask:

- Did you find any examples that did not agree with the conclusion in each activity? no
- What would finding one example that did not support the conclusion indicate? One example would be a counter example that would prove that the statement is false.

Practice Have students complete Exercises 1–3.

3 Assess

Formative Assessment

Use Exercises 1 and 2 to assess if students can investigate operations completed with specific types of numbers.

EXTEND 9-2 Algebra Lab: Rational and Irrational Numbers

A set is closed under an operation if for any numbers in the set, the result of the operation is also in the set. A set may be closed under one operation and not closed under another.

Mathematical Practices
Look for and make use of structure.

Activity 1 Closure of Rational Numbers and Irrational Numbers

Are the sets of rational and irrational numbers closed under multiplication? under addition?

Step 1 To determine if each set is closed under multiplication, examine several products of two rational factors and then two irrational factors.

Rational: $5 \times 2 = 10$; $-3 \times 4 = -12$; $3.7 \times 0.5 = 1.85$; $\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

Irrational: $\pi \times \sqrt{2} = \sqrt{2}\pi$; $\sqrt{3} \times \sqrt{7} = \sqrt{21}$; $\sqrt{5} \times \sqrt{5} = 5$

The product of each pair of rational numbers is rational. However, the products of pairs of irrational numbers are both irrational and rational. Thus, it appears that the set of rational numbers is closed under multiplication, but the set of irrational numbers is not.

Step 2 Repeat this process for addition.

Rational: $3 + 8 = 11$; $-4 + 7 = 3$; $3.7 + 5.82 = 9.52$; $\frac{2}{5} + \frac{1}{4} = \frac{13}{20}$

Irrational: $\sqrt{3} + \pi = \sqrt{3} + \pi$; $3\sqrt{5} + 6\sqrt{5} = 9\sqrt{5}$; $\sqrt{12} + \sqrt{50} = 2\sqrt{3} + 5\sqrt{2}$

The sum of each pair of rational numbers is rational, and the sum of each pair of irrational numbers is irrational. Both sets are closed under addition.

Activity 2 Rational and Irrational Numbers

What kind of numbers are the product and sum of a rational and irrational number?

Step 1 Examine the products of several pairs of rational and irrational numbers.

$3 \times \sqrt{8} = 6\sqrt{2}$; $\frac{3}{4} \times \sqrt{2} = \frac{3\sqrt{2}}{4}$; $1 \times \sqrt{7} = \sqrt{7}$; $0 \times \sqrt{5} = 0$

The product is rational only when the rational factor is 0. The product of each nonzero rational number and irrational number is irrational.

Step 2 Find the sums of several pairs of a rational and irrational number.

$5 + \sqrt{3} = 5 + \sqrt{3}$; $\frac{2}{3} + \sqrt{5} = \frac{2 + 3\sqrt{5}}{3}$; $-4 + \sqrt{6} = -1(4 - \sqrt{6})$

The sum of each rational and irrational number is irrational.

Analyze the Results

1. What kinds of numbers are the difference of two unique rational numbers, two unique irrational numbers, and a rational and an irrational number? **1. The difference of any two unique rational numbers is always a rational number. The difference of any two unique irrational numbers is always irrational. The difference of any rational and irrational numbers is always irrational.**
2. Is the quotient of every rational and irrational number always another rational or irrational number? If not, provide a counterexample. **No; sample answer: $\frac{\sqrt{5}}{0}$ does not equal a real number.**
3. **CHALLENGE** Recall that rational numbers are numbers that can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Using $\frac{a}{b}$ and $\frac{c}{d}$ show that the sum and product of two rational numbers must always be a rational number. **See margin.**

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From Concrete to Abstract

After students complete Exercise 3, ask the class for possible ways that irrational numbers could be represented and used to test the product prediction from Activity 1.

Additional Answer

3. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ and $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ The sum and product of two integers is always an integer, so $(ad + bc)$, ac, and bd are all integers. By definition of a rational number, the sum and product of the two rational numbers are rational.

LESSON 9-3 Operations with Radical Expressions

Then

- You simplified radical expressions.

Now

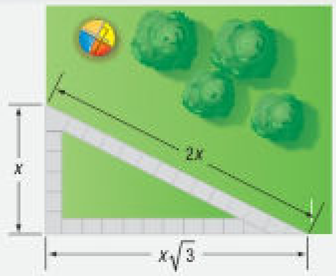
1. Add and subtract radical expressions.
2. Multiply radical expressions.

Why?

- Noura is going to run in her neighborhood to get ready for the soccer season. She plans to run the course that she has laid out three times each day.

  How far does Noura have to run to complete the course that she laid out?

  How far does she run every day?



Mathematical Practices
Reason abstractly and quantitatively.

1 Add or Subtract Radical Expressions

To add or subtract radical expressions, the radicands must be alike in the same way that monomial terms must be alike to add or subtract.

Monomials	Radical Expressions
$4a + 2a = (4 + 2)a$	$4\sqrt{5} + 2\sqrt{5} = (4 + 2)\sqrt{5}$
$= 6a$	$= 6\sqrt{5}$
$9b - 2b = (9 - 2)b$	$9\sqrt{3} - 2\sqrt{3} = (9 - 2)\sqrt{3}$
$= 7b$	$= 7\sqrt{3}$

Notice that when adding and subtracting radical expressions, the radicand does not change. This is the same as when adding or subtracting monomials.

Example 1 Add and Subtract Expressions with Like Radicands

Simplify each expression.

a. $5\sqrt{2} + 7\sqrt{2} - 6\sqrt{2}$

$5\sqrt{2} + 7\sqrt{2} - 6\sqrt{2} = (5 + 7 - 6)\sqrt{2}$ Distributive Property

$= 6\sqrt{2}$ Simplify.

b. $10\sqrt{7} + 5\sqrt{11} + 4\sqrt{7} - 6\sqrt{11}$

$10\sqrt{7} + 5\sqrt{11} + 4\sqrt{7} - 6\sqrt{11} = (10 + 4)\sqrt{7} + (5 - 6)\sqrt{11}$ Distributive Property

$= 14\sqrt{7} - \sqrt{11}$ Simplify.

Guided Practice **1C.** $4\sqrt{3} - 8\sqrt{5}$ **1D.** $-2\sqrt{3} + 6\sqrt{7}$

1A. $3\sqrt{2} - 5\sqrt{2} + 4\sqrt{2}$ $2\sqrt{2}$

1B. $6\sqrt{11} + 2\sqrt{11} - 9\sqrt{11}$ $-\sqrt{11}$

1C. $15\sqrt{3} - 14\sqrt{5} + 6\sqrt{5} - 11\sqrt{3}$

1D. $4\sqrt{3} + 3\sqrt{7} - 6\sqrt{3} + 3\sqrt{7}$

Not all radical expressions have like radicands. Simplifying the expressions may make it possible to have like radicands so that they can be added or subtracted.

1 Focus

Vertical Alignment

Before Lesson 9-3 Simplify radical expressions.

Lesson 9-3 Add and subtract radical expressions. Multiply radical expressions.

After Lesson 9-3 Solve radical equations.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What expression shows the length of the course? $x + 2x + x\sqrt{3}$
- Which terms in the expression can be combined? Explain. x and $2x$ because they are like terms
- Which term cannot be combined? Explain. $x\sqrt{3}$ because it contains a radical
- What expression shows how far Conchita will run each day? $x(3 + \sqrt{3})$

1 Add or Subtract Radical Expressions

Example 1 shows how to use the Distributive Property to add or subtract like radicands in a radical expression. **Example 2** shows how to simplify the radicals in a radical expression with unlike radicands.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Simplify each expression.

a. $6\sqrt{5} + 2\sqrt{5} - 5\sqrt{5}$ $3\sqrt{5}$

b. $7\sqrt{2} + 8\sqrt{11} - 4\sqrt{11} - 6\sqrt{2}$ $\sqrt{2} + 4\sqrt{11}$

2 Simplify $6\sqrt{27} + 8\sqrt{12} + 2\sqrt{75}$. $44\sqrt{3}$

2 Multiply Radical Expressions

Example 3 shows how to multiply radical expressions with different radicands. **Example 4** shows how to multiply radical expressions to find the area of a rectangle.

Additional Example

3 Simplify each expression.

a. $2\sqrt{3} \cdot 4\sqrt{6}$ $24\sqrt{2}$

b. $4\sqrt{2}(3\sqrt{2} + 2\sqrt{6})$ $24 + 16\sqrt{3}$

StudyTip

Simplify First Simplify each radical term first. Then perform the operations needed.

Example 2 Add and Subtract Expressions with Unlike Radicands

Simplify $2\sqrt{18} + 2\sqrt{32} + \sqrt{72}$.

$2\sqrt{18} + 2\sqrt{32} + \sqrt{72} = 2(\sqrt{3^2} \cdot \sqrt{2}) + 2(\sqrt{4^2} \cdot \sqrt{2}) + (\sqrt{6^2} \cdot \sqrt{2})$ Product Property

$= 2(3\sqrt{2}) + 2(4\sqrt{2}) + (6\sqrt{2})$ Simplify.

$= 6\sqrt{2} + 8\sqrt{2} + 6\sqrt{2}$ Multiply.

$= 20\sqrt{2}$ Simplify.

GuidedPractice

2A. $4\sqrt{54} + 2\sqrt{24}$ $16\sqrt{6}$

2B. $4\sqrt{12} - 6\sqrt{48}$ $-16\sqrt{3}$

2C. $3\sqrt{45} + \sqrt{20} - \sqrt{245}$ $4\sqrt{5}$

2D. $\sqrt{24} - \sqrt{54} + \sqrt{96}$ $3\sqrt{6}$

2 Multiply Radical Expressions

Multiplying radical expressions is similar to multiplying monomial algebraic expressions. Let $x \geq 0$.

Monomials	Radical Expressions
$(2x)(3x) = 2 \cdot 3 \cdot x \cdot x$	$(2\sqrt{x})(3\sqrt{x}) = 2 \cdot 3 \cdot \sqrt{x} \cdot \sqrt{x}$
$= 6x^2$	$= 6x$

You can also apply the Distributive Property to radical expressions.

WatchOut!

Multiplying Radicands Make sure that you multiply the radicands when multiplying radical expressions. A common mistake is to add the radicands rather than multiply.

Example 3 Multiply Radical Expressions

Simplify each expression.

a. $3\sqrt{2} \cdot 2\sqrt{6}$

$3\sqrt{2} \cdot 2\sqrt{6} = (3 \cdot 2)(\sqrt{2} \cdot \sqrt{6})$ Associative Property

$= 6(\sqrt{12})$ Multiply.

$= 6(2\sqrt{3})$ Simplify.

$= 12\sqrt{3}$ Multiply.

b. $3\sqrt{5}(2\sqrt{5} + 5\sqrt{3})$

$3\sqrt{5}(2\sqrt{5} + 5\sqrt{3}) = (3\sqrt{5} \cdot 2\sqrt{5}) + (3\sqrt{5} \cdot 5\sqrt{3})$ Distributive Property

$= [(3 \cdot 2)(\sqrt{5} \cdot \sqrt{5})] + [(3 \cdot 5)(\sqrt{5} \cdot \sqrt{3})]$ Associative Property

$= [6(\sqrt{25})] + [15(\sqrt{15})]$ Multiply.

$= [6(5)] + [15(\sqrt{15})]$ Simplify.

$= 30 + 15\sqrt{15}$ Multiply.

GuidedPractice

3A. $2\sqrt{6} \cdot 7\sqrt{3}$ $42\sqrt{2}$

3B. $9\sqrt{5} \cdot 11\sqrt{15}$ $495\sqrt{3}$

3C. $3\sqrt{2}(4\sqrt{3} + 6\sqrt{2})$ $12\sqrt{6} + 36$

3D. $5\sqrt{3}(3\sqrt{2} - \sqrt{3})$ $15\sqrt{6} - 15$

You can also multiply radical expressions with more than one term in each factor. This is similar to multiplying two algebraic binomials with variables.

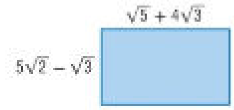
DifferentiatedInstruction AL OL

If students struggle with the FOIL method in Example 4,

Then have students rewrite the example in five steps to facilitate an understanding of First terms, Outer terms, Inner terms, and Last terms. For example, in Step 1, have students write the original expression, underline the first terms, and then multiply to find the product of the first terms. Students continue this process until they have found the products of all of the terms. Then for the fifth step, have them combine and simplify the results of the previous four steps.

Real-World Example 4 Multiply Radical Expressions

GEOMETRY Find the area of the rectangle in simplest form.



$A = (5\sqrt{2} - \sqrt{3})(\sqrt{5} + 4\sqrt{3})$ $\quad A = \ell \cdot w$

First Terms, Outer Terms, Inner Terms, Last Terms

$= (5\sqrt{2})(\sqrt{5}) + (5\sqrt{2})(4\sqrt{3}) + (-\sqrt{3})(\sqrt{5}) + (\sqrt{3})(4\sqrt{3})$

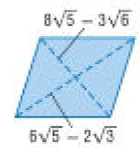
$= 5\sqrt{10} + 20\sqrt{6} - \sqrt{15} - 4\sqrt{9}$ Multiply.

$= 5\sqrt{10} + 20\sqrt{6} - \sqrt{15} - 12$ Simplify.

> **Review Vocabulary**
>
> **FOIL Method** Multiply two binomials by finding the sum of the products of the First terms, the Outer terms, the Inner terms, and the Last terms.

Guided Practice

4. **GEOMETRY** The area A of a rhombus can be found using the equation $A = \frac{1}{2}d_1d_2$, where d_1 and d_2 are the lengths of the diagonals. What is the area of the rhombus at the right? $A = 120 - 8\sqrt{15} - 9\sqrt{30} + 9\sqrt{2}$



Concept Summary Operations with Radical Expressions

Operation	Symbols	Example
addition, $b \ge 0$	$a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$ like radicands	$4\sqrt{3} + 6\sqrt{3} = (4 + 6)\sqrt{3}$ $= 10\sqrt{3}$
subtraction, $b \ge 0$	$a\sqrt{b} + c\sqrt{b} = (a - c)\sqrt{b}$ like radicands	$12\sqrt{5} - 8\sqrt{5} = (12 - 8)\sqrt{5}$ $= 4\sqrt{5}$
multiplication, $b \ge 0, g \ge 0$	$a\sqrt{b}(f\sqrt{g}) = af\sqrt{bg}$ Radicands do not have to be like radicands.	$3\sqrt{2}(5\sqrt{7}) = (3 \cdot 5)(\sqrt{2 \cdot 7})$ $= 15\sqrt{14}$

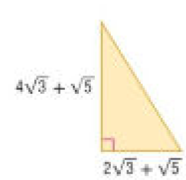
Check Your Understanding

Examples 1–3 Simplify each expression.

1. $3\sqrt{5} + 6\sqrt{5}$ $9\sqrt{5}$
2. $8\sqrt{3} + 5\sqrt{3}$ $13\sqrt{3}$
3. $\sqrt{7} - 6\sqrt{7}$ $-5\sqrt{7}$
4. $10\sqrt{2} - 6\sqrt{2}$ $4\sqrt{2}$
5. $4\sqrt{5} + 2\sqrt{20}$ $8\sqrt{5}$
6. $\sqrt{12} - \sqrt{3}$ $\sqrt{3}$
7. $\sqrt{8} + \sqrt{12} + \sqrt{18}$ $5\sqrt{2} + 2\sqrt{3}$
8. $\sqrt{27} + 2\sqrt{3} - \sqrt{12}$ $3\sqrt{3}$
9. $9\sqrt{2}(4\sqrt{6})$ $72\sqrt{3}$
10. $4\sqrt{3}(8\sqrt{3})$ 96
11. $\sqrt{3}(\sqrt{7} + 3\sqrt{2})$ $\sqrt{21} + 3\sqrt{6}$
12. $\sqrt{5}(\sqrt{2} + 4\sqrt{2})$ $5\sqrt{10}$

Example 4

13. **GEOMETRY** The area A of a triangle can be found by using the formula $A = \frac{1}{2}bh$, where b represents the base and h is the height. What is the area of the triangle at the right? $14.5 + 3\sqrt{15}$



Focus on Mathematical Content

Operations with Radical Expressions Radical expressions can be added or subtracted only if the radicands are the same. Radical expressions can be multiplied whether the radicands are the same or different.

Additional Example

4 **GEOMETRY** Find the area of a rectangle in simplest form with a width of $4\sqrt{6} - 2\sqrt{10}$ and a length of $5\sqrt{3} + 7\sqrt{5}$. $18\sqrt{30} - 10\sqrt{2}$

3 Practice

Formative Assessment

Use Exercises 1–13 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Teach with Tech

Interactive Whiteboard Write an expression on the board to add or subtract radical expressions. As you simplify the expression, drag the like terms to group them together. Then combine like terms and simplify the expression.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	14–26, 37–62	15–25 odd, 40–43	14–26 even, 37–39, 44–62
OL Core	15–33 odd, 34, 35, 37–62	14–26, 40–43	27–35, 37–39, 44–62
BL Advanced	27–62		

Teaching the Mathematical Practices

Arguments Mathematically proficient students make conjectures and build a logical progression of statements to explore the truth of their conjectures. In Exercise 37, suggest that students use a calculator to find several sums and products.

Practice and Problem Solving

Examples 1–3 Simplify each expression. **18.** $12\sqrt{3}+\sqrt{2}$ **23.** $3\sqrt{5}+6-\sqrt{30}-2\sqrt{6}$ **24.** $9\sqrt{33}-6\sqrt{22}+27\sqrt{5}-6\sqrt{30}$

14. $7\sqrt{5}+4\sqrt{5}$ $11\sqrt{5}$

15. $2\sqrt{6}+9\sqrt{6}$ $11\sqrt{6}$

16. $3\sqrt{5}-2\sqrt{20}$ $-\sqrt{5}$

17. $3\sqrt{50}-3\sqrt{32}$ $3\sqrt{2}$

18. $7\sqrt{3}-2\sqrt{2}+3\sqrt{2}+5\sqrt{3}$

19. $\sqrt{5}(\sqrt{2}+4\sqrt{2})$ $5\sqrt{10}$

20. $\sqrt{6}(2\sqrt{10}+3\sqrt{2})$ $4\sqrt{15}+6\sqrt{3}$

21. $4\sqrt{5}(3\sqrt{5}+8\sqrt{2})$ $60+32\sqrt{10}$

22. $5\sqrt{3}(6\sqrt{10}-6\sqrt{3})$ $30\sqrt{30}-90$

23. $(\sqrt{3}-\sqrt{2})(\sqrt{15}+\sqrt{12})$

24. $(3\sqrt{11}+3\sqrt{15})(3\sqrt{3}-2\sqrt{2})$

25. $(5\sqrt{2}+3\sqrt{5})(2\sqrt{10}-5)$ $5\sqrt{5}+5\sqrt{2}$

Example 4

26. GEOMETRY Find the perimeter and area of a rectangle with a width of $2\sqrt{7}-2\sqrt{5}$ and a length of $3\sqrt{7}+3\sqrt{5}$. $10\sqrt{7}+2\sqrt{5}$ units; 12 units^2

Simplify each expression. **30.** $8\sqrt{5}$

B

27. $\sqrt{\frac{1}{5}}-\sqrt{5}$ $\frac{-4\sqrt{5}}{5}$

28. $\sqrt{\frac{2}{3}}+\sqrt{6}$ $\frac{4\sqrt{6}}{3}$

29. $2\sqrt{\frac{1}{2}}+2\sqrt{2}-\sqrt{8}$ $\sqrt{2}$

30. $8\sqrt{\frac{5}{4}}+3\sqrt{20}-10\sqrt{\frac{1}{5}}$

31. $(3-\sqrt{5})^2$ $14-6\sqrt{5}$

32. $(\sqrt{2}+\sqrt{3})^2$ $5+2\sqrt{6}$

33. ROLLER COASTERS The velocity v in meters per second of a roller coaster at the bottom of a hill is related to the vertical drop h in meters and the velocity v_0 of the coaster at the top of the hill by the formula $v_0=\sqrt{v^2-20h}$.

a. What velocity must a coaster have at the top of a 68.8 meter hill to achieve a velocity of 37 meters per second at the bottom? 0 m/s

34. FINANCIAL LITERACY Ibrahim invests AED 225 in a savings account. In two years, Ibrahim has AED 232 in his account. You can use the formula $r=\sqrt{\frac{v_2}{v_0}}-1$ to find the average annual interest rate r that the account has earned. The initial investment is v_0, and v_2 is the amount in two years. What was the average annual interest rate that Ibrahim's account earned? about 1.5%

35. ELECTRICITY Electricians can calculate the electrical current in amps A by using the formula $A=\frac{\sqrt{w}}{\sqrt{r}}$, where w is the power in watts and r the resistance in ohms. How much electrical current is running through a microwave oven that has 850 watts of power and 5 ohms of resistance? Write the number of amps in simplest radical form, and then estimate the amount of current to the nearest tenth. $\sqrt{170}$; about 13 amps

H.O.T. Problems Use Higher-Order Thinking Skills

C

36. CHALLENGE Determine whether the following statement is *true* or *false*. Provide a proof or counterexample to support your answer. See margin.

$$x+y>\sqrt{x^2+y^2} \text{ when } x>0 \text{ and } y>0$$

37. ARGUMENTS Make a conjecture about the sum of a rational number and an irrational number. Is the sum *rational* or *irrational*? Is the product of a nonzero rational number and an irrational number *rational* or *irrational*? Explain your reasoning. See margin.

38. OPEN ENDED Write an equation that shows a sum of two radicals with different radicands. Explain how you could combine these terms. See margin.

39. WRITING IN MATH Describe step by step how to multiply two radical expressions, each with two terms. Write an example to demonstrate your description. See margin.

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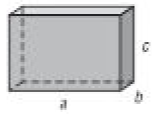
Standardized Test Practice

40. SHORT RESPONSE The population of a town is 13,000 and is increasing by about 250 people per year. This can be represented by the equation $p = 13{,}000 + 250y$, where y is the number of years from now and p represents the population. In how many years will the population of the town be 14,500? **6 years**

41. GEOMETRY Which expression represents the sum of the lengths of the 12 edges on this rectangular solid? **C**



A $2(a + b + c)$
B $3(a + b + c)$
C $4(a + b + c)$
D $12(a + b + c)$

42. Evaluate $\sqrt{n - 9}$ and $\sqrt{n} - \sqrt{9}$ for $n = 25$. **G**

F 4; 4 G 4; 2
H 2; 4 J 2; 2

43. The current I in a simple electrical circuit is given by the formula $I = \frac{V}{R}$, where V is the voltage and R is the resistance of the circuit. If the voltage remains unchanged, what effect will doubling the resistance of the circuit have on the current? **C**

A The current will remain the same.
B The current will double its previous value.
C The current will be half its previous value.
D The current will be two units more than its previous value.

Spiral Review

Simplify. (Lesson 9-2)

44. $\sqrt{18}$ **$3\sqrt{2}$** **45.** $\sqrt{24}$ **$2\sqrt{6}$** **46.** $\sqrt{60}$ **$2\sqrt{15}$**

47. $\sqrt{50a^3b^5}$ **$5ab^2\sqrt{2ab}$** **48.** $\sqrt{169x^4y^7}$ **$13x^2y^3\sqrt{y}$** **49.** $\sqrt{63c^3d^4f^5}$ **$3cd^2f^2\sqrt{7cf}$**

Graph each function. Compare to the parent graph. State the domain and range. (Lesson 9-1) **50–55. See Ch. 9 Answer Appendix.**

50. $y = 2\sqrt{x}$ **51.** $y = -3\sqrt{x}$ **52.** $y = \sqrt{x + 1}$

53. $y = \sqrt{x - 4}$ **54.** $y = \sqrt{x} + 3$ **55.** $y = \sqrt{x} - 2$

56. FINANCIAL LITERACY Determine the value of an investment if AED 400 is invested at an interest rate of 7.25% compounded quarterly for 7 years. **AED 661.44**

Skills Review

Solve each equation. Round each solution to the nearest tenth, if necessary.

57. $-4c - 1.2 = 0.8$ **−0.5** **58.** $-2.6q - 33.7 = 84.1$ **−45.3** **59.** $0.3m + 4 = 9.6$ **18.7**

60. $-10 - \frac{n}{5} = 6$ **−80** **61.** $\frac{-4h - (-5)}{-7} = 13$ **24** **62.** $3.6t + 6 - 2.5t = 8$ **1.8**

DifferentiatedInstruction OL BL

Extension Write $\sqrt{6} + \sqrt{3} = \sqrt{9} = 3$ and $\sqrt{40} = \sqrt{36} + \sqrt{4} = 8$ on the board. Ask students to explain the errors. You cannot add radical expressions unless they can be written with the same radicand, and you cannot write the square root of a sum as the sum of the square roots.

4 Assess

Crystal Ball Ask students to write how they think what they learned today about adding, subtracting, and multiplying radical expressions will connect with the next lesson on radical equations.

Additional Answers

36. True; $(x + y)^2 > \left(\sqrt{x^2 + y^2}\right)^2$
$x^2 + 2xy + y^2 > x^2 + y^2$
$2xy > 0$
Because $x > 0$ and $y > 0$, $2xy > 0$ is always true. so $x + y > \sqrt{x^2 + y^2}$ is true for all $x > 0$ and $y > 0$.

37. Irrational; irrational; no rational number could be added to or multiplied by an irrational number so that the result is rational.

38. Sample answer: $\sqrt{12} + \sqrt{27} = 5\sqrt{3}$; When you simplify $\sqrt{12}$, you get $2\sqrt{3}$. When you simplify $\sqrt{27}$, you get $3\sqrt{3}$. Because $2\sqrt{3}$ and $3\sqrt{3}$ have the same radicand, you can add them.

39. Sample answer: You can use the FOIL method. You multiply the first terms within the parentheses. Then you multiply the outer terms within the parentheses. Then you would multiply the inner terms within the parentheses. And, then you would multiply the last terms within each parentheses. Combine any like terms and simplify any radicals. For example, $(\sqrt{2} + \sqrt{3})(\sqrt{5} + \sqrt{7}) = \sqrt{10} + \sqrt{14} + \sqrt{15} + \sqrt{21}$.

1 Focus

VerticalAlignment

Before Lesson 9-4 Add, subtract, and multiply radical expressions.

Lesson 9-4 Solve radical equations. Solve radical equations with extraneous solutions.

After Lesson 9-4 Identify and use inverse variations.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- If you know the value for h, what do you need to isolate to solve the equation? $\sqrt{\ell}$
- How can you isolate it? Divide each side by 1.34.
- How can you remove the radical sign? Square each side of the equation.

LESSON 9-4 Radical Equations

Then

- You added, subtracted, and multiplied radical expressions.

Now

1. Solve radical equations.
2. Solve radical equations with extraneous solutions.

Why?

- The waterline length of a sailboat is the length of the line made by the water's edge when the boat is full. A sailboat's hull speed is the fastest speed that it can travel.

  You can estimate hull speed h by using the formula $h = 1.34\sqrt{\ell}$, where ℓ is the length of the sailboat's waterline.

NewVocabulary
radical equations
extraneous solutions

Mathematical Practices
Construct viable arguments and critique the reasoning of others.
Model with mathematics.

1 Radical Equations Equations that contain variables in the radicand, like $h = 1.34\sqrt{\ell}$, are called **radical equations**. To solve, isolate the desired variable on one side of the equation first. Then square each side of the equation to eliminate the radical.

KeyConcept Power Property of Equality

Words	If you square both sides of a true equation, the resulting equation is still true.
Symbols	If $a = b$, then $a^2 = b^2$.
Examples	If $\sqrt{x} = 4$, then $(\sqrt{x})^2 = 4^2$.

Real-World Example 1 Variable as a Radicand

SAILING Usama and Ismail are sailing in a friend's sailboat. They measure the hull speed at 9 kilometers per hour. Find the length of the sailboat's waterline. Round to the nearest meters.

Understand You know how fast the boat will travel and that it relates to the length.

Plan The boat travels at 9 nautical kilometers per hour. The formula for hull speed is $h = 1.34\sqrt{\ell}$.

Solve

$h = 1.34\sqrt{\ell}$ Formula for hull speed

$9 = 1.34\sqrt{\ell}$ Substitute 9 for h.

$\frac{9}{1.34} = \frac{1.34\sqrt{\ell}}{1.34}$ Divide each side by 1.34.

$6.72 \approx \sqrt{\ell}$ Simplify.

$(6.72)^2 \approx (\sqrt{\ell})^2$ Square each side of the equation.

$45.16 \approx \ell$ Simplify.

The sailboat's waterline length is about 45 meters.

Check Check by substituting the estimate into the original formula.

$h = 1.34\sqrt{\ell}$ Formula for hull speed

$9 \stackrel{?}{=} 1.34\sqrt{45}$ $h = 9$ and $\ell = 45$

$9 \approx 8.98899327$ ✓ Multiply.

Guided Practice

1. **DRIVING** The equation $v = \sqrt{21.4r}$ represents the maximum velocity that a car can travel safely on an unbanked curve when v is the maximum velocity in kilometers and r is the radius of the turn in meters. If a road is designed for a maximum speed of 505 kilometers per hour, what is the radius of the turn? 1690 m

To solve a radical equation, isolate the radical first. Then square both sides of the equation.

Example 2 Expression as a Radicand

Solve $\sqrt{a+5} + 7 = 12$.

$\sqrt{a+5} + 7 = 12$	Original equation
$\sqrt{a+5} = 5$	Subtract 7 from each side.
$(\sqrt{a+5})^2 = 5^2$	Square each side.
$a + 5 = 25$	Simplify.
$a = 20$	Subtract 5 from each side.

Watch Out!

Squaring Each Side Remember that when you square each side of the equation, you must square the entire side of the equation, even if there is more than one term on the side.

Guided Practice

Solve each equation.

2A. $\sqrt{c-3} - 2 = 4$ 39

2B. $4 + \sqrt{h+1} = 14$ 99

2 Extraneous Solutions

Squaring each side of an equation sometimes produces a solution that is not a solution of the original equation. These are called **extraneous solutions**. Therefore, you must check all solutions in the original equation.

Example 3 Variable on Each Side

Solve $\sqrt{k+1} = k - 1$. Check your solution.

$\sqrt{k+1} = k - 1$	Original equation
$(\sqrt{k+1})^2 = (k-1)^2$	Square each side.
$k + 1 = k^2 - 2k + 1$	Simplify.
$0 = k^2 - 3k$	Subtract k and 1 from each side.
$0 = k(k-3)$	Factor.
$k = 0$ or $k - 3 = 0$	Zero Product Property
$k = 3$	Solve.

CHECK

$\sqrt{k+1} = k - 1$	Original equation	$\sqrt{k+1} = k - 1$	Original equation
$\sqrt{0+1} \stackrel{?}{=} 0 - 1$	$k = 0$	$\sqrt{3+1} \stackrel{?}{=} 3 - 1$	$k = 3$
$\sqrt{1} \stackrel{?}{=} -1$	Simplify.	$\sqrt{4} \stackrel{?}{=} 2$	Simplify.
$1 \neq -1$ ✗	False	$2 = 2$ ✓	True

Since 0 does not satisfy the original equation, 3 is the only solution.

Study Tip

Extraneous Solutions When checking solutions for extraneous solutions, we are only interested in principal roots.

Guided Practice

Solve each equation. Check your solution.

3A. $\sqrt{t+5} = t + 3$ −1

3B. $x - 3 = \sqrt{x-1}$ 5

Differentiated Instruction OL BL

If students are familiar with graphing calculators,

Then have students graph the equation in Example 3 to check the solution. Have them subtract $k - 1$ from both sides of the equation and then enter the equation as $Y_1 = \sqrt{x+1} - x + 1$ [GRAPH]. Press [2nd] [CALC] 2 to calculate the zero point or x-intercept of the graph. Move the cursor to the left of the x-intercept for left bound, press [ENTER], and to the right of the x-intercept for right bound and press [ENTER]. Press [ENTER] to give the coordinates of the x-intercept.

1 Radical Equations

Example 1 shows how to solve a real-world problem with a variable in the radicand. **Example 2** shows how to solve a radical equation with a radical expression as the radicand.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 **FREE-FALL HEIGHT** An object is dropped from an unknown height and reaches the ground in 5 seconds. Use the equation $t = \sqrt{\frac{2h}{9.6}}$, where t is time in seconds and h is height in, to find the height from which the object was dropped. 120 m

2 Solve $\sqrt{x-3} + 8 = 15$. 52

2 Extraneous Solutions

Example 3 shows how to determine extraneous solutions when solving a radical equation with a variable on each side of the equal sign.

Additional Example

3 Solve $\sqrt{2-y} = y$. Check your solution. 1

Focus on Mathematical Content

Solutions to Radical Equations

When solving radical equations, it is always important to check all solutions in the *original* equation, since one or more of the solutions could be extraneous.

3 Practice

Formative Assessment

Use Exercises 1–7 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Multiple Representations

In Exercise 29, students solve a radical equation algebraically and by use of a graphing calculator and compare the solutions.

Exercise Alert

Graphing For Exercise 29, students will need a graphing calculator.

Teaching the Mathematical Practices

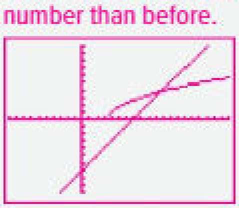
Critique Mathematically proficient students distinguish correct logic or reasoning from that which is flawed. In Exercise 31, have students carefully check each step in the two solutions. Have them identify the step in which the error was made and the likely cause. Have students discuss ways they can avoid similar errors.

Additional Answers

8c. Increases; sample answer: As the leg length increases, the value of the radicand also increases.

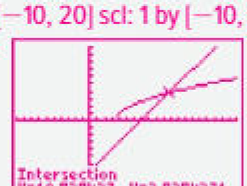
21b. Increases; sample answer: If the length is longer, the quotient and square root will be a greater number than before.

29a.



$[-10, 20]$ scl: 1 by $[-10, 10]$ scl: 1

29c.



$[-10, 20]$ scl: 1 by $[-10, 10]$ scl: 1

Check Your Understanding

Example 1

1. **GEOMETRY** The surface area of a basketball is x square centimeters. What is the radius of the basketball if the formula for the surface area of a sphere is $SA = 4\pi r^2$? $r = \frac{\sqrt{\pi x}}{2\pi}$

Examples 2–3 **Solve each equation. Check your solution.**

2. $\sqrt{10h} + 1 = 21$ **40**
3. $\sqrt{7r + 2} + 3 = 7$ **2**
4. $5 + \sqrt{g - 3} = 6$ **4**
5. $\sqrt{3x - 5} = x - 5$ **10**
6. $\sqrt{2n + 3} = n$ **3**
7. $\sqrt{a - 2} + 4 = a$ **6**

Practice and Problem Solving

Example 1

8. **EXERCISE** Suppose the function $S = \pi\sqrt{\frac{9.8\ell}{1.6}}$, where S represents speed in meters per second and ℓ is the leg length of a person in meters, can approximate the maximum speed that a person can run.
   a. What is the maximum running speed of a person with a leg length of 1.1 meters to the nearest tenth of a meter? **8.2 m/s**
   b. What is the leg length of a person with a running speed of 6.7 meters per second to the nearest tenth of a meter? **0.7 m**
   c. As leg length increases, does maximum speed increase or decrease? Explain. **See margin.**

Examples 2–3 **Solve each equation. Check your solution.**

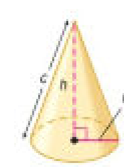
9. $\sqrt{a} + 11 = 21$ **100**
10. $\sqrt{t} - 4 = 7$ **121**
11. $\sqrt{n - 3} = 6$ **39**
12. $\sqrt{c + 10} = 4$ **6**
13. $\sqrt{h - 5} = 2\sqrt{3}$ **17**
14. $\sqrt{k + 7} = 3\sqrt{2}$ **11**
15. $y = \sqrt{12 - y}$ **3**
16. $\sqrt{u + 6} = u$ **3**
17. $\sqrt{r + 3} = r - 3$ **6**
18. $\sqrt{1 - 2t} = 1 + t$ **0**
19. $5\sqrt{a - 3} + 4 = 14$ **7**
20. $2\sqrt{x - 11} - 8 = 4$ **47**

B 21. **RIDES** The amount of time t, in seconds, that it takes a simple pendulum to complete a full swing is called the *period*. It is given by $t = 2\pi\sqrt{\frac{\ell}{9.8}}$, where ℓ is the length of the pendulum, in meters.
   a. The Giant Swing completes a period in about 8 seconds. About how long is the pendulum's arm? Round to the nearest meter. **16 m**
   b. Does increasing the length of the pendulum increase or decrease the period? Explain. **See margin.**

Solve each equation. Check your solution.

22. $\sqrt{6a - 6} = a + 1$ **no solution**
23. $\sqrt{x^2 + 9x + 15} = x + 5$ **no solution**
24. $6\sqrt{\frac{5k}{4}} - 3 = 0$ $\frac{1}{5}$
25. $\sqrt{\frac{5y}{6}} - 10 = 4$ **235.2**
26. $\sqrt{2a^2 - 121} = a$ **11**
27. $\sqrt{5x^2 - 9} = 2x$ **3**

28. **REASONING** The formula for the slant height c of a cone is $c = \sqrt{h^2 + r^2}$, where h is the height of the cone and r is the radius of its base. Find the height of the cone if the slant height is 4 units and the radius is 2 units. Round to the nearest tenth. $2\sqrt{3}$ **or ≈ 3.5 units**



Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	8–20, 31–35, 37–60	9–19 odd, 38–41	8–20 even, 31–35, 37, 42–60
OL Core	9–27 odd, 28–35, 37–60	8–20, 38–41	21–35, 37, 42–60
BL Advanced	21–60		

29. **MULTIPLE REPRESENTATIONS** Consider $\sqrt{2x - 7} = x - 7$.

a. **Graphical** Clear the Y= list. Enter the left side of the equation as $Y_1 = \sqrt{2x - 7}$. Enter the right side of the equation as $Y_2 = x - 7$. Press GRAPH. **See margin.**

b. **Graphical** Sketch what is shown on the screen. **See students' work.**

c. **Analytical** Use the **intersect** feature on the **CALC** menu to find the point of intersection. **See margin.**

d. **Analytical** Solve the radical equation algebraically. How does your solution compare to the solution from the graph? **About 10.83; they are the same.**

30. **PACKAGING** A cylindrical container of chocolate drink mix has a volume of 162 cubic centimeters. The radius r of the container can be found by using the formula $r = \sqrt{\frac{V}{\pi h}}$, where V is the volume of the container and h is the height.

a. If the radius is 2.5 centimeters, find the height of the container. Round to the nearest hundredth. **8.25 cm.**

b. If the height of the container is 10 centimeters, find the radius. Round to the nearest hundredth. **2.27 cm.**

32. B; The solution of the given equation is 2. Choice B is the only equation that is true when $x = 2$.

H.O.T. Problems Use Higher-Order Thinking Skills

31. **CRITIQUE** Asma and Eiman solved $\sqrt{6 - b} = \sqrt{b + 10}$. Is either of them correct? Explain. **Asma; Eiman had the wrong sign for $2b$ in the fourth step.**

Asma

$\sqrt{6 - b} = \sqrt{b + 10}$

$(\sqrt{6 - b})^2 = (\sqrt{b + 10})^2$

$6 - b = b + 10$

$-2b = 4$

$b = -2$

Check $\sqrt{6 - (-2)} \stackrel{?}{=} \sqrt{(-2) + 10}$

$\sqrt{8} = \sqrt{8}$ ✓

Eiman

$\sqrt{6 - b} = \sqrt{b + 10}$

$(\sqrt{6 - b})^2 = (\sqrt{b + 10})^2$

$6 - b = b + 10$

$2b = 4$

$b = 2$

Check $\sqrt{6 - (2)} \stackrel{?}{=} \sqrt{(2) + 10}$

$\sqrt{4} \neq \sqrt{12}$ ✗

no solution

33. Sample answer: In the first equation, you have to isolate the radical first by subtracting 1 from each side. Then square each side to find the value of x. In the second equation, the radical is already isolated, so square each side to start. Then subtract 1 from each side to solve for x.

32. **REASONING** Which equation has the same solution set as $\sqrt{4} = \sqrt{x + 2}$? Explain.

A. $\sqrt{4} = \sqrt{x} + \sqrt{2}$ **B.** $4 = x + 2$ **C.** $2 - \sqrt{2} = \sqrt{x}$

33. **REASONING** Explain how solving $5 = \sqrt{x} + 1$ is different from solving $5 = \sqrt{x + 1}$.

34. **OPEN ENDED** Write a radical equation with a variable on each side. Then solve the equation. **Sample answer: $\sqrt{2x - 1} = x$; 1**

35. **REASONING** Is the following equation *sometimes*, *always* or *never* true? Explain.

$$\sqrt{(x - 2)^2} = x - 2$$

Sometimes; the equation is true for $x \geq 2$, but false for $x < 2$.

36. **CHALLENGE** Solve $\sqrt{x + 9} = \sqrt{3} + \sqrt{x}$. **3**

37. **WRITING IN MATH** Write some general rules about how to solve radical equations. Demonstrate your rules by solving a radical equation. **See margin.**

Additional Answer

37. Sample answer: Add or subtract any expressions that are not in the radicand from each side. Multiply or divide any values that are not in the radicand to each side. Square each side of the equation. Solve for the variable as you did previously. See students' examples.

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4 Assess

Yesterday's News Have students explain how yesterday's lesson on operations with radical expressions helped with today's lesson on radical equations.

Teach with Tech

Blog On your secure classroom blog have students write a blog entry about extraneous solutions. Have them explain what extraneous solutions are and how they can check their solutions both graphically and algebraically.

Additional Answers

49. Yes; 12 is a real number and therefore a monomial.

50. Yes; $4x^3$ is the product of a number and three variables.

51. No; $a - 2b$ shows subtraction, not multiplication alone of numbers and variables.

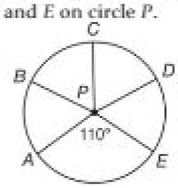
52. No; $4n + 5p$ shows addition, not multiplication alone of numbers and variables.

53. No; $\frac{x}{y^2}$ has a variable in the denominator.

54. Yes; $\frac{1}{5}abc^{14}$ is the product of a number, $\frac{1}{5}$, and several variables.

Standardized Test Practice

38. SHORT RESPONSE Hassan needs to drill a hole at A, B, C, D, and E on circle P.



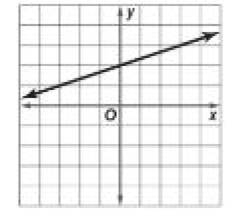
If Hassan drills holes so that $m\angle APE = 110°$ and the other four angles are congruent, what is $m\angle CPD$? **62.5°**

39. Which expression is undefined when $w = 3$? **C**

A $\frac{w-3}{w+1}$ C $\frac{w+1}{w^2-3w}$

B $\frac{w^2-3w}{3w}$ D $\frac{3w}{3w^2}$

40. What is the slope of a line that is parallel to the line? **H**



F -3 H $\frac{1}{3}$

G $-\frac{1}{3}$ J 3

41. What are the solutions of $\sqrt{x+3} - 1 = x - 4$? **D**

A 1, 6 C 1

B $-1, -6$ D 6

Spiral Review

42. ELECTRICITY The voltage V required for a circuit is given by $V = \sqrt{PR}$, where P is the power in watts and R is the resistance in ohms. How many more volts are needed to light a 100-watt light bulb than a 75-watt light bulb if the resistance of both is 110 ohms? (Lesson 9-3) **about 14**

Simplify each expression. (Lesson 9-2)

43. $\sqrt{6} \cdot \sqrt{8}$ **$4\sqrt{3}$**

44. $\sqrt{3} \cdot \sqrt{6}$ **$3\sqrt{2}$**

45. $7\sqrt{3} \cdot 2\sqrt{6}$ **$42\sqrt{2}$**

46. $\sqrt{\frac{27}{a^2}}$ **$\frac{3\sqrt{3}}{|a|}$**

47. $\sqrt{\frac{5c^5}{4d^5}}$ **$\frac{c^2\sqrt{5cd}}{2|d^3|}$**

48. $\frac{\sqrt{9x^3y}}{\sqrt{16x^2y^2}}$ **$\frac{3\sqrt{xy}}{4|y|}$**

Determine whether each expression is a monomial. Write *yes* or *no*. Explain. **49–54. See margin.**

49. 12 **50.** $4x^3$ **51.** $a - 2b$ **52.** $4n + 5p$ **53.** $\frac{x}{y^2}$ **54.** $\frac{1}{5}$

Skills Review

Simplify.

55. 9^2 **81**

56. 10^6 **1,000,000**

57. 4^5 **1024**

58. $(8v)^2$ **$64v^2$**

59. $\left(\frac{w^3}{9}\right)^2$ **$\frac{w^6}{81}$**

60. $(10y^2)^3$ **$1000y^6$**

Differentiated Instruction BL

Extension Explain that the *geometric mean* of two positive numbers is the positive square root of their product. Ask students to find a pair of consecutive positive even integers whose geometric mean is $4\sqrt{5}$. Since $4\sqrt{5} = \sqrt{x(x+2)}$, $x = 8$ or -10. Since the two numbers are positive, x must equal to 8 and, therefore, $x + 2 = 10$. The two numbers are 8 and 10.

CHAPTER 9 Mid-Chapter Quiz
Lessons 9-1 through 9-4

Graph each function. Compare to the parent graph. State the domain and range. (Lesson 9-1)

1. $y = 2\sqrt{x}$ **1–6. See Ch. 9 Answer Appendix.**
2. $y = -4\sqrt{x}$
3. $y = \frac{1}{2}\sqrt{x}$
4. $y = \sqrt{x} - 3$
5. $y = \sqrt{x - 1}$
6. $y = 2\sqrt{x - 2}$

7. **MULTIPLE CHOICE** The length of the side of a square is given by the function $s = \sqrt{A}$, where A is the area of the square. What is the length of the side of a square that has an area of 121 square centimeters? (Lesson 9-1) **B**

   A 121 centimeters
   B 11 centimeters
   C 44 centimeters
   D 10 centimeters

Simplify each expression. (Lesson 9-2)

8. $2\sqrt{25}$ **10**
9. $\sqrt{12} \cdot \sqrt{8}$ **$4\sqrt{6}$**
10. $\sqrt{72xy^5z^6}$ **$6y^2|z^3|\sqrt{2xy}$**
11. $\frac{3}{1+\sqrt{5}}$ **$\frac{3-3\sqrt{5}}{-4}$**
12. $\frac{1}{5-\sqrt{7}}$ **$\frac{5+\sqrt{7}}{18}$**

13. **SATELLITES** A satellite is launched into orbit 200 kilometers above Earth. The orbital velocity of a satellite is given by the formula $v = \sqrt{\frac{Gm_E}{r}}$. v is velocity in meters per second, G is a given constant, m_E is the mass of Earth, and r is the radius of the satellite's orbit in meters. (Lesson 9-2)

    a. The radius of Earth is 6,380,000 meters. What is the radius of the satellite's orbit in meters? **6,580,000 m**

    b. The mass of Earth is 5.97×10^{24} kilograms, and the constant G is 6.67×10^{-11} N $\cdot \frac{m^2}{kg^2}$ where N is in Newtons. Use the formula to find the orbital velocity of the satellite in meters per second. **about 7779 m/s**

14. **MULTIPLE CHOICE** Which expression is equivalent to $\sqrt{\frac{16}{32}}$? (Lesson 9-2) **G**

    F $\frac{1}{2}$
    G $\frac{\sqrt{2}}{2}$
    H 2
    J 4

Simplify each expression. (Lesson 9-3)

15. $3\sqrt{2} + 5\sqrt{2}$ **$8\sqrt{2}$**
16. $\sqrt{11} - 3\sqrt{11}$ **$-2\sqrt{11}$**
17. $6\sqrt{2} + 4\sqrt{50}$ **$26\sqrt{2}$**
18. $\sqrt{27} - \sqrt{48}$ **$-\sqrt{3}$**
19. $4\sqrt{3}(2\sqrt{6})$ **$24\sqrt{2}$**
20. $3\sqrt{20}(2\sqrt{5})$ **60**
21. $(\sqrt{5} + \sqrt{7})(\sqrt{20} + \sqrt{3})$ **$10 + \sqrt{15} + 2\sqrt{35} + \sqrt{21}$**

22. **GEOMETRY** Find the area of the rectangle. (Lesson 9-3) **$36\sqrt{5}$**

Solve each equation. Check your solution. (Lesson 9-4)

23. $\sqrt{5x} - 1 = 4$ **5**
24. $\sqrt{a - 2} = 6$ **38**
25. $\sqrt{15 - x} = 4$ **−1**
26. $\sqrt{3x^2 - 32} = x$ **4**
27. $\sqrt{2x - 1} = 2x - 7$ **5**
28. $\sqrt{x + 1} + 2 = 4$ **3**

29. **GEOMETRY** The lateral surface area S of a cone can be found by using the formula $S = \pi r\sqrt{r^2 + h^2}$, where r is the radius of the base and h is the height of the cone. Find the height of the cone. (Lesson 9-4) **about 12.5 cm.**

Formative Assessment

Use the Mid-Chapter Quiz to assess students' progress in the first half of the chapter.

For problems answered incorrectly, have students review the lessons indicated in parentheses.

FOLDABLES StudyOrganizer

Dinah Zike's Foldables®

Before students complete the Mid-Chapter Quiz, encourage them to review the information for Lessons 9-1 through 9-4 cm their Foldables.

1 Focus

Vertical Alignment

Before Lesson 9-5 Solve radical equations.

Lesson 9-5 Identify and use inverse variations. Graph inverse variations.

After Lesson 9-5 Identify and use rational functions.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- If a runner has an average pace of 5 kilometers an hour, how long will it take the runner to run 10 kilometers? 2 hours
- If a runner has an average pace of 6 kilometers an hour, how long will it take the runner to run 10 kilometers? 1 hour 40 min
- In both cases, what number does not change? the distance, 10 kilometers

LESSON 9-5 Inverse Variation

Then

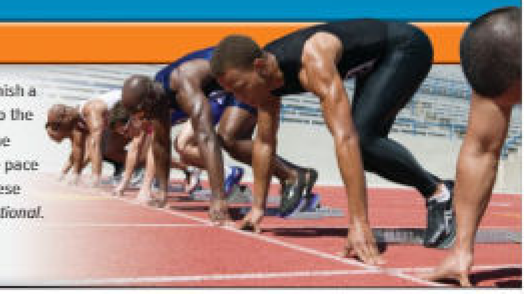
- You solved problems involving direct variation.

Now

1. Identify and use inverse variations.
2. Graph inverse variations.

Why?

- The time it takes a runner to finish a race is inversely proportional to the average pace of the runner. The runner's time decreases as the pace of the runner increases. So, these quantities are *inversely proportional.*



New Vocabulary
inverse variation
product rule

Mathematical Practices
Make sense of problems and persevere in solving them.

1 Identify and Use Inverse Variations

An **inverse variation** can be represented by the equation $y = \frac{k}{x}$ or $xy = k$.

Key Concept Inverse Variation

y varies inversely as x if there is some nonzero constant k such that $y = \frac{k}{x}$ or $xy = k$, where $x, y \neq 0$.

In an inverse variation, the product of two values remains constant. Recall that a relationship of the form $y = kx$ is a *direct variation*. The constant k is called the *constant of variation* or the *constant of proportionality*.

Example 1 Identify Inverse and Direct Variations

Determine whether each table or equation represents an *inverse* or a *direct variation*. Explain.

a.

x	y
1	16
2	8
4	4

In an inverse variation, xy equals a constant k. Find xy for each ordered pair in the table.

$1 \cdot 16 = 16 \quad 2 \cdot 8 = 16 \quad 4 \cdot 4 = 16$

The product is constant, so the table represents an inverse variation.

b.

x	y
1	3
2	6
3	9

Notice that xy is not constant. So, the table does not represent an indirect variation.

$3 = k(1) \quad 6 = k(2) \quad 9 = k(3)$

$3 = k \quad 3 = k \quad 3 = k$

The table of values represents the direct variation $y = 3x$.

c. $x = 2y$

The equation can be written as $y = \frac{1}{2}x$. Therefore, it represents a direct variation.

d. $2xy = 10$

$2xy = 10$ Write the equation.
$xy = 5$ Divide each side by 2.

The equation represents an inverse variation.

Guided Practice

1A.

x	1	2	5
y	10	5	2

Inverse; xy equals a constant.

1B. $-2x = y$ Direct; $-2x = y$ is equivalent to $y = -2x$.

You can use $xy = k$ to write an inverse variation equation that relates x and y.

Reading Math

Variation Equations For direct variation equations, you say that *y varies directly* as *x*. For inverse variation equations, you say that *y varies inversely* as *x*.

Example 2 Write an Inverse Variation

Assume that y varies inversely as x. If $y = 18$ when $x = 2$, write an inverse variation equation that relates x and y.

$xy = k$	Inverse variation equation
$2(18) = k$	$x = 2$ and $y = 18$
$36 = k$	Simplify.

The constant of variation is 36. So, an equation that relates x and y is $xy = 36$ or $y = \frac{36}{x}$.

Guided Practice

2. Assume that y varies inversely as x. If $y = 5$ when $x = -4$, write an inverse variation equation that relates x and y. $xy = -20$ or $y = \frac{-20}{x}$

If (x_1, y_1) and (x_2, y_2) are solutions of an inverse variation, then $x_1y_1 = k$ and $x_2y_2 = k$.

$x_1y_1 = k$ and $x_2y_2 = k$

$x_1y_1 = x_2y_2$ Substitute x_2y_2 for k.

The equation $x_1y_1 = x_2y_2$ is called the **product rule** for inverse variations.

Key Concept Product Rule for Inverse Variations

Words	If (x_1, y_1) and (x_2, y_2) are solutions of an inverse variation, then the products x_1y_1 and x_2y_2 are equal.
Symbols	$x_1y_1 = x_2y_2$ or $\frac{x_1}{x_2} = \frac{y_2}{y_1}$

Example 3 Solve for x or y

Assume that y varies inversely as x. If $y = 3$ when $x = 12$, find x when $y = 4$.

$x_1y_1 = x_2y_2$	Product rule for inverse variations
$12 \cdot 3 = x_2 \cdot 4$	$x_1 = 12$, $y_1 = 3$, and $y_2 = 4$
$36 = x_2 \cdot 4$	Simplify.
$\frac{36}{4} = x_2$	Divide each side by 4.
$9 = x_2$	Simplify.

So, when $y = 4$, $x = 9$.

Guided Practice

3. If y varies inversely as x and $y = 4$ when $x = -8$, find y when $x = -4$. 8

The product rule for inverse variations can be used to write an equation to solve real-world problems.

1 Identify and Use Inverse Variations

Example 1 shows how to identify inverse and direct variations. **Example 2** shows how to write an inverse variation equation to relate x and y. **Example 3** shows how to use the product rule for inverse variations to find a value for x or y. **Example 4** shows how to use the product rule for inverse variations to model a real-world situation.

Formative Assessment

Use the Guided Practice exercises after each Example to determine students' understanding of concepts.

Additional Examples

1 Determine whether each table or equation represents an *inverse* or a *direct* variation. Explain.

a.

x	6	8	10
y	3	4	5

Direct variation; $y = \frac{1}{2}x$.

b.

x	1	2	3
y	12	6	4

Inverse variation; xy is a constant.

c. $-2xy = 20$ Inverse variation; xy is a constant.

d. $x = 0.5y$ Direct variation; the equation can be written in the form $y = kx$.

2 Assume that y varies inversely as x. If $y = 5$ when $x = 3$, write an inverse variation equation that relates x and y. $xy = 15$ or $y = \frac{15}{x}$

3 Assume that y varies inversely as x. If $y = 5$ when $x = 12$, find x when $y = 15$. 4

Teach with Tech

Interactive Whiteboard Choose a student to work through an example in front of the class. Give the student an x-value and a y-value. Have the student find the constant of variation given that y varies inversely with x.

Additional Example

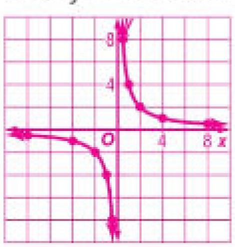
4 **PHYSICAL SCIENCE** When two people are balanced on a seesaw, their distances from the center of the seesaw are inversely proportional to their weights. How far should a 47 kilogram person sit from the center of the seesaw to balance a 29 kilogram person sitting 1.06 meters from the center? 0.65 m

2 Graph Inverse Variations

Example 5 shows how to graph an inverse variation in which there are negative values of *x*.

Additional Example

5 Graph an inverse variation in which $y = 1$ when $x = 4$.

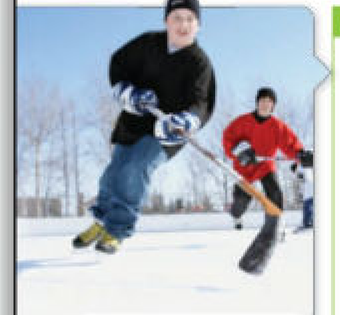


Focus on Mathematical Content

Inverse Variation When the product of two quantities remains constant, the quantities form an inverse variation. As one quantity increases, the other decreases, as long as $k > 0$. The nonzero product of the two quantities *xy* is called the constant of variation *k*.

Teaching the Mathematical Practices

Sense-Making Mathematically proficient students consider analogous problems, and try simpler forms of the original problem in order to gain insight into its solution. Point out that the procedure for making a table of values and graphing points is the same for various types of functions.



Real-World Link

A standard hockey puck is 2.5 centimeters thick and 7.6 centimeters in diameter. Its mass is between approximately 156 and 170 grams.

Source: *NHL Rulebook*

Real-World Example 4 Use Inverse Variations

PHYSICS The acceleration a of a hockey puck is inversely proportional to its mass m. Suppose a hockey puck with a mass of 164 grams is hit so that it accelerates 122 m/s². Find the acceleration of a 159-gram hockey puck if the same amount of force is applied.

Make a table to organize the information.

Puck	Mass	Acceleration
1	164 g	122 m/s²
2	158 g	a_2

Let $m_1 = 164$, $a_1 = 122$, and $m_2 = 164$. Solve for a_2.

$m_1a_1 = m_2a_2$	Use the product rule to write an equation.
$164 \cdot 122 = 158a_2$	$m_1 = 164$, $a_1 = 122$, and $m_2 = 158$
$20{,}008 = 158a_2$	Simplify.
$126.6 \approx a_2$	Divide each side by 158 and simplify.

The 159-gram puck has an acceleration of approximately 126.6 m/s².

Guided Practice

4. **RACING** Abdulaziz runs an average of 8 kilometers per hour and finishes a race in 0.39 hour. Huda finished the race in 0.35 hour. What was her average pace? about 8.9 kmph

2 Graph Inverse Variations

The graph of an inverse variation is not a straight line like the graph of a direct variation.

Problem-Solving Tip

Sense-Making Sometimes it is necessary to break a problem into parts, solve each part, and then combine them to find the solution to the problem.

Example 5 Graph an Inverse Variation

Graph an inverse variation equation in which $y = 8$ when $x = 3$.

Step 1 Write an inverse variation equation.

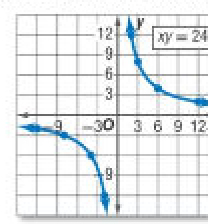
$xy = k$	Inverse variation equation
$3(8) = k$	$x = 3, y = 8$
$24 = k$	Simplify.

The inverse variation equation is $xy = 24$ or $y = \frac{24}{x}$.

Step 2 Choose values for x and y that have a product of 24.

Step 3 Plot each point and draw a smooth curve that connects the points.

x	y
−12	−2
−8	−3
−4	−6
−2	−12
0	undefined
2	12
3	8
6	4
12	2

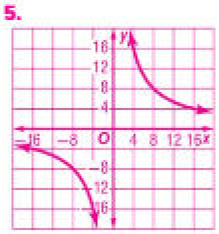


Notice that since y is undefined when $x = 0$, there is no point on the graph when $x = 0$. This graph is called a hyperbola.

Guided Practice

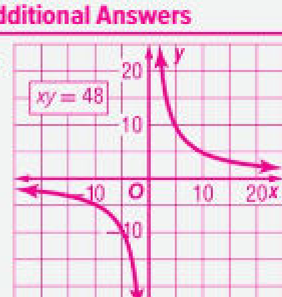
5. Graph an inverse variation equation in which $y = 16$ when $x = 4$.

5.



Additional Answers

5.



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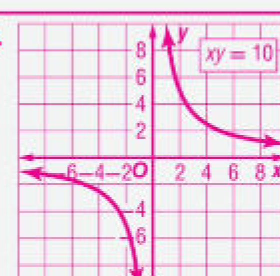

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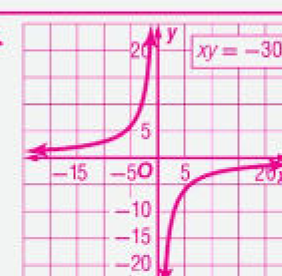
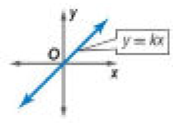
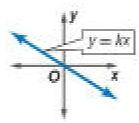
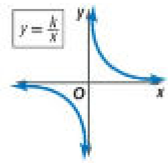
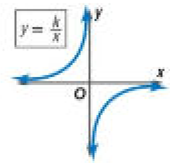


ConceptSummary Direct and Inverse Variations

Direct Variation	Inverse Variation
 $k > 0$  $k < 0$	 $k > 0$  $k < 0$
• $y = kx$ • y varies directly as x. • The ratio $\frac{y}{x}$ is a constant.	• $y = \frac{k}{x}$ • y varies inversely as x. • The product xy is a constant.

Check Your Understanding

Example 1 Determine whether each table or equation represents an *inverse* or a *direct* variation. Explain.

1.

x	1	4	8	12
y	2	8	16	24

1. Direct; the data in the table can be represented by the equation $y = 2x$.

2. Inverse; $xy = 24$.

x	1	2	3	4
y	24	12	8	6

3. $xy = 4$ Inverse; $xy = 4$.

4. $y = \frac{x}{10}$ Direct; $y = \frac{1}{10}x$.

Examples 2, 5 Assume that y varies inversely as x. Write an inverse variation equation that relates x and y. Then graph the equation. 5–8. See margin.

5. $y = 8$ when $x = 6$
6. $y = 2$ when $x = 5$
7. $y = 3$ when $x = -10$
8. $y = -1$ when $x = -12$

Example 3 Solve. Assume that y varies inversely as x.

9. If $y = 8$ when $x = 4$, find x when $y = 2$. 16
10. If $y = 7$ when $x = 6$, find y when $x = -21$. −2
11. If $y = -5$ when $x = 9$, find y when $x = 6$. −7.5

Example 4

12. **RACING** The time it takes to complete a go-cart race course is inversely proportional to the average speed of the go-cart. One rider has an average speed of 22.3 meters per second and completes the course in 30 seconds. Another rider completes the course in 25 seconds. What was the average speed of the second rider? 26.76 m/s

13. **OPTOMETRY** When a person does not have clear vision, an optometrist can prescribe lenses to correct the condition. The power P of a lens, in a unit called diopters, is equal to 1 divided by the focal length f, in meters, of the lens.
   a. Graph the inverse variation $P = \frac{1}{f}$. See margin.
   b. Find the powers of lenses with focal lengths +0.2 to −0.4 meters. 5 to −2.5 diopters

3 Practice

Formative Assessment

Use Exercises 1–13 to check for understanding.

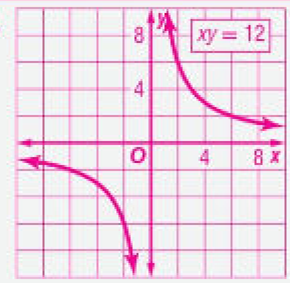
Use the chart at the bottom of the next page to customize assignments for your students.

Exercise Alert

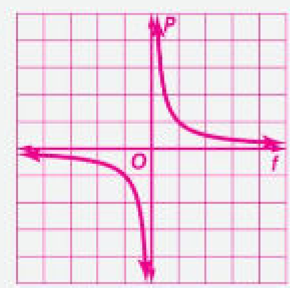
Grid Paper For Exercises 5–8, 13, 22–27, and 51 students will need grid paper.

Additional Answers

8.



13a.



DifferentiatedInstruction BL

If students need a challenge in this lesson,

Then borrow a fulcrum, lever, and weights from a science teacher to recreate Additional Example 4 or Exercise 44. After some experimentation, ask students to calculate where to place the weights for the lever to balance.

Teaching the Mathematical Practices

Critique Mathematically proficient students read the arguments of others and decide whether they make sense. In Exercise 52, point out that a single pair of x- and y-values could represent either a direct or indirect variation. In this problem, x and y vary inversely.

Additional Answers

14. Inverse; $xy = 30$

15. Direct; $y = -3x$

16. Direct; $y = -\frac{1}{2}x$

17. Inverse; $xy = -40$

18. Direct; $y = 5x$

19. Inverse; $xy = \frac{1}{4}$

20. Direct; $y = kx$

21. Direct; $y = 9x$

36. Inverse; the cost per wig times the number of wigs equals the total amount they can spend, AED20.

37. Direct; the number of lemonades times the cost per lemonade equals the total cost. So the ratio $\frac{\text{total cost}}{\text{number of lemonades}}$ is a constant AED1.50.

38. Direct; the number of hours times the rate per hour equals the total pay. The ratio total $\frac{\text{pay}}{\text{number of hours}}$ is a constant AED7.

39. Inverse; the number of friends times the number of tokens per person equals the constant 30.

40. Direct; $y = 0.2x$

41. Inverse; $xy = 21$

42. Inverse; $xy = 2$

43. Direct; $y = \frac{1}{2}x$

55. Sample answer: Newton's Law of Gravitational Force is an example of an inverse variation that models real-world situations. The gravitational force exerted on two objects is inversely proportional to the square of the distances between the two objects. The force exerted on the two objects, times the square of the distance between the two objects, is equal to the gravitational constant times the masses of the two objects.

Practice and Problem Solving

Example 1 **Determine whether each table or equation represents an *inverse* or a *direct* variation. Explain.** 14–21. See margin.

14.

x	y
1	30
2	15
5	6
6	5

15.

x	y
2	−6
3	−9
4	−12
5	−15

16.

x	y
−4	−2
−2	−1
2	1
4	2

17.

x	y
−5	8
−2	20
4	−10
8	−5

18. $5x - y = 0$ **19.** $xy = \frac{1}{4}$ **20.** $x = 14y$ **21.** $\frac{y}{x} = 9$

Examples 2, 5 **Assume that y varies inversely as x. Write an inverse variation equation that relates x and y. Then graph the equation.** 22–27. See Ch. 9 Answer Appendix.

22. $y = 2$ when $x = 20$ **23.** $y = 18$ when $x = 4$ **24.** $y = -6$ when $x = -3$

25. $y = -4$ when $x = -3$ **26.** $y = -4$ when $x = 16$ **27.** $y = 12$ when $x = -9$

Example 3 **Solve. Assume that y varies inversely as x.**

28. If $y = 12$ when $x = 3$, find x when $y = 6$. 6

29. If $y = 5$ when $x = 6$, find x when $y = 2$. 15

30. If $y = 4$ when $x = 14$, find x when $y = -5$. −11.2

31. If $y = 9$ when $x = 9$, find y when $x = -27$. −3

32. If $y = 15$ when $x = -2$, find y when $x = 3$. −10

33. If $y = -8$ when $x = -12$, find y when $x = 10$. 9.6

Example 4 **34.** **EARTH SCIENCE** The water level in a river varies inversely with air temperature. When the air temperature was 32° Celsius, the water level was 3.35 meters. If the air temperature was 43° Celsius, what was the level of water in the river? 2.49 m

35 **MUSIC** When under equal tension, the frequency of a vibrating string in a piano varies inversely with the string length. If a string that is 420 millimeters in length vibrates at a frequency of 523 cycles a second, at what frequency will a 707-millimeter string vibrate? approximately 311 cycles per second

Determine whether each situation is an example of an *inverse* or a *direct* variation. Justify your reasoning. 36–39. See margin.

36. The drama club can afford to purchase 10 wigs at AED 2 each or 5 wigs at AED 4 each.

37. The Spring family buys several lemonades for AED 1.50 each.

38. Amal earns AED 14 for providing child care for 2 hours, and AED 21 for providing child care for 3 hours.

B **39.** Thirty video game tokens are divided evenly among a group of friends.

Determine whether each table or graph represents an *inverse* or a *direct* variation. Explain. 40–43. See margin.

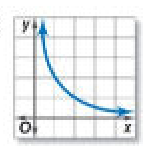
40.

x	y
5	1
8	1.6
11	2.2

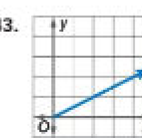
41.

x	y
−3	−7
−2	−10.5
4	5.25

42.



43.



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Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	14–35, 52, 54–74	15–35 odd, 57–60	14–34 even, 52, 54–56, 61–74
OL Core	15–43 odd, 44, 45–49 odd, 50–52, 54–74	14–35, 57–60	36–52, 54–56, 61–74
BL Advanced	36–70, (optional: 71–74)		

44. **PHYSICAL SCIENCE** When two people are balanced on a seesaw, their distances from the center of the seesaw are inversely proportional to their weights. If a 53.5 kilogram person sits 1.8 meters from the center of the seesaw, how far should a 56.7 kilogram person sit from the center to balance the seesaw? **about 1.7 m**

C **Solve. Assume that y varies inversely as x.**

45. If $y = 9.2$ when $x = 6$, find x when $y = 3$. **18.4**

46. If $y = 3.8$ when $x = 1.5$, find x when $y = 0.3$. **19**

47. If $y = \frac{1}{5}$ when $x = -20$, find y when $x = -\frac{8}{5}$. **2.5**

48. If $y = -6.3$ when $x = \frac{2}{3}$, find y when $x = 8$. **−0.525**

49. **SWIMMING** Badr and Hessa each bought a pool membership. Their average cost per day is inversely proportional to the number of days that they go to the pool. Badr went to the pool 25 days for an average cost per day of AED 5.60. Hessa went to the pool 35 days. What was her average cost per day? **AED 4**

50. **PHYSICAL SCIENCE** The amount of force required to do a certain amount of work in moving an object is inversely proportional to the distance that the object is moved. Suppose 90 N of force is required to move an object 10 meters. Find the force needed to move another object 15 meters if the same amount of work is done. **60 N**

51. **DRIVING** Buthaina must practice driving 40 hours with a parent or guardian before she is allowed to take the test to get her driver's license. She plans to practice the same number of hours each week. **a, c. See Ch. 9 Answer Appendix.**

   a. Let h represent the number of hours per week that she practices driving. Make a table showing the number of weeks w that she will need to practice for the following values of h: 1, 2, 4, 5, 8, and 10.

   b. Describe how the number of weeks changes as the number of hours per week increases. **The number of weeks decreases.**

   c. Write and graph an equation that shows the relationship between h and w.

H.O.T. Problems Use Higher-Order Thinking Skills

52. **CRITIQUE** Ahmed and Ayman found an equation such that x and y vary inversely, and $y = 10$ when $x = 5$. Is either of them correct? Explain.

Ayman; the constant of variation is 5(10) or 50, so the correct equation is $xy = 50$ or $y = \frac{50}{x}$. Ahmed found an equation for a direct variation.

Ahmed

$k = \frac{y}{x}$

$= \frac{10}{2}$ or 5

$y = 5x$

Ayman

$k = xy$

$= (5)(10)$ or 50

$y = \frac{50}{x}$

53. **CHALLENGE** Suppose f varies inversely with g, and g varies inversely with h. What is the relationship between f and h? **direct variation**

54. **REASONING** Does $xy = -k$ represent an inverse variation when $k \neq 0$? Explain. **Yes; the product of x and y is a nonzero constant.**

55. **OPEN ENDED** Give a real-world situation or phenomena that can be modeled by an inverse variation equation. Use the correct terminology to describe your example and explain why this situation is an inverse variation. **See margin.**

56. **WRITING IN MATH** Compare and contrast direct and inverse variation. Include a description of the relationship between slope and the graphs of a direct and inverse variation. **See margin.**

Additional Answer

56. Direct variation can be written as $y = kx$ where k is the constant of proportionality. The graph of a direct variation is a line through the origin with a slope of k. An inverse variation is written in the form $y = \frac{k}{x}$. The graph is a two-part curve (a hyperbola).

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4 Assess

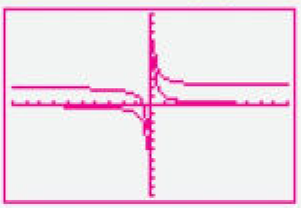
Name the Math Prepare two paper bags containing slips of paper: one containing a value for x on each slip, the other a value for y on each slip. Have each student select both an x-value and a y-value and write an inverse variation equation that relates x and y.

Additional Answers

61. Positive; it means the more you study, the better your test score.

Additional Answers (Explore 9-6)

2.



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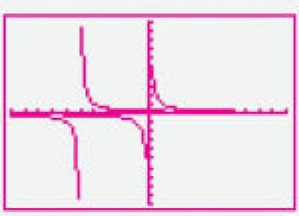
Both graphs have the same shape, but the graph of $y = \frac{1}{x} + 2$ is 2 units above the graph of $y = \frac{1}{x}$.

3.



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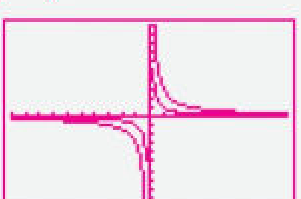
Both graphs have the same shape, but the graph of $y = \frac{1}{x+5}$ is 5 units to the left of the graph of $y = \frac{1}{x}$.

4.



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The graph of $y = \frac{3}{x}$ is farther from the axes than the graph of $y = \frac{1}{x}$.

Standardized Test Practice

57. Given a constant force, the acceleration of an object varies inversely with its mass. Assume that a constant force is acting on an object with a mass of 6 kilogram resulting in an acceleration of 10 m/s^2. The same force acts on another object with a mass of 12 kilogram. What would be the resulting acceleration? **B**

A 4 m/s^2
B 5 m/s^2
C 6 m/s^2
D 7 m/s^2

58. Houriyya had an average of 56% on her first seven tests. What would she have to make on her eighth test to average 60% on 8 tests? **G**

F 82%
G 88%
H 98%
J 100%

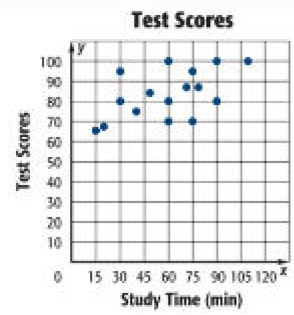
59. Hareb takes a picture of a 1-meter snake beside a brick wall. When he develops the pictures, the 1-meter snake is 2 centimeters long and the wall is 4.5 centimeters high. What was the actual height of the brick wall? **C**

A 2.25 cm
B 22.5 cm
C 225 cm
D 2250 cm

60. SHORT RESPONSE Find the area of the rectangle. $x^2 + 15x + 36$ cm^2

Spiral Review

61. TESTS Determine whether the graph at the right shows a *positive*, *negative*, or *no* correlation. If there is a correlation, describe its meaning. **See margin.**



Suppose y varies directly as x.

62. If $y = 2.5$ when $x = 0.5$, find y when $x = 20$. **100**

63. If $y = -6.6$ when $x = 9.9$, find y when $x = 6.6$. **−4.4**

64. If $y = 2.6$ when $x = 0.25$, find y when $x = 1.125$. **11.7**

65. If $y = 6$ when $x = 0.6$, find x when $y = 12$. **1.2**

66. FINANCIAL LITERACY A salesperson is paid AED 32,000 a year plus 5% of the amount in sales made. What is the amount of sales needed to have an annual income greater than AED 45,000? **The amount of sales must be more than AED 260,000.**

Skills Review

Simplify. Assume that no denominator is equal to zero.

67. $\frac{7^8}{7^6}$ 7^2 or 49

68. $\frac{x^8y^{12}}{x^2y^7}$ x^6y^5

69. $\frac{5pq^7}{10p^6q^3}$ $\frac{q^4}{2p^5}$

70. $\left(\frac{2c^3d}{7z^2}\right)^3$ $\frac{8c^9d^3}{343z^6}$

71. $\left(\frac{4a^2b}{2c^3}\right)^2$ $\frac{4a^4b^2}{c^6}$

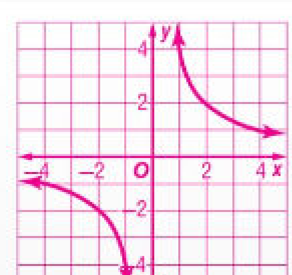
72. $y^0(y^5)(y^{-9})$ $\frac{1}{y^4}$

73. $\frac{(4m^{-3}n^5)^0}{mn}$ $\frac{1}{mn}$

74. $\frac{(3x^2y^5)^0}{(21x^5y^2)^0}$ 1

Differentiated Instruction OL BL

Extension Write $k = 4$ on the board. Ask students to write and graph an inverse variation equation that uses 4 as the constant of variation. $xy = 4$



EXPLORE 9-6 Graphing Technology Lab
Family of Rational Functions

You can use a graphing calculator to analyze how changing the parameters a and b in $y = \frac{a}{x-b} + c$ affects the graphs in the family of rational functions.

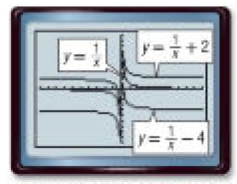
Activity **Change Parameters**

Graph each set of equations on the same screen in the standard viewing window. Describe any similarities and differences among the graphs.

a. $y = \frac{1}{x}$, $y = \frac{1}{x} + 2$, $y = \frac{1}{x} - 4$

Enter the equations in the Y= list and graph in the standard viewing window.

The graphs have the same shape. Each graph approaches the y-axis on both sides. However, the graphs have different vertical positions.



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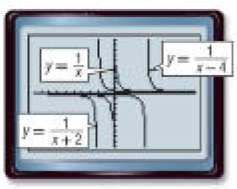
b. $y = \frac{1}{x}$, $y = \frac{1}{x+2}$, $y = \frac{1}{x-4}$



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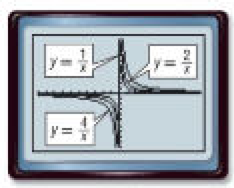
The graphs have the same shape, and all approach the x-axis from both sides. However, the graphs have different horizontal positions.

c. $y = \frac{1}{x}$, $y = \frac{2}{x}$, $y = \frac{4}{x}$



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The graphs all approach the x-axis and the y-axis from both sides. However, the graphs have different shapes.

Model and Analyze

1. How do a, b, and c affect the graph of $y = \frac{a}{x-b} + c$? Give examples. **See margin.**

Examine each pair of equations and predict the similarities and differences in their graphs. Use a graphing calculator to confirm your predictions. Write a sentence or two comparing the two graphs. **2–4. See margin.**

2. $y = \frac{1}{x}$, $y = \frac{1}{x} + 2$

3. $y = \frac{1}{x}$, $y = \frac{1}{x+5}$

4. $y = \frac{1}{x}$, $y = \frac{3}{x}$

1 Focus

Objective Use graphing technology to investigate families of rational functions.

Materials

- graphing calculator

Teaching Tips

- Remind students to use ZOOM 6 to show graphs in the standard viewing window.
- In parts b and c, students will need to clear the Y= lists. To do this, they can use CLEAR.
- Use TRACE and ▼ or ▲ to display the equation for a graph.

2 Teach

Working in Cooperative Groups

Divide the class into pairs. Work through the Activity. Then ask students to work with their partners to complete Exercise 1.

Practice Have students complete Exercises 2–4.

3 Assess

Formative Assessment

Use Exercises 2–4 to assess each student's ability to predict similarities and differences in rational functions.

From Concrete to Abstract

Ask students to summarize using technology to investigate families of rational functions.

Additional Answer

1. The value of c affects the vertical position of the graph. The value of b affects the horizontal position of the graph. The value of a affects the shape of the graph. Sample answer: The graph of $y = \frac{1}{x} + 5$ is 5 units above the x-axis. The graph of $y = \frac{1}{x+5}$ is 5 units to the left of the y-axis. The graph of $y = \frac{5}{x}$ is farther from the axes than the graph of $y = \frac{1}{x}$.

1 Focus

VerticalAlignment

Before Lesson 9-6 Write inverse variation equations.

Lesson 9-6 Identify excluded values. Identify and use asymptotes to graph rational functions.

After Lesson 9-6 Use rational equations to solve problems.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What does the average number of pages Halima reads each day depend on? the number of days she reads
- What happens to the value of y as x increases? It decreases.
- What values of x are excluded from consideration? x cannot equal 0 or any negative number.

LESSON 9-6 Rational Functions

Then

- You wrote inverse variation equations.

Now

- 1 Identify excluded values.
- 2 Identify and use asymptotes to graph rational functions.

Why?

- Halima is reading a 300-page book. The average number of pages she reads each day y is given by $y = \frac{300}{x}$, where x is the number of days that she reads.

NewVocabulary
rational function
excluded value
asymptote

Mathematical Practices
Construct viable arguments and critique the reasoning of others.
Look for and make use of structure.

1 Identify Excluded Values

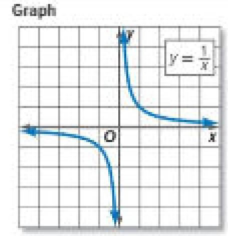
The function $y = \frac{300}{x}$ is an example of a **rational function**. This function is nonlinear.

KeyConcept Rational Functions

Words A rational function can be described by an equation of the form $y = \frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Parent function:	$f(x) = \frac{1}{x}$
Type of graph:	hyperbola
Domain:	$\{x \mid x \neq 0\}$
Range:	$\{y \mid y \neq 0\}$

Graph



Since division by zero is undefined, any value of a variable that results in a denominator of zero in a rational function is excluded from the domain of the function. These are called **excluded values** for the rational function.

Example 1 Find Excluded Values

State the excluded value for each function.

a. $y = -\frac{2}{x}$

The denominator cannot equal 0. So, the excluded value is $x = 0$.

b. $y = \frac{2}{x + 1}$

$x + 1 = 0$ Set the denominator equal to 0.

$x = -1$

The excluded value is $x = -1$.

c. $y = \frac{5}{4x - 8}$

$4x - 8 = 0$

$4x = 8$

$x = 2$

The excluded value is $x = 2$.

GuidedPractice

1A. $y = \frac{5}{2x}$ $x = 0$ **1B.** $y = \frac{x}{x - 7}$ $x = 7$ **1C.** $y = \frac{4}{3x + 9}$ $x = -3$

Depending on the real-world situation, in addition to excluding x-values that make a denominator zero from the domain of a rational function, additional values might have to be excluded from the domain as well.

Real-WorldLink

As the temperature of the gas inside a hot air balloon increases, the density of the gas decreases. A hot air balloon rises because the density of the air inside it is less than the density of the air outside.

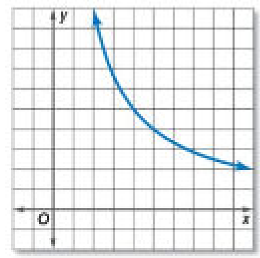
Source: Goddard Space Flight Center

Real-World Example 2 Graph Real-Life Rational Functions

BALLOONS **If there are x people in the basket of a hot air balloon, the function $y = \frac{20}{x}$ represents the number of square meters y per person. Graph this function.**

Since the number of people cannot be zero or less, it is reasonable to exclude negative values and only use positive values for x.

Number of People x	2	4	5	10
Square Meters per Person y	10	5	4	2



Notice that as x increases y approaches 0. This is reasonable since as the number of people increases, the space per person gets closer to 0.

GuidedPractice

2. **GEOMETRY** A rectangle has an area of 18 square centimeters. The function $\ell = \frac{18}{w}$ shows the relationship between the length and width. Graph the function. See Ch. 9 Answer Appendix.

2 Identify and Use Asymptotes

In Example 2, an excluded value is $x = 0$. Notice that the graph approaches the vertical line $x = 0$, but never touches it.

The graph also approaches but never touches the horizontal line $y = 0$. The lines $x = 0$ and $y = 0$ are called *asymptotes*. An **asymptote** is a line that the graph of a function approaches.

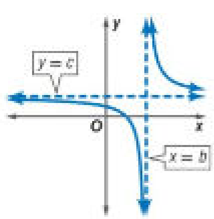
StudyTip

Use Asymptotes Asymptotes are helpful for graphing rational functions. However, they are not part of the graph.

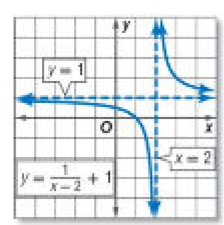
KeyConcept Asymptotes

Words A rational function in the form $y = \frac{a}{x - b} + c$, $a \neq 0$, has a vertical asymptote at the x-value that makes the denominator equal zero, $x = b$. It has a horizontal asymptote at $y = c$.

Model



Example



The domain of $y = \frac{a}{x - b} + c$ is all real numbers except $x = b$. The range is all real numbers except $y = c$. Rational functions cannot be traced with a pencil that never leaves the paper, so choose x-values on both sides of the vertical asymptote to graph both portions of the function.

1 Identify Excluded Values

Example 1 shows how to find excluded values for rational functions. **Example 2** shows how to graph real-life rational functions and determine which values should be excluded.

Formative Assessment

Use the Check Your Progress exercises after each example to determine students' understanding of concepts.

Additional Examples

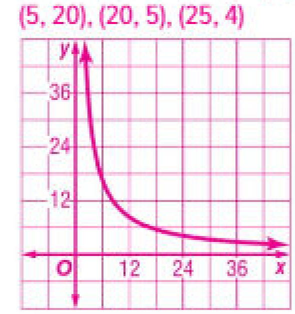
1 State the excluded value for each function.

a. $y = \frac{3}{x}$ $x = 0$

b. $y = \frac{3}{x + 2}$ $x = -2$

c. $y = \frac{8}{2x + 1}$ $x = -\frac{1}{2}$

2 **TALENT SHOW** If x students will compete in a talent show lasting 100 minutes, the function $y = \frac{100}{x}$ represents the number of minutes available for each act. Graph this function. graph of positive values for $y = \frac{100}{x}$ as shown below, through (10, 10) (5, 20), (20, 5), (25, 4)



DifferentiatedInstruction AL OL

Logical Learners Most students understand that a fraction with a denominator equal to 0 is undefined, thus the vertical asymptote. To help students understand that a horizontal asymptote occurs at $y = c$ in a function in the form $y = \frac{a}{x - b} + c$ (where $a \neq 0$), have students make a table of values for x and y and a graph using the example shown in the Key Concept, $y = \frac{1}{x - 2} + 1$. Have students extend the table of values and the graph until they all agree that, as x gets greater and greater, $y = \frac{1}{x - 2}$ approaches 0, and y gets closer and closer to 1, the value for c.

Tips for New Teachers

Discrete vs. Continuous The functions used in Real World Example 2 and Additional Example 2 are discrete, not continuous.

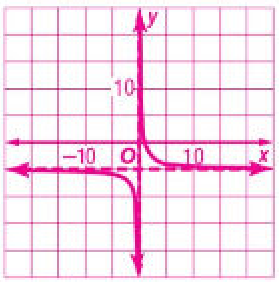
2 Identify and Use Asymptotes

Example 3 shows how to identify the asymptotes of a function and use them to graph the function.

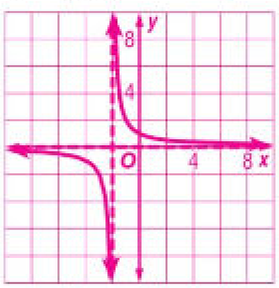
Additional Example

3 Identify the asymptotes for each function. Then graph the function.

a. $y = \frac{3}{x} - 4$ $x = 0; y = -4$

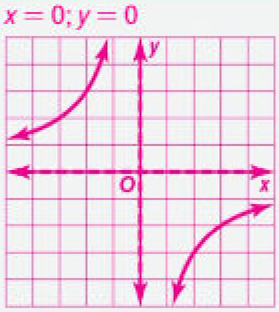


b. $y = \frac{2}{x+2}$ $x = -2; y = 0$

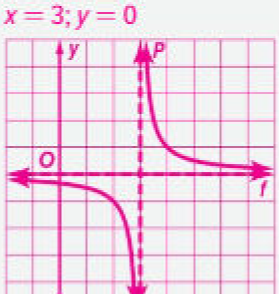


Additional Answers (Guided Practice)

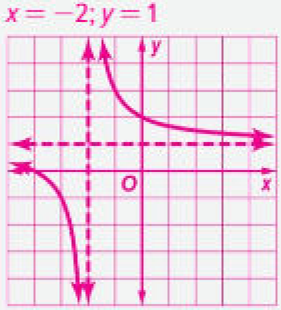
3A. $x = 0; y = 0$



3B. $x = 3; y = 0$



3C. $x = -2; y = 1$



Example 3 Identify and Use Asymptotes to Graph Functions

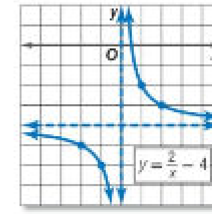
Identify the asymptotes of each function. Then graph the function.

a. $y = \frac{2}{x} - 4$

Step 1 Identify and graph the asymptotes using dashed lines.

vertical asymptote: $x = 0$

horizontal asymptote: $y = -4$



Step 2 Make a table of values and plot the points. Then connect them.

x	−2	−1	1	2
y	−5	−6	−2	−3

b. $y = \frac{1}{x+1}$

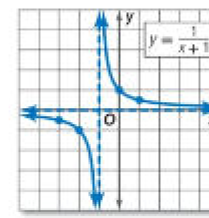
Step 1 To find the vertical asymptote, find the excluded value.

$x + 1 = 0$ Set the denominator equal to 0.

$x = -1$ Subtract 1 from each side.

vertical asymptote: $x = -1$

horizontal asymptote: $y = 0$



Step 2

x	−3	−2	0	1
y	−0.5	−1	1	0.5

Math History Link

Evelyn Boyd Granville

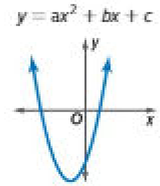
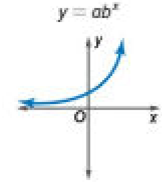
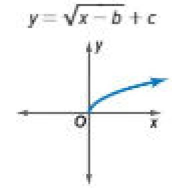
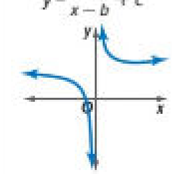
(1924–) Granville majored in mathematics and physics at Smith College in 1945, where she graduated summa cum laude. She earned an M.A. in mathematics and physics and a Ph.D. in mathematics from Yale University. Granville's doctoral work focused on functional analysis.

Guided Practice 3A–3C. See margin.

3A. $y = -\frac{6}{x}$ **3B.** $y = \frac{1}{x-3}$ **3C.** $y = \frac{2}{x+2} + 1$

Four types of nonlinear functions are shown below.

Concept Summary Families of Functions

Quadratic	Exponential	Radical	Rational
Parent function: $y = x^2$	Parent function: varies	Parent function: $y = \sqrt{x}$	Parent function: $y = \frac{1}{x}$
General form: $y = ax^2 + bx + c$	General form: $y = ab^x$	General form: $y = \sqrt{x - b} + c$	General form: $y = \frac{a}{x-b} + c$
			

Teach with Tech

Video Recording Have students create video recordings showing how to graph a rational function. Have students first describe how to find the horizontal and vertical asymptotes, and then use a table of values to draw the graph.

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Check Your Understanding

Example 1 State the excluded value for each function.

1. $y = \frac{5}{x}$ $x = 0$
2. $y = \frac{1}{x+3}$ $x = -3$
3. $y = \frac{x+2}{x-1}$ $x = 1$
4. $y = \frac{x}{2x-8}$ $x = 4$

Example 2

5. **PARTY PLANNING** The cost of decorations for a dinner party is AED 32. This is split among a group of friends. The amount each person pays y is given by $y = \frac{32}{x}$, where x is the number of people. Graph the function. See margin.

6–11. See Ch. 9 Answer Appendix.

Example 3 Identify the asymptotes of each function. Then graph the function.

6. $y = \frac{2}{x}$
7. $y = \frac{3}{x} - 1$
8. $y = \frac{1}{x-2}$
9. $y = \frac{-4}{x+2}$
10. $y = \frac{3}{x-1} + 2$
11. $y = \frac{2}{x+1} - 5$

Practice and Problem Solving

Example 1 State the excluded value for each function. 14. $x = -2$ 15. $x = -6$

17. $x = -5$
18. $x = 2$
19. $x = -7$

12. $y = \frac{-1}{x}$ $x = 0$
13. $y = \frac{8}{x-8}$ $x = 8$
14. $y = \frac{x}{x+2}$
15. $y = \frac{4}{x+6}$
16. $y = \frac{x+1}{x-3}$ $x = 3$
17. $y = \frac{2x+5}{x+5}$
18. $y = \frac{7}{5x-10}$
19. $y = \frac{x}{2x+14}$

Example 2

20. **ANTELOPES** A pronghorn antelope can run 40 kilometers without stopping. The average speed is given by $y = \frac{40}{x}$, where x is the time it takes to run the distance.
    a. Graph $y = \frac{40}{x}$. See Ch. 9 Answer Appendix.
    b. Describe the asymptotes. $x = 0$ and $y = 0$

21. **CYCLING** A cyclist rides 10 kilometers each morning. Her average speed y is given by $y = \frac{10}{x}$, where x is the time it takes her to ride 10 kilometers. Graph the function. See margin.

22–33. See Ch. 9 Answer Appendix.

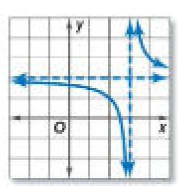
Example 3 Identify the asymptotes of each function. Then graph the function.

22. $y = \frac{5}{x}$
23. $y = \frac{-3}{x}$
24. $y = \frac{2}{x} + 3$
25. $y = \frac{1}{x} - 2$
26. $y = \frac{1}{x+3}$
27. $y = \frac{1}{x-2}$
28. $y = \frac{-2}{x+1}$
29. $y = \frac{4}{x-1}$
30. $y = \frac{1}{x-2} + 1$
31. $y = \frac{3}{x-1} - 2$
32. $y = \frac{2}{x+1} - 4$
33. $y = \frac{-1}{x+4} + 3$

B

34. **READING** Refer to the application at the beginning of the lesson.
    a. Graph the function. Interpret key features of the graph in terms of the situation. See Ch. 9 Answer Appendix.
    b. Choose a point on the graph, and describe what it means in the context of the situation. Sample answer: (6, 50); if she reads for 6 days, she will read 50 pages per day.

35. **STRUCTURE** The graph shows a translation of the graph of $y = \frac{1}{x}$.
    a. Describe the asymptotes. $x = 3$ and $y = 2$
    b. Write a possible function for the graph.
    $y = \frac{1}{x-3} + 2$



Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	12–33, 43–65	13–33 odd, 48–51	12–32 even, 43–47, 52–65
OL Core	13–33 odd, 34–37, 39, 41, 43–65	12–33, 48–51	34–41, 43–47, 52–65
BL Advanced	34–65		

3 Practice

Formative Assessment

Use Exercises 1–11 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

4 Assess

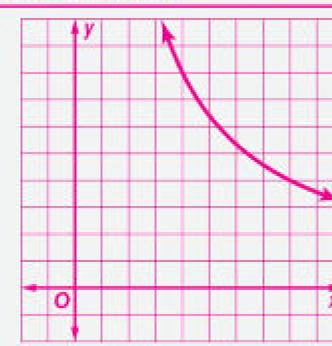
Ticket Out the Door On small pieces of paper, write one of five different rational functions similar to those in this lesson. Give one to each student. Ask them to tell you the horizontal and vertical asymptotes.

Teaching the Mathematical Practices

Structure Mathematically proficient students look closely to discern a pattern or structure. In Exercise 35, remind students that they have experience with transformations of the graphs of many types of functions.

Additional Answers

5.



21.

Teaching the Mathematical Practices

Arguments Mathematically proficient students are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. In Exercise 45, suggest that students graph a rational function and visualize moving the graph to test the conjecture.

36. BIRDS A long-tailed jaeger is a sea bird that can migrate 5000 kilometers or more each year. The average rate in kilometers per hour r can be given by the function $r = \frac{5000}{t}$, where t is the time in hours. Use the function to determine the average rate of the bird if it spends 250 hours flying. **20 km per hour**

37. CLASS TRIP Students are going to a science museum. As part of the trip, each person is also contributing an equal amount of money to name a star.

a. Write a verbal description for the cost per person.

b. Write an equation to represent the total cost y per person if p people go to the museum. $y = \frac{95}{p} + 8.50$

c. Use a graphing calculator to graph the equation. Interpret key features of the graph in terms of the situation. **See margin.**

d. Estimate the number of people needed for the total cost of the trip to be about AED 15. **Sample answer: 15 people**

37a. Sample answer: The total cost of the trip equals the cost of a ticket plus the cost of the star-naming package divided by the number of people.

Graph each function. Identify the asymptotes. **38–40. See Ch. 9 Answer Appendix.**

38. $y = \frac{4x + 3}{2x - 4}$ **39.** $y = \frac{x^2}{x^2 - 1}$ **40.** $y = \frac{x}{x^2 - 9}$

41. GEOMETRY The equation $h = \frac{2(64)}{b_1 + 8}$ represents the height h of a trapezoid with an area of 64 square units. The trapezoid has two opposite sides that are parallel and h units apart; one is b_1 units long and another is 8 units long.

a. Describe a reasonable domain and range for the function.

b. Graph the function in the first quadrant. **See margin.**

c. Use the graph to estimate the value of h when $b_1 = 10$. **about 7 units**

41a. D = {all positive real numbers}; R = {all positive real numbers}

43. The graph of $y = \frac{1}{x+5} - 2$ is the graph of $y = \frac{1}{x}$ translated 5 units to the left and 2 units down.

H.O.T. Problems Use Higher-Order Thinking Skills

42. CHALLENGE Graph $y = \frac{1}{x^2 - 4}$. State the domain and the range of the function. **See margin.**

43. REASONING Without graphing, describe the transformation that takes place between the graph of $y = \frac{1}{x}$ and the graph of $y = \frac{1}{x + 5} - 2$.

44. OPEN ENDED Write a rational function if the asymptotes of the graph are at $x = 3$ and $y = 1$. Explain how you found the function. **See margin.**

45. ARGUMENTS Is the following statement *true* or *false*? If false, give a counterexample. **False; sample answer: The graph of $y = \frac{1}{x}$ has no x- or y-intercepts.**

The graph of a rational function will have at least one intercept.

46. Sample answer: The graph of $y = \frac{8}{x} + 1$ is the only one that does not have $y = 0$ as an asymptote.

46. WHICH ONE DOESN'T BELONG Identify the function that does not belong with the other three. Explain your reasoning.

$y = \frac{4}{x}$	$y = \frac{6}{x + 1}$	$y = \frac{8}{x} + 1$	$y = \frac{10}{2x}$

47. WRITING IN MATH How are the properties of a rational function reflected in its graph? **See margin.**

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Standardized Test Practice

48. Simplify $\frac{2a^2d}{3bc} \cdot \frac{9b^2c}{16ad^2}$. **D**

A $\frac{abd}{c}$ C $\frac{6a}{4bd}$

B $\frac{ab}{d}$ D $\frac{3ab}{8d}$

49. SHORT RESPONSE One day Suha ran 100 meters in 15 seconds, 200 meters in 45 seconds, and 300 meters over low hurdles in one and a half minutes. How many more seconds did it take her to run 300 meters over low hurdles than the 200-meter dash? **45 seconds**

50. Khalid and Khalaf started a T-shirt printing business. The total start-up costs were AED 450. It costs AED 5.50 to print one T-shirt. Write a rational function $A(x)$ for the average cost of producing x T-shirts. **F**

F $A(x) = \frac{450 + 5.5x}{x}$ H $A(x) = 450x + 5.5$

G $A(x) = \frac{450}{x} + 5.5$ J $A(x) = 450 + 5.5x$

51. GEOMETRY Which of the following is a quadrilateral with exactly one pair of parallel sides? **D**

A parallelogram C square

B rectangle D trapezoid

Spiral Review

52. TRAVEL Khamis' family can drive to the beach, which is 352 km away, in 4 hours if they drive 88 kilometers per hour. Sumayya says that they would save at least a half an hour if they were to drive 105 kilometers per hour. Is Sumayya correct? Explain. (Lesson 9-5) **Yes; sample answer: At 105 kmph, the family can travel 352 kilometers in about 3 hours and 21 minutes, saving them about 39 minutes.**

53. SIGHT The formula $d = \sqrt{\frac{3h}{2}}$ represents the distance d in kilometers that a person h meters high can see. Shaima is standing on a cliff that is 310 meters above sea level. How far can Shaima see from the cliff? Write a simplified radical expression and a decimal approximation. (Lesson 9-2) **$\sqrt{465}$ or about 21.56 km**

54. $(x + 3)(x + 8)$ **55.** $(w + 16)(w - 3)$ **56.** $(p - 7)(p + 5)$
57. $(3 + a)(24 + a)$ **58.** $(c + 7)(c + 5)$ **59.** $(d - 2)(d - 5)$

Skills Review

60. $(g - 15)(g - 4)$ **61.** $(n + 9)(n - 6)$ **62.** $(5x + 2)(x + 5)$
63. $(4b - 3)(6b + 1)$ **64.** $(4a + 5)(3a - 7)$ **65.** $2(x - 3)(3x + 2)$

Factor each trinomial.

54. $x^2 + 11x + 24$ **55.** $w^2 + 13w - 48$ **56.** $p^2 - 2p - 35$ **57.** $72 + 27a + a^2$

58. $c^2 + 12c + 35$ **59.** $d^2 - 7d + 10$ **60.** $g^2 - 19g + 60$ **61.** $n^2 + 3n - 54$

62. $5x^2 + 27x + 10$ **63.** $24b^2 - 14b - 3$ **64.** $12a^2 - 13a - 35$ **65.** $6x^2 - 14x - 12$

Differentiated Instruction OL BL

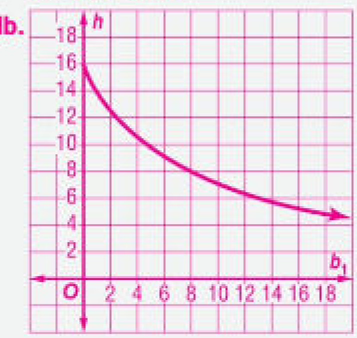
Extension Ask students to graph the function $y = \frac{x-1}{x-1}$. Then ask students to describe the graph. The graph is a horizontal line, $y = 1$, with a hole at (1, 1).

Additional Answers

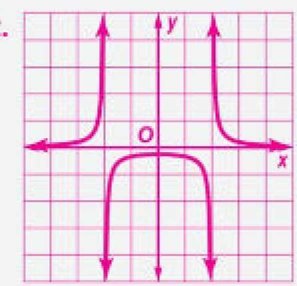
37c.

Sample answer: The end behavior indicates that as the number of people increases, the cost per person approaches 0. Since there is no x-intercept, the cost per person will never be 0.

41b.



42.



D = all real numbers except $x = -2$ and $x = 2$;
R = $\left\{y \mid y > 0 \text{ or } y \leq -\frac{1}{4}\right\}$

44. Sample answer: $y = \frac{1}{x-3} + 1$; use the equation $y = \frac{a}{x-b} + c$. The vertical asymptote is $x = b$. So, $b = 3$. The horizontal asymptote is $y = c$. So, $c = 1$. Substituting the values for b and c and letting $a = 1$, the equation becomes $y = \frac{1}{x-3} + 1$.

47. Sample answer: Vertical asymptotes occur at values that make the denominator 0; horizontal asymptotes occur at $y = c$ for any rational function of the form $y = \frac{a}{x-b} + c$.

1 Focus

VerticalAlignment

Before Lesson 9-7 Graph rational functions.

Lesson 9-7 Solve rational equations. Use rational equations to solve problems.

After Lesson 9-7 Use statistics to analyze data.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- When a coastal dolphin travels at x kilometers per hour, at what rate does an oceanic dolphin travel? $(x + 5)$ km/h
- When a coastal dolphin travels 2 kilometers, how far does an oceanic dolphin travel? 3 kilometers
- In the time it takes an oceanic dolphin to swim 6 kilometers, what do you know about the coastal dolphin? It can swim 4 kilometers.

LESSON 9-7 Rational Equations

Then

- You solved proportions.

Now

1. Solve rational equations.
2. Use rational equations to solve problems.

Why?

- Oceanic species of dolphins can swim 8 kilometers per hour faster than coastal species of dolphins. An oceanic dolphin can swim 4.8 kilometers in the same time that it takes a coastal dolphin to swim 3.2 kilometers.

Dolphins			
Species	Distance	Rate	Time
coastal		x kmph	t hours
oceanic		$x + 8$ kmph	t hours

Since time = $\frac{\text{distance}}{\text{rate}}$, the equation below represents this situation.

Time an oceanic dolphin swims 4.8 kilometers equals time a coastal dolphin swims 3.2 kilometers.

$$\frac{4.8}{x+8} = \frac{3.2}{x}$$

(distance → 4.8, rate → $x + 8$; 3.2 ← distance, x ← rate)

NewVocabulary
rational equation
extraneous solution
work problem
rate problem

Mathematical Practices
Reason abstractly and quantitatively.
Model with mathematics.

1 Solve Rational Equations

A **rational equation** contains one or more rational expressions. When a rational equation is a proportion, you can use cross products to solve it.

Real-World Example 1 Use Cross Products to Solve Equations

DOLPHINS Refer to the information above. Solve $\frac{4.8}{x+8} = \frac{3.2}{x}$ to find the speed of a coastal dolphin. Check the solution.

$\frac{4.8}{x+8} = \frac{3.2}{x}$	Original equation
$4.8x = 3.2x + 25.6$	Find the cross products.
$16x = 25.6$	Distributive Property
$x = 16$	Subtract 2x from each side.

So, a coastal dolphin can swim 16 kilometers per hour.

CHECK

$\frac{4.8}{x+8} = \frac{3.2}{x}$	Original equation
$\frac{4.8}{16+8} = \frac{3.2}{16}$	Replace x with 10.
$\frac{4.8}{24} = \frac{3.2}{16}$	Simplify.
$\frac{1}{5} = \frac{1}{5}$ ✓	Simplify.

GuidedPractice

Solve each equation. Check the solution.

1A. $\frac{7}{y-3} = \frac{3}{y+1}$ -4

1B. $\frac{13}{10} = \frac{2f+0.2}{7}$ 4.45

Another method that can be used to solve any rational equation is to find the LCD of all the fractions in the equation. Then multiply each side of the equation by the LCD to eliminate the fractions.

Example 2 Use the LCD to Solve Rational Equations

Solve $\frac{4}{y} + \frac{5y}{y+1} = 5$. Check the solution.

Step 1 Find the LCD.

The LCD of $\frac{4}{y}$ and $\frac{5y}{y+1}$ is $y(y+1)$.

Step 2 Multiply each side of the equation by the LCD.

$\frac{4}{y} + \frac{5y}{y+1} = 5$	Original equation
$y(y+1)\left(\frac{4}{y} + \frac{5y}{y+1}\right) = y(y+1)(5)$	Multiply each side by the LCD, $y(y+1)$.
$\left(\frac{y(y+1)}{1} \cdot \frac{4}{y}\right) + \left(\frac{y(y+1)}{1} \cdot \frac{5y}{y+1}\right) = y(y+1)(5)$	Distributive Property
$(y+1)4 + y(5y) = y(y+1)(5)$	Simplify.
$4y + 4 + 5y^2 = 5y^2 + 5y$	Multiply.
$4y + 4 + 5y^2 - 5y^2 = 5y^2 - 5y^2 + 5y$	Subtract $5y^2$ from each side.
$4y + 4 = 5y$	Simplify.
$4y - 4y + 4 = 5y - 4y$	Subtract $4y$ from each side.
$4 = y$	Simplify.

CHECK

$\frac{4}{y} + \frac{5y}{y+1} = 5$	Original equation
$\frac{4}{4} + \frac{5(4)}{4+1} \stackrel{?}{=} 5$	Replace y with 4.
$1 + 4 \stackrel{?}{=} 5$	Simplify.
$5 = 5$ ✓	Simplify.

Study Tip

Solutions It is important to check the solutions of rational equations to be sure that they satisfy the original equation.

Guided Practice

Solve each equation. Check your solutions.

2A. $\frac{2b-5}{b-2} - 2 = \frac{3}{b+2}$ **1**

2B. $1 + \frac{1}{c+2} = \frac{28}{c^2+2c}$ **4, −7**

2C. $\frac{y+2}{y-2} - \frac{2}{y+2} = -\frac{7}{3}$ **$-1, \frac{2}{5}$**

2D. $\frac{n}{3n+6} - \frac{n}{5n+10} = \frac{2}{5}$ **−3**

Vocabulary Link

extraneous
Everyday Use irrelevant or unimportant

extraneous solution
Math Use a result that is not a solution of the original equation

Recall that any value of a variable that makes the denominator of a rational expression zero must be excluded from the domain.

In the same way, when a solution of a rational equation results in a zero in the denominator, that solution must be excluded. Such solutions are also called **extraneous solutions**.

$$\frac{4+x}{x-5} + \frac{1}{x} = \frac{2}{x+1} \quad \text{5, 0, and −1 cannot be solutions.}$$

1 Solve Rational Equations

Example 1 shows how to use cross products to solve rational expressions when both sides of the equation are single fractions. **Example 2** shows how to solve rational equations by multiplying each side of the equation by the LCD to eliminate fractions. **Example 3** shows how to determine which solutions are extraneous solutions when both sides of a rational equation are multiplied by the LCD of the rational expressions that make up the equation.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 **FRIENDS** Saeed can run 3 kilometers an hour faster than Sultan. Saeed can run 5 kilometers in the same time it takes Sultan to run 3 kilometers. Solve $\frac{5}{x+3} = \frac{3}{x}$ to find how fast Sultan can run. Check the solution. $4\frac{1}{2}$ kilometers per hour

2 Solve $\frac{5}{x+1} - \frac{1}{x} = \frac{2}{x^2+x}$. Check the solution. $\frac{3}{4}$

Watch Out!

Preventing Errors Suggest students make a mental note of the values for the variable that make the denominator equal to zero.

Suggest that students check their solutions by substituting them back into the original equation.

Teach with Tech

Interactive Whiteboard Display the graph of the radical function on the board. Solve the equation algebraically, and show students how this solution relates to the graph. Save your notes as a PDF and post on your class Web site.

Additional Example

3 Solve $\frac{3x}{x-1} + \frac{6x-9}{x-1} = 6.$ State any extraneous solutions. no solution; 1

WatchOut!

Student Misconceptions Emphasize that the equation must be solved in order to find the solution(s) and to determine whether the solutions are extraneous. The value of the variable that would make the denominator zero is not necessarily an extaneous solution of the rational equation.

Focus on Mathematical Content

Extraneous Roots It is always important to check a solution to an equation in the original equation, but it is especially important when each side of an equation has been multiplied by a variable. If your solution is an approximation, it is sometimes difficult to determine whether a discrepancy is due to rounding or if it is an incorrect solution.

2 Use Rational Equations to Solve Problems

Example 4 shows how to use rational equations to solve real-world work problems. **Example 5** shows how to use rational equations to solve real-world rate problems.

Additional Example

4 **TV INSTALLATION** On Saturdays, Khawla helps her father install satellite TV systems. The jobs normally take Khawla's father about $2\frac{1}{2}$ hours. But when Khawla helps, the jobs only take them $1\frac{1}{2}$ hours. If Khawla were installing a satellite system herself, how long would the job take? $3\frac{3}{4}$ hours

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Example 3 Extraneous Solutions

Solve $\frac{2n}{n-5} + \frac{4n-30}{n-5} = 5$. State any extraneous solutions.

$$\frac{2n}{n-5} + \frac{4n-30}{n-5} = 5 \quad \text{Original equation}$$

$$(n-5)\left(\frac{2n}{n-5} + \frac{4n-30}{n-5}\right) = (n-5)5 \quad \text{Multiply each side by the LCD, } n-5.$$

$$\left(\frac{\cancel{n-5}^{1}}{1} \cdot \frac{2n}{\cancel{n-5}_{1}}\right) + \left(\frac{\cancel{n-5}^{1}}{1} \cdot \frac{4n-30}{\cancel{n-5}_{1}}\right) = (n-5)5 \quad \text{Distributive Property}$$

$$2n + 4n - 30 = 5n - 25 \quad \text{Simplify.}$$

$$6n - 30 = 5n - 25 \quad \text{Add like terms.}$$

$$6n - 5n - 30 = 5n - 5n - 25 \quad \text{Subtract } 5n \text{ from each side.}$$

$$n - 30 = -25 \quad \text{Simplify.}$$

$$n - 30 + 30 = -25 + 30 \quad \text{Add 30 to each side.}$$

$$n = 5 \quad \text{Simplify.}$$

Since $n = 5$ results in a zero in the denominator of the original equation, it is an extraneous solution. So, the equation has no solution.

StudyTip

Solutions It is possible to get both a valid solution and an extraneous solution when solving a rational equation.

GuidedPractice

3\. Solve $\frac{n^2 - 3n}{n^2 - 4} - \frac{10}{n^2 - 4} = 2$. State any extraneous solutions. −1; extraneous solution: −2

2 Use Rational Equations to Solve Problems

You can use rational equations to solve **work problems**, or problems involving work rates.

Real-World Example 4 Work Problem

JOBS At his part-time job at the zoo, Rasheed can clean the bird area in 2 hours. Reham can clean the same area in 1 hour and 15 minutes. How long would it take them if they worked together?

Understand It takes Rasheed 2 hours to complete the job and Reham $1\frac{1}{4}$ hours.

You need to find the rate that each person works and the total time t that it will take them if they work together.

Plan Find the fraction of the job that each person can do in an hour.

$$\text{Rasheed's rate} \longrightarrow \frac{1 \text{ job}}{2 \text{ hours}} = \frac{1}{2} \text{ job per hour}$$

$$\text{Reham's rate} \longrightarrow \frac{1 \text{ job}}{1\frac{1}{4} \text{ hours}} \text{ or } \frac{1 \text{ job}}{\frac{5}{4} \text{ hours}} = \frac{4}{5} \text{ job per hour}$$

Since rate · time = fraction of job done, multiply each rate by the time t to represent the amount of the job done by each person.

StudyTip

Reasoning When solving work problems, remember that each term should represent the portion of a job completed in one unit of time.

Solve

Fraction of job Rasheed completes	plus	fraction of job Reham completes	equals	1 job.
$\frac{1}{2}t$	$+$	$\frac{4}{5}t$	$=$	1

$10\left(\frac{1}{2}t + \frac{4}{5}t\right) = 10(1)$ — Multiply each side by the LCD, 10.

$10\left(\frac{1}{2}t\right) + 10\left(\frac{4}{5}t\right) = 10$ — Distributive Property

$5t + 8t = 10$ — Simplify.

$t = \frac{10}{13}$ — Add like terms and divide each side by 13.

So, it would take them $\frac{10}{13}$ hour or about 46 minutes to complete the job if they work together.

Check In $\frac{10}{13}$ hour, Rasheed would complete $\frac{1}{2} \cdot \frac{10}{13}$ or $\frac{5}{13}$ of the job and Reham would complete $\frac{4}{5} \cdot \frac{10}{13}$ or $\frac{8}{13}$ of the job. Together, they complete $\frac{5}{13} + \frac{8}{13}$ or 1 whole job. So, the answer is reasonable. ✓

GuidedPractice

4. **RAKING** Alia can rake the leaves in 2 hours. It takes her brother Zayed 3 hours. How long would it take them if they worked together? $1\frac{1}{5}$ hour

Rational equations can also be used to solve **rate problems**.

Real-WorldLink

The longest nonstop commercial flight was 21,600 kilometers from Hong Kong Airport in China to London Heathrow in the United Kingdom. It took 22 hours and 42 minutes.

Source: *Guinness Book of World Records*

Real-World Example 5 Rate Problem

AIRPLANES An airplane takes off and flies an average of 772 kilometers per hour. Another plane leaves 15 minutes later and flies to the same city traveling 900 kilometers per hour. How long will it take the second plane to pass the first plane?

Record the information that you know in a table.

Plane	Distance	Rate	Time
1	d kilometers	772 kilometers	t hours
2	d kilometers	900 kilometers	$t - \frac{1}{4}$ hours

← Plane 2 took off 15 minutes, or $\frac{1}{4}$ hour, after Plane 1

Since both planes will have traveled the same distance when Plane 2 passes Plane 1, you can write the following equation.

Distance for Plane 1 = Distance for Plane 2

$772 \cdot t = 900 \cdot \left(t - \frac{1}{4}\right)$ — distance = rate • time

$772t = (900 \cdot t) - \left(900 \cdot \frac{1}{4}\right)$ — Distributive Property

$772t = 900t - 225$ — Simplify.

$-80t = -225$ — Subtract 560t from each side.

$t = 1.75$ — Divide each side by −80.

So, the second plane passes the first plane after 1.75 hours.

GuidedPractice

5. Huda leaves the house walking at 3 kilometers per hour. After 10 minutes, her mother leaves the house riding a bicycle at 10 kilometers per hour. In how many minutes will Huda's mother catch her? $14\frac{2}{7}$ min

Additional Example

5 BUS A bus leaves a station and travels an average of 80 kilometers per hour towards a city. Another bus leaves the same station 20 minutes later and travels to the same city traveling 96 kilometers per hour. How long will it take the second bus to pass the first bus? 1 hour 40 minutes, or $1\frac{2}{3}$ hours

Teaching the Mathematical Practices

Reasoning Mathematically proficient students make sense of quantities and their relationships in problem situations. Encourage students to start problems they work on their own by planning and writing an equation as shown in Example 4.

DifferentiatedInstruction AL OL

If you think students would benefit by acting out a concept in this lesson,

Then have students refer to Example 5. Ask students to design a rate problem that two students can act out, such as walking a certain distance. Have the two students start at different times. The student who starts later should take more steps per second than the other. Have another student record the time the one student passes the other. Then work out the problem to see if the calculations reflect the actual time.

3 Practice

Formative Assessment

Use Exercises 1–8 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Teaching the Mathematical Practices

Reasoning Mathematically proficient students create a coherent representation of the problem at hand. In Exercise 28, remind students to examine their solution in terms of the situation.

Arguments Mathematically proficient students can recognize and use counterexamples. In Exercise 39, remind students that a counterexample is a specific case in which the statement is false. So they need to write a rational equation that has a solution of 0.

Additional Answers

25a. line

25b. $f(x) = \frac{x(x+5)(x-6)}{x-6}$
$= x + 5$

25c. -5

26a. parabola

26b. $f(x) = \frac{x(x+2)(x-1)}{x+2}$
$= x(x-1)$

26c. 0, 1

27a. parabola

27b. $f(x) = x^2 + 6x + 12$

27c. no real zeros

Check Your Understanding

Examples 1–3 **Solve each equation. State any extraneous solutions.** 6. $-\frac{4}{3}$; extraneous: 1

1. $\frac{2}{x+1} = \frac{4}{x}$ -2
2. $\frac{t+3}{5} = \frac{2t+3}{9}$ 12
3. $\frac{a+3}{a} - \frac{6}{5a} = \frac{1}{a}$ $-\frac{4}{5}$
4. $4 - \frac{p}{p-1} = \frac{2}{p-1}$ 2
5. $\frac{2t}{t+1} + \frac{4}{t-1} = 2$ -3
6. $\frac{x+3}{x^2-1} - \frac{2x}{x-1} = 1$

Example 4

7. **WEEDING** Sultan can weed the garden in 45 minutes. His sister Abeer can weed the garden in 50 minutes. How long would it take them to weed the garden if they work together? $\frac{15}{38}$ hour or about 0.4 hour

Example 5

8. **LANDSCAPING** Amer is filling a 13.2 liter bucket to water plants at a faucet that flows at a rate of 6.6 liters a minute. If he were to add a hose that flows at a rate of 5.4 liters per minute, how many minutes would it take him to fill the bucket? Round to the nearest tenth. 1.1 min

Practice and Problem Solving

Examples 1–3 **Solve each equation. State any extraneous solutions.**

9. $\frac{8}{n} = \frac{3}{n-5}$ 8
10. $\frac{6}{t+2} = \frac{4}{t}$ 4
11. $\frac{3g+2}{12} = \frac{g}{2}$ $\frac{2}{3}$
12. $\frac{5h}{4} + \frac{1}{2} = \frac{3h}{8}$ $-\frac{4}{7}$
13. $\frac{2}{3w} = \frac{2}{15} + \frac{12}{5w}$ -13
14. $\frac{c-4}{c+1} = \frac{c}{c-1}$ $\frac{2}{3}$
15. $\frac{x-1}{x+1} - \frac{2x}{x-1} = -1$ 0
16. $\frac{y+4}{y-2} + \frac{6}{y-2} = \frac{1}{y+3}$ $-4, -8$
17. $\frac{a}{a+3} + \frac{a^2}{a+3} = 2$ $-2, 3$
18. $\frac{12}{a+3} + \frac{6}{a^2-9} = \frac{8}{a+3}$ $\frac{3}{2}$
19. $\frac{3n}{n-1} + \frac{6n-9}{n-1} = 6$ no solution; extraneous: 1
20. $\frac{n^2-n-6}{n^2-n} - \frac{n-5}{n-1} = \frac{n-3}{n^2-n}$ no solution; extraneous: 1

Example 4

21. **PAINTING** It takes Saeed 3 hours to paint one side of a fence. It takes Tarek 5 hours. How long would it take them if they worked together? $\frac{15}{8}$ hours or $1\frac{7}{8}$ hours

22. **DISHWASHING** Obaid works as a dishwasher and can wash 500 plates in two hours and 15 minutes. Abdulrahman can finish the 500 plates in 3 hours. About how long would it take them to finish all of the plates if they work together? $\frac{9}{7}$ hours or $1\frac{2}{7}$ hours

Example 5

23. **ICE** A hotel has two ice machines in its kitchen. How many hours would it take both machines to make 60 kg of ice? Round to the nearest tenth. 26.2 hours

24. **CYCLING** Two cyclists travel in opposite directions around a 5.6-kilometer circular trail. They start at the same time. The first cyclist completes the trail in 22 minutes and the second in 28 minutes. At what time do they pass each other? 12.32 min

25–27. See margin.

B **GRAPHING CALCULATOR** **For each function, a) describe the shape of the graph, b) use factoring to simplify the function, and c) find the zeros of the function.**

25. $f(x) = \frac{x^2 - x - 30}{x - 6}$
26. $f(x) = \frac{x^3 + x^2 - 2x}{x + 2}$
27. $f(x) = \frac{x^3 + 6x^2 + 12x}{x}$

28. **REASONING** Abdulkarim can paint a standard-sized house in about 5 days. For his latest job, Abdulkarim hires two assistants. At what rate must these assistants work for Abdulkarim to meet a deadline of two days? They must paint $\frac{3}{10}$ of the house each day for 2 days.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	9–24, 37–57	9–23 odd, 41–44	10–24 even, 37–40, 45–57
OL Core	9–27 odd, 28–31, 33, 35, 37–57	9–24, 41–44	25–35, 37–40, 45–57
BL Advanced	25–57		

594 | Lesson 9-7 | Rational Equations

29. **AIRPLANES** Headwinds push against a plane and reduce its total speed, while tailwinds push on a plane and increase its total speed. Let w equal the speed of the wind, r equal the speed set by the pilot, and s equal the total speed.

a. Write an equation for the total speed with a headwind and an equation for the total speed with a tailwind. $s = r - w$; $s = r + w$

b. Use the rate formula to write an equation for the distance traveled by a plane with a headwind and another equation for the distance traveled by a plane with a tailwind. Then solve each equation for time instead of distance. $d = t(r - w)$, $d = t(r + w)$; $t = \frac{d}{r-w}$, $t = \frac{d}{r+w}$

30. **MIXTURES** A pitcher of fruit juice has 3 liters of pineapple juice and 2 liters of orange juice. Fatheya wants to add more orange juice so that the fruit juice mixture is 60% orange juice. Let x equal the liters of orange juice that she needs to add.

a. Copy and complete the table below.

Juice	Liters of Orange Juice	Total liters of Juice	Percent of Orange Juice
original mixture	2	5	0.4
final mixture	$2 + x$	$5 + x$	0.6

b. Write and solve an equation to find the liters of orange juice to add. $\frac{2+x}{5+x} = 0.6$; 2.5 pt

31. **DORMITORIES** The number of hours h it takes to clean a dormitory varies inversely with the number of people cleaning it c and directly with the number of people living there p.

a. Write an equation showing how h, c, and p are related. (*Hint:* Include the constant k.) $h = \frac{kp}{c}$

b. It takes 8 hours for 5 people to clean the dormitory when there are 100 people there. How long will it take to clean the dormitory if there are 10 people cleaning and the number of people living in the dorm stays the same? 4 hours

Solve each equation. State any extraneous solutions.

32. $\frac{4b+2}{b^2-3b} + \frac{b+2}{b} = \frac{b-1}{b}$ 1

33. $\frac{x^2-x-6}{x+2} + \frac{x^3+x^2}{x} = 3$ $-1 \pm \sqrt{7}$; extraneous: 0, −2

34. $\frac{y^2+5y-6}{y^3-2y^2} = \frac{5}{y} - \frac{6}{y^3-2y^2}$ $\frac{15}{4}$; extraneous: 0

35. $\frac{x-\frac{6}{5}}{x} - \frac{x-10\frac{1}{2}}{x-5} = \frac{x+21}{x^2-5x}$ $\frac{50}{11}$

H.O.T. Problems Use Higher-Order Thinking Skills

36. **CHALLENGE** Solve $\frac{2x}{x-2} + \frac{x^2+3x}{(x+1)(x-2)} = \frac{2}{(x+1)(x-2)}$. $-2, \frac{1}{3}$

37. **REASONING** How is an excluded value of a rational expression related to an extraneous solution of a corresponding rational equation? Explain. 37. The extraneous solution of a rational equation is an excluded value of one of the expressions in the equation.

38. **WRITING IN MATH** Why should you check solutions of rational equations? See margin.

39. **ARGUMENTS** Find a counterexample for the following statement.

The solution of a rational equation can never be zero. Sample answer: $\frac{x}{x-8} = 0$

40. **WRITING IN MATH** Describe the steps for solving a rational equation that is not a proportion. Sample answer: First, find the LCD of the fractions in the equation. Then multiply each side of the equation by the LCD. Simplify and use the order of operations to solve for the variable.

Additional Answer

38. Sample answer: Multiplying each side of a rational equation by the LCD can result in extraneous solutions. Therefore, all solutions should be checked to make sure that they satisfy the original equation.

4 Assess

Yesterday's News Have students write how knowing how to simplify mixed expressions helped them with today's lesson.

Standardized Test Practice

41. It takes Ali 4 hours to build a fence. If he hires Omar to help him, they can do the job in 3 hours. If Omar built the same fence alone, how long would it take him? D

A $1\frac{5}{7}$ hours C 8 hours

B $3\frac{2}{3}$ hours D 12 hours

42. In the 1000-meter race, Adnan finished 35 meters ahead of Mansour and 53 meters ahead of Ayoub. How far was Mansour ahead of Ayoub? F

F 18 m G 35 m H 53 m J 88 m

43. Twenty liters of lemonade were poured into two containers of different sizes. Express the amount of lemonade poured into the smaller container in terms of g, the amount poured into the larger container. D

A $g + 20$ C $g - 20$

B $20 + g$ D $20 - g$

44. **GRIDDED RESPONSE** The gym has 2-kilogram and 5-kilogram disks for weight lifting. They have fourteen disks in all. The total weight of the 2-kilogram disks is the same as the total weight of the 5-kilogram disks. How many 2-kilogram disks are there? 10

Spiral Review

45. **POPULATION** The country of Latvia has been experiencing a 1.1% annual decrease in population. In 2009, its population was 2,261,294. If the trend continues, predict Latvia's population in 2019. about 2,172,453

46. **TOMATOES** There are more than 10,000 varieties of tomatoes. One seed company produces seed packages for 200 varieties of tomatoes. For how many varieties do they not provide seeds?
Sample answer: Let t = the number of tomato varieties for which they do not produce seeds, $t + 200 > 10{,}000$; $\{t \mid t > 9800\}$.

47. **DRIVING** Tires should be kept within 2 kilograms per square centimeter (psi) of the manufacturer's recommended tire pressure. If the recommendation for a tire is 30 psi, what is the range of acceptable pressures? $\{p \mid 28 \le p \le 32\}$

Express each number in scientific notation.

48. 12,300 1.23×10^4

49. 0.0000375 3.75×10^{-5}

50. 1,255,000 1.255×10^6

51. **FINANCIAL LITERACY** Mohammad has AED 13 to order pizza. The pizza costs AED 7.50 plus AED 1.25 per topping. He plans to tip 15% of the total cost. Write and solve an inequality to find out how many toppings he can order.
$7.50 + 1.25t + 0.15(7.50 + 1.25t) \le 13$; 3 or fewer toppings

Solve each inequality. Check your solution.

52. $\frac{b}{10} \le 5$ $\{b \mid b \le 50\}$

53. $-7 > -\frac{r}{7}$ $\{r \mid r > 49\}$

54. $\frac{5}{8}y \ge -15$ $\{y \mid y \ge -24\}$

Skills Review

Determine the probability of each event if you randomly select a marble from a bag containing 9 red marbles, 6 blue marbles, and 5 yellow marbles.

55. P(blue) 0.3

56. P(red) 0.45

57. P(not yellow) 0.75

Differentiated Instruction OL BL

Extension Solve $\frac{a}{x} = \frac{2}{3}$ for x in terms of a. $x = \frac{3a}{2}$

EXTEND 9-7 Graphing Technology Lab: Solving Rational Equations

You can use a graphing calculator to solve rational equations by graphing, by using tables, and by using a computer algebra system (CAS).

Mathematical Practices
Use appropriate tools strategically.

To solve by graphing, graph both sides of the equation and locate the point(s) of intersection.

Activity 1 Solve a Rational Equation by Graphing

Solve $\frac{5}{x+2} = \frac{3}{x}$ by graphing.

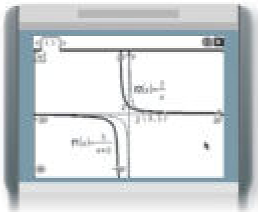
Step 1 Add a new **Graphs** page.

Step 2 Use the **Window Settings** option from the **Window/Zoom** menu to adjust the window to -20 to 20 for both x and y. Set both scales to 2.

Step 3 Enter $\frac{5}{x+2}$ into **f1(*x*)** and $\frac{3}{x}$ into **f2(*x*)**.

Step 4 Change the thickness of the graph of **f1(*x*)** by selecting the graph of **f1(*x*)** and the **ctrl menu Attributes** option.

Step 5 Use the **Intersection Point(s)** tool from the **Points & Lines** menu to find the intersection of the two graphs. Select the graph of **f1(*x*) enter** and then the graph of **f2(*x*) enter**.



[−20, 20] scl: 2 by [−20, 20] scl: 2

The graphs intersect at (3, 1). This means that $\frac{5}{x+2}$ and $\frac{3}{x}$ both equal 1 when $x = 3$. Thus, the solution of $\frac{5}{x+2} = \frac{3}{x}$ is $x = 3$.

Exercises

Use a graphing calculator to solve each equation.

1. $\frac{5}{x} + \frac{4}{x} = 10$ $\frac{9}{10}$
2. $\frac{12}{x} + \frac{3}{4} = \frac{3}{2}$ 16
3. $\frac{6}{x} + \frac{3}{2x} = 12$ $\frac{5}{8}$
4. $\frac{4}{x} + \frac{3}{4x} = \frac{1}{8}$ 38
5. $\frac{4}{x} + \frac{x-2}{2x} = x$ $-\frac{3}{2}$ or 2
6. $\frac{3}{3x-2} + \frac{5}{x} = 0$ $\frac{5}{9}$
7. $\frac{2x+1}{2} + \frac{3}{2x} = \frac{2}{x}$ -1 or $\frac{1}{2}$
8. $\frac{x}{x+2} + x = \frac{5x+8}{x+2}$ 4
9. $\frac{1}{2x} + \frac{5}{x} = \frac{3}{x-1}$ $\frac{11}{5}$
10. $\frac{4x-3}{x-2} + \frac{2x+5}{x-2} = 6$ no solution

1 Focus

Objective Use graphing technology to solve rational equations.

Materials

- graphing calculator

Teaching Tips

- For Activity 1 step 3, remind students that to enter $\frac{5}{x+2}$, they will need to put parentheses around the denominator. Then they will need to use the **tab** key to move the curser to the entry line to type $\frac{3}{x}$ into ***f2(x).***
- When changing the windows settings, use the **tab** key to move from one field to another.
- A spreadsheet program can also be used to complete Activity 2.
- For Activity 3, remind students that they cannot edit a line once they press **enter**. However, they can use **ctrl c** and **ctrl x** to copy and paste a line, and then make edits.

2 Teach

Working in Cooperative Groups

Divide the class into pairs. Work through Activity 1 as a class. Then ask students to work with their partners to complete Exercises 1 to 10 and Activities 2 and 3.

Practice Have students complete Exercises 11 to 15.

3 Assess

Tips for New Teachers

Using Tables Point out to students that the table method only works when their table includes the x-value(s) of the solution(s). If a solution is not found with a table with integer values of x, students should adjust their x-values or use another method to find any solutions.

Formative Assessment

Use Exercises 16–19 to assess each student's knowledge of solving rational equations.

From Concrete to Abstract

Ask students to summarize the use of technology to find the solutions to rational equations.

Graphing Technology Lab
Solving Rational Equations *Continued*

Activity 2 Solve a Rational Equation by Using a Table

Solve $\frac{2x+1}{3} = \frac{x+2}{2}$ using a table.

Step 1 Add a new **Lists & Spreadsheet** page.

Step 2 Label column A as x. Enter values from -4 to 4 in cells A1 to A9.

Step 3 In column B in the formula row, enter the left side of the rational equation, with parenthesis around the binomials. In column C in the formula row, enter the right side of the rational equation, with parenthesis around the binomials. Specify **Variable Reference** when prompted.

Scroll until you see where the values in Columns B and C are equal. This occurs at $x = 4$. Therefore the solution of $\frac{2x+1}{3} = \frac{x+2}{2}$ is 4.

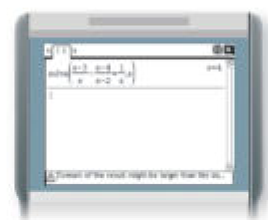
You can also use a computer algebra system (CAS) to solve rational equations.

Activity 3 Solve a Rational Equation by Using a CAS

Solve $\frac{x-3}{x} - \frac{x-4}{x-2} = \frac{1}{x}$ using a CAS.

Step1 Add a new **Calculator** page.

Step 2 To solve, select the **Solve** tool from the **Algebra** menu. Enter the left side of the equation with parenthesis around the binomials. Enter = and the right side of the equation. Then type a comma, followed by x, and then **enter**.



The solution of 4 is displayed.

Exercises

Use a table or CAS to solve each equation.

11. $\frac{2}{x} + \frac{2+x}{2} = \frac{x+3}{2}$ **4**

12. $\frac{4}{x-2} = -\frac{1}{x+3}$ **−2**

13. $\frac{3}{x+2} + \frac{4}{x-1} = 0$ **$-\frac{5}{7}$**

14. $\frac{1}{x+1} + \frac{2}{x-1} = 0$ **$-\frac{1}{3}$**

15. $\frac{2}{x+4} + \frac{4}{x-1} = 0$ **$-\frac{7}{3}$**

16. $\frac{1}{x-2} + \frac{x+2}{4} = 2x$ **0 or $\frac{16}{7}$**

17. $\frac{2x}{x+3} + \frac{x+1}{2} = x$ **−1 or 3**

18. $\frac{2}{x-3} + \frac{3}{x-2} = \frac{4}{x}$ **$\frac{-7 \pm \sqrt{145}}{2}$**

19. $\frac{x^2}{x+1} + \frac{x}{x-1} = x$ **0**

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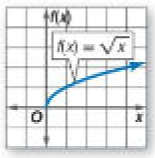
CHAPTER 9 Study Guide and Review

Study Guide

KeyConcepts

Square Root Functions (Lesson 9-1)

- A square root function contains the square root of a variable.
- The parent function of the family of square root functions is $f(x) = \sqrt{x}$.



Simplifying Radical Expressions (Lesson 9-2)

- A radical expression is in simplest form when
  - no radicands have perfect square factors other than 1,
  - no radicals contain fractions,
  - and no radicals appear in the denominator of a fraction.

Operations with Radical Expressions and Equations (Lessons 9-3 and 9-4)

- Radical expressions with like radicals can be added or subtracted.
- Use the FOIL method to multiply radical expressions.

Inverse Variation (Lesson 9-5)

- You can use $\frac{x_1}{x_2} = \frac{y_2}{y_1}$ to solve problems involving inverse variation.

Rational Functions (Lesson 9-6)

- Excluded values are values of a variable that result in a denominator of zero.
- If vertical asymptotes occur, it will be at excluded values.

Solving Rational Equations (Lesson 9-7)

- Use cross products to solve rational equations with a single fraction on each side of the equals sign.

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.

KeyVocabulary

asymptote	radical function
closed	radicand
conjugate	rate problem
excluded value	rationalize the denominator
extraneous solution	rational function
inverse variation	rational equation
product rule	square root function
radical equations	work problem
radical expression	

VocabularyCheck

State whether each sentence is *true* or *false*. If *false*, replace the underlined word, phrase, expression, or number to make a true sentence.

1. The expressions $\underline{12\sqrt{4}}$ and $\sqrt{288}$ are equivalent. **false; $12\sqrt{2}$**
2. The expressions $2 + \sqrt{5}$ and $\underline{2 - \sqrt{5}}$ are conjugates. **true**
3. In the expression $-5\sqrt{2}$, the radicand is $\underline{2}$. **true**
4. If the product of two variables is a nonzero constant, the relationship is an <u>inverse variation</u>. **true**
5. If the line $x = a$ is a vertical <u>asymptote</u> of a rational function, then a is an excluded value. **true**
6. The excluded values for $\frac{x}{x^2 + 5x + 6}$ are $\underline{-2 \text{ and } -3}$. **true**
7. The equation $\frac{3x}{x-2} = \frac{6}{x-2}$ has an extraneous solution, $\underline{2}$. **true**

Formative Assessment

KeyVocabulary If students have difficulty answering questions 1–7, remind them that they can review the lessons to refresh their memories about the vocabulary terms.

FOLDABLES StudyOrganizer

Dinah Zike's Foldables®

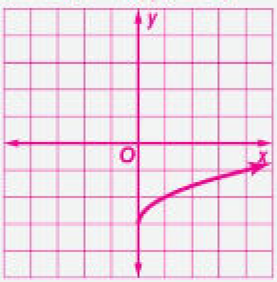
Have students look through the chapter to make sure they have included examples in their Foldables for each lesson of the chapter. Suggest that students keep their Foldables handy while completing the Study Guide and Review pages. Point out that their Foldables can serve as a quick review when studying for the chapter test.

Lesson-by-Lesson Review

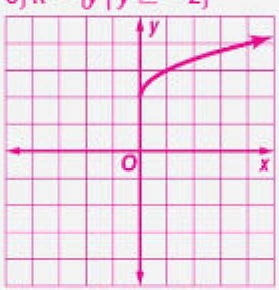
Intervention If the given examples are not sufficient to review the topics covered by the questions, remind students that the lesson references tell them where to review that topic in their textbooks.

Additional Answers

8. translated down 3 units; D = {x | $x \geq 0$} R = {y | $y \geq 3$}



9. translated up 2 units; D = {x | $x \geq 0$} R = {y | $y \geq -2$}



CHAPTER 9 Study Guide and Review *Continued*

Lesson-by-Lesson Review

9-1 Square Root Functions

Graph each function. Compare to the parent graph. State the domain and range. **8–13. See margin.**

8. $y = \sqrt{x} - 3$
9. $y = \sqrt{x} + 2$
10. $y = -5\sqrt{x}$
11. $y = \sqrt{x} - 6$
12. $y = \sqrt{x - 1}$
13. $y = \sqrt{x} + 5$

14. **GEOMETRY** The function $s = \sqrt{A}$ can be used to find the length of a side of a square given its area. Use this function to determine the length of a side of a square with an area of 90 square centimeters. Round to the nearest tenth if necessary. **9.5 cm.**

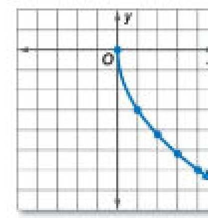
Example 1

Graph $y = -3\sqrt{x}$. Compare to the parent graph. State the domain and range.

Make a table. Choose nonnegative values for x.

x	0	1	2	3	4
y	0	−3	≈−4.2	≈−5.2	−6

Plot points and draw a smooth curve.



The graph of $y = \sqrt{x}$ is stretched vertically and is reflected across the x-axis.

The domain is {x|$x \geq 0$}.

The range is {y|$y \leq 0$}.

9-2 Simplifying Radical Expressions

Simplify.

15. $\sqrt{36x^2y^7}$ **$6|x|y^3\sqrt{y}$**
16. $\sqrt{20ab^3}$ **$2b\sqrt{5ab}$**
17. $\sqrt{3} \cdot \sqrt{6}$ **$3\sqrt{2}$**
18. $2\sqrt{3} \cdot 3\sqrt{12}$ **36**
19. $(4 - \sqrt{5})^2$ **$21 - 8\sqrt{5}$**
20. $(1 + \sqrt{2})^2$ **$2\sqrt{2} + 3$**
21. $\sqrt{\frac{50}{a^2}}$ **$\frac{5\sqrt{2}}{|a|}$**
22. $\sqrt{\frac{2}{5}} \cdot \sqrt{\frac{3}{4}}$ **$\frac{\sqrt{30}}{10}$**
23. $\frac{3}{2 - \sqrt{5}}$ **$-6 - 3\sqrt{5}$**
24. $\frac{5}{\sqrt{7} + 6}$ **$\frac{5\sqrt{7} - 30}{-29}$**

25. **WEATHER** To estimate how long a thunderstorm will last, use $t = \sqrt{\frac{d^3}{216}}$, where t is the time in hours and d is the diameter of the storm in kilometers. A storm is 10 kilometers in diameter. How long will it last? **about 2.15 hours or 2 hours and 9 minutes**

Example 2

Simplify $\frac{2}{4 + \sqrt{3}}$.

$\frac{2}{4 + \sqrt{3}}$ — Original expression

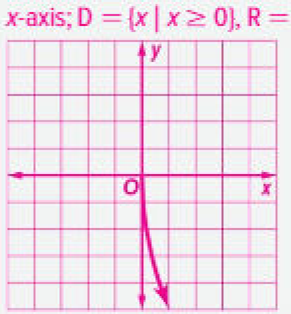
$= \frac{2}{4 + \sqrt{3}} \cdot \frac{4 - \sqrt{3}}{4 - \sqrt{3}}$ — Rationalize the denominator.

$= \frac{2(4) - 2\sqrt{3}}{4^2 - (\sqrt{3})^2}$ — $(a - b)(a + b) = a^2 - b^2$

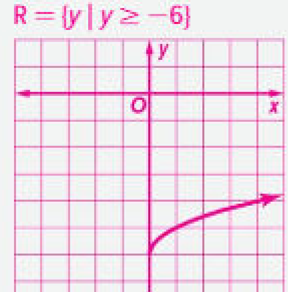
$= \frac{8 - 2\sqrt{3}}{16 - 3}$ — $(\sqrt{3})^2 = 3$

$= \frac{8 - 2\sqrt{3}}{13}$ — Simplify.

10. stretched vertically and reflected across the x-axis; D = {x | $x \geq 0$}, R = {y | $y \leq 0$}



11. translated down 6 units; D = {x | $x \geq 0$}, R = {y | $y \geq -6$}



9-3 Operations with Radical Expressions

Simplify each expression.

26. $\sqrt{6} - \sqrt{54} + 3\sqrt{12} + 5\sqrt{3}$ $-2\sqrt{6} + 11\sqrt{3}$

27. $2\sqrt{6} - \sqrt{48}$ $2\sqrt{6} - 4\sqrt{3}$

28. $4\sqrt{3x} - 3\sqrt{3x} + 3\sqrt{3x}$ $4\sqrt{3x}$

29. $\sqrt{50} + \sqrt{75}$ $5\sqrt{2} + 5\sqrt{3}$

30. $\sqrt{2}(5 + 3\sqrt{3})$ $5\sqrt{2} + 3\sqrt{6}$

31. $(2\sqrt{3} - \sqrt{5})(\sqrt{10} + 4\sqrt{6})$ $-2\sqrt{30} + 19\sqrt{2}$

32. $(6\sqrt{5} + 2)(4\sqrt{2} + \sqrt{3})$ $24\sqrt{10} + 8\sqrt{2} + 6\sqrt{15} + 2\sqrt{3}$

33. **MOTION** The velocity of a dropped object when it hits the ground can be found using $v = \sqrt{2gd}$, where v is the velocity in meters per second, g is the acceleration due to gravity, and d is the distance in meters the object drops. Find the speed of a penny when it hits the ground, after being dropped from 984 meters. Use 9.8 meters per second squared for g. **about 76.68 m/s**

Example 3

Simplify $2\sqrt{6} - \sqrt{24}$.

$2\sqrt{6} - \sqrt{24} = 2\sqrt{6} - \sqrt{4 \cdot 6}$	Product Property
$= 2\sqrt{6} - 2\sqrt{6}$	Simplify.
$= 0$	Simplify.

Example 4

Simplify $(\sqrt{3} - \sqrt{2})(\sqrt{3} + 2\sqrt{2})$.

$(\sqrt{3} - \sqrt{2})(\sqrt{3} + 2\sqrt{2})$

$= (\sqrt{3})(\sqrt{3}) + (\sqrt{3})(2\sqrt{2}) + (-\sqrt{2})(\sqrt{3}) + (\sqrt{2})(2\sqrt{2})$

$= 3 + 2\sqrt{6} - \sqrt{6} + 4$

$= 7 + \sqrt{6}$

9-4 Radical Equations

Solve each equation. Check your solution.

34. $10 + 2\sqrt{x} = 0$ no solution

35. $\sqrt{5 - 4x} - 6 = 7$ -41

36. $\sqrt{a + 4} = 6$ 32

37. $\sqrt{3x} = 2$ $\frac{4}{3}$

38. $\sqrt{x + 4} = x - 8$ 12

39. $\sqrt{3x - 14} + x = 6$ 5

40. **FREE FALL** Assuming no air resistance, the time t in seconds that it takes an object to fall h meters can be determined by $t = \sqrt{\frac{2h}{4}}$. If a skydiver jumps from an airplane and free falls for 10 seconds before opening the parachute, how many meters does she free fall? **480.5 m**

Example 5

Solve $\sqrt{7x + 4} - 18 = 5$.

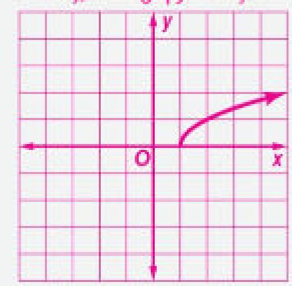
$\sqrt{7x + 4} - 18 = 5$	Original equation
$\sqrt{7x + 4} = 23$	Add 18 to each side.
$(\sqrt{7x + 4})^2 = 23^2$	Square each side.
$7x + 4 = 529$	Simplify.
$7x = 525$	Subtract 4 from each side.
$x = 75$	Divide each side by 7.

CHECK

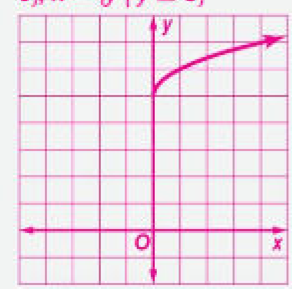
$\sqrt{7x + 4} - 18 = 5$	Original equation
$\sqrt{7(75) + 4} - 18 \stackrel{?}{=} 5$	$x = 75$
$\sqrt{525 + 4} - 18 \stackrel{?}{=} 5$	Multiply.
$\sqrt{529} - 18 \stackrel{?}{=} 5$	Add.
$23 - 18 \stackrel{?}{=} 5$	Simplify.
$5 = 5$ ✓	True.

Additional Answers

12. translated right 1 unit; $D = \{x \mid x \geq 1\}$, $R = \{y \mid y \geq 0\}$

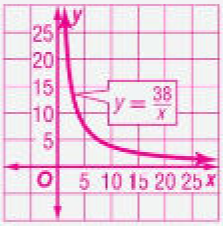


13. translated up 5 units; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq 5\}$



Additional Answer

49.



The vertical asymptote is at $x = 0$ and the horizontal asymptote is at $y = 0$.

CHAPTER 9

Study Guide and Review *Continued*

Lesson-by-Lesson Review

9-5 Inverse Variation

Solve. Assume that y varies inversely as x.

41. If $y = 4$ when $x = 1$, find x when $y = 12$ $\frac{1}{3}$

42. If $y = -1$ when $x = -3$, find y when $x = -9$ $-\frac{1}{3}$

43. If $y = 1.5$ when $x = 6$, find y when $x = -16$ $-\frac{9}{16}$

44. PHYSICS A 61 kilograms person sits 1.5 m from the center of a seesaw. How far from the center should a 49 kilograms person sit to balance the seesaw? **1.87 m**

Example 6

If y varies inversely as x and $y = 28$ when $x = 42$, find y when $x = 56$.

Let $x_1 = 42$, $x_2 = 56$, and $y_1 = 28$. Solve for y_2.

$\frac{x_1}{x_2} = \frac{y_2}{y_1}$ Proportion for inverse variation

$\frac{42}{56} = \frac{y_2}{28}$ Substitution

$1176 = 56y_2$ Cross multiply.

$21 = y_2$

Thus, $y = 21$ when $x = 56$.

9-6 Rational Functions

State the excluded value for each function.

45. $y = \frac{1}{x-3}$ **3**

46. $y = \frac{2}{2x-5}$ $\frac{5}{2}$

47. $y = \frac{3}{3x-6}$ **2**

48. $y = \frac{-1}{2x+8}$ **−4**

49. PIZZA PARTY Hiyam ordered pizza and soda for her study group for AED 38. The cost per person y is given by $y = \frac{38}{x}$, where x is the number of people in the study group. Graph the function and describe the asymptotes. **See margin.**

Example 7

State the excluded value for the function $y = \frac{1}{4x+16}$.

Set the denominator equal to zero.

$4x + 16 = 0$

$4x + 16 - 16 = 0 - 16$ Subtract 16 from each side.

$4x = -16$ Simplify.

$x = -4$ Divide each side by 4.

9-7 Rational Equations

Solve each equation. State any extraneous solutions.

50. $\frac{5n}{6} + \frac{1}{n-2} = \frac{n+1}{3(n-2)}$ $\frac{2}{5}$, **extraneous: 2**

51. $\frac{4x}{3} + \frac{7}{2} = \frac{7x}{12} - 14$ $-\frac{70}{3}$

52. $\frac{11}{2x} + \frac{2}{4x} = \frac{1}{4}$ **24**

53. $\frac{1}{x+4} - \frac{1}{x-1} = \frac{2}{x^2+3x-4}$ **no solution**

54. $\frac{1}{n-2} = \frac{n}{8}$ **−2, 4**

55. PAINTING Wafa can paint a room in 6 hours. Hana can paint a room in 4 hours. How long will it take them to paint the room working together? $\frac{12}{5}$ **or** $2\frac{2}{5}$ **hours**

Example 8

Solve $\frac{3}{x^2+3x} + \frac{x+2}{x+3} = \frac{1}{x}$.

$$\frac{3}{x^2+3x} + \frac{x+2}{x+3} = \frac{1}{x}$$

$$x(x+3)\left(\frac{3}{x(x+3)}\right) + x(x+3)\left(\frac{x+2}{x+3}\right) = x(x+3)\left(\frac{1}{x}\right)$$

$$3 + x(x+2) = 1(x+3)$$

$$3 + x^2 + 2x = x + 3$$

$$x^2 + x = 0$$

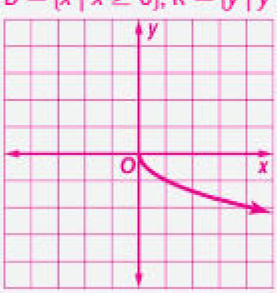
$$x(x+1) = 0$$

$$x = 0 \text{ or } x = -1$$

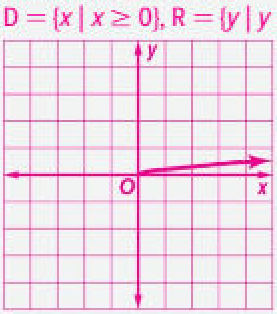
The solution is −1, and there is an extraneous solution of 0.

Additional Answers (Practice Test)

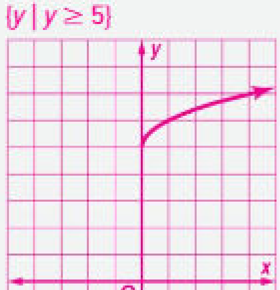
1. reflected across the x-axis; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \leq 0\}$



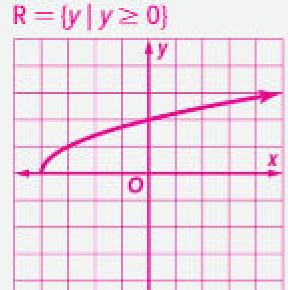
2. compressed vertically; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq 0\}$



3. translated up 5 units; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq 5\}$



4. translated left 4 units; $D = \{x \mid x \geq -4\}$, $R = \{y \mid y \geq 0\}$



CHAPTER 9 Practice Test

Graph each function, and compare to the parent graph. State the domain and range. **1–4. See margin.**

1. $y = -\sqrt{x}$
2. $y = \frac{1}{4}\sqrt{x}$
3. $y = \sqrt{x} + 5$
4. $y = \sqrt{x + 4}$

5. **MULTIPLE CHOICE** The length of the side of a square is given by the function $s = \sqrt{A}$, where A is the area of the square. What is the perimeter of a square that has an area of 64 square centimeters? **C**

   A 64 centimeters

   B 8 centimeters

   C 32 centimeters

   D 16 centimeters

Simplify each expression.

6. $5\sqrt{36}$ **30**
7. $\frac{3}{1 - \sqrt{2}}$ $-3 - 3\sqrt{2}$
8. $2\sqrt{3} + 7\sqrt{3}$ $9\sqrt{3}$
9. $3\sqrt{6}(5\sqrt{2})$ $30\sqrt{3}$

10. **MULTIPLE CHOICE** Find the area of the rectangle. **H**

   F $7\sqrt{2}$

   G 14

   H $14\sqrt{2}$

   J $98\sqrt{2}$

Solve each equation. Check your solution.

11. $\sqrt{10x} = 20$ **40**
12. $\sqrt{4x - 3} = 6 - x$ **3**

13. **PACKAGING** A cylindrical container of chocolate drink mix has a volume of about 2564.7 m^3. The radius of the container can be found by using the formula $r = \sqrt{\frac{V}{\pi h}}$, where r is the radius and h is the height. If the height is 21 centimeters, find the radius of the container. **about 6.24 cm.**

Determine whether each table represents an inverse variation. Explain.

14.

x	y
2	10
4	12
8	14

No; the products are not the same.

Solve. Assume that y varies inversely as x.

15. If $y = 3$ when $x = 9$, find x when $y = 1$. **27**
16. If $y = 2$ when $x = 0.5$, find y when $x = 3$. $\frac{1}{3}$

Assume that y varies inversely as x. Write an inverse variation equation that relates x and y.

17. $y = 2$ when $x = 8$ $y = \frac{16}{x}$
18. $y = -3$ when $x = 1$ $y = -\frac{3}{x}$

19. **MULTIPLE CHOICE** Humaid can shovel the driveway in 3 hours, and Hamad can shovel the driveway in 2 hours. How long will it take them working together? **J**

   F 6 hours

   G 5 hours

   H $\frac{3}{2}$ hours

   J $\frac{6}{5}$ hours

20. **PAINTING** Jassim can paint a 60-square meter wall in 40 minutes. Working with his friend Jamal, the two of them can paint the wall in 25 minutes. How long would it take Jamal to do the job himself? $\frac{10}{9}$ **or** $1\frac{1}{9}$ **hours**

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1 Focus

Objective Use the strategy of drawing a picture to solve standardized test problems.

2 Teach

Scaffolding Questions

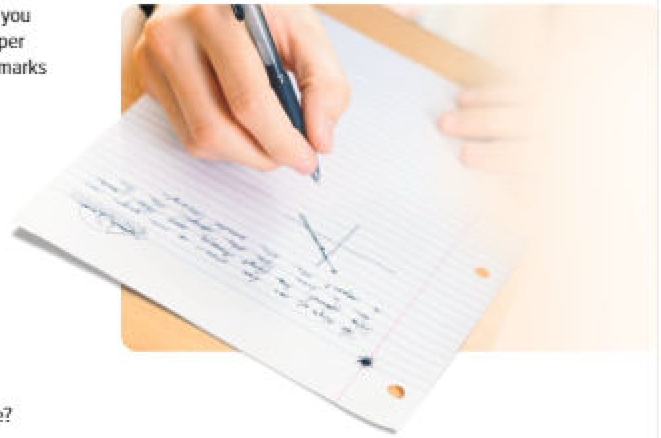
Ask:

- Have you ever drawn a picture to help solve a problem? Answers will vary.
- What kind of information did you include in the picture? Answers will vary.
- Why do you think it is helpful to draw pictures when solving problems? Sample answer: It allows you to see relationships in a different way, which makes it easier to solve the problem.

CHAPTER 9 Preparing for Standardized Tests

Draw a Picture

Sometimes it is easier to visualize how to solve a problem if you draw a picture first. You can sketch your picture on scrap paper or in your test booklet (if allowed). Be careful not make any marks on your answer sheet other than your answers.



Strategies for Drawing a Picture

Step 1

Read the problem statement carefully.

Ask yourself:

- What am I being asked to solve?
- What information is given in the problem?
- What is the unknown quantity for which I need to solve?

Step 2

Sketch and label your picture.

- Draw your picture as clearly and accurately as possible.
- Label the picture carefully. Be sure to include all of the information given in the problem statement.

Step 3

Solve the problem.

- Use your picture to help you model the problem situation with an equation. Then solve the equation.
- Check your answer to make sure it is reasonable.

Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve. Show your work.

An 5.5 meter ladder is leaning against a building. For stability, the base of the ladder must be 100 centimeters away from the wall. How far up the wall does the ladder reach?

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Read the problem statement carefully. You know the height of the ladder leaning against the building and you know that the base of the ladder must be 100 centimeters away from the wall. You need to find how far up the wall the ladder reaches.

Scoring Rubric	
Criteria	**Score**
Full Credit: The answer is correct and a full explanation is provided that shows each step.	2
Partial Credit: • The answer is correct, but the explanation is incomplete. • The answer is incorrect, but the explanation is correct.	1
No Credit: Either an answer is not provided or the answer does not make sense.	0

Example of a 2-point response:

First convert all measurements to meters.

100 centimeters = 1 meter

Use a right triangle to find how high the ladder reaches. Draw and label a triangle to represent the situation.

You know the measures of a leg and the hypotenuse, and need to know the length of the other leg. So you can use the Pythagorean Theorem.

$$5.5^2 = 1^2 + b^2$$
$$30.25 = 1 + b^2$$
$$29.25 = b^2$$
$$\pm 5.4 = b$$
$$5.4 \approx b$$

100 centimeters = 1 meter

The ladder reaches about 5.4 meters.

Exercises

Read each problem. Identify what you need to know. Then use the information in the problem to solve. Show your work.

1. A building casts a 4.6 meter shadow, while a billboard casts a 1.4 meter shadow. If the billboard is 7.9 meters high, what is the height of the building? Round to the nearest tenth if necessary. **about 25.95 meters**

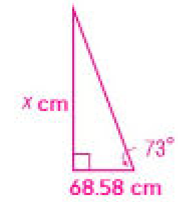
2. A space shuttle is directed toward the Moon, but drifts 1.2° from its intended course. The distance from Earth to the Moon is about 386,200 kilometers. If the pilot doesn't get the shuttle back on course, how far will the shuttle have drifted from its intended landing position? **8165.4 kilometers**

Additional Example

SHORT RESPONSE Bilal is standing next to a purple martin birdhouse that is on the ground. The bird house casts a shadow that is 68.58 centimeters long. The Sun is at an elevation of 73°. What is the height of the purple martin bird house? Round to the nearest tenth of a meter.

Sample 2-point response:

Draw and label a right triangle to represent the situation.



Set up and solve an equation.

$$\tan 73° = \frac{x}{68.58}$$
$$68.58 \tan 73° = x$$
$$2.24 \approx x$$

The purple martin bird house is about 2.2 high.

3 Assess

Use Exercises 1 and 2 to assess students' understanding.

CHAPTER 9

Standardized Test Practice

Cumulative, Chapters 1 through 9

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Each year a local country club sponsors a tennis tournament. Play starts with 256 participants. During each round, half of the players are eliminated. How many players remain after 6 rounds? **D**

A 128

B 64

C 16

D 4

2. Evaluate $\frac{5^6 - 5^5}{4}$. **G**

F 5^6

G 5^5

H $\frac{5}{4}$

J $\frac{25}{4}$

3. Which of the following numbers is less than zero? **D**

A 1.03×10^{-21}

B 7.5×10^2

C 8.21543×10^{10}

D none of the above

4. Write an equation in slope-intercept form with a slope of $\frac{9}{10}$ and y-intercept of 3. **G**

F $y = 3x + \frac{9}{10}$

G $y = \frac{9}{10}x + 3$

H $y = \frac{9}{10}x - 3$

J $y = 3x - \frac{9}{10}$

5. Bilal is playing games at a family fun center. So far he has won 38 prize tickets. How many more tickets would he need to win to place him in the gold prize category? **G**

Number of Tickets	Prize Category
1–20	bronze
21–40	silver
41–60	gold
61–80	platinum

F $2 \le t \le 22$

G $3 \le t \le 22$

H $1 \le t \le 20$

J $3 \le t \le 20$

6. Which of the following is an equation of the line perpendicular to $4x - 2y = 6$ and passing through $(4, -4)$? **J**

F $y = -\frac{3}{4}x + 3$

G $y = -\frac{3}{4}x - 1$

H $y = -\frac{1}{2}x - 4$

J $y = -\frac{1}{2}x - 2$

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Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

7. **GRIDDED RESPONSE** Mr. Abdalla bought a total of 9 tickets to the zoo. He bought children tickets at the rate of AED 6.50 and adult tickets for AED 9.25 each. If he spent AED 69.50 altogether, how many adult tickets did Mr. Abdalla purchase? **4**

8. What is the domain of the following relation? $\{(2, -1), (4, 3), (7, 6)\}$ **$\{2, 4, 7\}$**

9. Rashid just added 15 more songs to his digital media player, making the total number of songs more than 84. Draw a number line that represents the original number of songs he had on his digital media player. **See margin.**

10. Khalifa bought a rare painting in 1995 for AED 14,200. By 2003, the painting was worth AED 17,120. Assuming that a linear relationship exists, write an equation in slope-intercept form that represents the value V of the painting after t years. **$V(t) = 365t + 14{,}200$**

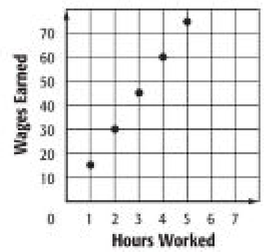
11. Ahmed spent AED 24.50 on peanuts and walnuts for a dinner party. He bought 1.5 kilograms more peanuts than walnuts. How many kilograms of peanuts and walnuts did he buy?

Product	Price per kilograms
Peanuts p	AED 3.80
Cashews c	AED 6.90
Walnuts w	AED 5.60

3.5 kgs of peanuts and 2 kgs of walnuts

12. **GRIDDED RESPONSE** Moza purchased a car several years ago for AED 21,459. The value of the car depreciated at a rate of 15% annually. What was the value of the car after 5 years? Round your answer to the nearest whole dirham. **9521**

13. **GRIDDED RESPONSE** The amount of money that Nasser earns varies directly as the number of hours that he works as shown in the graph. How much money will he earn for working 40 hours next week? Express your answer in dirhams. **AED 600**



Extended Response

Record your answers on a sheet of paper. Show your work.

14. The fare charged by a taxi drive is a AED 3 fixed charge plus AED 0.35 per kilometer. Maysoun pays AED 10 for a ride of m kilometers.

Part A Write an equation that can be used to find m. Show your work. **$3 + 0.35m = 10$**

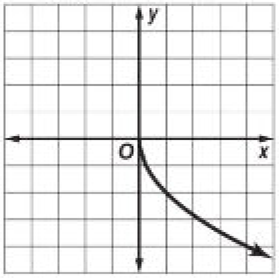
Part B Use the equation in Part A to find how many kilometers Maysoun rode. Show your work. **20 km**

Homework Option

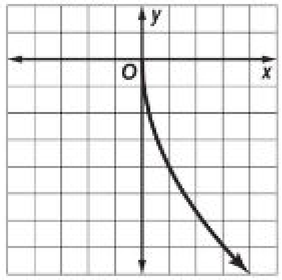
Get Ready for Chapter 10 Assign students the exercises on p. 609 as homework to assess whether they possess the prerequisite skills needed for the next chapter.

Lesson 9-1 (Guided Practice)

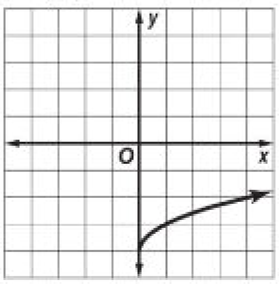
2A. stretched vertically and reflected across the x-axis; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \leq 0\}$



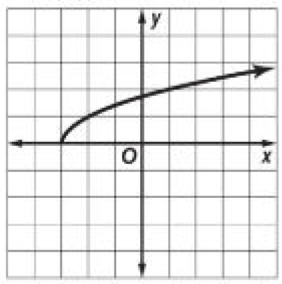
2B. stretched vertically and reflected across the x-axis; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \leq 0\}$



3A. translated down 4 units; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq -4\}$

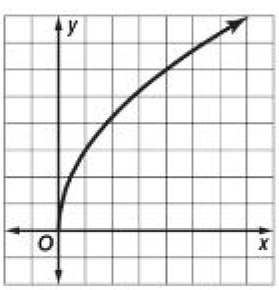


3B. translated left 3 units; $D = \{x \mid x \geq -3\}$, $R = \{y \mid y \geq 0\}$

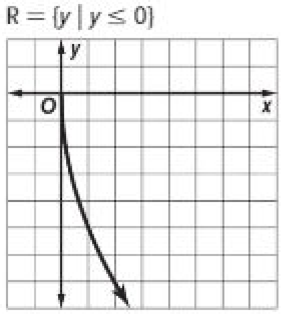


Lesson 9-1

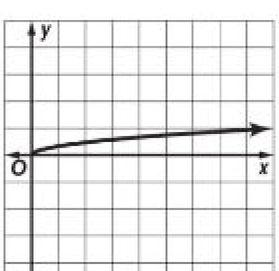
1. vertical stretch of $y = \sqrt{x}$; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq 0\}$



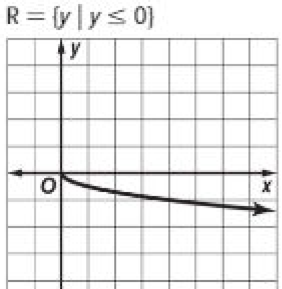
2. vertical stretch of $y = \sqrt{x}$ and a reflection across the x-axis; $D = \{x \mid x \leq 0\}$, $R = \{y \mid y \leq 0\}$



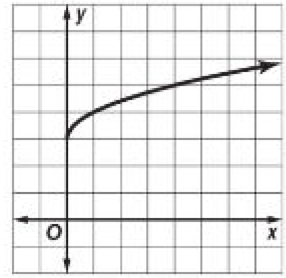
3. vertical compression of $y = \sqrt{x}$; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq 0\}$



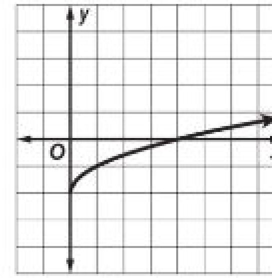
4. vertical compression of $y = \sqrt{x}$ and a reflection across the x-axis; $D = \{x \mid x \geq 0\}$; $R = \{y \mid y \leq 0\}$



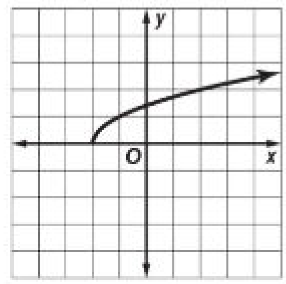
5. translated up 3 units; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq 3\}$



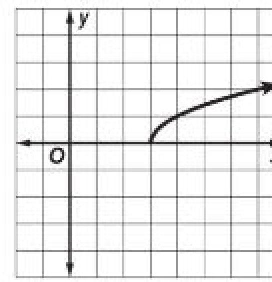
6. translated down 2 units; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq -2\}$



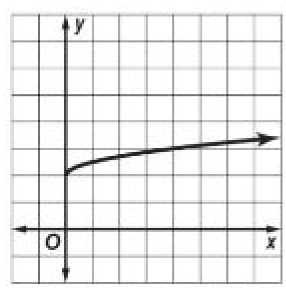
7. translated left 2 units; $D = \{x \mid x \geq -2\}$, $R = \{y \mid y \geq 0\}$



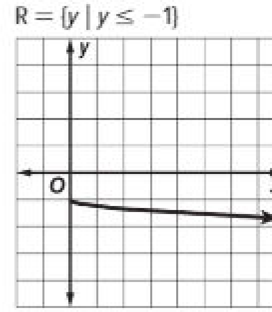
8. translated right 3 units; $D = \{x \mid x \geq 3\}$, $R = \{y \mid y \geq 0\}$



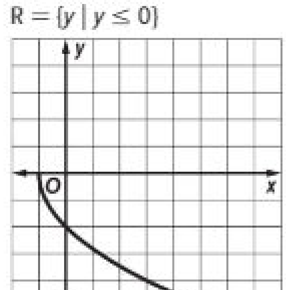
10. vertical compression of $\sqrt{x}$ and translated up 2 units; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq 2\}$



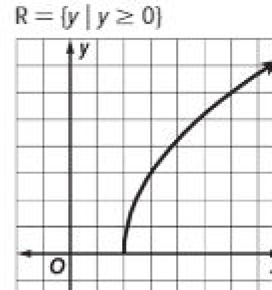
11. vertical compression of $\sqrt{x}$, and reflected across the x-axis and translated down 1 unit; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \leq -1\}$



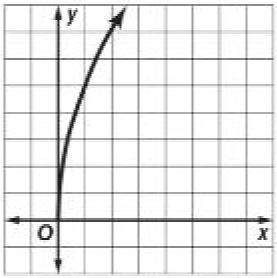
12. translated left 1 unit, vertical stretch and reflected across x-axis; $D = \{x \mid x \geq -1\}$, $R = \{y \mid y \leq 0\}$



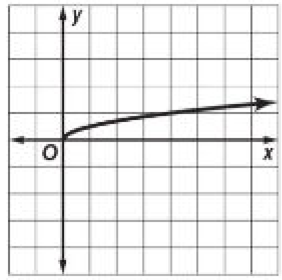
13. translated right 2 units and vertical stretch of $\sqrt{x}$; $D = \{x \mid x \geq 2\}$, $R = \{y \mid y \geq 0\}$



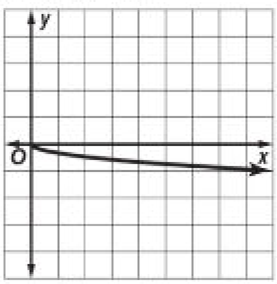
14. vertical stretch of $\sqrt{x}$;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \geq 0\}$



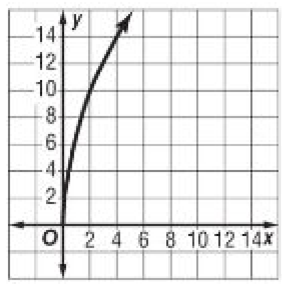
15. vertical compression of $\sqrt{x}$;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \geq 0\}$



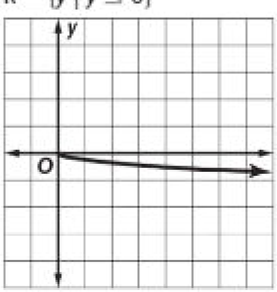
16. vertical compression of $\sqrt{x}$ and reflected across the x-axis; $D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \leq 0\}$



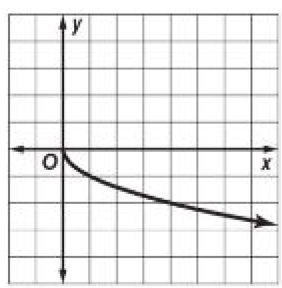
17. vertical stretch of $\sqrt{x}$;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \geq 0\}$



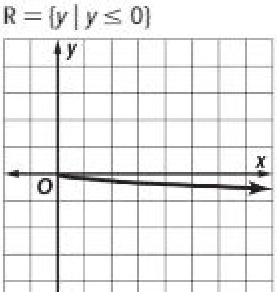
18. vertical compression of $\sqrt{x}$ and reflected across the x-axis; $D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \leq 0\}$



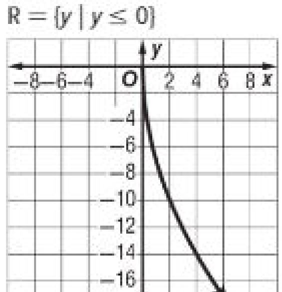
19. reflected across the x-axis;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \leq 0\}$



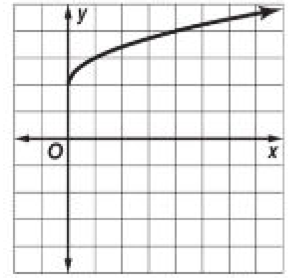
20. vertical compression of $\sqrt{x}$ and reflected across x-axis;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \leq 0\}$



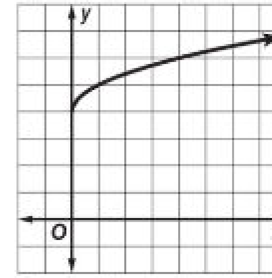
21. vertical stretch of $\sqrt{x}$ and reflected across the x-axis;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \leq 0\}$



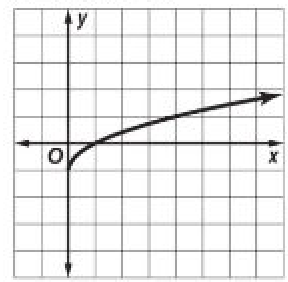
22. translated up 2 units;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \geq 2\}$



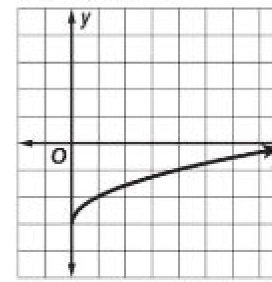
23. translated up 4 units;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \geq 4\}$



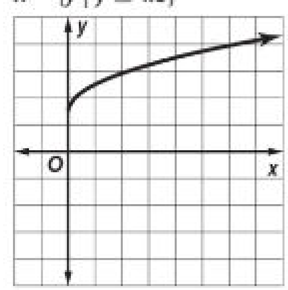
24. translated down 1 unit;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \geq -1\}$



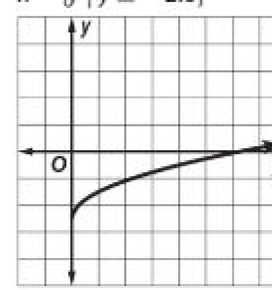
25. translated down 3 units;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \geq -3\}$



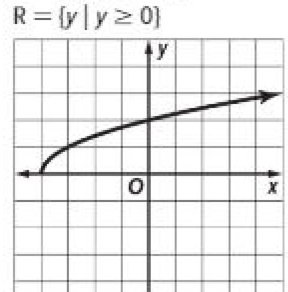
26. translated up 1.5 units;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \geq 1.5\}$



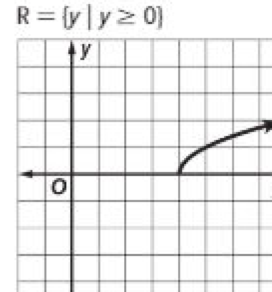
27. translated down 2.5 units;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \geq -2.5\}$



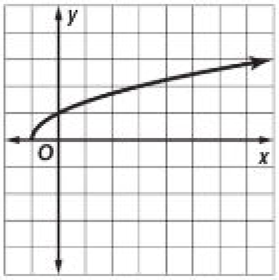
28. translated left 4 units;
$D = \{x \mid x \geq -4\}$,
$R = \{y \mid y \geq 0\}$



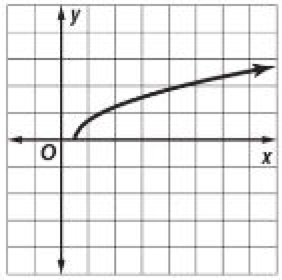
29. translated right 4 units;
$D = \{x \mid x \geq 4\}$,
$R = \{y \mid y \geq 0\}$



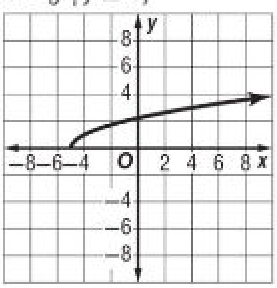
30. translated left 1 unit;
$D = \{x \mid x \geq -1\}$,
$R = \{y \mid y \geq 0\}$



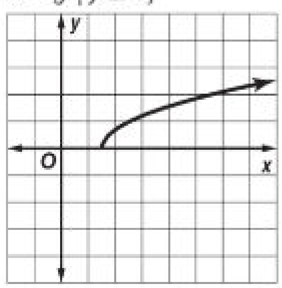
31. translated right 0.5 unit;
$D = \{x \mid x \geq 0.5\}$,
$R = \{y \mid y \geq 0\}$



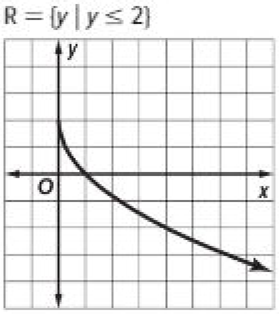
32. translated left 5 units;
$D = \{x \mid x \geq -5\}$,
$R = \{y \mid y \geq 0\}$



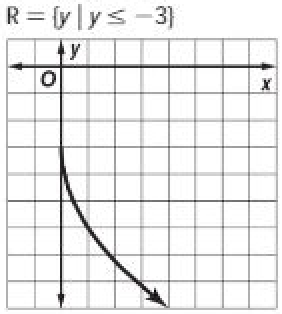
33. translated right 1.5 units;
$D = \{x \mid x \geq 1.5\}$,
$R = \{y \mid y \geq 0\}$



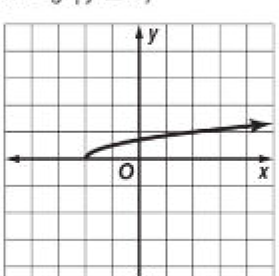
35. vertical stretch of $y = \sqrt{x}$, reflected across the x-axis, and translated up 2 units;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \leq 2\}$



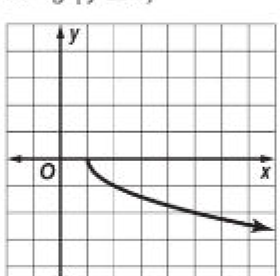
36. vertical stretch of $y = \sqrt{x}$, reflected across the x-axis, and translated down 2 units;
$D = \{x \mid x \geq 0\}$,
$R = \{y \mid y \leq -3\}$



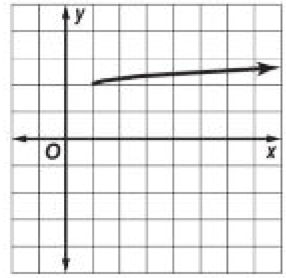
37. vertical compression of $y = \sqrt{x}$ and translated left 2 units;
$D = \{x \mid x \geq -2\}$,
$R = \{y \mid y \geq 0\}$



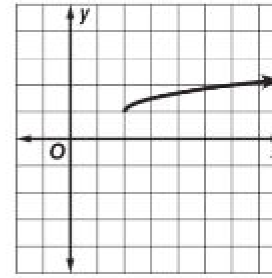
38. reflected across the x-axis and translated right 1 unit;
$D = \{x \mid x \geq 1\}$,
$R = \{y \mid y \leq 0\}$



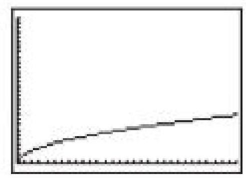
39. vertical compression of $y = \sqrt{x}$ and translated up 2 units and right 1 unit;
$D = \{x \mid x \geq 1\}$, $R = \{y \mid y \geq 2\}$



40. vertical compression of $y = \sqrt{x}$ and translated up 1 unit and right 2 units;
$D = \{x \mid x \geq 2\}$, $R = \{y \mid y \geq 1\}$

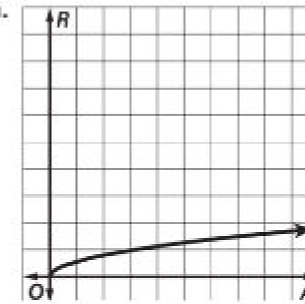


41.

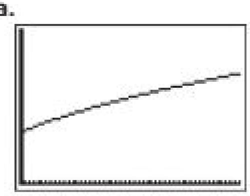


[0, 28] scl: 1 by [0, 28] scl: 1

42a.

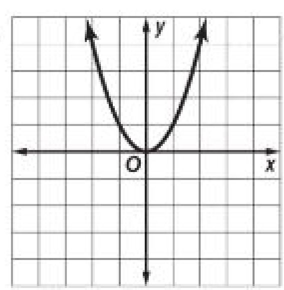


43a.



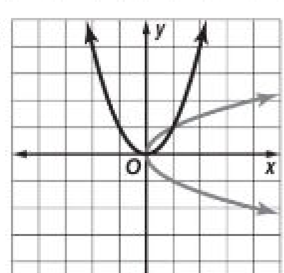
[0, 1000] scl: 20 by [0, 1000] scl: 0.1

44a.

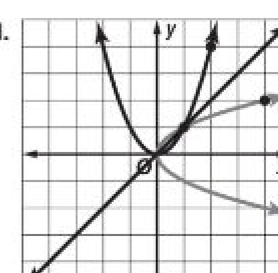


44b. $y = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{x} & \text{if } x \geq 0 \end{cases}$

44c.



44d.



44e. The combined graphs of the square root functions are the same size and shape as the parabola. It is a reflection in the line $y = x$.

45. False; sample answer: The domain of $y = \sqrt{x + 3}$ includes -1, -2, and -3.

46. False; sample answer: -6 and -5 are in the range of $y = \sqrt{x} - 6$.

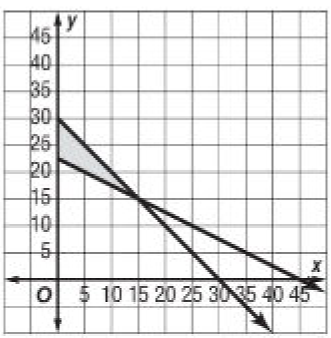
47. Sample answer: The domain is limited because square roots of negative numbers are imaginary; therefore the radicand must be nonnegative. Since the principal square root of a nonnegative number is a nonnegative number, the range will be nonnegative.

49. $y = \sqrt{x} + 3$; it is a translation of $y = \sqrt{x}$; the other equations represent vertical streches or compressions.

51. The value of a is negative. For the function to have negative y-values, the value of a must be negative.

52. Both functions are translations of the square root function, but $f(x) = \sqrt{x} + 2$ is a translation 2 units up and $g(x) = \sqrt{x + 2}$ is a translation 2 units to the left.

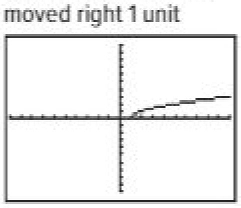
57a.



57b. Sample answer: walk: 15 min, jog: 15 min; walk: 10 min, jog: 20 min; walk: 5 min, jog: 25 min

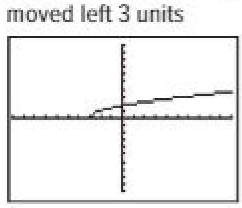
Extend 9-1

1. $\{x \mid x \geq 1\}$, $\{y \mid y \geq 0\}$; moved right 1 unit



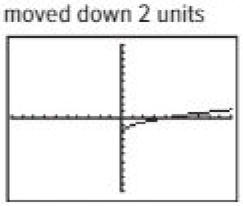
$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

2. $\{x \mid x \geq -3\}$, $\{y \mid y \geq 0\}$; moved left 3 units



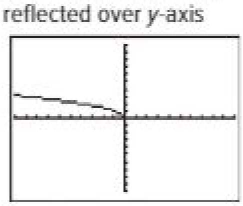
$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

3. $\{x \mid x \geq 0\}$, $\{y \mid y \geq -2\}$; moved down 2 units



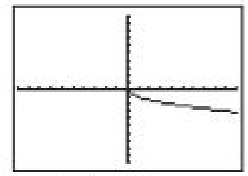
$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

4. $\{x \mid x \leq 0\}$, $\{y \mid y \geq 0\}$; reflected over y-axis



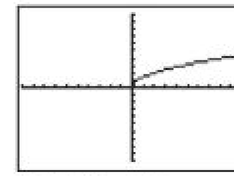
$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

5. $\{x \mid x \geq 0\}$, $\{y \mid y \leq 0\}$; reflected across x-axis



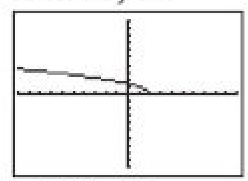
$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

6. $\{x \mid x \geq 0\}$, $\{y \mid y \geq 0\}$; vertical stretch of $y = \sqrt{x}$



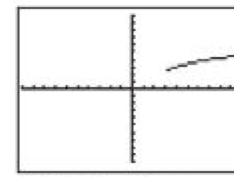
$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

7. $\{x \mid x \leq 2\}$, $\{y \mid y \geq 0\}$; moved right 2 units and reflected across the y-axis



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

8. $\{x \mid x \geq 3\}$, $\{y \mid y \geq 2\}$; moved right 3 units and up 2 units



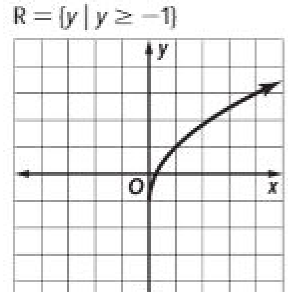
$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

Lesson 9-2

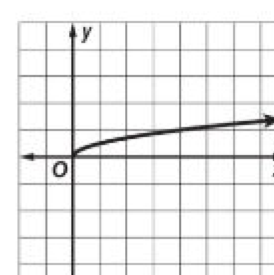
56.

$ax^2 + bx + c = 0$	Original equation
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	Divide each side by a.
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Subtract $\frac{c}{a}$ from each side.
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$	Complete the square.
$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$	Factor $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$.
$\left\lvert x + \frac{b}{2a} \right\rvert = \sqrt{\frac{b^2 - 4ac}{4a^2}}$	Take the square root of each side.
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Remove the absolute value symbols and insert $\pm$.
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Quotient Property of Square Roots
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	$\sqrt{4a^2} = 2a$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Subtract $\frac{b}{2a}$ from each side.

62. vertical stretch of $y = \sqrt{x}$ and translated down 1 unit; $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq -1\}$

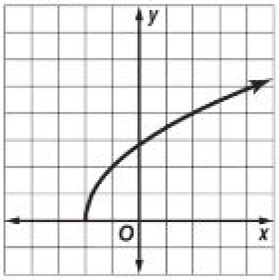


63. vertical compression of $y = \sqrt{x}$, $D = \{x \mid x \geq 0\}$, $R = \{y \mid y \geq 0\}$

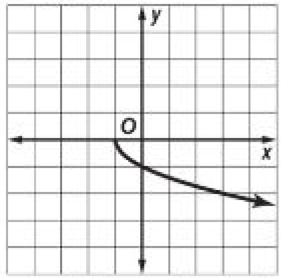


Chapter 9 Answer Appendix

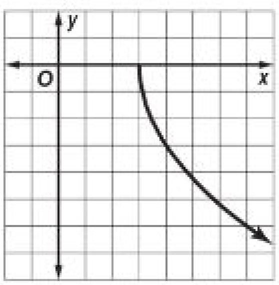
64. stretched vertically and translated left 2 units; D = {x | $x \geq -2$}, R = {y | $y \geq 0$}



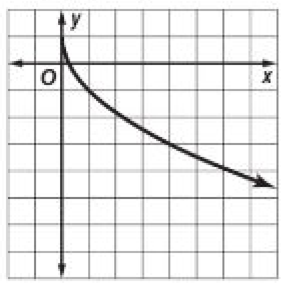
65. reflected across the x-axis and translated left 1 unit; D = {x | $x \geq -1$}, R = {y | $y \leq 0$}



66. stretched vertically, reflected across the x-axis, and translated right 3 units; D = {x | $x \geq 3$}, R = {y | $y \leq 0$}

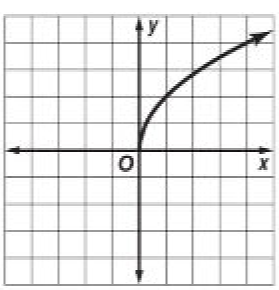


67. stretched vertically, reflected across the x-axis, and translated up 1 unit; D = {x | $x \geq 0$}, R = {y | $y \leq 1$}

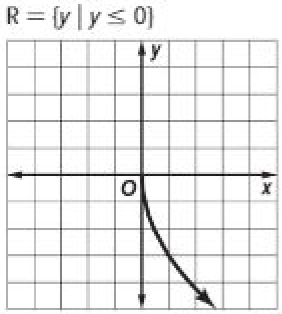


Lesson 9-3

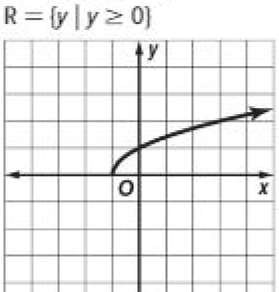
50. stretched vertically; D = {x | $x \geq 0$}, R = {y | $y \geq 0$}



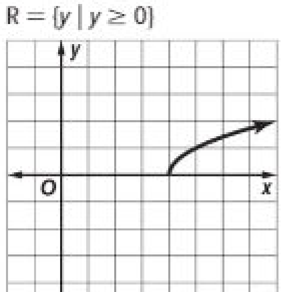
51. stretched vertically, reflected across the x-axis, D = {x | $x \geq 0$}, R = {y | $y \leq 0$}



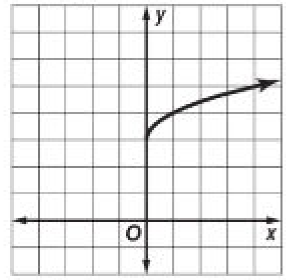
52. translated left 1 unit; D = {x | $x \geq -1$}, R = {y | $y \geq 0$}



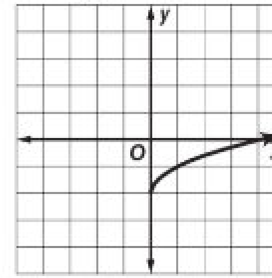
53. translated right 4 units; D = {x | $x \geq 4$}, R = {y | $y \geq 0$}



54. translated up 3 units; D = {x | $x \geq 0$}, R = {y | $y \geq 3$}

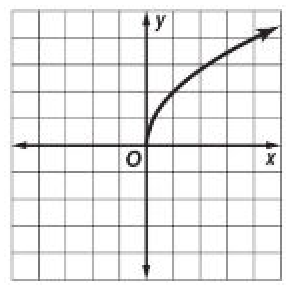


55. translated down 2 units; D = {x | $x \geq 0$}, R = {y | $y \geq -2$}

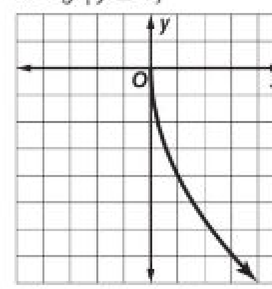


Mid-Chapter Quiz

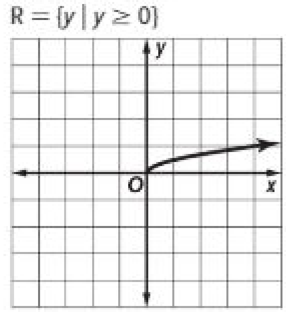
1. stretched vertically; D = {x | $x \geq 0$}, R = {y | $y \geq 0$}



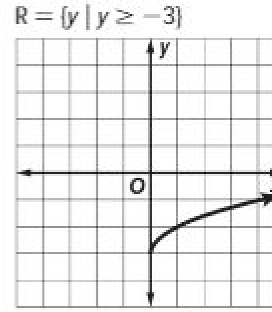
2. vertical stretch of $y = \sqrt{x}$ and reflection across the x-axis D = {x | $x \geq 0$}, R = {y | $y \leq 0$}



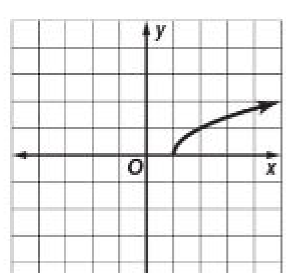
3. compressed vertically; D = {x | $x \geq 0$}, R = {y | $y \geq 0$}



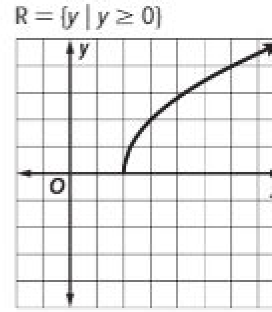
4. translation down 3 units; D = {x | $x \geq 0$}, R = {y | $y \geq -3$}



5. translation right 1 unit; D = {x | $x \geq 1$}, R = {y | $y \geq 0$}

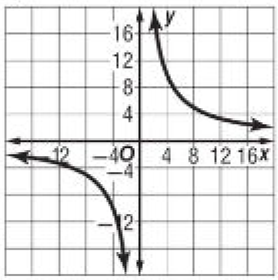


6. stretched vertically and translated right 2 units; D = {x | $x \geq 2$}, R = {y | $y \geq 0$}

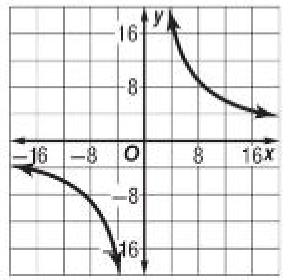


Lesson 9-5

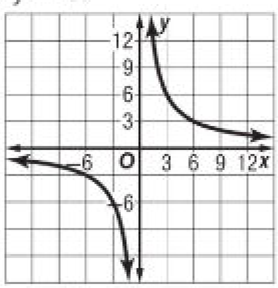
22. $xy = 40$



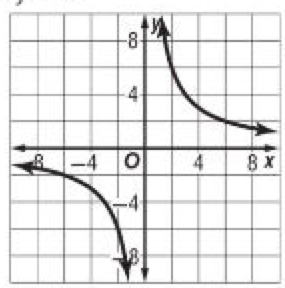
23. $xy = 72$



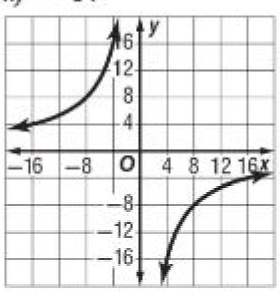
24. $xy = 18$



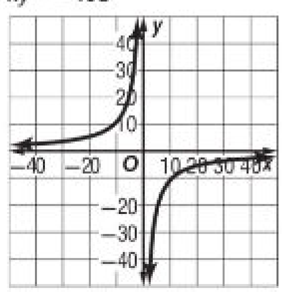
25. $xy = 12$



26. $xy = -64$



27. $xy = -108$

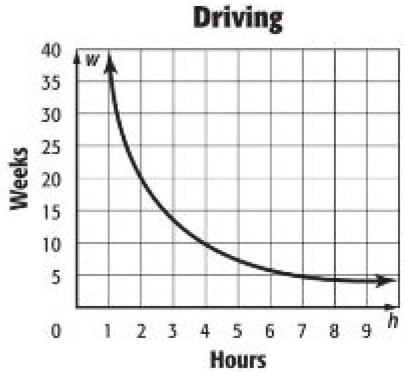


51a.

Hours per Week *h*	Number of Weeks *w*
1	40
2	20
4	10
5	8
8	5
10	4

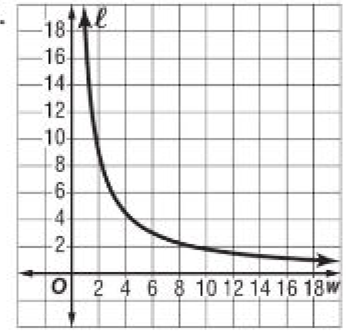
51c. $hw = 40$ or $w = \frac{40}{h}$

Driving



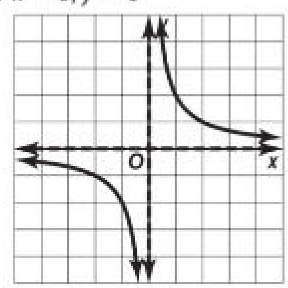
Lesson 9-6 (Guided Practice)

2.

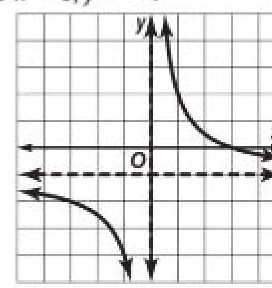


Lesson 9-6

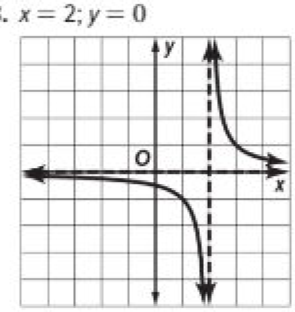
6. $x = 0; y = 0$



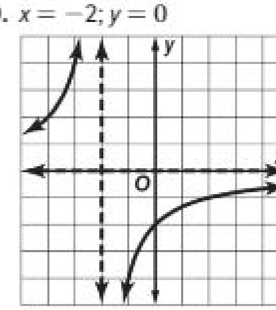
7. $x = 0; y = -1$



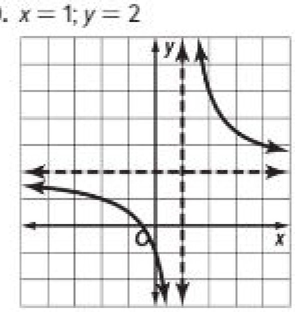
8. $x = 2; y = 0$



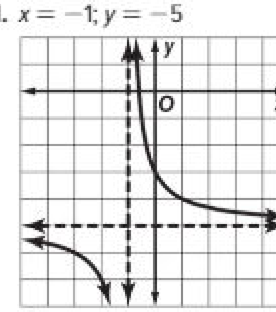
9. $x = -2; y = 0$



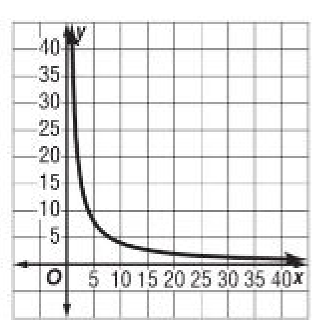
10. $x = 1; y = 2$



11. $x = -1; y = -5$



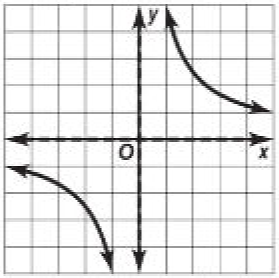
20a.



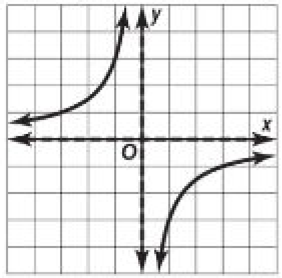
Chapter 9 Answer Appendix

Chapter 9 Answer Appendix

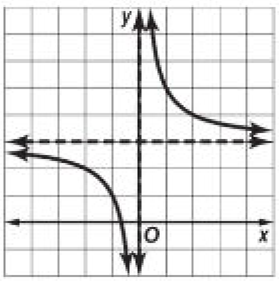
22. $x = 0; y = 0$



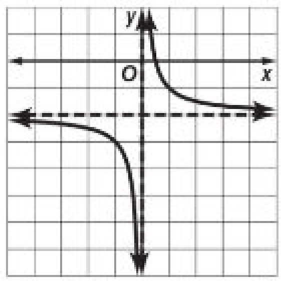
23. $x = 0; y = 0$



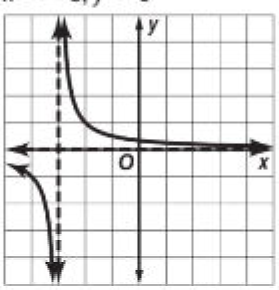
24. $x = 0; y = 3$



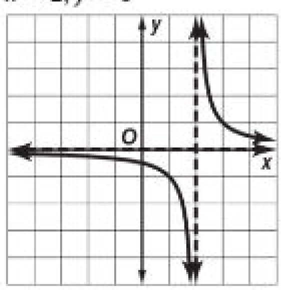
25. $x = 0; y = -2$



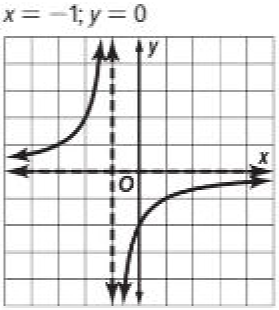
26. $x = -3; y = 0$



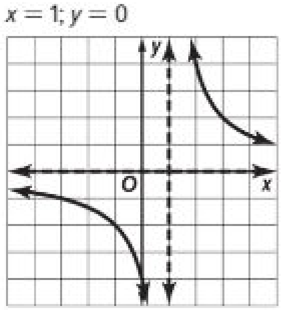
27. $x = 2; y = 0$



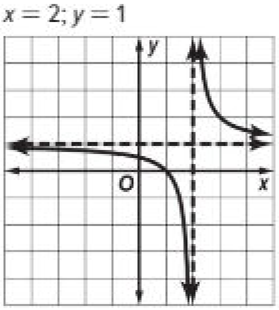
28. $x = -1; y = 0$



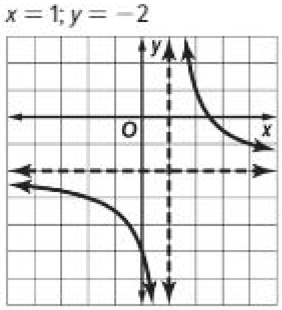
29. $x = 1; y = 0$



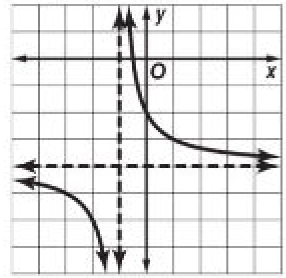
30. $x = 2; y = 1$



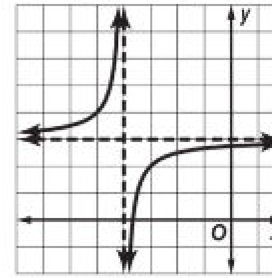
31. $x = 1; y = -2$



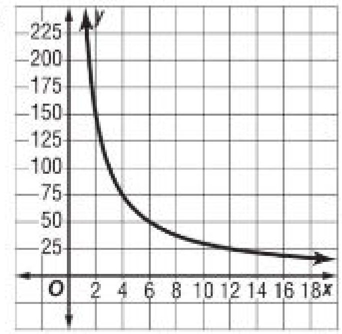
32. $x = -1; y = -4$



33. $x = -4; y = 3$

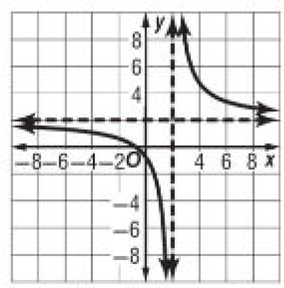


34a.



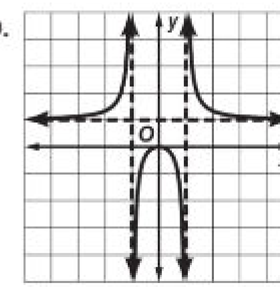
Sample answer: The end behavior indicates that as the number of days increases, the number of pages per day approaches 0. Since there is no x-intercept the number of pages per day will never be 0.

38.



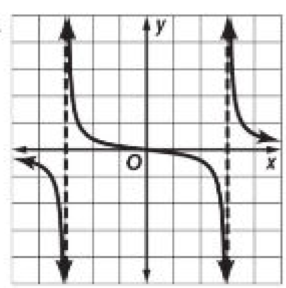
$x = 2; y = 2$

39.



$x = -1, x = 1; y = 1$

40.



$x = 3, x = -3; y = 0$

NOTES

Chapter Planner

	Diagnostic Assessment Quick Check			
	LESSON 10-1 45 min: 1.5 days 90 min: 0.75 day	EXTEND 10-1 45 min: 0.5 day 90 min: 0.25 day	LESSON 10-2 45 min: 1.5 days 90 min: 0.75 day	EXTEND 10-2 45 min: 0.5 day 90 min: 0.25 day
Title	**Points, Lines, and Planes**	**Geometry Lab: Describing What You See**	**Linear Measure**	**Extension Lesson: Precision and Accuracy**
Objectives	■ Identify and model points, lines, and planes. ■ Identify intersecting lines and planes.	■ Use correct mathematical terminology to describe geometric figures.	■ Measure segments. ■ Calculate with measures.	■ Determine precision of measurements. ■ Determine accuracy of measurements.
Key Vocabulary	undefined term point line plane collinear coplanar space		line segment betweenness of points between congruent segments construction	precision absolute error significant digits accuracy

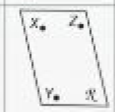
	LESSON 10-3 45 min: 1 day 90 min: 0.5 day	LESSON 10-4 45 min: 2 days 90 min: 0.75 day
	Distance and Midpoints	**Proving Theorems about Line Segments**
	▪ Find the distance between two points. ▪ Find the midpoint of a segment.	▪ Write proofs involving segment addition. ▪ Write proofs involving congruence.
	distance irrational number midpoint segment bisector	

Reading and Writing in Mathematics

Project CRISS

STUDY SKILL

The content frame can be used to help students compare and contrast information. The information at the right is from Lesson 10-1. Have students continue the frame for additional terms and geometric figures defined and used in this chapter.

Geometric Figure or Term	Definition	Dimension	Picture	Name	Special Features
point	Undefined; Description: a figure with no size or shape	0	A •	*A*	
line	Undefined Description: a set points	1		$\overleftrightarrow{LM}$	
plane	Undefined. Description: a set of points that create a flat surface	2		plane *R* plane *XYZ*	

Creating Independence through Student-owned Strategies

NOTES

CHAPTER 10

Tools of Geometry

Assessment

SE = Student Edition, TE = Teacher Edition

	Diagnosis	Prescription
DIAGNOSTIC ASSESSMENT	**Beginning Chapter 10**	
	Get Ready for Chapter 10 **SE**	Response to Intervention **TE**
	Beginning Every Lesson	
	Then, Now, Why? **SE**	Chapter 0 **SE**
FORMATIVE ASSESSMENT	**During/After Every Lesson**	
	Guided Practice **SE**, every example Check Your Understanding **SE** H.O.T. Problems **SE** Spiral Review **SE** Additional Examples **TE** Watch Out! **TE** Step 4, Assess **TE**	Differentiated Instruction **TE** Differentiated Homework Options **TE**
	Mid-Chapter	
	Mid-Chapter Quiz **SE**	
SUMMATIVE ASSESSMENT	**Before Chapter Test**	
	Chapter Study Guide and Review **SE** Practice Test **SE** Standardized Test Practice **SE**	

Differentiated Instruction

Option 1 Reaching All Learners AL OL BL

Logical *Give students the following problem, explaining that they can use any means they need to solve it. (i.e. drawing a diagram, trial and error, using manipulatives, or any other problem solving strategy they choose.)*

Your classroom will receive 20 new desks. Each desk is 61 centimeters wide, 84 centimeters deep, and 76 centimeters high. They will be placed in four rows, spaced one meter apart, and 15 centimeters from the desk in front of them. How much surface area will the new desks need?

Naturalist Have students use disposable or digital cameras to take at least 12 pictures of various angles found on school grounds. Print or develop the pictures and have students measure and classify the angles in their photographs. You can have students share their most interesting photos with the class.

Option 2 Approaching Level AL

Use poster board or chart paper to draw several angles of different measures. Label each point on your drawing and display these angles in the front of the classroom. Separate students into two teams, and have them form two single file lines facing the angle drawings. Explain that each team will take turns sending a member to the front of the room. Tell them that they will classify and measure the angle you call out. Call out an angle and have the player measure and classify the angle. If either the classification or the measurement is incorrect, the other team can "steal it." The team that answers correctly earns a point. Play for as long as desired.

Option 3 Beyond Level BL

Have students work in small groups to create a scale map of their school. They should plot important points and label them (i.e., bathrooms, specific classrooms, office, cafeteria, and so on) using coordinates. Tell students to provide a map scale so that it is possible to calculate distances from point to point.

Focus on Mathematical Content

Vertical Alignment

Before Chapter 10

Related Topics

- Interpret situations in terms of given graphs or create situations that fit given graphs.
- Use symbols to represent unknowns and variables.

Chapter 10

Topics

- Develop an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.
- Use construction to explore attributes of geometric figures and to make conjectures about geometric relationships.
- Use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures.
- Find areas of regular polygons, circles, and composite figures.

After Chapter 10

Preparation

- Analyze situations modeled by square root functions, formulate equations and inequalities, select a method, and solve problems.

Lesson-by-Lesson Preview

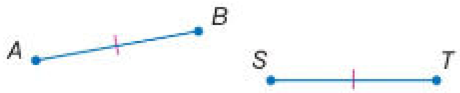
10-1 Points, Lines, and Planes

In geometry, a *point* is a location without shape or size. A *line* contains points and has no thickness or width. Points on the same line are collinear, and there is exactly one line through any two points. The intersection of two lines is a point.

A *plane* is a flat surface made of points. A plane has no depth and extends infinitely in all directions. Points on the same plane are coplanar, and the intersection of two planes is a line.

10-2 Linear Measure

A line cannot be measured because it extends infinitely in each direction. A line segment, however, has two endpoints and can be measured. Two segments with the same measure are said to be congruent. The symbol for congruence is $\cong$. An equal number of tic marks also indicates that segments are congruent.



10-3 Distance and Midpoints

The coordinates of the endpoints of a segment can be used to find the length of the segment. On a number line, the distance between the endpoints is the absolute value of their difference. On a coordinate plane, you can use the Distance Formula or the Pythagorean Theorem to calculate the distance between two points.

The *midpoint* of a segment is the point halfway between its endpoints. On a number line, the coordinate of the midpoint of a segment with endpoints a and b is the sum of a and b divided by 2. To find the midpoint of a segment on the coordinate plane, use the Midpoint Formula.

10-4 Proving Theorems about Line Segments

As you learned in Chapter 1, a segment can be measured, and measures can be used in calculations because they are real numbers. The Ruler Postulate states that the points on any line or line segment can be paired with real numbers so that, given any two points A and B on a line, A corresponds to 0, and B corresponds to a positive real number. That number is the length of the segment. Another postulate states that if point B lies between points A and C on the same line, $AB + BC = AC$. The converse statement holds true as well.

$$AB + BC = AC$$

The Reflexive, Symmetric, and Transitive Properties of Equality can be used to write proofs about segment congruence. The theorem resulting from the proofs states that congruence of segments is reflexive, symmetric, and transitive.

Chapter Project

Map Your Town

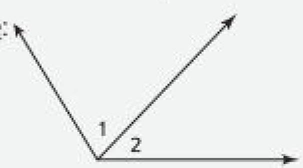
Students use what they have learned about points, lines, planes, and polygons to create a map of a fictitious town and connect the parts of the map to the geometric figures they represent.

- Have students create a map grid and label the horizontal and vertical axes. Students should also name the town.
- Students place important landmarks of their town on the grid represented by a 3-dimensional polygon. Each of the polygons discussed in this chapter should be used at least one time. Landmarks may include the school, Post Office, library, park, home, etc. and should be indicated by a labeled point at its location.
- Next, students use line segments to build the road system of their town and name the major roadways.
- Lastly, students should write a short paper that describes all of the geometric figures and concepts used to create the map. You may want to remind students that the map itself is a plane.

KeyVocabulary Introduce the key vocabulary in the chapter using the method below.

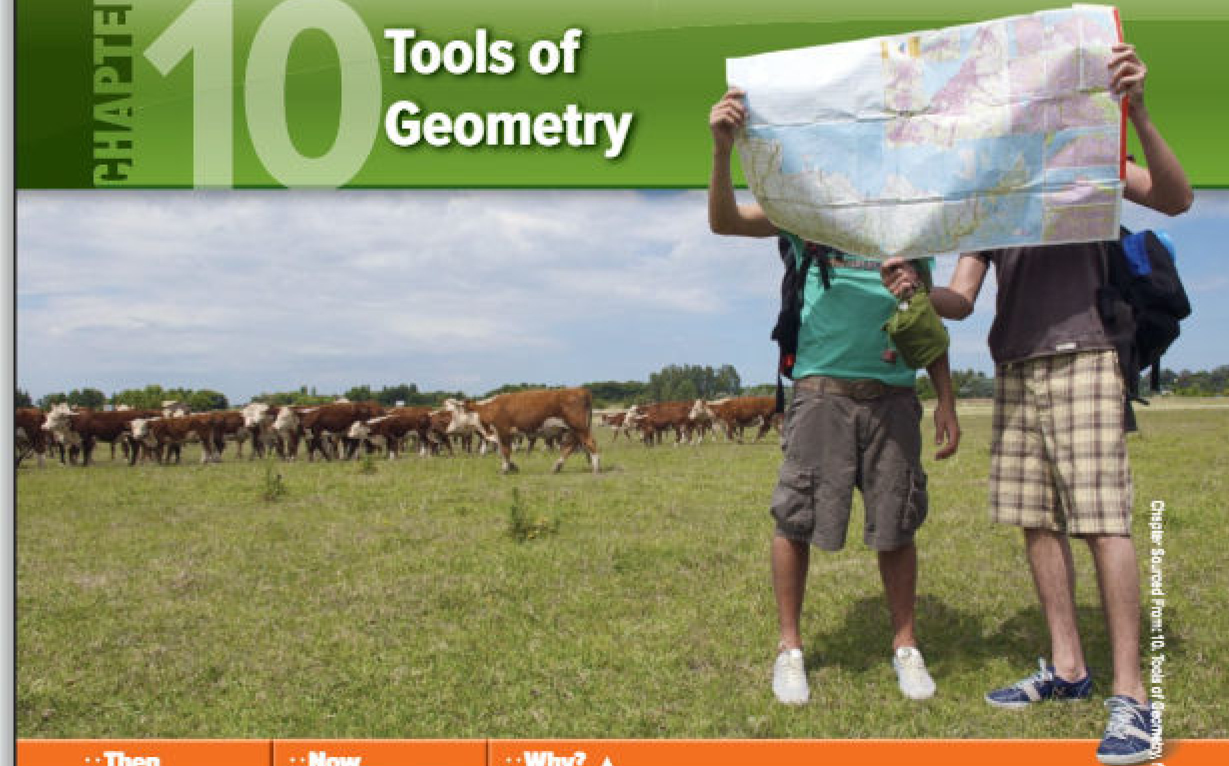
Define: Adjacent angles are angles that lie in the same plane and have a common vertex and a common side, but no common interior points.

Example:



Ask: How do the angles relate to one another? What is their sum? Angles 1 and 2 are adjacent angles. The sum of their measures is about 120°

CHAPTER 10 Tools of Geometry



Then

You graphed points on the coordinate plane and evaluated mathematical expressions.

Now

In chapter, you will:

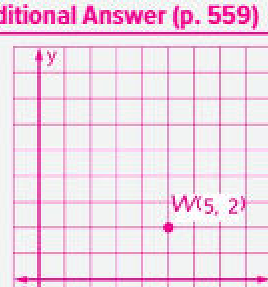
- Find distances between points and midpoints of line segments.
- Identify angle relationships.
- Find perimeters, areas, surface areas, and volumes.

Why?

MAPS Geometric figures and terms can be used to represent and describe real-world situations. On a map, locations of cities can be represented by points, highways or streets by lines, and national parks by polygons that have both perimeter and area. The map itself is representative of a plane.

Additional Answer (p. 559)

1.



2–4. See p. 560.

Get Ready for the Chapter

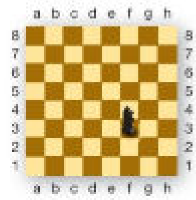
1 Textbook Option Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Graph and label each point in the coordinate plane.

1. $W(5, 2)$
2. $X(0, 6)$
3. $Y(-3, -1)$
4. $Z(4, -2)$ **1–4. See margin.**

5. **GAMES** Suhaila is using the diagram to record her chess moves. She moves her knight 2 spaces up and 1 space to the left from f3. What is the location of the knight after Suhaila completes her turn? **e5**



Find each sum or difference.

6. $\frac{2}{3} + \frac{5}{6}$ **$1\frac{1}{2}$**
7. $2\frac{1}{18} + 4\frac{3}{4}$ **$6\frac{29}{36}$**
8. $\frac{13}{18} - \frac{5}{9}$ **$\frac{1}{6}$**
9. $14\frac{3}{5} - 9\frac{7}{15}$ **$5\frac{2}{15}$**

10. **FOOD** Mazen ate $\frac{1}{3}$ of a pizza for dinner and took $\frac{1}{6}$ of it for lunch the next day. How much of the pizza does he have left? **$\frac{1}{2}$**

Evaluate each expression.

11. $(-4 - 5)^2$ **81**
12. $(6 - 10)^2$ **16**
13. $(8 - 5)^2 + [9 - (-3)]^2$ **153**

Solve each equation.

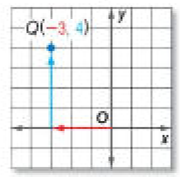
14. $6x + 5 + 2x - 11 = 90$ **12**
15. $8x - 7 = 53 - 2x$ **6**

QuickReview

Example 1

Graph and label the point $Q(-3, 4)$ in the coordinate plane.

Start at the origin. Since the *x*-coordinate is negative, move 3 units to the left. Then move 4 units up since the *y*-coordinate is positive. Draw a dot and label it *Q*.



Example 2

Find $3\frac{1}{6} + 2\frac{3}{4}$.

$3\frac{1}{6} + 2\frac{3}{4} = \frac{19}{6} + \frac{11}{4}$ — Write as improper fractions.

$= \frac{19}{6}\left(\frac{2}{2}\right) + \frac{11}{4}\left(\frac{3}{3}\right)$ — The LCD is 12.

$= \frac{38}{12} + \frac{33}{12}$ — Multiply.

$= \frac{71}{12}$ or $5\frac{11}{12}$ — Simplify.

Example 3

Evaluate the expression $[-2 - (-7)]^2 + (1 - 8)^2$.

Follow the order of operations.

$[-2 - (-7)]^2 + (1 - 8)^2$

$= 5^2 + (-7)^2$ — Subtract.

$= 25 + 49$ — $5^2 = 25, (-7)^2 = 49$

$= 74$ — Add.

Essential Question

- Why are geometry and measurement important in the real-world? **Sample answer: Geometry can be used to represent and describe objects and relationships in the real-world. Measurement, including linear, square, and cubic measures, provide a quantifiable way to describe objects in the real-world.**

FOLDABLES **Study**Organizer

Dinah Zike's Foldables®

Focus Note-taking is a skill that is based upon listening or reading for main ideas and then recording those ideas for future reference.

Teach Under the tabs of their Foldable, have students take notes about the chapter.

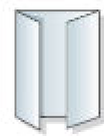
Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 10. To get ready, identify important terms and organize your resources.

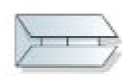
FOLDABLES **Study**Organizer

Tools of Geometry Make this Foldable to help you organize your Chapter 10 notes about points, lines, and planes; angles and angle relationships; and formulas and notes for distance, midpoint, perimeter, area, and volume. Begin with a sheet of 11″ × 17″ paper.

1 **Fold** the short sides to meet in the middle.



2 **Fold** the booklet in thirds lengthwise.



3 **Open and cut** the booklet in thirds lengthwise.

4 **Label** the tabs as shown.

NewVocabulary

- collinear
- coplanar
- congruent
- midpoint
- segment bisector
- angle
- vertex
- angle bisector
- polygon
- perimeter

ReviewVocabulary

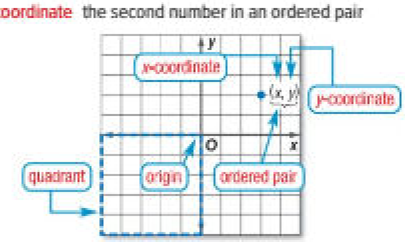
ordered pair a set of numbers or coordinates used to locate any point on a coordinate plane, written in the form (x, y)

origin the point where the two axes intersect at their zero points

quadrants the four regions into which the x-axis and y-axis separate the coordinate plane

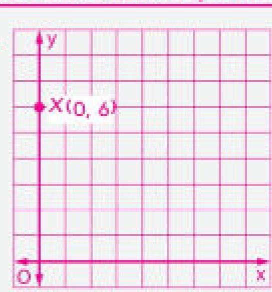
x-coordinate the first number in an ordered pair

y-coordinate the second number in an ordered pair



Additional Answers (Get Ready p. 559)

2.



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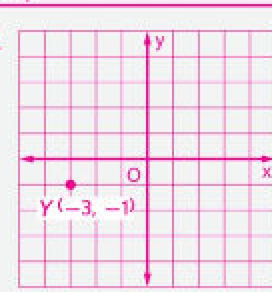
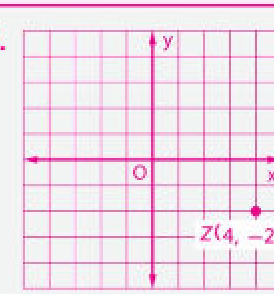


4.



LESSON 10-1 Points, Lines, and Planes

:: Then	:: Now	:: Why?
• You used basic geometric concepts and properties to solve problems.	• 1 Identify and model points, lines, and planes. 2 Identify intersecting lines and planes.	• On a subway map, the locations of stops are represented by *points*. The route the train can take is modeled by a series of connected paths that look like *lines*. The flat surface of the map on which these points and lines lie is representative of a *plane*.

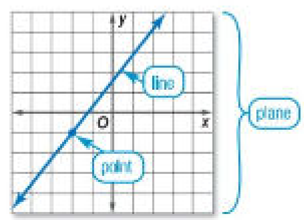
NewVocabulary
undefined term
point
line
plane
collinear
coplanar
intersection
definition
defined term
space

Mathematical Practices
Model with mathematics.
Attend to precision.

1 Points, Lines, and Planes

Unlike the real-world objects that they model, shapes, points, lines, and planes do not have any actual size. In geometry, *point*, *line*, and *plane* are considered **undefined terms** because they are only explained using examples and descriptions.

You are already familiar with the terms point, line, and plane from algebra. You graphed on a coordinate *plane* and found ordered pairs that represented *points* on *lines*. In geometry, these terms have a similar meaning.



The phrase *exactly one* in a statement such as, "There is exactly one line through any two points," means that there is *one and only one*.

KeyConcept Undefined Terms

A **point** is a location. It has neither shape nor size.

Named by a capital letter

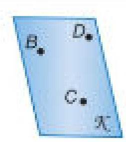
Example point A

A **line** is made up of points and has no thickness or width. There is exactly one line through any two points.

Named by the letters representing two points on the line or a lowercase script letter

Example line m, line PQ or $\overleftrightarrow{PQ}$, line QP or $\overleftrightarrow{QP}$

A **plane** is a flat surface made up of points that extends infinitely in all directions. There is exactly one plane through any three points not on the same line.



Named by a capital script letter or by the letters naming three points that are not all on the same line

Example plane $\mathcal{K}$, plane BCD, plane CDB, plane DCB, plane DBC, plane CBD, plane BDC

Collinear points are points that lie on the same line. *Noncollinear* points do not lie on the same line. **Coplanar** points are points that lie in the same plane. *Noncoplanar* points do not lie in the same plane.

1 Focus

VerticalAlignment

Before Lesson 10-1 Use geometric concepts and properties to solve problems.

Lesson 10-1 Identify and model points, lines, and planes. Identify intersecting lines and planes.

After Lesson 10-1 Use numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What are some other objects that could represent a point, a line, and a plane? Sample response: Points can be represented by stars, lines by the segments that form the constellations, and a plane by the sky.
- What are some other ways that combinations of points, lines, and planes are used? networks and maps

(continued on next page)

- Describe a point, a line, and a plane. Can you clearly define these geometric terms? What is the difference between a description and a definition? A point is like a dot, a line is like a long, straight road, and a plane is like a desktop. There are no clear definitions of these terms; a description simply describes, whereas a definition is a detail of specific criteria required for a figure.

1 Points, Lines, and Planes

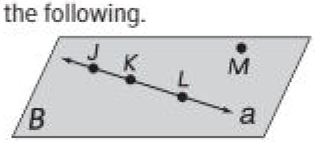
Examples 1 and 2 show how to name and model points, lines, and planes by using the key concepts provided in this lesson.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Use the figure to name each of the following.



a. a line containing point *K*
line *a*, $\overleftrightarrow{JK}$, $\overleftrightarrow{JL}$, $\overleftrightarrow{KJ}$, $\overleftrightarrow{KL}$, $\overleftrightarrow{LJ}$, or $\overleftrightarrow{LK}$

b. a plane containing point *L*
plane *B*, plane *JKM*, plane *KLM*, plane *JLM*. Reorder the letters in these names to create 15 other acceptable names.

2 Name the geometric shape modeled by each object.

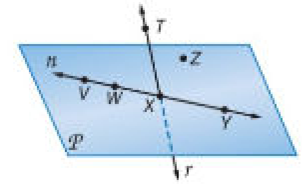
a. a 10 × 12 patio plane

b. a button on a table point

Example 1 Name Lines and Planes

Use the figure to name each of the following.

a. a line containing point *W*



The line can be named as line *n*, or any two of the four points on the line can be used to name the line.

$\overleftrightarrow{VW}$ $\overleftrightarrow{WV}$ $\overleftrightarrow{VX}$ $\overleftrightarrow{XV}$ $\overleftrightarrow{VY}$ $\overleftrightarrow{YV}$
$\overleftrightarrow{WX}$ $\overleftrightarrow{XW}$ $\overleftrightarrow{WY}$ $\overleftrightarrow{YW}$ $\overleftrightarrow{XY}$ $\overleftrightarrow{YX}$

b. a plane containing point *X*

One plane that can be named is plane *P*. You can also use the letters of any three *noncollinear* points to name this plane.

plane *XZY*	plane *VZW*	plane *VZX*
plane *VZY*	plane *WZX*	plane *WZY*

The letters of each of these names can be reordered to create other acceptable names for this plane. For example, *XZY* can also be written as *XYZ*, *ZXY*, *ZYX*, *YXZ*, and *YZX*. In all, there are 36 different three-letter names for this plane.

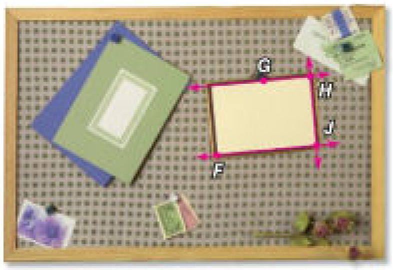
StudyTip
Additional Planes Although not drawn in Example 1b, there is another plane that contains point *X*. Since points *W*, *T*, and *X* are noncollinear, point *X* is also in plane *WTX*.

GuidedPractice

1A. a plane containing points *T* and *Z* *TZX* **1B.** a line containing point *T* *r*

Real-World Example 2 Model Points, Lines, and Planes

MESSAGE BOARD **Name the geometric terms modeled by the objects in the picture.**

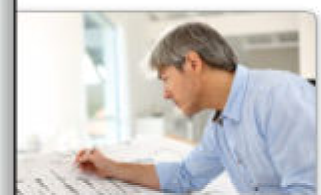


The push pin models point *G*.

The maroon border on the card models line *GH*.

The edge of the card models line *HJ*.

The card itself models plane *FGJ*.



Real-WorldCareer
Drafter Drafters use perspective to create drawings to build everything from toys to school buildings. Drafters need skills in math and computers. They get their education at trade schools, community colleges, and some 4-year colleges. Refer to Exercises 50 and 51.

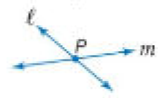
GuidedPractice

Name the geometric term modeled by each object.

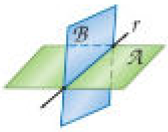
2A. stripes on a sweater lines **2B.** the corner of a box point

2 Intersections of Lines and Planes

The **intersection** of two or more geometric figures is the set of points they have in common. Two lines intersect in a point. Lines can intersect planes, and planes can intersect each other.



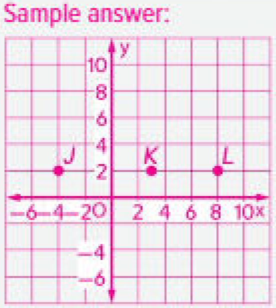
P represents the intersection of lines *ℓ* and *m*.



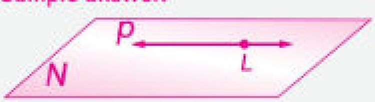
Line *r* represents the intersection of planes *A* and *B*.

Additional Answers (Guided Practice)

3A. Sample answer:



3B. Sample answer:



Example 3 Draw Geometric Figures

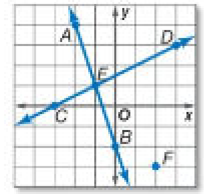
Draw and label a figure for each relationship.

a. ALGEBRA Lines *AB* and *CD* intersect at *E* for *A*(−2, 4), *B*(0, −2), *C*(−3, 0), and *D*(3, 3) on a coordinate plane. Point *F* is coplanar with these points, but not collinear with $\overleftrightarrow{AB}$ or $\overleftrightarrow{CD}$.

Graph each point and draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Label the intersection point as *E*.

An infinite number of points are coplanar with *A*, *B*, *C*, *D* and *E* but not collinear with $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. In the graph, one such point is *F*(2, −3).

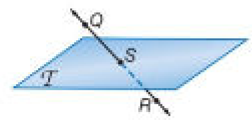


b. *QR* intersects plane *T* at point *S*.

Draw a surface to represent plane *T* and label it.

Draw a dot for point *S* anywhere on the plane and a dot that is not on plane *T* for point *Q*.

Draw a line through points *Q* and *S*. Dash the line to indicate the portion hidden by the plane. Then draw another dot on the line and label it *R*.



StudyTip

Three-Dimensional Drawings Because it is impossible to show an entire plane in a figure, edged shapes with different shades of color are used to represent planes.

GuidedPractice **3A–3B. See margin.**

3A. Points *J*(−4, 2), *K*(3, 2), and *L* are collinear.

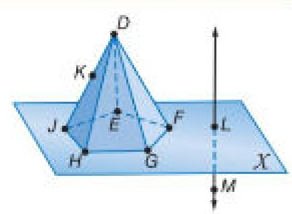
3B. Line *p* lies in plane *N* and contains point *L*.

Definitions or **defined terms** are explained using undefined terms and/or other defined terms. **Space** is defined as a boundless, three-dimensional set of all points. Space can contain lines and planes.

Example 4 Interpret Drawings

a. How many planes appear in this figure?

Six: plane *X*, plane *JDH*, plane *JDE*, plane *EDF*, plane *FDG*, and plane *HDG*.



b. Name three points that are collinear.

Points *J*, *K*, and *D* are collinear.

c. Name the intersection of plane *HDG* with plane *X*.

Plane *HDG* intersects plane *X* in $\overleftrightarrow{HG}$.

d. At what point do $\overleftrightarrow{LM}$ and $\overleftrightarrow{EF}$ intersect? Explain.

It does not appear that these lines intersect. $\overleftrightarrow{EF}$ lies in plane *X*, but only point *L* of $\overleftrightarrow{LM}$ lies in *X*.

StudyTip

Precision A point has no dimension. A line exists in one dimension. However, a circle is two-dimensional, and a pyramid is three-dimensional.

GuidedPractice

4A. Points *E*, *F*, and *G* lie in plane *X*, but point *D* does not lie in plane *X*. Thus, they are not coplanar.

4B. Point *D* is the only point the planes have in common.

Explain your reasoning.

4A. Are points *E*, *D*, *F*, and *G* coplanar?

4B. At what point or in what line do planes *JDH*, *JDE*, and *EDF* intersect?

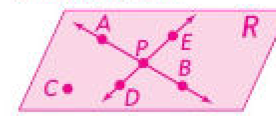
2 Intersection of Lines and Planes

Examples 3 and 4 show how to draw, label, and identify points, lines, and planes in space.

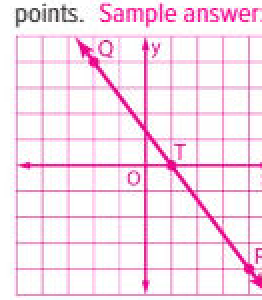
Additional Examples

3 Draw and label a figure for each relationship.

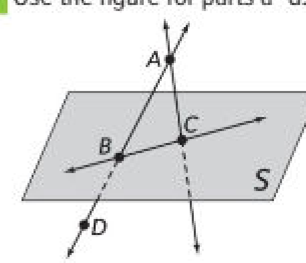
a. ALGEBRA Plane *R* contains lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$, which intersect at point *P*. Add point *C* on plane *R* so that it is not collinear with $\overleftrightarrow{AB}$ or $\overleftrightarrow{DE}$. Sample answer:



b. $\overleftrightarrow{QR}$ on a coordinate plane contains *Q*(−2, 4) and *R*(4, −4). Add point *T* so that *T* is collinear with these points. Sample answer:



4 Use the figure for parts a–d.



a. How many planes appear in this figure? two

b. Name three points that are collinear. *A*, *B*, and *D*

c. Are points *A*, *B*, *C*, and *D* coplanar? Explain. Points *A*, *B*, *C*, and *D* all lie in plane *ABC*, so they are coplanar.

d. At what point do $\overleftrightarrow{DB}$ and $\overleftrightarrow{CA}$ intersect? *A*

Tips for New Teachers

Using Notation Chapter 10 includes many new vocabulary terms and geometric notations. Be sure to stress the importance of understanding and using the new vocabulary and notation. Students will be expected to proficiently read, understand, and use these terms and notations.

Teach with Tech

Interactive Whiteboard Draw a plane on the board. Select students and give them specific points and lines to draw that are either in the plane or not in the plane.

3 Practice

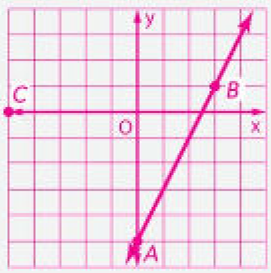
Formative Assessment

Use Exercises 1–12 to check for understanding.

Then use the chart at the bottom of this page to customize assignments for your students.

Additional Answers

6. Sample answer:

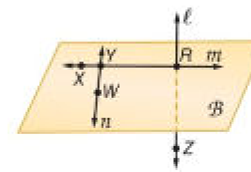


7. Sample answer:

11. Yes; points *B*, *D*, and *F* lie in plane *BDF*.

Check Your Understanding

Example 1 **Use the figure to name each of the following.**



1. a line containing point *X* **Sample answer: *m***

2. a line containing point *Z* **Sample answer: ℓ**

3. a plane containing points *W* and *R* ***B***

Example 2 **Name the geometric term modeled by each object.**

4. a beam from a laser **line**

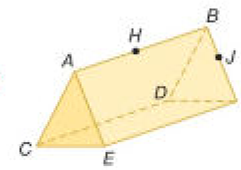
5. a floor **plane**

Example 3 **Draw and label a figure for each relationship.**

6. A line in a coordinate plane contains $A(0, -5)$ and $B(3, 1)$ and a point *C* that is not collinear with $\overleftrightarrow{AB}$. **See margin.**

7. Plane *Z* contains lines *x*, *y*, and *w*. Lines *x* and *y* intersect at point *V* and lines *x* and *w* intersect at point *P*. **See margin.**

Example 4 **Refer to the figure.**



10. No; points *A*, *H*, and *J* lie in plane *ABF*, but point *D* does not lie in plane *ABF*.

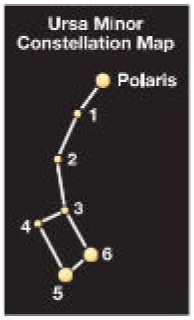
8. How many planes are shown in the figure? **5**

9. Name three points that are collinear. **Sample answer: *A*, *H*, and *B***

10. Are points *A*, *H*, *J*, and *D* coplanar? Explain.

11. Are points *B*, *D*, and *F* coplanar? Explain. **See margin.**

12. **ASTRONOMY** Ursa Minor, or the Little Dipper, is a constellation made up of seven stars in the northern sky including the star Polaris.



a. What geometric figures are modeled by the stars? **points**

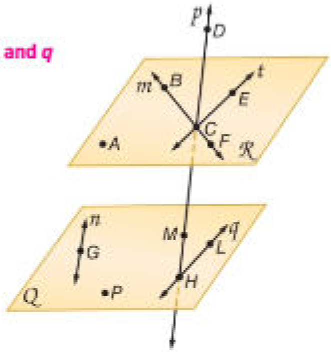
b. Are Star 1, Star 2, and Star 3 collinear on the constellation map? Explain.

c. Are Polaris, Star 2, and Star 6 coplanar on the map? **Yes; it appears that all three points lie in the same plane. However they are probably not coplanar in reality.**

12b. No; there is not a straight line through all three points. So the points are not collinear.

Practice and Problem Solving

Example 1 **Refer to the figure.**



13. Name the lines that are only in plane *Q*. **Sample answer: *n* and *q***

14. How many planes are labeled in the figure? **2**

15. Name the plane containing the lines *m* and *t*. ***R***

16. Name the intersection of lines *m* and *t*. ***C***

17. Name a point that is not coplanar with points *A*, *B*, and *C*. **Sample answer: point *P***

18. Are points *F*, *M*, *G*, and *P* coplanar? Explain.

19. Name the points not contained in a line shown. **points *A* and *P***

20. What is another name for line *t*? $\overleftrightarrow{CE}$

21. Does line *n* intersect line *q*? Explain. **Yes; line *n* intersects line *q* when the lines are extended.**

18. No; point *F* lies in plane *R*, while points *G* and *P* lie in plane *Q* and *M* lies between planes *R* and *Q*.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	13–48, 57, 58, 60–87	13–47 odd, 62–65	14–48 even, 57, 58, 60, 61, 66–87
OL Core	13–47 odd, 49–58, 60–87	13–48, 62–65	49–58, 60, 61, 66–87
BL Advanced	49–79, (optional: 80–87)		

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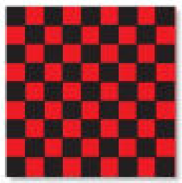
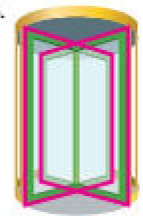
Example 2 **Name the geometric term(s) modeled by each object.**

22. 
point

23. 
intersecting lines

24. 
plane; intersecting lines

25. 
two planes intersecting in a line

26. a blanket **plane**
27. a knot in a rope **point**
28. a telephone pole **line**
29. the edge of a desk **line**
30. two connected walls **intersecting planes**
31. a partially opened folder **intersecting planes**

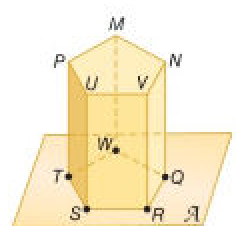
Example 3 **Draw and label a figure for each relationship.** **32–39. See Ch. 10 Answer Appendix.**

32. Line m intersects plane R at a single point.
33. Two planes do not intersect.
34. Points X and Y lie on $\overleftrightarrow{CD}$.
35. Three lines intersect at point J but do not all lie in the same plane.
36. Points $A(2, 3)$, $B(2, -3)$, C and D are collinear, but A, B, C, D, and F are not.
37. Lines $\overleftrightarrow{LM}$ and $\overleftrightarrow{NP}$ are coplanar but do not intersect.
38. $\overleftrightarrow{FG}$ and $\overleftrightarrow{JK}$ intersect at $P(4, 3)$, where point F is at $(-2, 5)$ and point J is at $(7, 9)$.
39. Lines s and t intersect, and line v does not intersect either one.

Example 4 **MODELING** **When packing breakable objects such as glasses, movers frequently use boxes with inserted dividers like the one shown.**

40. How many planes are modeled in the picture? **15**
41. What parts of the box model lines? **edges**
42. What parts of the box model points? **vertices**

Refer to the figure at the right.



43. Name two collinear points. **Sample answer: *M* and *N***
44. How many planes appear in the figure? **7**
45. Do plane A and plane MNP intersect? Explain. **No; they do not have any lines in common.**
46. In what line do planes A and QRV intersect? $\overleftrightarrow{QR}$
47. Are points T, S, R, Q, and V coplanar? Explain. **No; *V* does not lie in the same plane.**
48. Are points T, S, R, Q, and W coplanar? Explain. **Yes; all of the points lie in plane *TSR*.**

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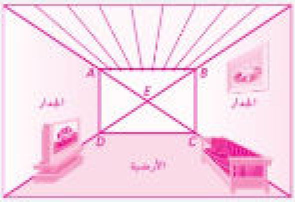
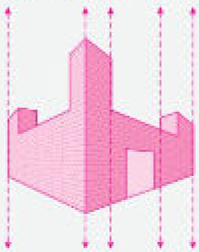
Additional Answers

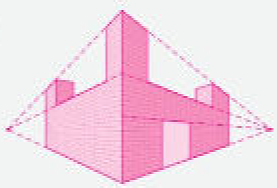
50b.



51a. Sample answer:



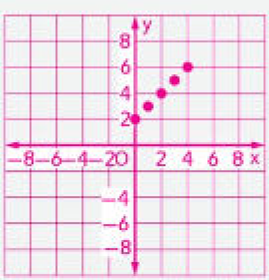
51b.



54a. Sample answer:

x	y
0	2
1	3
2	4
3	5
4	6

54b.

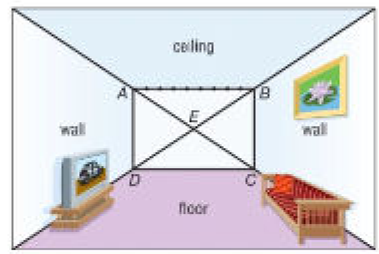


49. **FINITE PLANES** A *finite plane* is a plane that has boundaries, or does not extend indefinitely. The street signs shown are finite planes.

a. If the pole models a line, name the geometric term that describes the intersection between the signs and the pole. **point**

b. What geometric term(s) describes the intersection between the two finite planes? Explain your answer with a diagram if necessary. **line**

50. **ONE-POINT PERSPECTIVE** One-point perspective drawings use lines to convey depth. Lines representing horizontal lines in the real object can be extended to meet at a single point called the *vanishing point*. Suppose you want to draw a tiled ceiling in the room below with nine tiles across.

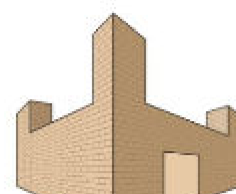


a. What point represents the vanishing point in the drawing? **Point *E***

b. Trace the figure. Then draw lines from the vanishing point through each of the eight points between *A* and *B*. Extend these lines to the top edge of the drawing. **See margin.**

c. How could you change the drawing to make the back wall of the room appear farther away? **Sample answer: Draw points *A*, *B*, *C*, and *D* closer to the vanishing point.**

51. **TWO-POINT PERSPECTIVE** Two-point perspective drawings use two vanishing points to convey depth.



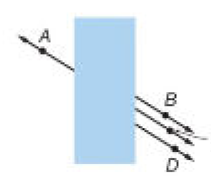
a. Trace the drawing of the castle shown. Draw five of the vertical lines used to create the drawing. **a–b. See margin.**

b. Draw and extend the horizontal lines to locate the vanishing points and label them.

c. What do you notice about the vertical lines as they get closer to the vanishing point? **Sample answer: They get closer together.**

d. Draw a two-point perspective of a home or a room in a home. **See students' work.**

52. **ARGUMENTS** Name two points on the same line in the figure. How can you support your assertion? **Sample answer: *A* and *C*; use a ruler to figure out which point would be the end of the line if it continued.**



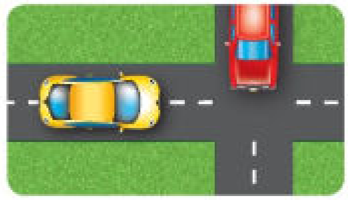
Differentiated Instruction AL OL BL

Extension Challenge students to develop three-dimensional models to demonstrate difficult geometric concepts related to points, lines, and planes. Some examples include:

- Develop a model to show that three points can be noncollinear.
- Develop a demonstration to show that three points are coplanar, but four points can be noncoplanar.
- Develop a three-dimensional model of lines that are not parallel and do not intersect.

53. **TRANSPORTATION** When two cars enter an intersection at the same time on opposing paths, one of the cars must adjust its speed or direction to avoid a collision. Two airplanes, however, can cross paths while traveling in different directions without colliding. Explain how this is possible.

53. Sample answer: The airplanes are in different horizontal planes.



54. **MULTIPLE REPRESENTATIONS** Another way to describe a group of points is called a locus. A **locus** is a set of points that satisfy a particular condition. In this problem, you will explore the locus of points that satisfy an equation.

a. **Tabular** Represent the locus of points satisfying the equation $2 + x = y$ using a table of at least five values. **See margin.**

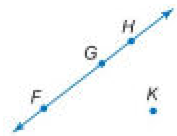
b. **Graphical** Represent this same locus of points using a graph. **See margin.**

c. **Verbal** Describe the geometric figure that the points suggest. **a line**

55. **PROBABILITY** Three of the labeled points are chosen at random.

a. What is the probability that the points chosen are collinear? $\frac{1}{4}$

b. What is the probability that the points chosen are coplanar? **1**



56. **MULTIPLE REPRESENTATIONS** In this problem, you will explore the locus of points that satisfy an inequality.

a. **Tabular** Represent the locus of points satisfying the inequality $y < -3x - 1$ using a table of at least ten values. **See margin.**

b. **Graphical** Represent this same locus of points using a graph. **See margin.**

c. **Verbal** Describe the geometric figure that the points suggest. **the part of the coordinate plane below the line $y = -3x - 1$**

H.O.T. Problems Use Higher-Order Thinking Skills

57. **OPEN ENDED** Sketch three planes that intersect in a line. **See margin.**

58. **ERROR ANALYSIS** Yasmin and Wafa are trying to determine the most number of lines that can be drawn using any two of four random points. Is either correct? Explain.

58. Sample answer: Yasmin is correct; after you draw the line from the first point to the other three, one of the lines from the second point is already drawn.

> **Yasmin**
> Since there are four points, 4 · 3 or 12 lines can be drawn between the points.

> **Wafa**
> You can draw 3 · 2 · 1 or 6 lines between the points.

59. **ARGUMENTS** What is the greatest number of planes determined using any three of the points A, B, C, and D if no three points are collinear? **4**

60. **REASONING** Is it possible for two points on the surface of a prism to be neither collinear nor coplanar? Justify your answer.

60. No. Sample answer: There is exactly one line through any two points and exactly one plane through any three points not on the same line. Therefore, any two points on the prism must be collinear and coplanar.

61. **WRITING IN MATH** Refer to Exercise 49. Give a real-life example of a finite plane. Is it possible to have a real-life object that is an infinite plane? Explain your reasoning.
Sample answer: A table is a finite plane. It is not possible to have a real-life object that is an infinite plane because all real-life objects have boundaries.

Multiple Representations

In Exercises 54 and 56, students use a table and a graph to investigate a locus of points.

WatchOut!

Error Analysis In Exercise 58, students should see that a line drawn between two points counts only one time. Camille counted each line twice, once from each point.

DifferentiatedInstruction AL OL

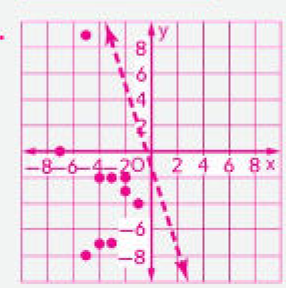
Naturalist Learners Explain how points, lines, and planes exist in nature. For example, planes can model leaves, lily pads, and the surface of a pond; lines can model spider webs, sunbeams, tree trunks, the edge of a riverbed, and the veins of a leaf.

Additional Answers

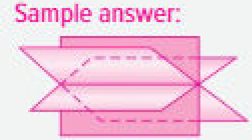
56a. Sample answer:

x	y
−1	−4
−2	−3
−2	2
−3	−2
−3	−7
−4	−2
−4	−7
−5	−8
−5	9
−7	0

56b.



57. Sample answer:

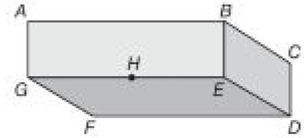


4 Assess

Name the Math Discuss how points, lines, and planes are modeled by the objects students see and use every day.

Standardized Test Practice

62. Which statement about the figure below is *not* true? **D**



A Point H lies in planes AGE and GED.
B Planes GAB, GFD and BED intersect at point E.
C Points F, E, and B are coplanar.
D Points A, H, and D are collinear.

63. ALGEBRA What is the value of x if $3x + 2 = 8$? **H**

F -2 G 0 H 2 J 6

64. GRIDDED RESPONSE An ice chest contains 3 types of drinks: 10 apple juices, 15 grape juices, and 15 bottles of water. What is the probability that a drink selected randomly from the ice chest does *not* contain fruit juice? **3/8 or 0.375**

65. SAT/ACT A certain school's enrollment increased 6% this year over last year's enrollment. If the school now has 1378 students enrolled, how many students were enrolled last year? **B**

A 1295 C 1350 E 1500
B 1300 D 1460

Spiral Review

Use elimination to solve each system of equations.

66. $2x + y = 5$, $3x - 2y = 4$ **(2, 1)**

67. $4x - 3y = 12$, $x + 2y = 14$ **(6, 4)**

68. $2x - 3y = 2$, $5x + 4y = 28$ **(4, 2)**

69. HEALTH About 20% of the time you sleep is spent in rapid eye movement (REM), which is associated with dreaming. If an adult sleeps 7 to 8 hours, how much time is spent in REM sleep? **between 1.4 and 1.6 hours, inclusive**

Simplify. Assume that no denominator is equal to zero.

70. $\frac{a^6}{a^3}$ a^3

71. $\frac{4^7}{4^5}$ **16**

72. $\frac{c^3d^4}{cd^7}$ $\frac{c^2}{d^3}$

73. $\left(\frac{4h^{-2}g}{2g^5}\right)^0$ **1**

74. $\frac{5q^{-2}t^6}{10q^2t^{-4}}$ $\frac{t^{10}}{2q^4}$

75. $b^3(m^{-3})(b^{-6})$ $\frac{1}{m^3b^3}$

Solve each open sentence.

76. $|y - 2| > 7$ $\{y \mid y > 9 \text{ or } y < -5\}$

77. $|z + 5| < 3$ $\{z \mid -8 < z < -2\}$

78. $|2b + 7| \leq -6$ $\varnothing$

79. $|3 - 2y| \geq 8$ $\{y \mid y \geq 5.5 \text{ or } y \leq -2.5\}$

80. $|9 - 4m| < -1$ $\varnothing$

81. $|5c - 2| \leq 13$ $\{c \mid -2.2 \leq c \leq 3\}$

Skills Review

Replace each ● with >, <, or = to make a true statement.

82. $\frac{1}{4}$ cm ● $\frac{1}{2}$ cm **<**

83. $\frac{3}{4}$ cm ● $\frac{5}{8}$ cm **>**

84. $\frac{3}{8}$ cm ● $\frac{6}{16}$ cm **=**

85. 18 mm ● 2 cm **<**

86. 32 mm ● 3.2 cm **=**

87. 0.8 m ● 8 cm **>**

Follow-up

Students have explored points, lines, and planes.

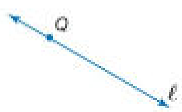
Ask:

- What are the undefined terms in geometry, and how do we use them to represent and describe real-world objects or situations? Sample answer: Points, lines, and planes: points can describe locations on a map; a line can represent the path along which an object can travel; a plane can represent a tabletop.

EXTEND 10-1 Geometry Lab
Describing What You See

When you are learning geometric concepts, it is critical to have accurate drawings to represent the information. It is helpful to know what words and phrases can be used to describe figures. Likewise, it is important to know how to read a geometric description and be able to draw the figure it describes.

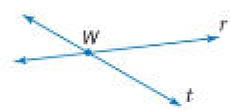
The figures and descriptions below help you visualize and write about points, lines, and planes.



Point Q is on ℓ.

Line ℓ contains Q.

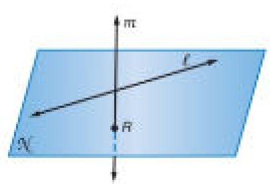
Line ℓ passes through Q.



Lines r and t intersect at W.

Point W is the intersection of r and t.

Point W is on r. Point W is on t.



Line ℓ and point R are in N.

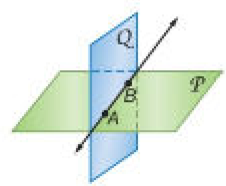
Point R lies in N.

Plane N contains R and ℓ.

Line m intersects N at R.

Point R is the intersection of m with N.

Lines ℓ and m do not intersect.



$\overleftrightarrow{AB}$ is in P and Q.

Points A and B lie in both P and Q.

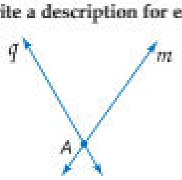
Planes P and Q both contain $\overleftrightarrow{AB}$.

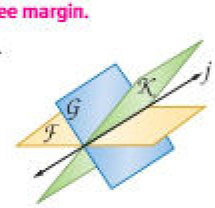
Planes P and Q intersect in $\overleftrightarrow{AB}$.

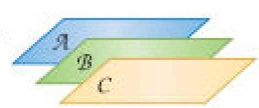
$\overleftrightarrow{AB}$ is the intersection of P and Q.

Exercises

Write a description for each figure. **1–4. See margin.**

1. 

2. 

3. 

4. Draw and label a figure for the statement Planes N and P contain line a.

1 Focus

Objective Use correct mathematical terminology to describe geometric figures.

Materials for Each Group

- straightedge

Tips for New Teachers

Stress the importance of learning definitions and using the proper notation. A clear understanding of terms and the correct use of standard geometric notation is necessary for a strong start in geometry.

2 Teach

Working in Cooperative Groups

Organize students in groups of 2, mixing abilities. Then have students read and verify each of the descriptions. Next, have students draw and label a geometric figure that includes points, lines, and planes. Next, have students exchange figures and write descriptions, using proper terminology and notation.

Practice Have students complete Exercises 1–4.

3 Assess

Formative Assessment

Use Exercises 1–3 to assess whether students correctly use definitions and proper notations to describe geometric figures containing points, lines, and planes.

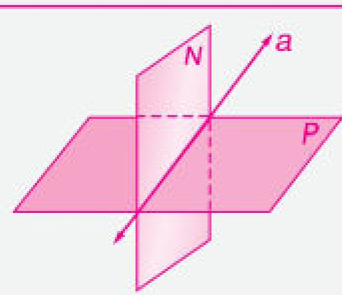
From Concrete to Abstract

Have students draw a figure that is described by the following statement.

Line p intersects line m and plane R at point U.

Next, compare the drawings and discuss whether different drawings accurately depict the same description.

Additional Answers

1. Lines q and m intersect at A. Point A is the intersection of m and q. Point A is on m.
2. Line j lies in planes F, G, and K.
3. Planes A, B, and C do not intersect.
4. 

619

1 Focus

VerticalAlignment

Before Lesson 10-2 Identify and model points, lines, and planes.

Lesson 10-2 Measure segments. Calculate with measures.

After Lesson 10-2 Develop an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What is another part of the human body that is used as a unit of measure? meter
- Is a meter a customary unit or metric unit? customary unit
- Which system of measure uses fractions and whole numbers? customary
- Which system of measure uses decimals? metric

LESSON 10-2 Linear Measure

Then

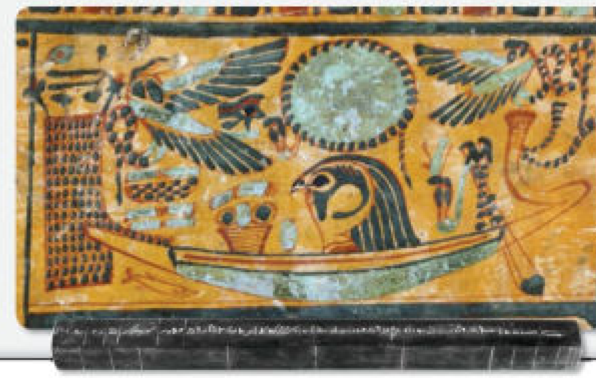
- You identified and modeled points, lines, and planes.

Now

- 1 Measure segments.
- 2 Calculate with measures.

Why?

- When the ancient Egyptians found a need for a measurement system, they used the human body as a guide. The cubit was the length of an arm from the elbow to the fingertips. Eventually the Egyptians standardized the length of a cubit, with ten *royal cubits* equivalent to one *rod*.



NewVocabulary
line segment
betweenness of points
between
congruent segments
construction

Mathematical Practices
Attend to precision.

1 Measure Line Segments

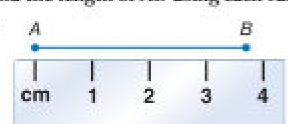
Unlike a line, a **line segment**, or *segment*, can be measured because it has two endpoints. A segment with endpoints A and B can be named as $\overline{AB}$ or $\overline{BA}$. The *measure* of $\overline{AB}$ is written as AB. The length or measure of a segment always includes a unit of measure, such as meter or centimeter.

All measurements are approximations dependent upon the smallest unit of measure available on the measuring instrument.

Example 1 Length in Metric Units

Find the length of $\overline{AB}$ using each ruler.

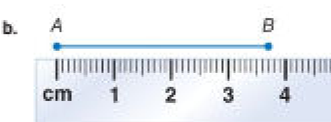
a.



The ruler is marked in centimeters. Point B is closer to the 4-centimeter mark than to 3 centimeters.

Thus, $\overline{AB}$ is about 4 centimeters long.

b.



The long marks are centimeters, and the shorter marks are millimeters. There are 10 millimeters for each centimeter.

Thus, $\overline{AB}$ is about 3.7 centimeters long.

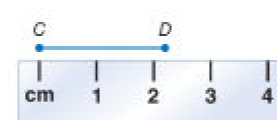
1B. Sample answer: 170 mm

GuidedPractice

1A. Measure the length of a 5-dirham bill in centimeters. **15.6 cm**

1B. Measure the length of a pencil in millimeters.

1C. Find the length of $\overline{CD}$. **2 cm or 20 mm**



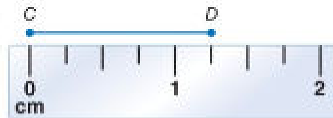
StudyTip

Using a Ruler The zero point on a ruler may not be clearly marked. For some rulers, zero is the left edge of the ruler. On others, it may be a fine line farther in on the scale. If it is not clear where the zero is, align one endpoint on 1 and subtract 1 from the measurement at the other endpoint.

Example 2 Length in Standard Units

Find the length of $\overline{CD}$ using each ruler.

a.



Each centimeter is divided into fourths.

Point D is closer to the $1\frac{1}{4}$-centimeter mark.

$\overline{CD}$ is about $1\frac{1}{4}$ centimeters long.

b.

Each centimeter is divided into sixteenths.

Point D is closer to the $1\frac{4}{16}$-centimeter mark.

$\overline{CD}$ is about $1\frac{4}{16}$ or $1\frac{1}{4}$ centimeters long.

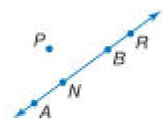
GuidedPractice

2A. Measure the length of a 5 dirham bill in centimeters. $6\frac{3}{16}$ cm

2B. Measure the length of a pencil in centimeters. Sample answer: $6\frac{3}{4}$ cm

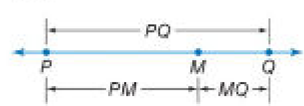
2 Calculate Measures

Recall that for any two real numbers a and b, there is a real number n that is *between* a and b such that $a < n < b$. This relationship also applies to points on a line and is called **betweenness of points**. In the figure, point N is between points A and B, but points R and P are not.



Measures are real numbers, so all arithmetic operations can be used with them. You know that the whole usually equals the sum of its parts. That is also true of line segments in geometry.

KeyConcept Betweenness of Points

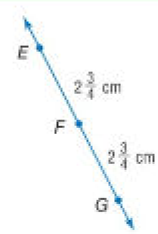
Words	Model
Point M is **between** points P and Q if and only if P, Q, and M are collinear and $PM + MQ = PQ$.	

StudyTip

Comparing Measures Because measures are real numbers, you can compare them. If points X, Y, and Z are collinear in that order, then one of these statements is true: $XY = YZ$, $XY > YZ$, or $XY < YZ$.

Example 3 Find Measurements by Adding

Find EG. Assume that the figure is not drawn to scale.



EG is the measure of $\overline{EG}$. Point F is between E and G. Find EG by adding EF and FG.

$EF + FG = EG$	Betweenness of points
$2\frac{3}{4} + 2\frac{3}{4} = EG$	Substitution
$5\frac{1}{2}$ cm $= EG$	Add.

GuidedPractice

3. Find JL. Assume that the figure is not drawn to scale. 12.3 cm

1 Measure Line Segments

Examples 1 and 2 show how to use a ruler to measure a line segment.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Find the length of $\overline{AB}$ using each ruler.

a.

42 mm

b.

4.5 cm

2 Use a customary ruler to draw each segment.

a. $\overline{DE}$, 3 centimeters long
See students' work.

b. $\overline{FG}$, $2\frac{3}{4}$ centimeters long
See students' work.

3 Find XZ. Assume that the figure is not drawn to scale. $7\frac{1}{8}$ cm

DifferentiatedInstruction AL OL BL

Extension What do you think would happen if the divisions on a ruler were not evenly spaced? Could you be sure that measures are congruent? Explain. Sample answer: The measured length of a segment would be different, depending on which part of the ruler is used.

2 Calculate Measures

Examples 3–6 show how to calculate measures by using arithmetic and algebraic operations

Additional Examples

4 Find LM. Assume that the figure is not drawn to scale. 1.4 cm

5 **ALGEBRA** Find the value of x and ST if T is between S and U, $ST = 7x$, $SU = 45$, and $TU = 5x - 3$. $x = 4$, $ST = 28$

Focus on Mathematical Content

Congruence Provide examples of segments with one, two, or three tick marks indicating congruence. Involve students by having them use the ≅ symbol to categorize the congruent segments for the class. This exercise reviews writing correct names and symbols.

Teach with Tech

Interactive Whiteboard Give students several exercises where the measurements of sections of segments are given in terms of a variable. Choose several students to explain to the class how they found the value of the variable and each of the measurements.

Example 4 Find Measurements by Subtracting

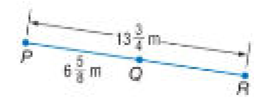
Find AB. Assume that the figure is not drawn to scale.

Point B is between A and C.

$AB + BC = AC$	Betweenness of points
$AB + 5.8 = 13.2$	Substitution
$AB + 5.8 - 5.8 = 13.2 - 5.8$	Subtract 5.8 from each side.
$AB = 7.4$ m	Simplify.

Guided Practice

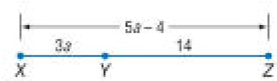
4. Find QR. Assume that the figure is not drawn to scale. $7\frac{1}{8}$ m



Example 5 Write and Solve Equations to Find Measurements

ALGEBRA **Find the value of a and XY if Y is between X and Z, $XY = 3a$, $XZ = 5a - 4$, and $YZ = 14$.**

Draw a figure to represent this information.



$XZ = XY + YZ$	Betweenness of points
$5a - 4 = 3a + 14$	Substitution
$5a - 4 - 3a = 3a + 14 - 3a$	Subtract 3a from each side.
$2a - 4 = 14$	Simplify.
$2a - 4 + 4 = 14 + 4$	Add 4 to each side.
$2a = 18$	Simplify.
$\frac{2a}{2} = \frac{18}{2}$	Divide each side by 2.
$a = 9$	Simplify.

Now find XY.

$XY = 3a$	Given
$= 3(9)$ or 27	$a = 9$

Guided Practice

5. Find x and BC if B is between A and C, $AC = 4x - 12$, $AB = x$, and $BC = 2x + 3$. $x = 15$, $BC = 33$

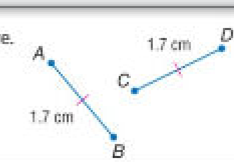
Segments that have the same measure are called **congruent segments**.

Watch Out!

Equal vs. Congruent Lengths are equal and segments are congruent. It is correct to say that $AB = CD$ and $\overline{AB} \cong \overline{CD}$. However, it is *not* correct to say that $\overline{AB} = \overline{CD}$ or that $AB \cong CD$.

Key Concept Congruent Segments

Words	Congruent segments have the same measure.
Symbols	≅ is read *is congruent to*. Red slashes on the figure also indicate congruence.
Example	$\overline{AB} \cong \overline{CD}$



Differentiated Instruction AL OL BL

Kinesthetic Learners Students can physically participate in techniques of measuring, accuracy, and the betweenness of points by grouping in threes, standing in designated spots, and using a yardstick or meterstick to measure distances between them, add distances together, and find unknown distances. They can model examples in the book or create new scenarios. If the floors or walls are tiled, they can also measure distances with one tile representing one unit increment.

Real-WorldLink

The first commercial skateboard was introduced in 1959. Now there are more than 500 skate parks in the United States.

Source: *Encyclopaedia Britannica*

Real-World Example 6 Congruent Segments

SKATE PARKS In the graph, suppose a segment was drawn along the top of each bar. Which emirates would have segments that are congruent? Explain.

Source: SITE Design Group, Inc.

The segments on the bars for Dubai and Ajman would be congruent because they both represent the same number of skate parks.

GuidedPractice

6A. Suppose Abu Dhabi added another skate park. The segment drawn along the bar representing Abu Dhabi would be congruent to which other segment? **Ras Al Khaimah**

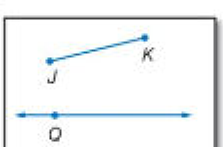
6B. Name the congruent segments in the sign shown.
$\overline{AB} \cong \overline{AG}$, $\overline{BC} \cong \overline{GF}$, $\overline{CD} \cong \overline{FE}$

Drawings of geometric figures are created using measuring tools such as a ruler and protractor. **Constructions** are methods of creating these figures without the benefit of measuring tools. Generally, only a pencil, straightedge, and compass are used in constructions. *Sketches* are created without the use of any of these tools.

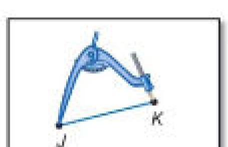
You can construct a segment that is congruent to a given segment.

Construction Copy a Segment

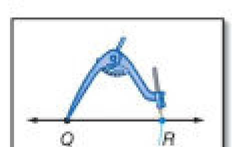
Step 1 Draw a segment $\overline{JK}$. Elsewhere on your paper, draw a line and a point on the line. Label the point Q.



Step 2 Place the compass at point J and adjust the compass setting so that the pencil is at point K.



Step 3 Using that setting, place the compass point at Q and draw an arc that intersects the line. Label the point of intersection R. $\overline{JK} \cong \overline{QR}$



DifferentiatedInstruction AL OL BL

Extension Tap into students' creativity by having them create a game that requires measurement to determine a winner. Many real-life games and sports are decided by measurement. Examples include bocce ball and javelin throw.

Additional Example

6 **FONTS** The Arial font is often used because it is easy to read. Study the word *time* shown in Arial type. Each letter can be broken into individual segments. The letter T has two segments, a short horizontal segment, and a longer vertical segment. Assume that all segments overlap where they meet. Which segments are congruent?

TIME

The five vertical segments in the letters *T*, *I*, *M*, and *E* are congruent. The four horizontal segments in *T* and *E* are congruent. The two diagonal segments in the letter *M* are congruent.

3 Practice

Formative Assessment

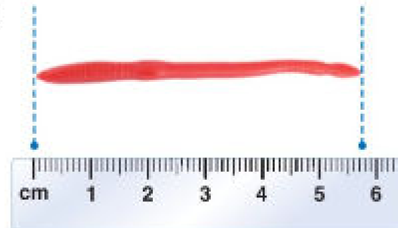
Use Exercises 1–9 to check for understanding.

Then use the chart at the bottom of this page to customize assignments for your students.

Check Your Understanding

Example 1 Find the length of each line segment or object. **1. 5.7 cm or 57 mm**

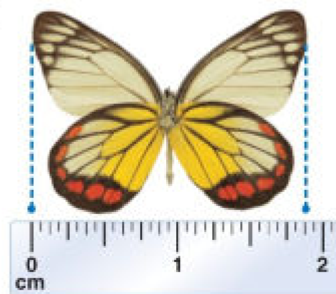
1.



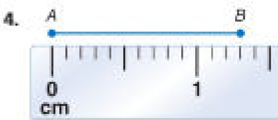
2. **3.5 cm or 35 mm**

Example 2

3. **$1\frac{7}{8}$ cm**



4. **$1\frac{3}{10}$ cm**



Examples 3–4 Find the measurement of each segment. Assume that each figure is not drawn to scale.

5. $\overline{CD}$ **3.8 cm**

6. $\overline{RS}$ **2.4 cm**

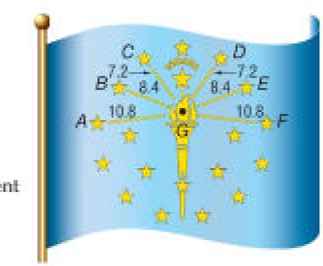
Example 5 **ALGEBRA** Find the value of x and BC if B is between C and D.

7. $CB = 2x$, $BD = 4x$, and $BD = 12$ **$x = 3$; $BC = 6$**

8. $CB = 4x - 9$, $BD = 3x + 5$, and $CD = 17$ **$x = 3$; $BC = 3$**

Example 6

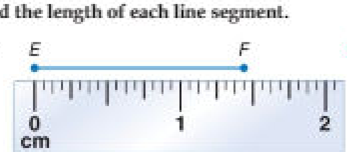
9. **STRUCTURE** The Indiana State Flag was adopted in 1917. The measures of the segments between the stars and the flame are shown on the diagram in centimeters. List all of the congruent segments in the figure.
$\overline{AG} \cong \overline{FG}$, $\overline{BG} \cong \overline{EG}$, $\overline{CG} \cong \overline{DG}$



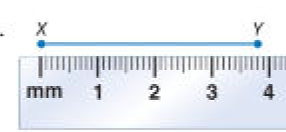
Practice and Problem Solving

Examples 1–2 Find the length of each line segment.

10. **$1\frac{7}{16}$ cm**



11. **3.8 mm**



Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	10–32, 37, 38, 40–60	11–31 odd, 42–45	10–32 even, 37, 38, 40, 41, 46–60
OL Core	11–19 odd, 20, 21–33 odd, 34–38, 40–60	10–32, 42–45	33–38, 40, 41, 46–60
BL Advanced	34–55, (optional: 56–60)		

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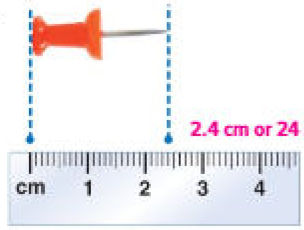
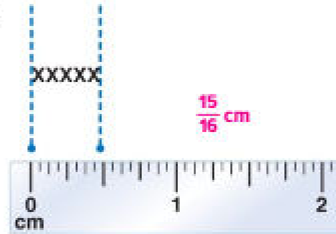
12.


2.4 cm or 24 mm

13.


$\frac{15}{16}$ cm

Examples 3–4 **Find the measurement of each segment. Assume that each figure is not drawn to scale.**

14. $\overline{EF}$ 5.3 cm

15. $\overline{JL}$ 1.1 cm

16. $\overline{PR}$ 2.1 mm

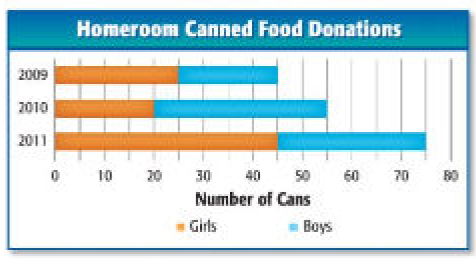
20. 2008: 20 cans, 2009: 35 cans, 2010: 30 cans; Subtract the number of cans the girls brought in from the total number of cans brought in by the girls and the boys.

17. $\overline{SV}$ 1.5 cm

18. $\overline{WY}$ 4.4 mm

19. $\overline{FG}$ 4.2 cm

20. **SENSE-MAKING** The stacked bar graph shows the number of canned food items donated by the girls and the boys in a homeroom class over three years. Use the concept of betweenness of points to find the number of cans donated by the boys for each year. Explain your method.



21. $c = 18$; $YZ = 72$ 22. $b = 12.5$; $YZ = 100$
23. $a = 4$; $YZ = 20$ 24. $d = 2$; $YZ = 16$
25. $n = 4\frac{1}{3}$; $YZ = 1\frac{2}{3}$ 26. $a = 6$; $YZ = 38$

Example 5 **ALGEBRA** **Find the value of the variable and YZ if Y is between X and Z.**

21. $XY = 11$, $YZ = 4c$, $XZ = 83$

22. $XY = 6b$, $YZ = 8b$, $XZ = 175$

23. $XY = 7a$, $YZ = 5a$, $XZ = 6a + 24$

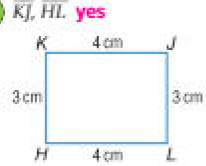
24. $XY = 11d$, $YZ = 9d - 2$, $XZ = 5d + 28$

25. $XY = 4n + 3$, $YZ = 2n - 7$, $XZ = 22$

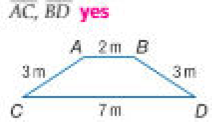
26. $XY = 3a - 4$, $YZ = 6a + 2$, $XZ = 5a + 22$

Example 6 **Determine whether each pair of segments is congruent.**

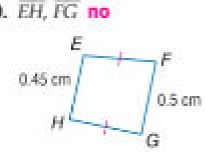
27. $\overline{KJ}$, $\overline{HL}$ yes



28. $\overline{AC}$, $\overline{BD}$ yes

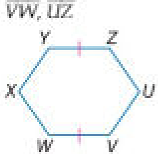


29. $\overline{EH}$, $\overline{FG}$ no

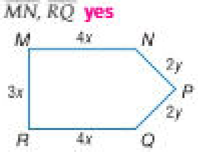


30. $\overline{VW}$, $\overline{UZ}$

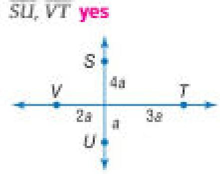
30. cannot be determined from the information given



31. $\overline{MN}$, $\overline{RQ}$ yes



32. $\overline{SU}$, $\overline{VT}$ yes



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Exercise Alert

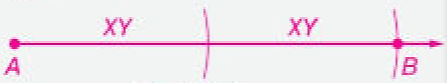
Ruler Exercises 34 and 35 require the use of a ruler.

Multiple Representations

In Exercise 36 students investigate the concept of betweenness. They should see that between any two points on a line another point exists.

Additional Answers

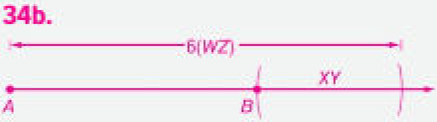
34a.



$AB = 2(XY)$

Sample answer: I measured $\overline{XY}$ with my compass and used this measurement to construct $\overline{XY}$ two times. Since I used the same arc measure to construct $\overline{XY}$ two times, the segment is $2(XY)$.

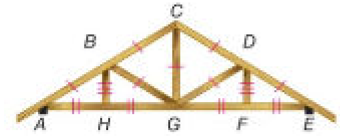
34b.



$AB = 6WZ - XY$

Sample answer: I measured $\overline{WZ}$ with my compass and constructed 6 segments equal to WZ. I measured $\overline{XY}$ and subtracted this measurement from the segment just constructed. Since I used the same arc measure to construct six WZ segments, the segment is $6(WZ)$. I then used the arc measure of XY to subtract from my previous segment. So, the resulting segment is $6(WZ) - XY$.

33. **TRUSSES** A truss is a structure used to support a load over a span, such as a bridge or the roof of a house. List all of the congruent segments in the figure. $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE} \cong \overline{DG} \cong \overline{BG} \cong \overline{CG}$, $\overline{AH} \cong \overline{HG} \cong \overline{GF} \cong \overline{FE}$, $\overline{BH} \cong \overline{DF}$, $\overline{AC} \cong \overline{EC}$, $\overline{AG} \cong \overline{HF} \cong \overline{GE}$



B 34. **CONSTRUCTION** For each expression:

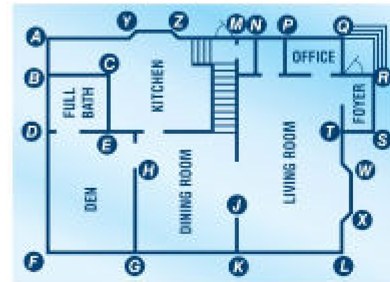
- construct a segment with the given measure,
- explain the process you used to construct the segment, and
- verify that the segment you constructed has the given measure.

a. $2(XY)$ See margin.

b. $6(WZ) - XY$ See margin.

35. **BLUEPRINTS** Use a ruler to determine at least five pairs of congruent segments with labeled endpoints in the blueprint at the right.

Sample answer: $\overline{BD} \cong \overline{CE}$; $\overline{BD} \cong \overline{PQ}$; $\overline{YZ} \cong \overline{JK}$; $\overline{PQ} \cong \overline{RS}$; $\overline{GK} \cong \overline{KL}$



C 36. **MULTIPLE REPRESENTATIONS** Betweenness of points ensures that a line segment may be divided into an infinite number of line segments. a–b. See margin.

a. **Geometric** Use a ruler to draw a line segment 3 centimeters long. Label the endpoints A and D. Draw two more points along the segment and label them B and C. Draw a second line segment 6 centimeters long. Label the endpoints K and P. Add four more points along the line and label them L, M, N, and O.

b. **Tabular** Use a ruler to measure the length of the line segment between each of the points you have drawn. Organize the lengths of the segments in $\overline{AD}$ and $\overline{KP}$ into a table. Include a column in your table to record the sum of these measures.

c. **Algebraic** Give an equation that could be used to find the lengths of $\overline{AD}$ and $\overline{KP}$. Compare the lengths determined by your equation to the actual lengths. $AD = AB + BC + CD$; $KP = KL + LM + MN + NO + OP$; the lengths of each segment add up to the length of the whole segment.

H.O.T. Problems Use Higher-Order Thinking Skills

37. **WRITING IN MATH** If point B is between points A and C, explain how you can find AC if you know AB and BC. Explain how you can find BC if you know AB and AC.

37. If point B is between points A and C, and you know AB and BC, add AB and BC to find AC. If you know AB and AC, subtract AB from AC to find BC.

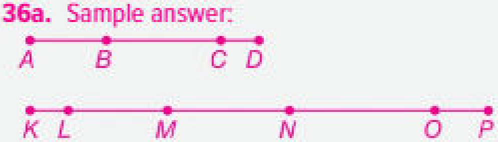
38. **OPEN ENDED** Draw a segment $\overline{AB}$ that measures between 2 and 3 centimeters long. Then sketch a segment $\overline{CD}$ congruent to $\overline{AB}$, draw a segment $\overline{EF}$ congruent to $\overline{AB}$, and construct a segment $\overline{GH}$ congruent to $\overline{AB}$. Compare your methods. See margin.

39. **CHALLENGE** Point K is between points J and L. If $JK = x^2 - 4x$, $KL = 3x - 2$, and $JL = 28$, write and solve an equation to find the lengths of JK and KL. $JK = 12$, $KL = 16$

40. **REASONING** Determine whether the statement *If point M is between points C and D, then CD is greater than either CM or MD* is sometimes, never, or always true. Explain. See margin.

41. **WRITING IN MATH** Why is it important to have a standard of measure? See margin.

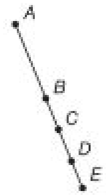
36a. Sample answer:



Standardized Test Practice

42. SHORT RESPONSE A 36-meter-long ribbon is cut into three pieces. The first piece of ribbon is half as long as the second piece of ribbon. The third piece is 1 meter longer than twice the length of the second piece of ribbon. How long is the longest piece of ribbon? **21 m**

43. In the figure, points A, B, C, D, and E are collinear. If $AE = 38$, $BD = 15$, and $\overline{BC} \cong \overline{CD} \cong \overline{DE}$, what is the length of $\overline{AD}$? **D**



A 7.5
B 15
C 22.5
D 30.5

44. SAT/ACT If $f(x) = 7x^2 - 4x$, what is the value of $f(2)$? **K**

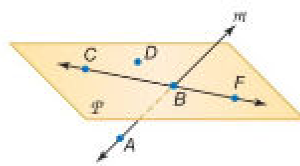
F −8
G 2
H 6
J 17
K 20

45. ALGEBRA
Simplify $(3x^2 - 2)(2x + 4) - 2x^2 + 6x + 7$. **D**

A $4x^2 + 14x - 1$
B $4x^2 - 14x + 15$
C $6x^3 + 12x^2 + 2x - 1$
D $6x^3 + 10x^2 + 2x - 1$

Spiral Review

Refer to the figure. (Lesson 10-1)



46. What are two other names for $\overleftrightarrow{AB}$? **$\overleftrightarrow{BA}$ or line *m***

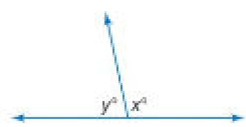
47. Give another name for plane $\mathcal{P}$. **Sample answer: plane *CDF***

48. Name the intersection of plane $\mathcal{P}$ and $\overleftrightarrow{AB}$. **point *B***

49. Name three collinear points. **points *C*, *B*, and *F***

50. Name two points that are not coplanar. **Sample answer: points *A* and *D***

51. GEOMETRY Supplementary angles are two angles with measures that have a sum of 180°. For the supplementary angles in the figure, the measure of the larger angle is 24° greater than the measure of the smaller angle. Write and solve a system of equations to find these measures.
$x + y = 180$; $x = y + 24$; 102°, 78°



Write an equation in point-slope form for the line that passes through each point with the given slope.

52. $(2, 5)$, $m = 3$ **$y - 5 = 3(x - 2)$**

53. $(-3, 6)$, $m = -7$ **$y - 6 = -7(x + 3)$**

54. $(-1, -2)$, $m = -\frac{1}{2}$ **$y + 2 = -\frac{1}{2}(x + 1)$**

Skills Review

Evaluate each expression if $a = -7$, $b = 4$, $c = -3$, and $d = 5$.

55. $b - c$ **7**

56. $|a - d|$ **12**

57. $|d - c|$ **8**

58. $\frac{b - a}{2}$ **5.5**

59. $(a - c)^2$ **16**

60. $\sqrt{(a - b)^2 + (c - d)^2}$ **$\sqrt{185}$**

40. Always; if point M is between points C and D, then $CM + MD = CD$. Since measures cannot be negative, CD, which represents the whole, must always be greater than either of the lengths of its parts, CM or MD.

41. Units of measure are used to differentiate between size and distance, as well as for precision. An advantage is that the standard of measure of a cubit is always available. A disadvantage is that a cubit would vary in length depending on whose arm was measured.

4 Assess

Ticket Out the Door Have students draw and label lines, planes, simple geometric shapes. Have students turn in their drawings before they leave the classroom.

Additional Answers

36b. Sample answer:

$\overline{AD}$	
Segment	**Length (cm)**
$\overline{AB}$	1.0
$\overline{BC}$	1.5
$\overline{CD}$	0.5
Total	3.0

$\overline{KP}$	
Segment	**Length (cm)**
$\overline{KL}$	0.5
$\overline{LM}$	1.3
$\overline{MN}$	1.6
$\overline{NO}$	1.9
$\overline{OP}$	0.7
Total	6.0

38.

Both $\overline{AB}$ and $\overline{EF}$ were created using a ruler, while $\overline{GH}$ was created using a straightedge and compass and $\overline{CD}$ was created without any of these tools. $\overline{AB}$, $\overline{EF}$, and $\overline{GH}$ have the same measure, but $\overline{CD}$ not only does not have the same length, it isn't even a straight line.

1 Focus

Materials for Each Group

- rulers

Teaching Tip

This lesson includes topics that most students find confusing. Be diligent when discussing the definitions of absolute and relative error. Also, tell students that these concepts as well as significant digits are heavily used in science courses.

2 Teach

Working in Cooperative Groups

Allow students to work individually as the lesson content and examples are discussed. Organize students in groups of 3, mixing abilities, and have them complete the Guided Practice Examples provided in the lesson. Facilitate the group work as you monitor students' progress and understanding.

Practice Have students complete Exercises 1–23.

EXTEND 10-2 Extension Lesson: Precision and Accuracy

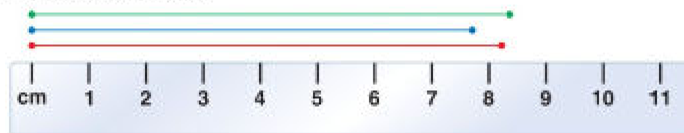
Objective

1. Determine precision of measurements.
2. Determine accuracy of measurements.

As stated in Lesson 10-2, all measurements are approximations. Two main factors are considered when determining the quality of such an approximation.

- How *precise* is the measure?
- How *accurate* is the measure?

1 Precision **Precision** refers to the clustering of a group of measurements. It depends only on the smallest unit of measure available on a measuring tool. Suppose you are told that a segment measures 8 centimeters. The length, to the nearest centimeter, of each segment shown below is 8 centimeters.



Notice that the exact length of each segment above is between 7.5 and 8.5 centimeters, or within 0.5 centimeter of 8 centimeters. The **absolute error** of a measurement is equal to one half the unit of measure. The smaller the unit of measure, the more precise the measurement.

StudyTip

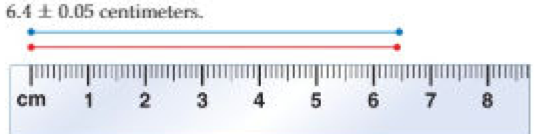
Precision The absolute error of a measurement in customary units is determined before reducing the fraction. For example, if you measure the length of an object to be $1\frac{4}{16}$ centimeters, then the absolute error measurement is precise to within $\frac{1}{32}$ centimeter.

Example 1 Find Absolute Error

Find the absolute error of each measurement. Then explain its meaning.

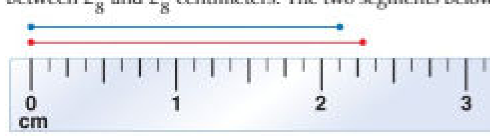
a. 6.4 centimeters

The measure is given to the nearest 0.1 centimeter, so the absolute error of this measurement is $\frac{1}{2}(0.1)$ or 0.05 centimeter. Therefore, the exact measurement could be between 6.35 and 6.45 centimeters. The two segments below measure 6.4 ± 0.05 centimeters.



b. $2\frac{1}{4}$ centimeters

The measure is given to the nearest $\frac{1}{4}$ centimeters, so the absolute error of this measurement is $\frac{1}{2}\left(\frac{1}{4}\right)$ or $\frac{1}{8}$ centimeters. Therefore, the exact measurement could be between $2\frac{1}{8}$ and $2\frac{3}{8}$ centimeters. The two segments below measure $2\frac{1}{4} \pm \frac{1}{8}$ centimeters.



Guided Practice

1A. $1\frac{1}{2}$ centimeters

1B. 4 centimeters

1A. $\frac{1}{4}$ cm; between $1\frac{1}{4}$ cm and $1\frac{3}{4}$ cm

1B. 0.5 cm; between 3.5 and 4.5 cm

Precision in a measurement is usually expressed by the number of **significant digits** reported. Reporting that the measure of AB is 4 centimeters is *less precise* than reporting that the measure of $\overline{AB}$ is 4.1 centimeters.

To determine whether digits are considered significant, use the following rules.

- Nonzero digits are always significant.
- In whole numbers, zeros are significant if they fall between nonzero digits.
- In decimal numbers greater than or equal to 1, every digit is significant.
- In decimal numbers less than 1, the first nonzero digit and every digit to the right are significant.

Real-WorldLink

Precision in measurement in the real world usually comes at a price.

- Precision in a process to 3 significant digits, commercial quality, can cost AED 100.
- Precision in a process to 4 significant digits, industrial quality, can cost AED 500.
- Precision in a process to 5 significant digits, scientific quality, can cost AED 2,500.

Source: Southwest Texas Junior College

Example 2 Significant Digits

Determine the number of significant digits in each measurement.

a. 430.008 meters

Since this is a decimal number greater than 1, every digit is significant. So, this measurement has 6 significant digits.

b. 0.00750 centimeter

This is a decimal number less than 1. The first nonzero digit is 7, and there are two digits to the right of 7, 5 and 0. So, this measurement has 3 significant digits.

GuidedPractice

2A. 779,000 km **3** **2B.** 50,008 m **5** **2C.** 230.004500 m **9**

2 Accuracy

Accuracy refers to how close a measured value comes to the actual or desired value. Consider the target practice results shown below.

The relative error of a measure is the ratio of the absolute error to the expected measure. A measurement with a smaller relative error is said to be more accurate.

StudyTip

Accuracy The accuracy or relative error of a measurement depends on both the absolute error and the size of the object being measured.

Example 3 Find Relative Error

MANUFACTURING A manufacturer measures each part for a piece of equipment to be 23 centimeters in length. Find the relative error of this measurement.

$$\text{relative error} = \frac{\text{absolute error}}{\text{expected measure}} = \frac{0.5 \text{ cm}}{23 \text{ cm}} \approx 0.022 \text{ or } 2.2\%$$

GuidedPractice

Find the relative error of each measurement.

3A. 3.2 km **about 1.6%** **3B.** 1 m **50%** **3C.** 26 m **about 1.9%**

WatchOut!

Accuracy vs. Precision
Students have trouble determining the difference between the accuracy and the precision of a measurement. Be sure to stress the definitions of these terms as well as the differences between the terms.

3 Assess

Formative Assessment

Use Exercises 1–21 to assess whether students understand the concepts of absolute error, relative error, precision, and the proper use of significant digits.

From Concrete to Abstract

It is important that students understand that a solution to a problem or a measurement cannot be more accurate than the level of accuracy of the given values, measurements, or the measurement tools used. Ask students how this applies to reporting the solution of a basic algebraic equation like:

$$4.2x = 36.$$

The highest number of significant digits given in the problems is 2; therefore, the value of x reported should not have more than 2 significant digits.

Additional Answers

1. 0.5 m; The exact measurement could be between 11.5 and 12.5 m.
2. $\frac{1}{32}$ cm; The exact measurement could be between $50\frac{7}{32}$ and $50\frac{9}{32}$.
3. 0.005 m; The exact measurement could be between 3.275 and 3.285 m.
4. 0.0005 cm; The exact measurement could be between 2.7585 and 2.7595 cm.
5. Neither is correct; sample answer: Since the ruler is marked in $\frac{1}{16}$-centimeter increments, the measurement can only be precise to $\frac{1}{32}$ of an centimeter.

23. Sample answer: An absolute error of 0.5 kilometer in measuring the driving distance from one city to another would likely not cause a problem. An absolute error of 0.5 kilometer in measuring the length of a bridge or overpass would likely cause a problem.

Extension Lesson

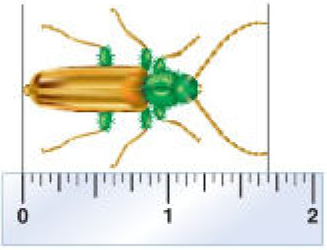
Precision and Accuracy *Continued*

Practice and Problem Solving

Find the absolute error of each measurement. Then explain its meaning. 1–5. See margin.

1. 12 m
2. $50\frac{4}{16}$ cm
3. 3.28 m
4. 2.759 cm

5. **ERROR ANALYSIS** In biology class, Ayoub and Saeed measure a beetle as shown. Ayoub says that the beetle measures between $1\frac{5}{8}$ and $1\frac{3}{4}$ centimeters. Saeed says that it measures between $1\frac{9}{16}$ and $1\frac{5}{8}$ centimeters. Is either of their statements about the beetle's measure correct? Explain your reasoning.

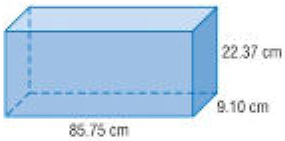


6. **PYRAMIDS** Research suggests that the design dimensions of the Great Pyramid of Giza in Egypt were 440 by 440 royal cubits. The sides of the pyramid are precise within 0.05%. What are the greatest and least possible lengths of the sides? 439.78 cubits and 440.22 cubits

Determine the number of significant digits in each measurement.

7. 4.05 cm 3
8. 53,000 km 2
9. 0.0005 mm 1
10. 750,001 m 6

11. **VOLUME** When multiplying or dividing measures, the product or quotient should have only as many significant digits as the multiplied or divided measurement showing the least number of significant digits. To how many significant digits should the volume of the rectangle prism shown be reported? Report the volume to this number of significant digits.



11. 9.10 shows the least number of significant digits—three. The product is 17,455.87025. It shows 10 significant digits, so we must round to the hundreds place in order to show only three significant digits. The final answer is 17,500 cm^3.

Find the relative error of each measurement.

12. 48 cm about 1.0%
13. 2.0 km about 2.5%
14. 11.14 cm about 0.04%
15. 0.6 m about 8.3%

Determine which measurement is more precise and which is more accurate. Explain your reasoning. 16–19. See Ch. 10 Answer Appendix for explanations.

16. 22.4 m; 5.82 m 5.82 m; 5.82 m
17. 25 km; 8 km Same precision; 25 km
18. 9.2 cm; 42 mm Same precision; 9.2 cm
19. $18\frac{1}{4}$ cm; 125 m 19. $18\frac{1}{4}$ cm; 125 m

For each situation, determine the level of accuracy needed. Explain.

20. You are estimating the height of a person. Which unit of measure should you use: 1 meter, 1 centimeter, or $\frac{1}{16}$ centimeter? 1 cm; See Ch.10 Answer Appendix for explanation.
21. You are estimating the height of a mountain. Which unit of measure should you use: 1 meter, 1 centimeter, or $\frac{1}{16}$ centimeter? 1 m; See Ch.10 Answer Appendix for explanation.
22. **PERIMETER** The *perimeter* of a geometric figure is the sum of the lengths of its sides. Hiyam uses a ruler divided into centimeters and measures the sides of a rectangle to be $2\frac{1}{4}$ centimeters and $4\frac{3}{4}$ centimeters. What are the least and greatest possible perimeters of the rectangle? Explain. 13.5 cm; 14.5 cm; See Ch. 10 Answer Appendix for explanation.
23. **WRITING IN MATH** How precise is precise enough? See margin.

WatchOut!

Error Analysis In Exercise 5, students should look at the increments on the ruler carefully. The ruler is marked in $\frac{1}{16}$-centimeter increments, so the absolute error is $\frac{1}{32}$. The beetle's measure is between $1\frac{21}{32}$ and $1\frac{23}{32}$, so neither is correct.

LESSON 10-3 Distance and Midpoints

Then

- You graphed points on the coordinate plane.

Now

1. Find the distance between two points.
2. Find the midpoint of a segment.

Why?

- The location of a city on a map is given in degrees of latitude and longitude. For short distances, the Pythagorean Theorem can be used to approximate the distance between two locations.

NewVocabulary
distance
irrational number
midpoint
segment bisector

Mathematical Practices
Reason abstractly and quantitatively.
Look for and make use of structure.

1 Distance Between Two Points

The **distance** between two points is the length of the segment with those points as its endpoints. The coordinates of the points can be used to find this length. Because $\overline{PQ}$ is the same as $\overline{QP}$, the order in which you name the endpoints is not important when calculating distance.

KeyConcept Distance Formula (on Number Line)

Words The distance between two points is the absolute value of the difference between their coordinates.

Symbols If P has coordinate x_1 and Q has coordinate x_2, $PQ = |x_2 - x_1|$ or $|x_1 - x_2|$.

Example 1 Find Distance on a Number Line

Use the number line to find *BE*.

The coordinates of B and E are -6 and 2.

$BE = |x_2 - x_1|$ Distance Formula

$= |2 - (-6)|$ $x_1 = -6$ and $x_2 = 2$

$= 8$ Simplify.

GuidedPractice

Use the number line above to find each measure.

1A. AC **3** **1B.** CF **9** **1C.** FB **11**

1 Focus

VerticalAlignment

Before Lesson 10-3 Use graphed points on the coordinate plane.

Lesson 10-3 Find the distance between two points. Find the midpoint of a segment.

After Lesson 10-3 Use distances and midpoints to solve problems and write geometric proofs.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- There are 60′ in 1° longitude and 60′ in 1° latitude. What are the coordinates, in degrees, for Orlando? Miami? Orlando: 23.55°N, 81.38°W; Miami: 25.8°N, 80.27°W
- What is the length, in degrees, of each leg of the right triangle shown on the map? long leg: 2.25°; short leg: 1.11°
- Use the Pythagorean Theorem to find the distance, in degrees, between Orlando and Miami. ≈2.51°

(continued on next page)

- Why must it be specified that the triangle side lengths and the distance between Orlando and Miami are measured in degrees? It must be specified that the lengths and distance are measured in degrees because the coordinates used to find the lengths are given in degrees. It is necessary to indicate that the small area of the map of Florida is transposed onto a coordinate plane so that 2-dimensional, plane figure calculations using the Pythagorean Theorem can be applied to locations (indicated by degrees latitude and longitude) that actually lie on a sphere (the Earth).

1 Distance Between Two Points

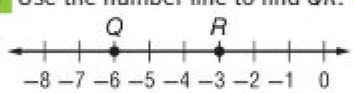
Example 1 shows how to find the distance between two points on a number line. **Example 2** shows how to use the Pythagorean Theorem or the Distance Formula to find the distance between two points on a coordinate plane.

Formative Assessment

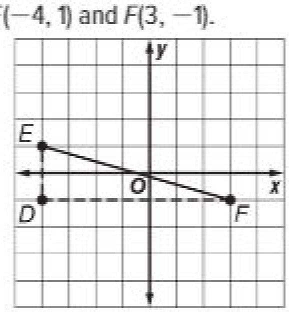
Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Use the number line to find QR. 3

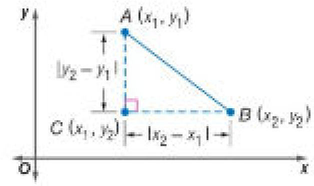


2 Find the distance between $E(-4, 1)$ and $F(3, -1)$.



$\sqrt{53} \approx 7.28$

To find the distance between two points A and B in the coordinate plane, you can form a right triangle with $\overline{AB}$ as its hypotenuse and point C as its vertex as shown. Then use the Pythagorean Theorem to find AB.



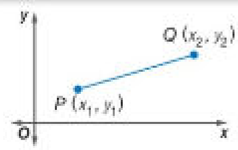
$(CB)^2 + (AC)^2 = (AB)^2$	Pythagorean Theorem
$(\lvert x_2 - x_1\rvert)^2 + (\lvert y_2 - y_1\rvert)^2 = (AB)^2$	$CB = \lvert x_2 - x_1\rvert, AC = \lvert y_2 - y_1\rvert$
$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (AB)^2$	The square of a number is always positive.
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = AB$	Take the positive square root of each side.

This gives us a Distance Formula for points in the coordinate plane. Because this formula involves taking the square root of a real number, distances can be irrational. Recall that an **irrational number** is a number that cannot be expressed as a terminating or repeating decimal.

KeyConcept Distance Formula (in Coordinate Plane)

If P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2), then

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



The order of the x- and y-coordinates in each set of parentheses is not important.

Example 2 Find Distance on a Coordinate Plane

Find the distance between $C(-4, -6)$ and $D(5, -1)$.

$CD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{[5 - (-4)]^2 + [-1 - (-6)]^2}$	$(x_1, y_1) = (-4, -6)$ and $(x_2, y_2) = (5, -1)$
$= \sqrt{9^2 + 5^2}$ or $\sqrt{106}$	Subtract.

The distance between C and D is $\sqrt{106}$ units. Use a calculator to find that $\sqrt{106}$ units is approximately 10.3 units.

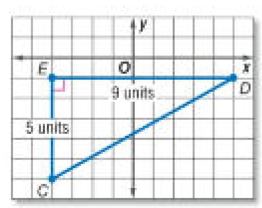
CHECK Graph the ordered pairs and check by using the Pythagorean Theorem.

$(CD)^2 \stackrel{?}{=} (EC)^2 + (ED)^2$

$(CD)^2 \stackrel{?}{=} 5^2 + 9^2$

$(CD)^2 \stackrel{?}{=} 106$

$CD = \sqrt{106}$ ✔



GuidedPractice

Find the distance between each pair of points.

2A. $E(-5, 6)$ and $F(8, -4)$ $\sqrt{269}$ or about 16.4 units

2B. $J(4, 3)$ and $K(-3, -7)$ $\sqrt{149}$ or about 12.2 units

Tips for New Teachers

Finding Distance Encourage students to use the Pythagorean Theorem to find the distance between two points several times before introducing the Distance Formula. Students tend to have difficulty remembering the Distance Formula, and this affords them an alternative to find the distance between two points.

Teach with Tech

Blog On your secure class blog, have students write a blog entry explaining how the distance formula and Pythagorean Theorem are similar to each other.

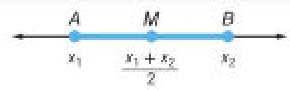
2 Midpoint of a Segment

Midpoint of a Segment The **midpoint** of a segment is the point halfway between the endpoints of the segment. If X is the midpoint of $\overline{AB}$, then $AX = XB$ and $\overline{AX} \cong \overline{XB}$. You can find the midpoint of a segment on a number line by finding the *mean*, or the average, of the coordinates of its endpoints.

KeyConcept Midpoint Formula (on Number Line)

If $\overline{AB}$ has endpoints at x_1 and x_2 on a number line, then the midpoint M of $\overline{AB}$ has coordinate

$$\frac{x_1 + x_2}{2}.$$

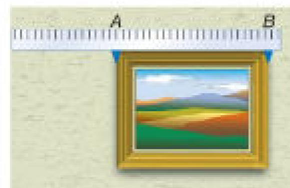


StudyTip

Alternative Method In Example 3, the coordinate of the midpoint could also have been located by first finding the length of $\overline{AB}$, which is 37.5 − 15 or 22.5 centimeters. Half of this measure is the distance from one endpoint to the point midway between A and B, $\frac{22.5}{2}$ or 11.25. Add this distance to point A's distance from the left wall. So the midpoint between A and B is 15 + 11.25 or 26.25 centimeters from the left wall.

Real-World Example 3 Find Midpoint on a Number Line

DECORATING **Hana hangs a picture 15 centimeters from the left side of a wall. How far from the edge of the wall should she mark the location for the nail the picture will hang on if the right edge is 37.5 centimeters from the wall's left side?**



The coordinates of the endpoints of the top of the picture frame are 15 centimeters and 37.5 centimeters. Let M be the midpoint of $\overline{AB}$.

$M = \frac{x_1 + x_2}{2}$ — Midpoint Formula

$= \frac{15 + 37.5}{2}$ — $x_1 = 15, x_2 = 37.5$

$= \frac{52.5}{2}$ or 26.25 — Simplify.

The midpoint is located at 26.25 or $26\frac{1}{4}$ centimeters from the left edge of the wall.

GuidedPractice

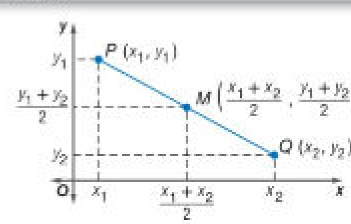
3. **TEMPERATURE** The temperature on a thermometer dropped from a reading of 25° to −8°. Find the midpoint of these temperatures. **8.5°**

You can find the midpoint of a segment on the coordinate plane by finding the average of the *x*-coordinates and of the *y*-coordinates of the endpoints.

KeyConcept Midpoint Formula (in Coordinate Plane)

If $\overline{PQ}$ has endpoints at $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the coordinate plane, then the midpoint M of $\overline{PQ}$ has coordinates

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



When finding the midpoint of a segment, the order of the coordinates of the endpoints is not important.

2 Midpoint of a Segment

The midpoint of a segment is the point halfway between the endpoints of the segment. **Examples 3–6** show how to find the midpoint of a segment arithmetically and algebraically on a number line and on a coordinate plane.

Additional Example

3 **DECORATING** Rashid places a couch so that its end is perpendicular and 76.2 centimeters away from the wall. The couch is 228 centimeters wide. How far is the midpoint of the couch back from the wall in meter? 1.91 m

Tips for New Teachers

Midpoint Formula You may want students to develop their own midpoint formulas by testing several examples.

WatchOut!

Unlocking Misconceptions

A common mistake is that students subtract the coordinates in the Midpoint Formula because subtraction is used in the distance and the slope formulas. Remind students that the midpoint is the *mean* of each coordinate and to find the mean or average, the sum is divided by the number of terms.

DifferentiatedInstruction OL BL

Extension Have students sketch three different segments that each have (0, 0) as a midpoint. Write the coordinates of the endpoints of each segment. What do you notice about the coordinates? Sample answers: In each pair, the *x*-coordinates and the *y*-coordinates are opposites.

Tips for New Teachers

Using Symbols Explain and demonstrate that a segment with points, *A*, *B*, and *C*, and tick marks on *AB* and *BC* indicates that *B* is the *midpoint* and *bisector* of $\overline{AC}$.

Additional Examples

4 Find the coordinates of *M*, the midpoint of *GH*, for *G*(8, −6) and *H*(−14, 12). (−3, 3)

5 Find the coordinates of *D* if *E*(−6, 4) is the midpoint of $\overline{DF}$ and *F* has coordinates (−5, −3). (−7, 11)

Focus on Mathematical Content

Segment Addition Traditionally, segment addition is shown in order of the points, as in $AB + BC = AC$. However, because addition is commutative, $BC + AB = AC$ is also correct.

Example 4 Find Midpoint in Coordinate Plane

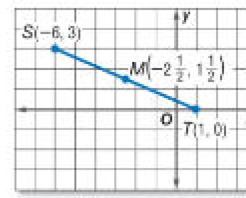
Find the coordinates of *M*, the midpoint of $\overline{ST}$, for *S*(−6, 3) and *T*(1, 0).

$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ Midpoint Formula

$= \left(\frac{-6 + 1}{2}, \frac{3 + 0}{2}\right)$ $(x_1, y_1) = S(-6, 3), (x_2, y_2) = T(1, 0)$

$= \left(\frac{-5}{2}, \frac{3}{2}\right)$ or $M\left(-2\frac{1}{2}, 1\frac{1}{2}\right)$ Simplify.

CHECK Graph *S*, *T*, and *M*. The distance from *S* to *M* does appear to be the same as the distance from *M* to *T*, so our answer is reasonable.



Guided Practice

Find the coordinates of the midpoint of a segment with the given coordinates.

4A. *A*(5, 12), *B*(−4, 8) $\left(\frac{1}{2}, 10\right)$

4B. *C*(−8, −2), *D*(5, 1) $\left(-1\frac{1}{2}, -\frac{1}{2}\right)$

You can also find the coordinates of the endpoint of a segment if you know the coordinates of its other endpoint and its midpoint.

Example 5 Find the Coordinates of an Endpoint

Find the coordinates of *J* if *K*(−1, 2) is the midpoint of $\overline{JL}$ and *L* has coordinates (3, −5).

Step 1 Let *J* be (x_1, y_1) and *L* be (x_2, y_2) in the Midpoint Formula.

$K\left(\frac{x_1 + 3}{2}, \frac{y_1 + (-5)}{2}\right) = K(-1, 2)$ $(x_2, y_2) = (3, -5)$

Step 2 Write two equations to find the coordinates of *J*.

$\frac{x_1 + 3}{2} = -1$ Midpoint Formula

$x_1 + 3 = -2$ Multiply each side by 2.

$x_1 = -5$ Subtract 3 from each side.

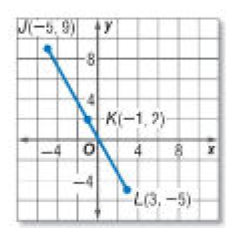
$\frac{y_1 + (-5)}{2} = 2$ Midpoint Formula

$y_1 - 5 = 4$ Multiply each side by 2.

$y_1 = 9$ Add 5 to each side.

The coordinates of *J* are (−5, 9).

CHECK Graph *J*, *K*, and *L*. The distance from *J* to *K* does appear to be the same as the distance from *K* to *L*, so our answer is reasonable.



Study Tip

Check for Reasonableness Always graph the given information and the calculated coordinates of the third point to check the reasonableness of your answer.

Guided Practice

Find the coordinates of the missing endpoint if *P* is the midpoint of $\overline{EG}$.

5A. *E*(−8, 6), *P*(−5, 10) *G*(−2, 14)

5B. *P*(−1, 3), *G*(5, 6) *E*(−7, 0)

Differentiated Instruction AL OL

Visual/Spatial Learners Hold a meterstick up for students to see so that the marked side is facing away from them. Ask a volunteer to mark on the back of the stick about where they visualize the middle of the meterstick to be. Have a second volunteer verify the first student's mark or add another mark. Place a pen upright on the 50-cm mark so that it shows exactly where the midpoint of the meterstick is and compare to the students' marks. Explain how people can use spatial skills to very closely identify the exact middle of many objects.

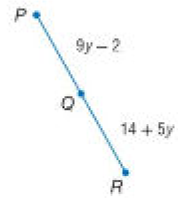
You can use algebra to find a missing measure or value in a figure that involves the midpoint of a segment.

StudyTip

Sense-Making and Perseverance The four-step problem solving plan is a tool for making sense of any problem. When making and executing your plan, continually ask yourself, "Does this make sense?" Monitor and evaluate your progress and change course if necessary.

Example 6 Use Algebra to Find Measures

ALGEBRA **Find the measure of $\overline{PQ}$ if Q is the midpoint of $\overline{PR}$.**



Understand You know that Q is the midpoint of $\overline{PR}$.
You are asked to find the measure of $\overline{PQ}$.

Plan Because Q is the midpoint, you know that $PQ = QR$. Use this equation to find a value for y.

Solve

$PQ = QR$	Definition of midpoint
$9y - 2 = 14 + 5y$	$PQ = 9y - 2$, $QR = 14 + 5y$
$4y - 2 = 14$	Subtract $5y$ from each side.
$4y = 16$	Add 2 to each side.
$y = 4$	Divide each side by 4.

Now substitute 4 for y in the expression for PQ.

$PQ = 9y - 2$	Original measure
$= 9(4) - 2$	$y = 4$
$= 36 - 2$ or 34	Simplify.

The measure of $\overline{PQ}$ is 34.

Check Since $PQ = QR$, when the expression for QR is evaluated for 4, it should also be 34.

$QR = 14 + 5y$	Original measure
$\stackrel{?}{=} 14 + 5(4)$	$y = 4$
$= 34$ ✔	Simplify.

GuidedPractice

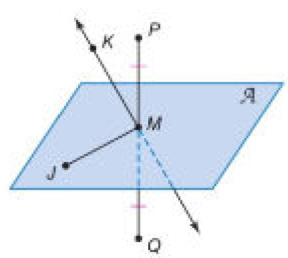
6A. Find the measure of $\overline{YZ}$ if Y is the midpoint of $\overline{XZ}$ and $XY = 2x - 3$ and $YZ = 27 - 4x$. **7**

6B. Find the value of x if C is the midpoint of $\overline{AB}$, $AC = 4x + 5$, and $AB = 78$. **8.5**

StudyTip

Segment Bisectors There can be an infinite number of bisectors and each must contain the midpoint of the segment.

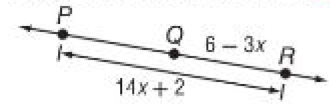
Any segment, line, or plane that intersects a segment at its midpoint is called a **segment bisector**. In the figure at the right, M is the midpoint of $\overline{PQ}$. Plane $\mathcal{A}$, $\overline{MJ}$, $\overleftrightarrow{KM}$, and point M are all bisectors of $\overline{PQ}$. We say that they *bisect* $\overline{PQ}$.



The construction on the following page shows how to construct a line that bisects a segment to find the midpoint of a given segment.

Additional Example

6 **ALGEBRA** Find the measure of $\overline{PR}$ if Q is the midpoint of $\overline{PR}$. **9**



3 Practice

Formative Assessment

Use Exercises 1–12 to check for understanding.

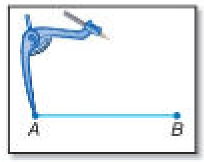
Then use the chart at the bottom of this page to customize assignments for your students.

Additional Answers

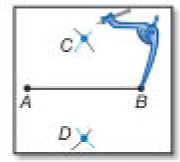
22. 5 units
23. $\sqrt{208}$ or about 14.4 units
24. $\sqrt{200}$ or about 14.1 units
25. $\sqrt{65}$ or about 8.1 units
26. $\sqrt{20}$ or about 4.5 units
27. $\sqrt{53}$ or about 7.3 units
28. $\sqrt{37}$ or about 6.1 units
29. $\sqrt{18}$ or about 4.2 units
30. $\sqrt{29}$ or about 5.4 units

Construction Bisect a Segment

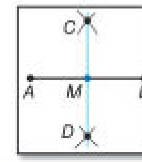
Step 1 Draw a segment and name it $\overline{AB}$. Place the compass at point A. Adjust the compass so that its width is greater than $\frac{1}{2}AB$. Draw arcs above and below $\overline{AB}$.



Step 2 Using the same compass setting, place the compass at point B and draw arcs above and below $\overline{AB}$ so that they intersect the two arcs previously drawn. Label the points of the intersection of the arcs as C and D.



Step 3 Use a straightedge to draw $\overline{CD}$. Label the point where it intersects $\overline{AB}$ as M. Point M is the midpoint of $\overline{AB}$, and $\overline{CD}$ is a bisector of $\overline{AB}$.

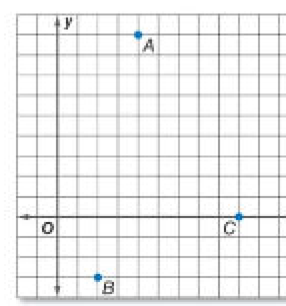


Check Your Understanding

Example 1 **Use the number line to find each measure.**

1. XY **8** **2.** WZ **9**

Example 2 **TIME CAPSULE** Graduating classes have buried time capsules on the campus of East Side High School for over twenty years. The points on the diagram show the position of three time capsules. Find the distance between each pair of time capsules.



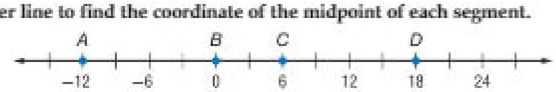
3. $A(4, 9)$, $B(2, -3)$ $\sqrt{148}$ **or about 12.2 units**

4. $A(4, 9)$, $C(9, 0)$ $\sqrt{106}$ **or about 10.3 units**

5. $B(2, -3)$, $C(9, 0)$ $\sqrt{58}$ **or about 7.6 units**

6. **REASONING** Which two time capsules are the closest to each other? Which are farthest apart? **closest: *B* and *C*; farthest: *A* and *B***

Example 3 **Use the number line to find the coordinate of the midpoint of each segment.**



7. $\overline{AC}$ **−3** **8.** $\overline{BD}$ **9**

Example 4 **Find the coordinates of the midpoint of a segment with the given endpoints.**

9. $J(5, -3)$, $K(3, -8)$ **(4, −5.5)** **10.** $M(7, 1)$, $N(4, -1)$ **(5.5, 0)**

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	13–56, 68, 69, 71–86	13–55 odd, 73–76	14–56 even, 64–69, 71, 72, 77–86
OL Core	13–55 odd, 57–60, 61–65 odd, 66–69, 71–86	13–56, 73–76	57–69, 71, 72, 77–86
BL Advanced	57–81, (optional: 82–86)		

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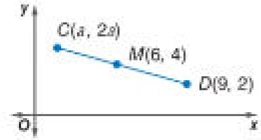
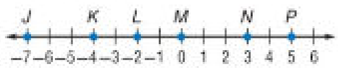
Example 5 **11.** Find the coordinates of G if $F(1, 3.5)$ is the midpoint of $\overline{GJ}$ and J has coordinates $(6, -2)$. **$(-4, 9)$**

Example 6 **12.** **ALGEBRA** Point M is the midpoint of $\overline{CD}$. What is the value of a in the figure? **3**



Practice and Problem Solving

Example 1 **Use the number line to find each measure.**



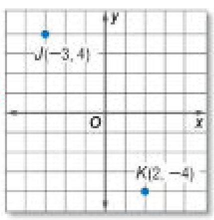
13. JL **5** **14.** JK **3** **15.** KP **9**

16. NP **2** **17.** JP **12** **18.** LN **5**

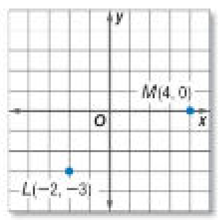
Example 2 **Find the distance between each pair of points.** **22–30. See margin.**

19. $\sqrt{89}$ or about 9.4 units
20. $\sqrt{45}$ or about 6.7 units
21. $\sqrt{58}$ or about 7.6 units

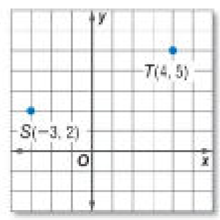
19.



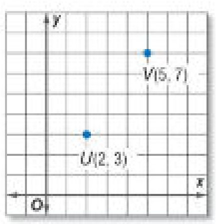
20.



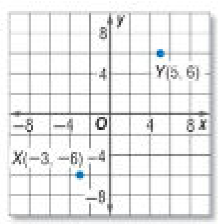
21.



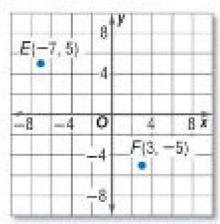
22.



23.



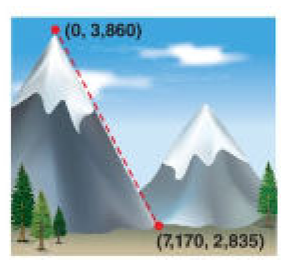
24.



25. $X(1, 2)$, $Y(5, 9)$ **26.** $P(3, 4)$, $Q(7, 2)$ **27.** $M(-3, 8)$, $N(-5, 1)$

28. $Y(-4, 9)$, $Z(-5, 3)$ **29.** $A(2, 4)$, $B(5, 7)$ **30.** $C(5, 1)$, $D(3, 6)$

31. **REASONING** Huda is planning to hike to the top of Humphreys Peak on her family vacation. The coordinates of the peak of the mountain and of the base of the trail are shown. If the trail can be approximated by a straight line, estimate the length of the trail. (*Hint:* 1 km = 5280 m) **7.24 km**



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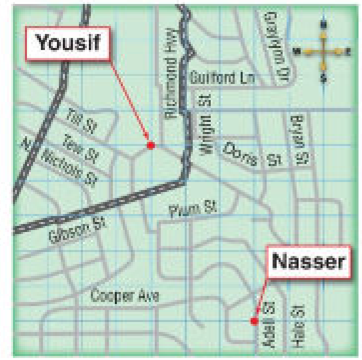
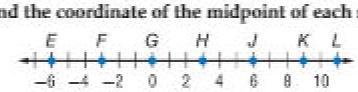
32. **MODELING** Yousif and Nasser live in the locations shown on the map below.



a. If each square on the grid represents one block and the bottom left corner of the grid is the location of the origin, what is the straight-line distance from Yousif's house to Nasser's? **5.8 blocks**

b. If Yousif moves three blocks to the north and Nasser moves 5 blocks to the west, how far apart will they be? **8.2 blocks**

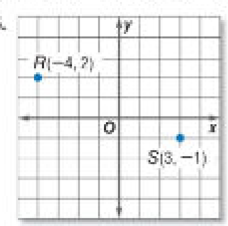
Example 3 **Use the number line to find the coordinate of the midpoint of each segment.**



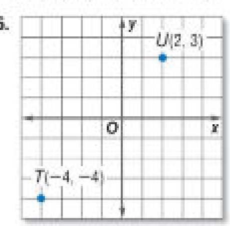
33. $\overline{HK}$ **6**
34. $\overline{JL}$ **8.5**
35. $\overline{EF}$ **−4.5**
36. $\overline{FG}$ **−1.5**
37. $\overline{FK}$ **3**
38. $\overline{EL}$ **2.5**

Example 4 **Find the coordinates of the midpoint of a segment with the given endpoints.**

39. $C(22, 4)$, $B(15, 7)$ **(18.5, 5.5)**
40. $W(12, 2)$, $X(7, 9)$ **(9.5, 5.5)**
41. $D(-15, 4)$, $E(2, -10)$ **(−6.5, −3)**
42. $V(-2, 5)$, $Z(3, -17)$ **(0.5, −6)**
43. $X(-2.4, -14)$, $Y(-6, -6.8)$ **(−4.2, −10.4)**
44. $J(-11.2, -3.4)$, $K(-5.6, -7.8)$ **(−8.4, −5.6)**

45. 

$\left(-\frac{1}{2}, \frac{1}{2}\right)$

46. 

$\left(-1, -\frac{1}{2}\right)$

Example 5 **Find the coordinates of the missing endpoint if B is the midpoint of $\overline{AC}$.**

47. $C(-5, 4)$, $B(-2, 5)$ **A(1, 6)**
48. $A(1, 7)$, $B(-3, 1)$ **C(−7, −5)**
49. $A(-4, 2)$, $B(6, -1)$ **C(16, −4)**
50. $C(-6, -2)$, $B(-3, -5)$ **A(0, −8)**
51. $A(4, -0.25)$, $B(-4, 6.5)$ **C(−12, 13.25)**
52. $C\left(\frac{5}{3}, -6\right)$, $B\left(\frac{8}{3}, 4\right)$ **$A\left(\frac{11}{3}, 14\right)$**

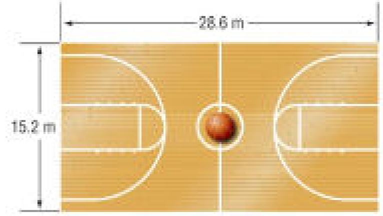
Example 6 **ALGEBRA** **Suppose M is the midpoint of $\overline{FG}$. Use the given information to find the missing measure or value.**

53. $FM = 3x - 4$, $MG = 5x - 26$, $FG = ?$ **58**
54. $FM = 5y + 13$, $MG = 5 - 3y$, $FG = ?$ **16**
55. $MG = 7x - 15$, $FG = 33$, $x = ?$ **4.5**
56. $FM = 8a + 1$, $FG = 42$, $a = ?$ **2.5**

DifferentiatedInstruction OL BL

Extension Have students investigate the difference between driving distances and "as the crow flies" distances. Which are found using the Distance Formula? Which are found by technologies such as mapping programs?

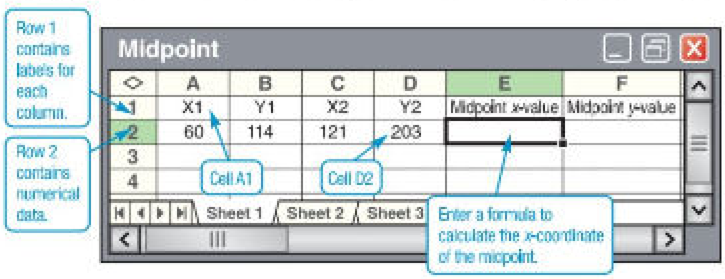
B **57** **BASKETBALL** The dimensions of a basketball court are shown below. Suppose a player throws the ball from a corner to a teammate standing at the center of the court.



a. If center court is located at the origin, find the ordered pair that represents the location of the player in the bottom right corner. **(14.3, 7.6)**

b. Find the distance that the ball travels. **≈16.2 m**

TOOLS Spreadsheets can be used to perform calculations quickly. The spreadsheet below can be used to calculate the distance between two points. Values are used in formulas by using a specific cell name. The value of x_1 is used in a formula using its cell name, A2.



Write a formula for the indicated cell that could be used to calculate the indicated value using the coordinates (x_1, y_1) and (x_2, y_2) as the endpoint of a segment.

58. E2; the x-value of the midpoint of the segment **=AVERAGE(A2,C2)**

59. F2; the y-value of the midpoint of the segment **=AVERAGE(B2,D2)**

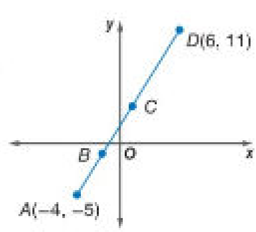
60. G2; the length of the segment **=SQRT((C2−A2)^2+(D2−B2)^2)**

Name the point(s) that satisfy the given condition.

61. two points on the x-axis that are 10 units from (1, 8) **(−5, 0), (7, 0)**

62. two points on the y-axis that are 25 units from (−24, 3) **(0, −4), (0, 10)**

C **63.** **COORDINATE GEOMETRY** Find the coordinates of B if B is the midpoint of $\overline{AC}$ and C is the midpoint of $\overline{AD}$. **$\left(-1\frac{1}{2}, -1\right)$**



ALGEBRA Determine the value(s) of n.

64. $J(n, n + 2)$, $K(3n, n - 1)$, $JK = 5$ **±2**

65. $P(3n, n - 7)$, $Q(4n, n + 5)$, $PQ = 13$ **±5**

Follow-up

Students have explored linear measure, distance, and midpoints.

Ask:

- How are segments measured and classified? Sample answer: Segments can be measured using a ruler in linear units such as centimeters , or using the Distance Formula if they are on the coordinate plane. Segments that have the same measure are congruent.
- When do we use the Distance and Midpoint Formulas? Sample answer: The Distance Formula is used to find the distance between two points. The Midpoint Formula is used to find the point that is halfway between the two endpoints of a segment.

Teach with Tech

Spreadsheets Tell students that spreadsheets often use special commands to perform operation. For example, $\sqrt{x_2 - x_1}$ would be written as SQRT(A2–C2). To find the average of the numbers in a range of cells, use AVERAGE (range of cells). Use the symbol ^ to raise a number to a power.

Exercise Alerts

Internet or Atlas Exercise 66 requires the use of the Internet or an atlas.

Ruler Exercise 67 requires the use of a ruler.

Compass and Straightedge Exercise 71 requires the use of a compass and a straightedge.

Multiple Representations

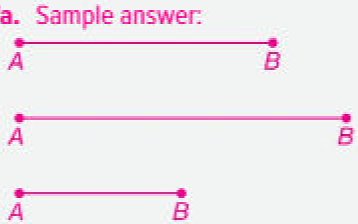
In Exercise 67, students use geometric figures, a table, and algebraic expressions, to investigate and conjecture about the relationship of the midpoint of the segment and the midpoint of the segment between the endpoint and the midpoint.

Tips for New Teachers

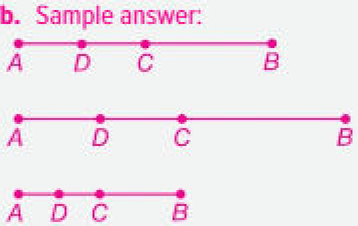
Challenge You may want to use a CHALLENGE problem, like Exercise 70, as a beginning-of-class exercise the day after the lesson is taught. This allows for an opener to homework discussion or a transition into the next lesson, and keeps students on task while daily "classroom maintenance" is performed.

Additional Answers

67a. Sample answer:



67b. Sample answer:



67c. Sample answer:

line	AB (cm)	AC (cm)	AD (cm)
1	4	2	1
2	6	3	1.5
3	3	1.5	0.75

66. PERSEVERANCE Wilmington, North Carolina, is located at (34.3°, 77.9°), which represents north latitude and west longitude. Winston-Salem is in the northern part of the state at (36.1°, 80.2°).

a. Find the latitude and longitude of the midpoint of the segment between Wilmington and Winston-Salem. **(35.2°, 79.1°)**

b. Use an atlas or the Internet to find a city near the location of the midpoint. **Sample answer: Fayetteville**

c. If Winston-Salem is the midpoint of the segment with one endpoint at Wilmington, find the latitude and longitude of the other endpoint. **(37.9°, 82.5°)**

d. Use an atlas or the Internet to find a city near the location of the other endpoint. **Sample answer: Prestonburg, Kentucky**

72. Sample answer: Divide each coordinate of the endpoint that is not located at the origin by 2. For example, if the segment has coordinates (0, 0) and (−10, 6), the midpoint is located at $\left(\frac{-10}{2}, \frac{6}{2}\right)$ or (−5, 3). Using the midpoint formula, if the endpoints of the segment are (0, 0) and (a, b), the midpoint is $\left(\frac{a-0}{2}, \frac{b-0}{2}\right)$ or $\left(\frac{a}{2}, \frac{b}{2}\right)$.

67 **MULTIPLE REPRESENTATIONS** In this problem, you will explore the relationship between a midpoint of a segment and the midpoint between the endpoint and the midpoint. **a–c. See margin.**

a. Geometric Use a straightedge to draw three different line segments. Label the endpoints A and B.

b. Geometric On each line segment, find the midpoint of $\overline{AB}$ and label it C. Then find the midpoint of $\overline{AC}$ and label it D.

c. Tabular Measure and record AB, AC, and AD for each line segment. Organize your results into a table.

d. Algebraic If $AB = x$, write an expression for the measures AC and AD. $AC = \frac{1}{2}x$, $AD = \frac{1}{4}x$

e. Verbal Make a conjecture about the relationship between AB and each segment if you were to continue to find the midpoints of a segment and a midpoint you previously found. **Sample answer: If n midpoints are found, then the smallest segment will have a measure of $\frac{1}{2^n}x$.**

H.O.T. Problems Use Higher-Order Thinking Skills

68. WRITING IN MATH Explain how the Pythagorean Theorem and the Distance Formula are related. **See margin.**

69. REASONING Is the point one third of the way from (x_1, y_1) to (x_2, y_2) *sometimes*, *always*, or *never* the point $\left(\frac{x_1 + x_2}{3}, \frac{y_1 + y_2}{3}\right)$? Explain. **Sample answer: Sometimes; when the point (x_1, y_1) has coordinates (0, 0)**

70. CHALLENGE Point P is located on the segment between point $A(1, 4)$ and point $D(7, 13)$. The distance from A to P is twice the distance from P to D. What are the coordinates of point P? **(5, 10)**

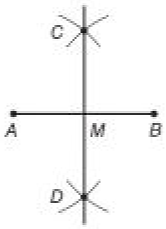
71. OPEN ENDED Draw a segment and name it $\overline{AB}$. Using only a compass and a straightedge, construct a segment $\overline{CD}$ such that $CD = 5\frac{1}{4}AB$. Explain and then justify your construction. **See margin.**

72. WRITING IN MATH Describe a method of finding the midpoint of a segment that has one endpoint at (0, 0). Give an example using your method, and explain why your method works.

68. Sample answer: The Pythagorean Theorem relates the lengths of the legs of a right triangle to the length of the hypotenuse using the formula $c^2 = a^2 + b^2$. If you take the square root of the formula, you get $c = \sqrt{a^2 + b^2}$. Think of the hypotenuse of the triangle as the distance between the two points, the a value as the horizontal distance $x_2 - x_1$, and the b value as the vertical distance $y_2 - y_1$. If you substitute, the Pythagorean Theorem becomes the Distance Formula, $c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Standardized Test Practice

73. Which of the following best describes the first step in bisecting $\overline{AB}$? **C**



A From point A, draw equal arcs on $\overline{CD}$ using the same compass width.

B From point A, draw equal arcs above and below $\overline{AB}$ using a compass width of $\frac{1}{3}\overline{AB}$.

C From point A, draw equal arcs above and below $\overline{AB}$ using a compass width greater than $\frac{1}{2}\overline{AB}$.

D From point A, draw equal arcs above and below $\overline{AB}$ using a compass width less than $\frac{1}{2}\overline{AB}$.

74. ALGEBRA Hidaya paid AED 74.88 for 3 pairs of jeans. All 3 pairs of jeans were the same price. How much did each pair of jeans cost? **F**

F AED 24.96   **H** AED 74.88

G AED 37.44   **J** AED 224.64

75. SAT/ACT If $5^{2x-3} = 1$, then $x =$ **C**

A 0.4   **D** 1.6

B 0.6   **E** 2

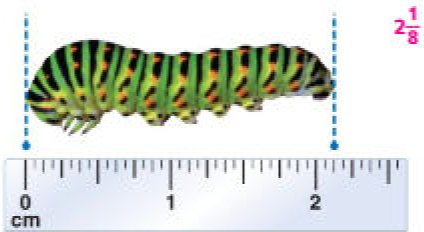
C 1.5

76. GRIDDED RESPONSE One endpoint of $\overline{AB}$ has coordinates $(-3, 5)$. If the coordinates of the midpoint of $\overline{AB}$ are $(2, -6)$, what is the approximate length of $\overline{AB}$? **24.2**

Spiral Review

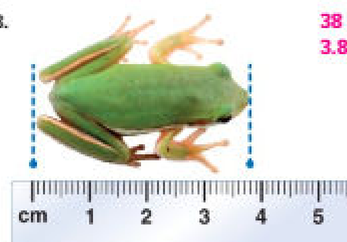
Find the length of each object. (Lesson 10-2)

77.



$2\frac{1}{8}$ cm

78.



38 mm or 3.8 cm

Draw and label a figure for each relationship. (Lesson 10-1) **79–80. See margin.**

79. $\overleftrightarrow{FG}$ lies in plane M and contains point H.

80. Lines r and s intersect at point W.

Skills Review

Solve each equation.

81. $8x - 15 = 5x$ **5**

82. $5y - 3 + y = 90$ **15.5**

83. $16a + 21 = 20a - 9$ **7.5**

84. $9k - 7 = 21 - 3k$ $\mathbf{2\frac{1}{3}}$

85. $11z - 13 = 3z + 17$ $\mathbf{3\frac{3}{4}}$

86. $15 + 6n = 4n + 23$ **4**

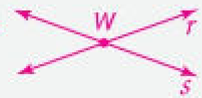
4 Assess

Yesterday's News Have students write a paragraph that explains how the lesson about linear measures helped them in the lesson about the Distance Formula, the Pythagorean Theorem, and the Midpoint Formula.

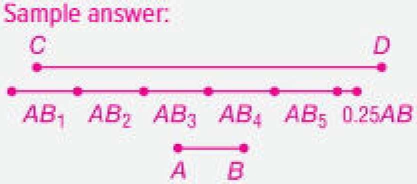
Additional Answers

79.

80.



71. Sample answer:



Draw $\overline{AB}$. Next, draw a construction line and place point C on it. From C, strike 6 arcs in succession of length AB. On the sixth $\overline{AB}$ length, perform a segment bisector two times to create a $\frac{1}{4}AB$ length. Label the endpoint D.

Formative Assessment

Use the Mid-Chapter Quiz to assess students' progress in the first half of the chapter.

Have students review the lesson indicated for the problems that they answered incorrectly.

FOLDABLES StudyOrganizer

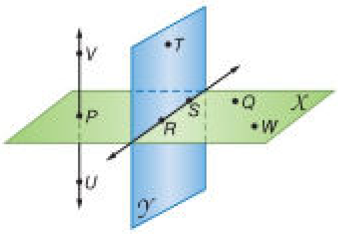
Dinah Zike's Foldables®

Before students complete the Mid-Chapter Quiz, encourage them to review the information from Lessons 10-1 through 10-4 in their Foldables.

CHAPTER 10 Mid-Chapter Quiz

Lessons 10-1 through 10-4

Use the figure to complete each of the following. (Lesson 10-1)



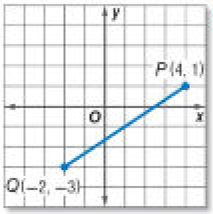
1. Name another point that is collinear with points U and V. **P**
2. What is another name for plane Y? **plane RST**
3. Name a line that is coplanar with points P, Q, and W. **$\overleftrightarrow{RS}$**

Find the value of x and AC if B is between points A and C. (Lesson 10-2)

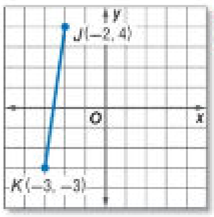
4. $AB = 12$, $BC = 8x - 2$, $AC = 10x$ **$x = 5$; $AC = 50$**
5. $AB = 5x$, $BC = 9x - 2$, $AC = 11x + 7.6$ **$x = 3.2$; $AC = 42.8$**
6. Find CD and the coordinate of the midpoint of $\overline{CD}$. **14; −2**

Find the coordinates of the midpoint of each segment. Then find the length of each segment. (Lesson 10-3)

7. **$(1, -1)$; $2\sqrt{13} \approx 7.2$**



8. **$\left(-\frac{5}{2}, \frac{1}{2}\right)$; $5\sqrt{2} \approx 7.1$**

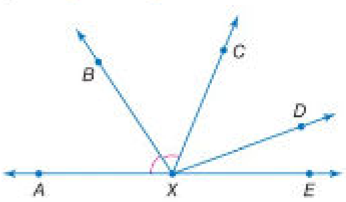


Find the coordinates of the midpoint of a segment with the given endpoints. Then find the distance between each pair of points. (Lesson 10-3)

9. $P(26, 12)$ and $Q(8, 42)$ **$(17, 27)$; $\sqrt{1224}$ or 35.0**
10. $M(6, -41)$ and $N(-18, -27)$ **$(-6, -34)$; $\sqrt{772}$ or 27.8**
11. **MAPS** A map of a town is drawn on a coordinate grid. The high school is found at point (3, 1) and town hall is found at (−5, 7). (Lesson 10-3)
    a. If the high school is at the midpoint between the town hall and the town library, at which ordered pair should you find the library? **(11, −5)**
    b. If one unit on the grid is equivalent to 50 meters, how far is the high school from town hall? **500 m**
12. **MULTIPLE CHOICE** The vertex of $\angle ABC$ is located at the origin. Point A is located at (5, 0) and Point C is located at (0, 2). How can $\angle ABC$ be classified? **C**

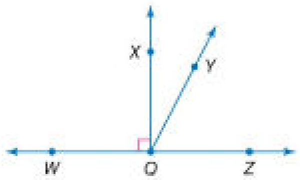
    A acute   **C** right
    B obtuse   **D** scalene

In the figure, $\overrightarrow{XA}$ and $\overrightarrow{XE}$ are opposite rays, and $\angle AXC$ is bisected by $\overrightarrow{XB}$. (Lesson 10-4)



13. If $m\angle AXC = 8x - 7$ and $m\angle AXB = 3x + 10$, find $m\angle AXC$. **101**
14. If $m\angle CXD = 4x + 6$, $m\angle DXE = 3x + 1$, and $m\angle CXE = 8x - 2$, find $m\angle DXE$. **28**

Classify each angle as *acute*, *right*, or *obtuse*. (Lesson 10-4)



15. $\angle WQY$ **obtuse**
16. $\angle YQZ$ **acute**

MY COMMENTS

1 Focus

Vertical Alignment

Before Lesson 10-4 Identify and name polygons.

Lesson 10-4 Write proofs involving segment addition and segment congruence.

After Lesson 10-4 Use deductive reasoning to prove a statement.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- Why did Emma need to measure the fabric in this way? Sample: The fabric was longer than the yardstick.
- Describe how measuring 36 and then 3 centimeters gives a length of 39 centimeters. The two lengths added together give the total length.
- How many times would Emma mark the fabric if she wanted to measure 120 centimeters? 3

LESSON 10-4 Proving Theorems about Line Segments

Then

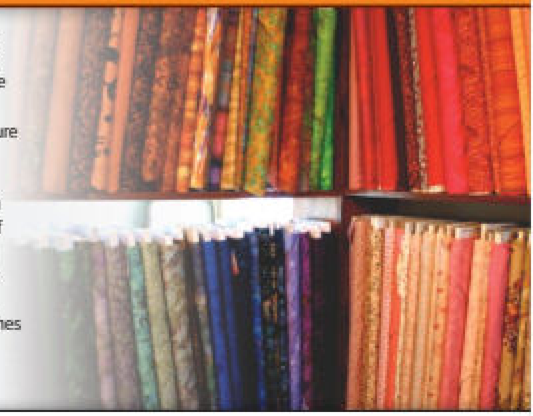
- You wrote algebraic and two-column proofs.

Now

1. Write proofs involving segment addition.
2. Write proofs involving segment congruence.

Why?

- Najla works at a fabric store after school. She measures a length of fabric by holding the straight edge of the fabric against a yardstick. To measure lengths such as 39 inches, which is longer than the yardstick, she marks a length of 36 inches. From the end of that mark, she measures an additional length of 3 inches. This ensures that the total length of fabric is 36 + 3 inches or 39 inches.



Mathematical Practices
Reason abstractly and quantitatively.
Construct viable arguments and critique the reasoning of others.

1 Ruler Postulate In Lesson 10-2, you measured segments with a ruler by matching the mark for zero with one endpoint and then finding the number on the ruler that corresponded to the other endpoint. This illustrates the Ruler Postulate.

Postulate 10.1 Ruler Postulate

Words The points on any line or line segment can be put into one-to-one correspondence with real numbers.

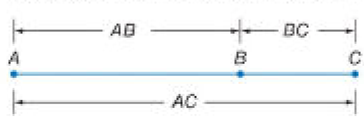
Symbols Given any two points A and B on a line, if A corresponds to zero, then B corresponds to a positive real number.

In Lesson 10-2, you also learned about what it means for a point to be *between* two other points. This relationship can be expressed as the Segment Addition Postulate.

Postulate 10.2 Segment Addition Postulate

Words If A, B, and C are collinear, then point B is between A and C if and only if $AB + BC = AC$.

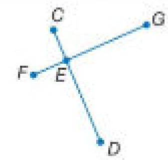
Symbols



The Segment Addition Postulate is used as a justification in many geometric proofs.

Example 1 Use the Segment Addition Postulate

Prove that if $\overline{CE} \cong \overline{FE}$ and $\overline{ED} \cong \overline{EG}$ then $\overline{CD} \cong \overline{FG}$.



Given: $\overline{CE} \cong \overline{FE}$; $\overline{ED} \cong \overline{EG}$

Prove: $\overline{CD} \cong \overline{FG}$

Proof:

Statements	Reasons
1. $\overline{CE} \cong \overline{FE}$; $\overline{ED} \cong \overline{EG}$	1. Given
2. $CE = FE$; $ED = EG$	2. Definition of congruence
3. $CE + ED = CD$	3. Segment Addition Postulate
4. $FE + EG = CD$	4. Substitution (Steps 2 & 3)
5. $FE + EG = FG$	5. Segment Addition Postulate
6. $CD = FG$	6. Substitution (Steps 4 & 5)
7. $\overline{CD} \cong \overline{FG}$	7. Definition of congruence

ReadingMath

Substitution Property The Substitution Property of Equality is often just written as *Substitution*.

GuidedPractice

Copy and complete the proof.

1. **Given:** $\overline{JL} \cong \overline{KM}$

   Prove: $\overline{JK} \cong \overline{LM}$

Proof:

Statements	Reasons
a. $\overline{JL} \cong \overline{KM}$	a. Given
b. $JL = KM$	b. ___?___ **Def. of ≅**
c. $JK + KL = $ ___?___ (**JL**); $KL + LM = $ ___?___ (**KM**)	c. Segment Addition Postulate
d. $JK + KL = KL + LM$	d. ___?___ **Subs.**
e. $JK + KL - KL = KL + LM - KL$	e. Subtraction Property of Equality
f. ___?___ **$JK = LM$**	f. Substitution
g. $\overline{JK} \cong \overline{LM}$	g. Definition of congruence

2 Segment Congruence

Segments with the same measure are congruent. Congruence of segments is also reflexive, symmetric, and transitive.

Theorem 10.1 Properties of Segment Congruence

Reflexive Property of Congruence	$\overline{AB} \cong \overline{AB}$
Symmetric Property of Congruence	If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.
Transitive Property of Congruence	If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.

You will prove the Symmetric and Reflexive Properties in Exercises 6 and 7, respectively.

VocabularyLink

Symmetric

Everyday Use balanced or proportional

Math Use If $a = b$, then $b = a$.

1 Segment Addition

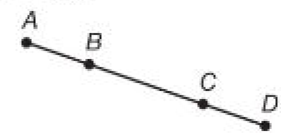
Example 1 shows how to use properties and postulates to prove segment addition.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Example

1 Prove that if $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$.



Proof:

Statements (Reasons)

1. $\overline{AB} \cong \overline{CD}$ (Given)
2. $AB = CD$ (Def. of ≅ segs.)
3. $BC = BC$ (Refl. Prop. of =)
4. $AB + BC = AC$ (Seg. Add. Post.)
5. $CD + BC = AC$ (Substi. Prop. of =)
6. $CD + BC = BD$ (Seg. Add. Post.)
7. $AC = BD$ (Trans. Prop. of =)
8. $\overline{AC} \cong \overline{BD}$ (Def. of ≅ segs.)

WatchOut!

End it Right In Example 1, the question asks to prove that two segments are congruent. Explain to students that the last line of the proof is needed to accurately describe the relationship of the segments as the question asks.

DifferentiatedInstruction AL OL

If students have difficulty identifying the given information and information implicit in a given figure,

Then encourage students to use their spatial skills to locate obvious and hidden congruent segments. Have students mark the figures so they can easily refer to the relationships in the figures while they are writing their proofs.

Tips for New Teachers

Knowledge Building Point out that with each new lesson, students are accumulating knowledge of more postulates and theorems that they can use for writing proofs. Encourage students to practice using these concepts as much as possible before moving on to the next lesson to strengthen their ability to recall important facts for proof-writing.

2 Segment Congruence

Example 2 shows how to use properties and postulates to prove segment congruence.

Additional Example

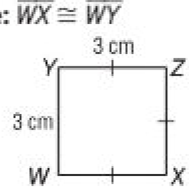
2 **BADGE** Maha is designing a badge for her club. The length of the top edge of the badge is equal to the length of the left edge of the badge. The top edge of the badge is congruent to the right edge of the badge, and the right edge of the badge is congruent to the bottom edge of the badge. Prove that the bottom edge of the badge is congruent to the left edge of the badge.

Given: $WY = YZ$

$\overline{YZ} \cong \overline{XZ}$

$\overline{XZ} \cong \overline{WX}$

Prove: $\overline{WX} \cong \overline{WY}$



Proof:

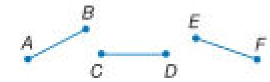
Statements (Reasons)

1. $WY = YZ$ (Given)
2. $\overline{WY} \cong \overline{YZ}$ (Def. of ≅ segs.)
3. $\overline{YZ} \cong \overline{XZ}$, $\overline{XZ} \cong \overline{WX}$ (Given)
4. $\overline{YZ} \cong \overline{WX}$ (Trans. Prop.)
5. $\overline{WX} \cong \overline{WY}$ (Substitution)

Proof Transitive Property of Congruence

Given: $\overline{AB} \cong \overline{CD}$; $\overline{CD} \cong \overline{EF}$

Prove: $\overline{AB} \cong \overline{EF}$



Paragraph Proof:

Since $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, $AB = CD$ and $CD = EF$ by the definition of congruent segments. By the Transitive Property of Equality, $AB = EF$. Thus, $\overline{AB} \cong \overline{EF}$ by the definition of congruence.

Real-WorldLink

According to a recent poll, 70% of teens who volunteer began doing so before age 12. Others said they would volunteer if given more opportunities to do so.

Source: Youth Service America

Real-World Example 2 Proof Using Segment Congruence

VOLUNTEERING **The route for a charity fitness run is shown. Checkpoints X and Z are the midpoints between the starting line and Checkpoint Y and Checkpoint Y and the finish line F, respectively. If Checkpoint Y is the same distance from Checkpoints X and Z, prove that the route from Checkpoint Z to the finish line is congruent to the route from the starting line to Checkpoint X.**

Given: X is the midpoint of $\overline{SY}$. Z is the midpoint of $\overline{YF}$. $XY = YZ$

Prove: $\overline{ZF} \cong \overline{SX}$

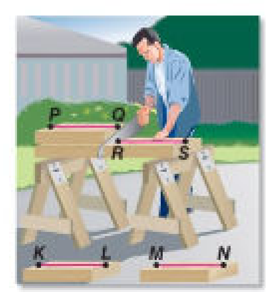
Two-Column Proof:

Statements	Reasons
1. X is the midpoint of $\overline{SY}$. Z is the midpoint of $\overline{YF}$. $XY = YZ$	1. Given
2. $\overline{SX} \cong \overline{XY}$; $\overline{YZ} \cong \overline{ZF}$	2. Definition of midpoint
3. $\overline{XY} \cong \overline{YZ}$	3. Definition of congruence
4. $\overline{SX} \cong \overline{YZ}$	4. Transitive Property of Congruence
5. $\overline{SX} \cong \overline{ZF}$	5. Transitive Property of Congruence
6. $\overline{ZF} \cong \overline{SX}$	6. Symmetric Property of Congruence

2. Given: $\overline{KL} \cong \overline{MN}$, $\overline{MN} \cong \overline{PQ}$, $\overline{PQ} \cong \overline{RS}$
Prove: $RS = KL$
Proof: If $\overline{KL} \cong \overline{MN}$ and $\overline{MN} \cong \overline{PQ}$, then by the Transitive Property of Congruence, $\overline{KL} \cong \overline{PQ}$. If $\overline{PQ} \cong \overline{RS}$, then $\overline{KL} \cong \overline{RS}$ by the Transitive Property of Congruence. $\overline{RS} \cong \overline{KL}$ by the Symmetric Property of Congruence and $RS = KL$ by the definition of congruence. So, the first board cut has the same measure as the last board cut.

GuidedPractice

2. **CARPENTRY** A carpenter cuts a 2″ × 4″ board to a desired length. He then uses this board as a pattern to cut a second board congruent to the first. Similarly, he uses the second board to cut a third board and the third board to cut a fourth board. Prove that the last board cut has the same measure as the first.



Focus on Mathematical Content

Line Segments It is important to know that the art and diagrams that go along with problems may not always be to scale. Two segments may be assumed to be congruent in a question, but if measured with a ruler they could be different. On the other hand, certain combinations of lines can create an optical illusion and lines that are the same length may not appear so.

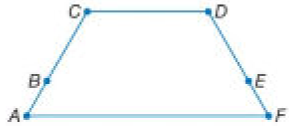
Teach with Tech

Interactive Whiteboard Work through several proofs on the board and save your work. Post your notes on a class Web page so students will have an additional reference outside of class.

Check Your Understanding

Example 1

1. Copy and complete the proof.



Given: $\overline{AB} \cong \overline{FE}$, $\overline{BC} \cong \overline{ED}$

Prove: $\overline{AC} \cong \overline{FD}$

Proof:

Statements	Reasons
a. $\overline{AB} \cong \overline{FE}$, $\overline{BC} \cong \overline{ED}$	a. **Given**
b. **$AB \cong FE$, $BC \cong ED$**	b. Definition of congruent segments
c. $AB + FE = BC + ED$	c. **Addition Property of Equality**
d. **$AB + BC = AC$ $FE + ED = FD$**	d. Segment Addition Postulate
e. $AC = FD$	e. **Substitution**
f. $\overline{AC} \cong \overline{FD}$	f. **Definition of Congruence**

Example 2

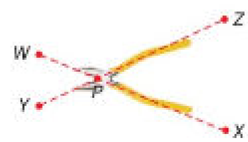
2. **PROOF** Prove the following. **See margin.**

Given: $\overline{JK} \cong \overline{LM}$

Prove: $\overline{JL} \cong \overline{KM}$

3. **PLIERS** Refer to the diagram shown. $\overline{WP}$ is congruent to $\overline{YP}$, $\overline{ZP}$ is congruent to $\overline{XP}$. Prove that $WP + ZP = YP + XP$

See margin.



Practice and Problem Solving

Example 1

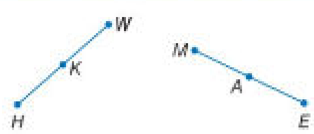
4. Copy and complete the proof.

Given: K is the midpoint of $\overline{HW}$

A is the midpoint of $\overline{ME}$

$\overline{HW} \cong \overline{ME}$

Prove: $\overline{HK} \cong \overline{MA}$



Statements	Reasons
a. **K is the midpoint of $\overline{HW}$ A is the midpoint of $\overline{ME}$ $\overline{HW} \cong \overline{ME}$**	a. Given
b. $HK = KW$, $MA = AE$	b. **Definition of Midpoint**
c. $HW = ME$	c. **Definition of Congruence**
d. **$HK + KW = HW$ $MA + AE = ME$**	d. Segment Addition Postulate
e. $HK + KW = MA + AE$	e. **Substitution**
f. $HK + HK = MA + MA$	f. **Substitution**
g. **$2HK = 2MA$**	g. Simplify.
h. **$HK = MA$**	h. Division Property of Equality
i. $\overline{HK} \cong \overline{MA}$	i. **Definition of Congruence**

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	4–13, 17, 19–29	5–13 odd, 23–26	4–12 even, 17, 19–22, 27–29
OL Core	5–13 odd, 15–17, 19–29	4–13, 23–26	14–17, 19–22, 27–29
BL Advanced	14–29		

3 Practice

Formative Assessment

Use Exercises 1–3 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Additional Answers

2. Given: $\overline{JK} \cong \overline{LM}$

Prove: $\overline{JL} \cong \overline{KM}$

Proof:

Statements (Reasons)

1. $\overline{JK} \cong \overline{LM}$ (Given)
2. $JK = LM$ (Definition of Congruence)
3. $KL = KL$ (Reflexive Property)
4. $JK + KL = KL + LM$ (Addition Property of Equality)
5. $JK + KL = JL$
$KL + LM = KM$ (Segment Addition Postulate)
6. $JL = KM$ (Substitution)
7. $\overline{JL} \cong \overline{KM}$ (Definition of Congruence)

3. Given: $\overline{WP} \cong \overline{YP}$, $\overline{ZP} \cong \overline{XP}$

Prove: $WP + ZP = YP + XP$

Proof:

Statements (Reasons)

1. $\overline{WP} \cong \overline{YP}$; $\overline{ZP} \cong \overline{XP}$ (Given)
2. $WP = YP$, $ZP = XP$ (Definition of Congruence)
3. $WP + ZP = YP + XP$ (Addition Property of Equality)

Prove each theorem. **5, 6. See Ch. 10 Answer Appendix.**

5. Symmetric Property of Congruence Theorem 10.1

6. Reflexive Property of Congruence Theorem 10.1

7. **TRAVEL** Kadoka, Rapid City, Sioux Falls, Alexandria, South Dakota are all connected by Interstate 90. **a, b. See Ch. 10 Answer Appendix.**

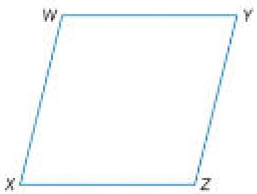
- Sioux Falls is 256 kilometers from Kadoka and 352 kilometers from Rapid City
- Rapid City is 96 kilometers from Kadoka and 292 kilometers from Alexandria

a. Draw a diagram to represent the locations of the cities in relation to each other and the distances between each city. Assume that Interstate 90 is straight.

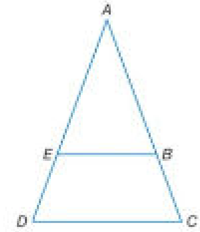
b. Write a paragraph proof to support your conclusion.

PROOF **Prove the following.** **8–12. See Ch. 10 Answer Appendix.**

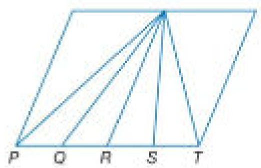
8. If $\overline{XW} \cong \overline{YZ}$ and $\overline{YZ} \cong \overline{ZX}$, then $\overline{XW} \cong \overline{ZX}$.



9. If $\overline{AC} \cong \overline{AD}$ and $\overline{ED} \cong \overline{BC}$, then $\overline{AE} \cong \overline{AB}$.



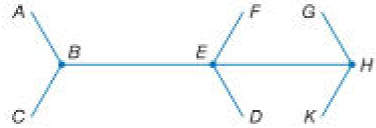
10. If R is the midpoint of $\overline{QS}$ and $\overline{PQ} \cong \overline{ST}$, then $\overline{PA} \cong \overline{RT}$.



11. If Q is the midpoint of $\overline{PR}$, S is the midpoint of $\overline{RT}$, and $\overline{QR} \cong \overline{RS}$, then $PT = 4QR$.

Example 1

12.



Given: $\overline{AB} \cong \overline{FE}$, $\overline{ED} \cong \overline{HK}$ and $AB + BE + ED = EF + EH + HK$

Prove: $\overline{BE} \cong \overline{EH}$

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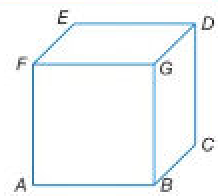
13. **CONSTRUCTION** Construct a segment that is twice as long as $\overline{PQ}$. Explain how the Segment Addition Postulate can be used to justify your construction. **See Ch. 10 Answer Appendix.**

14. **MULTIPLE REPRESENTATIONS** A is the midpoint of $\overline{PQ}$, B is the midpoint of $\overline{PA}$, and C is the midpoint of $\overline{PB}$. **a–e. See Ch. 10 Answer Appendix.**
   a. **Geometric** Make a sketch to represent this situation.
   b. **Algebraic** Make a conjecture as to the algebraic relationship between PC and PQ.
   c. **Geometric** Copy segment $\overline{PQ}$ from your sketch. Then construct points B and C on $\overline{PQ}$. Explain how you can use your construction to support your conjecture.
   d. **Concrete** Use a ruler to draw a segment congruent to $\overline{PQ}$ from your sketch and to draw points B and C on $\overline{PQ}$. Use your drawing to support your conjecture.
   e. **Logical** Prove your conjecture.

H.O.T. Problems Use Higher-Order Thinking Skills

15. **ERROR ANALYSIS** In the diagram, $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{DG}$. Examine the conclusions made by Najat and Nabila. Is either of them correct? Explain your reasoning. **See margin.**



Najat
Since $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{DG}$, then $\overline{AB} \cong \overline{DE}$ by the Transitive Property of Congruence.

Nabila
Since $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{DG}$, then $\overline{AB} \cong \overline{DG}$ by the Reflexive Property of Congruence.

16. **CHALLENGE** $ABCD$ is a rectangle. Prove that $\overline{AC} \cong \overline{BD}$. **See Ch. 10 Answer Appendix.**

17. **WRITING IN MATH** Does there exist a Subtraction Property of Congruence? Explain. **See margin.**

18. **REASONING** Classify the following statement as true or false. If false, provide a counterexample. **See Ch. 10 Answer Appendix.**

If, A, B, C, D, and E are collinear with B being the midpoint between A and C, C being the midpoint between B and D, and D being the midpoint between C and E, then AB = BC = DE.

19. **OPEN ENDED** Draw a representation of the Segment Addition Postulate in which the segment is $1\frac{1}{2}$ centimeters long, contains four collinear points, and contains no congruent segments. **See Ch. 10 Answer Appendix.**

20. **WRITING IN MATH** Compare and contrast paragraph proofs and two-column proofs. **See Ch. 10 Answer Appendix.**

Exercise Alert

Compass and Straightedge Exercise 14 requires students to use a compass and straightedge.

Compass and Ruler Exercise 16 requires students to use a compass and a ruler.

Multiple Representations

In Exercise 16, students use geometric sketches, algebraic conjecture, and direct measurement to investigate midpoints of a line.

WatchOut!

Error Analysis Leslie applied the property correctly but incorrectly stated that $\overline{AB} \cong \overline{AF}$. Shantice also applied the property correctly but mistakenly cited the Reflexive Property.

Additional Answers

15. Neither are correct. Mary stated the correct property but incorrectly stated that $\overline{AB} \cong \overline{DE}$, when it should have been $\overline{AB} \cong \overline{DG}$. Susan stated the correct congruence but gave the wrong reason.

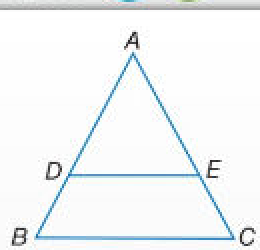
17. Student answers will vary, but should convey their understanding that there is no Subtraction Property of Congruence.

DifferentiatedInstruction OL BL

Extension

Given: $BD = EC$
$DA = AE$

Prove: $BA = AC$



Proof:

Statement (reason)

1. $BD = EC$; $DA = AE$ (Given)
2. $BD + DA = EC + AE$ (Add. Prop. of =)
3. $BD + DA = BA$; $EC + AE = AC$ (Seg. Add. Post)
4. $BA = AC$ (Substitution)

4 Assess

Name the Math Give each student a ruler to measure a finger from the tip to the first knuckle. Next, have each student measure from the first knuckle to the second knuckle. Have students find the length from the tip of the finger to second knuckle and determine whether the measurements taken on the corresponding finger of the other hand are congruent. Have them write the postulates or theorems that they used.

Additional Answer

27. Given: $AC = DF, AB = DE$

Prove: $BC = EF$

Proof:

Statements (Reasons)

1. $AC = DF, AB = DE$ (Given)
2. $AC = AB + BC; DF = DE + EF$ (Seg. Add. Post.)
3. $AB + BC = DE + EF$ (Subs.)
4. $BC = EF$ (Subt. Prop.)

Standardized Test Practice

21. ALGEBRA The chart below shows annual recycling by material in the United States. About how many kilograms of aluminum are recycled each year? **D**

A 7.5
B 15,000
C 7,500,000
D 15,000,000,000

22. ALGEBRA Which expression is equivalent to $\frac{12x^{-4}}{4x^{-8}}$? **G**

F $\frac{1}{3x^4}$
G $3x^4$
H $8x^2$
J $\frac{x^4}{3}$

23. SHORT RESPONSE The measures of two complementary angles are in the ratio 4:1. What is the measure of the smaller angle? **18**

24. SAT/ACT Maysoun can word process 40 words per minute. How many minutes will it take Maysoun to word process 200 words? **C**

A 0.5
B 2
C 5
D 10
E 12

Spiral Review

25. GEOMETRY If the side length of a cube is s, the volume is represented by s^3, and the surface area is represented by $6s^2$.

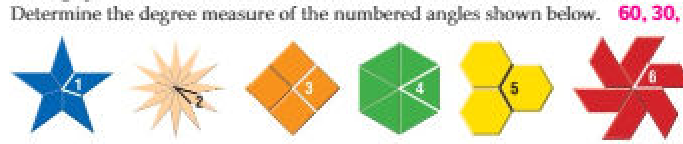
a. Are the expressions for volume and surface area monomials? Explain. **Yes; each is the product of variables and/or a real number.**

b. If the side of a cube measures 3 meters, find the volume and surface area. **27 m³; 54 m²**

c. Find a side length s such that the volume and surface area have the same measure. **6 units**

d. The volume of a cylinder can be found by multiplying the radius squared times the height times π, or $V = \pi r^2 h$. Suppose you have two cylinders. Each measure of the second is twice the measure of the first, so $V = \pi(2r)^2(2h)$. What is the ratio of the volume of the first cylinder to the second cylinder? **1:8**

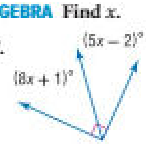
26. PATTERN BLOCKS Pattern blocks can be arranged to fit in a circular pattern without leaving spaces. Remember that the measurement around a full circle is 360°. Determine the degree measure of the numbered angles shown below. **60, 30, 90, 60, 120, 60**



Skills Review

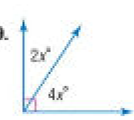
ALGEBRA Find x.

27. **7**



28. **8**

29. **15**



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NOTES

Formative Assessment

KeyVocabulary If students have difficulty completing Exercises 1–4, remind them that they can review the lessons to refresh their memories about the vocabulary terms.

FOLDABLES StudyOrganizer

Dinah Zike's Foldables®

Have students look through the chapter to make sure they have included examples in their Foldables for each lesson of the chapter. Suggest that students keep their Foldables handy while completing the Study Guide and Review pages. Point out that their Foldables can serve as a quick review when studying for the chapter test.

CHAPTER 10 Study Guide and Review

Study Guide

KeyConcepts

Points, Lines, and Planes (Lesson 10-1)

- There is exactly one line through any two points.
- There is exactly one plane through any three noncollinear points.

Distance and Midpoints (Lesson 10-3)

- On a number line, the measure of a segment with endpoint coordinates a and b is $|a - b|$.
- In the coordinate plane, the distance between two points (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- On a number line, the coordinate of the midpoint of a segment with endpoints a and b is $\frac{a + b}{2}$.
- In the coordinate plane, the coordinates of the midpoint of a segment with endpoints that are (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Angles (Lessons 10-3, 10-4, and 10-5)

- An angle is formed by two noncollinear rays that have a common endpoint, called its vertex. Angles can be classified by their measures.
- Adjacent angles are two coplanar angles that lie in the same plane and have a common vertex and a common side but no common interior points.
- Vertical angles are two nonadjacent angles formed by two intersecting lines.
- A linear pair is a pair of adjacent angles with noncommon sides that are opposite rays.
- Complementary angles are two angles with measures that have a sum of 90.
- Supplementary angles are two angles with measures that have a sum of 180.

Proof (Lessons 10-7 and 10-8)

Step 1 List the given information and draw a diagram, if possible.

Step 2 State what is to be proved.

Step 3 Create a deductive argument.

Step 4 Justify each statement with a reason.

Step 5 State what you have proved.

KeyVocabulary

acute angle
angle
angle bisector
area
between
circumference
collinear
concave
congruent
construction
convex
coplanar
degree
distance
equiangular polygon
equilateral polygon
exterior
interior
intersection
line
line segment
midpoint
n-gon
obtuse angle
opposite rays
perimeter
plane
point
polygon
ray
regular polygon
right angle
segment bisector
side
space
undefined term
vertex
vertex of a polygon

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.

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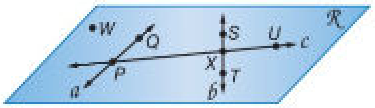
Lesson-by-Lesson Review

Intervention If the given examples are not sufficient to review the topics covered by the questions, remind students that the page references tell them where to review that topic in their textbooks.

Lesson-by-Lesson Review

10-1 Points, Lines, and Planes

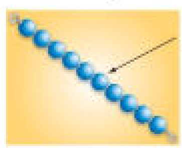
Use the figure to complete each of the following.



1. Name the intersection of lines a and c. **point P**
2. Give another name for line b. **$\overleftrightarrow{ST}$**
3. Name a point that is not contained in any of the three lines a, b, or c. **point W**
4. Give another name for plane WPX. **plane $\mathcal{R}$**

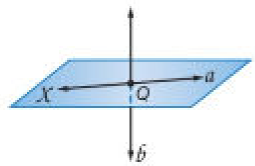
Name the geometric term that is best modeled by each item.

5.  **line**

6.  **point**

Example 1

Draw and label a figure for the relationship below.



Plane X contains line a, line b intersects line a at point Q, but line b is not in plane X.

Draw a surface to represent plane X and label it.

Draw a line in plane X and label it line a.

Draw a line b intersecting both the plane and line a and label the point of intersection Q.

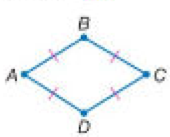
10-2 Linear Measure

Find the value of the variable and XP, if X is between P and Q.

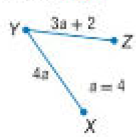
7. $XQ = 13$, $XP = 5x - 3$, $PQ = 40$ **$x = 6$, $XP = 27$**
8. $XQ = 3k$, $XP = 7k - 2$, $PQ = 6k + 16$ **$k = 4.5$, $XP = 29.5$**

Determine whether each pair of segments is congruent.

9. $\overline{AB}$, $\overline{CD}$ **yes**



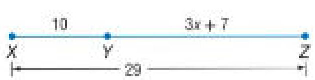
10. $\overline{XY}$, $\overline{YZ}$ **no**



11. **DISTANCE** The distance from Faris's job to his house is 3 times greater than the distance from his house to school. If his house is between his job and school and the distance from his job to school is 6 kilometers, how far is it from Faris's house to school? **1.5 km**

Example 2

Use the figure to find the value of the variable and the length of $\overline{YZ}$.



$XZ = XY + YZ$	Betweenness of points
$29 = 10 + 3x + 7$	Substitution
$29 = 3x + 17$	Simplify.
$12 = 3x$	Subtract 17 from each side.
$4 = x$	Divide each side by 3.
$YZ = 3x + 7$	Given
$= 3(4) + 7$ or 19	Substitution

So, $x = 4$ and $YZ = 19$.

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Additional Answer

28. Statements (Reasons)

1. X is the midpoint of $\overline{WY}$ and $\overline{VZ}$. (Given)
2. $\overline{WX} \cong \overline{YX}$, $\overline{VX} \cong \overline{ZX}$ (Def. of midpoint)
3. $WX = YX$, $VX = ZX$ (Def. of $\cong$)
4. $VX = VW + WX$, $ZX = ZY + YX$ (Seg. Add. Post.)
5. $VW + WX = ZY + YX$ (Subs.)
6. $VW = ZY$ (Subt. Prop.)

29. Statements (Reasons)

1. $AB = DC$ (Given)
2. $BC = BC$ (Refl. Prop.)
3. $AB + BC = DC + BC$ (Add. Prop.)
4. $AB + BC = AC$, $DC + BC = DB$ (Seg. Add. Post.)
5. $AC = DB$ (Subs.)

10-3 Distance and Midpoints

Find the distance between each pair of points.

12. $A(-3, 1)$, $B(7, 13)$ $\sqrt{244} \approx 15.6$

13. $P(2, -1)$, $Q(10, -7)$ **10**

Find the coordinates of the midpoint of a segment with the given endpoints.

14. $L(-3, 16)$, $M(17, 4)$ **(7, 10)**

15. $C(32, -1)$, $D(0, -12)$ **(16, −6.5)**

Find the coordinates of the missing endpoint if M is the midpoint of $\overline{XY}$.

16. $X(-11, -6)$, $M(15, 4)$ **(41, 14)**

17. $M(-4, 8)$, $Y(19, 0)$ **(−27, 16)**

18. HIKING Moza and Maha are hiking in a state park and decide to take separate trails. The map of the park is set up on a coordinate grid. Moza's location is at the point (7, 13) and Maha is at (3, 5).

a. Find the distance between them. **≈8.9 units**

b. Find the coordinates of the point midway between their locations. **(5, 9)**

Example 3

Find the distance between $X(5, 7)$ and $Y(-7, 2)$.

Let $(x_1, y_1) = (5, 7)$ and $(x_2, y_2) = (-7, 2)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-7 - 5)^2 + (2 - 7)^2}$$
$$= \sqrt{(-12)^2 + (-5)^2}$$
$$= \sqrt{169} \text{ or } 13$$

The distance from X to Y is 13 units.

Example 4

Find the coordinates of the midpoint between $P(-4, 13)$ and $Q(6, 5)$.

Let $(x_1, y_1) = (-4, 13)$ and $(x_2, y_2) = (6, 5)$.

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = M\left(\frac{-4 + 6}{2}, \frac{13 + 5}{2}\right)$$
$$= M(1, 9)$$

The coordinates of the midpoint are (1, 9).

10-4 Proving Segment Relationships

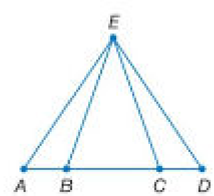
Write a two-column proof. **28, 29. See margin.**

28. Given: X is the midpoint of $\overline{WY}$ and $\overline{VZ}$.

Prove: $VW = ZY$

29. Given: $AB = DC$

Prove: $AC = DB$



30. GEOGRAPHY Eissa is planning to drive from Kansas City to Minneapolis along Interstate 35. The map he is using gives the distance from Kansas City to Des Moines as 194 kilometers and from Des Moines to Minneapolis as 243 kilometers. What allows him to conclude that the distance he will be driving is 437 kilometers from Kansas City to Minneapolis? Assume that Interstate 35 forms a straight line. **Seg. Add. Post.**

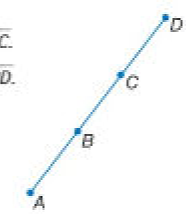
Example 9

Write a two-column proof.

Given: B is the midpoint of $\overline{AC}$.

C is the midpoint of $\overline{BD}$.

Prove: $\overline{AB} \cong \overline{CD}$

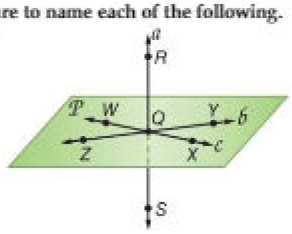


Proof:

Statements	Reasons
1. B is the midpoint of $\overline{AC}$.	1. Given
2. $\overline{AB} \cong \overline{BC}$	2. Definition of midpoint
3. C is the midpoint of $\overline{BD}$.	3. Given
4. $\overline{BC} \cong \overline{CD}$	4. Definition of midpoint
5. $\overline{AB} \cong \overline{CD}$	5. Transitive Property of Equality

CHAPTER 10 Practice Test

Use the figure to name each of the following.



1. the line that contains points Q and Z **line b**
2. two points that are coplanar with points W, X, and Y **points Q and Z**
3. the intersection of lines a and b **point Q**

Find the value of the variable if P is between J and K.

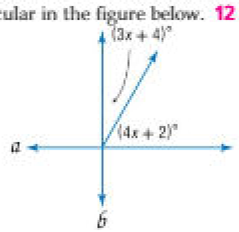
4. $JP = 2x$, $PK = 7x$, $JK = 27$ **3**
5. $JP = 3y + 1$, $PK = 12y - 4$, $JK = 75$ **5.2**
6. $JP = 8z - 17$, $PK = 5z + 37$, $JK = 17z - 4$ **6**

Find the coordinates of the midpoint of a segment with the given endpoints.

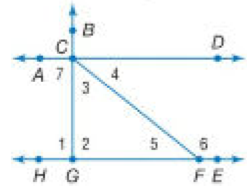
7. (16, 5) and (28, −13) **(22, −4)**
8. (−11, 34) and (47, 0) **(18, 17)**
9. (−4, −14) and (−22, 9) **(−13, −2.5)**

Find the distance between each pair of points.

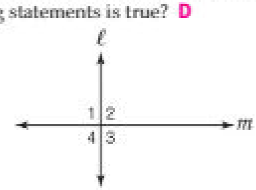
10. (43, −15) and (29, −3) **$\sqrt{340}$ or 18.4 units**
11. (21, 5) and (28, −1) **$\sqrt{85}$ or 9.2 units**
12. (0, −5) and (18, −10) **$\sqrt{349}$ or 18.7 units**
13. **ALGEBRA** The measure of $\angle X$ is 18 more than three times the measure of its complement. Find the measure of $\angle X$. **72**
14. Find the value of x that will make lines a and b perpendicular in the figure below. **12**



For Exercises 15–18, use the figure below.



15. Name the vertex of $\angle 3$. **point C**
16. Name the sides of $\angle 1$. **$\overrightarrow{GH}$ and $\overrightarrow{GB}$**
17. Write another name for $\angle 6$. **$\angle EFC$ or $\angle CFE$**
18. Name a pair of angles that share exactly one point. **$\angle 7$ and $\angle 4$**
19. **MULTIPLE CHOICE** If $m\angle 1 = m\angle 2$, which of the following statements is true? **D**



A $\angle 2 \cong \angle 4$

B $\angle 2$ is a right angle.

C $\ell \perp m$

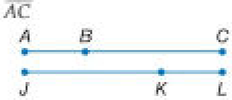
D All of the above

Find the perimeter of each polygon.

20. triangle XYZ with vertices $X(3, 7)$, $Y(-1, -5)$, and $Z(6, -4)$ **31.1 units**
21. rectangle $PQRS$ with vertices $P(0, 0)$, $Q(0, 7)$, $R(12, 7)$, and $S(12, 0)$ **38 units**
22. **SAFETY** A severe weather siren in a local city can be heard within a radius of 1.3 kilometers. If the mayor of the city wants a new siren that will cover double the area of the old siren, what should the radius of the new siren be? Round to the nearest tenth of a kilometer. **1.8 km**
23. **PROOF** Write a paragraph proof. **See margin.**

Given: $\overline{JK} \cong \overline{CB}$, $\overline{KL} \cong \overline{AB}$

Prove: $\overline{JL} \cong \overline{AC}$



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1 Focus

Objective Understand and use key strategies to solve problems.

2 Teach

Scaffolding Questions

Ask:

- How can you organize the information given in a problem? You can underline the important given information, cross out any extraneous information, or even make a table or chart.
- What are some different ways that information can be given in a problem? Information can be given directly in the problem, in a chart or graph, or in a diagram.
- What are some key words to look for to determine what a question is asking? Look for any arithmetic terms like sum, difference, product, or quotient, and for words like least, greatest, all, or none. Be sure to answer the question that is being asked as many problems require more than one step and the solutions to those intermediate steps are frequently answer choices.
- Why is it important to check your answer? It is important that your answer makes sense and is reasonable; very minor arithmetic errors can result in solutions that are completely unreasonable.

CHAPTER 10 Preparing for Standardized Tests

Solving Math Problems

Strategies for Solving Math Problems

The first step to solving any math problem is to read the problem. When reading a math problem to get the information you need to solve, it is helpful to use special reading strategies.

Step 1

Read the problem to determine what information is given.

- **Analyze:** Determine what exactly the problem is asking you to solve.
- **Underline:** If you are able to write in your test book, underline any important information.

Step 2

Reread the problem to determine what information is needed to solve the problem.

- **Think:** How does the information fit together?
- **Key Words:** Are there any key words, variables or mathematical terms in the problem?
- **Diagrams:** Do you need to use a diagram, list or table?
- **Formulas:** Do you need a formula or an equation to solve the problem?

Step 3

Devise a plan and solve the problem. Use the information you found in Steps 1 and 2.

- **Question:** What problem are you solving?
- **Estimate:** Estimate an answer.
- **Eliminate:** Eliminate all answers that do not make sense and/or vary greatly from your estimate.

Step 4

Check your answer.

- **Reread:** Quickly reread the problem to make sure you solved the whole problem.
- **Reasonableness:** Is your answer reasonable?
- **Units:** Make sure your answer has the correct units of measurement.

Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

Muna is using a coordinate grid to make a map of her backyard. She plots the swing set at point $S(2, 5)$ and the big oak tree at point $O(-3, -6)$. If each unit on the grid represents 5 meters, what is the distance between the swing set and the oak tree? Round your answer to the nearest whole meter.

A 12 m B 25 m C 60 m D 74 m

Determine what exactly the problem is asking you to solve. Underline any important information.

Muna is using a coordinate grid to make a map of her backyard. She plots the swing set at point $S(2, 5)$ and the big oak tree at point $O(-3, -6)$. If each unit on the grid represents 5 meters, what is the distance between the swing set and the oak tree? Round your answer to the nearest whole meter.

The problem is asking for the distance between the swing set and the oak tree. The key word is distance, so you know you will need to use the Distance Formula.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$= \sqrt{(-3 - 2)^2 + (-6 - 5)^2}$ $(x_1, y_1) = (2, 5)$ and $(x_2, y_2) = (-3, -6)$

$= \sqrt{(-5)^2 + (-11)^2}$ Subtract.

$= \sqrt{25 + 121}$ or $\sqrt{146}$ Simplify.

The distance between swing set and the oak tree is $\sqrt{146}$ units. Use a calculator to find that $\sqrt{146}$ units is approximately 12.08 units.

Since each unit on the grid represents 5 meter, the distance is $(12.08) \cdot (5)$ or 60.4 m. Therefore, the correct answer is C.

Check your answer to make sure it is reasonable, and that you have used the correct units.

Exercises

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. A regular pentagon has a perimeter of 24 centimeters. What is the measure of each side? D

   A 3 centimeters C 4 centimeters

   B 3.8 centimeters D 4.8 centimeters

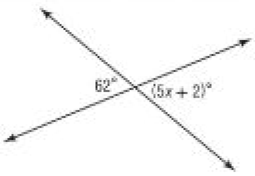
2. What is the value of x in the figure at the right? G

   F 10

   G 12

   H 14

   J 15



Additional Example

Omar uses a coordinate grid to make a map of his classroom. He plots the teacher's desk at point $A(1, 4)$ and the globe at point $B(-2, -3)$. If each unit of the grid represents 3 meter, what is the distance between the desk and the globe? Round your answer to the nearest whole meter. D

A 8

B 12

C 21

D 23

3 Assess

Use Exercises 1 and 2 to assess students' understanding.

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CHAPTER 10 Standardized Test Practice

Cumulative, Chapters 1 through 10

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Omar Rentals rented 12 more bicycles than scooters last weekend for a total revenue of AED 2,125. How many scooters were rented? **B**

Item	Rental Fee
Bicycle	AED 20
Scooter	AED 45

A 26 C 37

B 29 D 41

2. Find the distance between $M(-3, 1)$ and $N(2, 8)$ on a coordinate plane. **J**

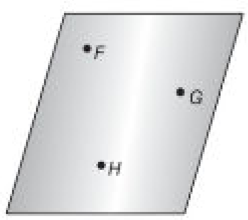
F 6.1 units

G 6.9 units

H 7.3 units

J 8.6 units

3. Which of the following terms best describes points *F*, *G*, and *H*? **C**



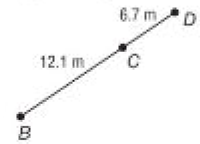
A collinear C coplanar

B congruent D skew

Test-Taking Tip

Question 3 Understanding the terms of geometry can help you solve problems. The term *congruent* refers to geometric figures, and *skew* refers to lines, therefore both answers can be eliminated.

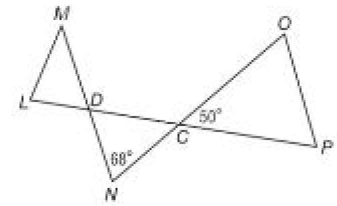
4. What is the length of segment *BD*? **H**



F 17.4 m H 18.8 m

G 18.3 m J 19.1 m

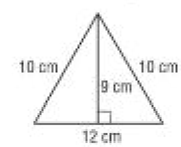
5. In the figure below, what is the measure of angle *CDN*? **B**



A 58° C 68°

B 62° D 70°

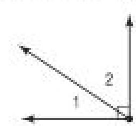
6. Find the perimeter of the figure below. **H**



F 20 cm H 32 cm

G 29 cm J 41 cm

7. What is the relationship of $\angle 1$ and $\angle 2$? **A**



A complementary angles

B linear pair

C supplementary angles

D vertical angles

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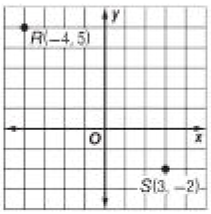
Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

8. Find the distance between points *R* and *S* on the coordinate grid below. Round to the nearest tenth. **9.9 units**



9. SHORT RESPONSE Find the value of *x* and *AB* if *B* is between *A* and *C*, $AB = 2x$, $AC = 6x - 5$, and $BC = 7$. **$x = 3$; $AB = 6$**

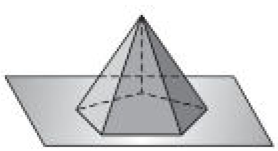
10. Suppose two lines intersect in a plane.

a. What do you know about the two pairs of vertical angles formed?

b. What do you know about the pairs of adjacent angles formed? **They are supplementary.**

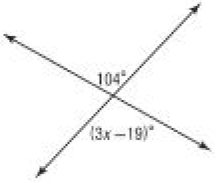
10a. They have the same measure.

11. GRIDDED RESPONSE How many planes are shown in the figure below? **6**



12. Omar received a AED 50 gift certificate for his graduation. He wants to buy a DVD and a poster from a media store. (Assume that sales tax is included in the prices.) Write and solve a linear inequality to show how much he would have left to spend after making these purchases.
$x + 14.95 + 10.99 \leq 50$; $x \leq 24.06$

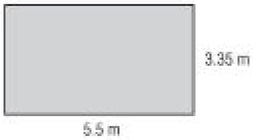
13. GRIDDED RESPONSE What is the value of *x* in the figure? **41**



Extended Response

Record your answers on a sheet of paper. Show your work.

14. Maysoun's room has the dimensions shown in the figure.



a. Find the perimeter of her room. **58 m**

b. Find the area of her room. **198 m^2**

c. If the length and width doubled, what effect would it have on the perimeter? **The perimeter would double.**

d. What effect would it have on the area?
a. 17.7 m
b. 18.425 m^2

Homework Option

Get Ready for Chapter 11 Assign students the exercises on p. 641 as homework to assess whether they possess the prerequisite skills needed for the next chapter.

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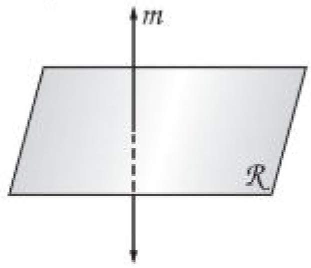
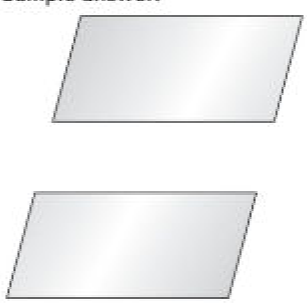
Lesson 10-1

32. Sample answer:

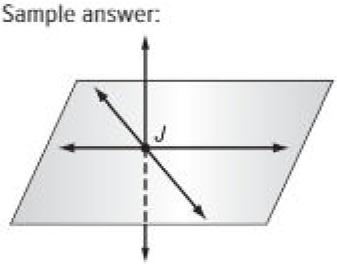


33. Sample answer:

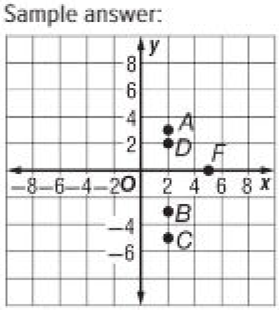


34. Sample answer:

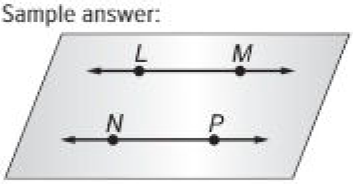
35. Sample answer:



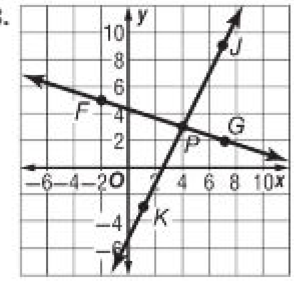
36. Sample answer:



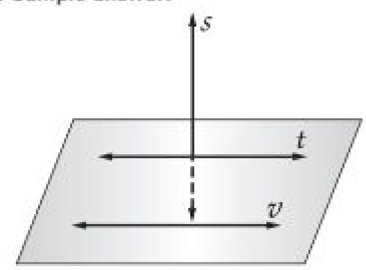
37. Sample answer:



38.



39. Sample answer:



Extend Lesson 10-2

16. 22.4 m is precise to within nearest 0.05 m and 5.82 m is precise to within 0.005, so 5.82 is the more precise measurement. The relative error of 22.4 m is $\frac{0.05\text{ m}}{22.4}$ or about 0.2%, while the relative error of 5.82 is $\frac{0.005}{5.82}$ or about 0.09%, so 5.82 is also more accurate.

17. Each measure is precise to within 0.5, so one measure is not more precise than the other. The relative error of 13 is $\frac{0.5}{25}$ or 2%, while the relative error of 83 is $\frac{0.5}{8}$ or about 6.3%, so 8 is more accurate.

18. 9.2 cm is precise to within 0.05 cm or 0.5 mm and 42 mm is precise to within 0.5 mm, so one measure is not more precise than the other. The relative error of 9.2 cm m is $\frac{0.05\text{ cm}}{9.2\text{ cm}}$ or about 0.5%, while the relative error of 42 mm is $\frac{0.5\text{ mm}}{42\text{ mm}}$ or about 1.2%, so 9.2 cm is more accurate.

19. $18\frac{1}{4}$ cm is precise to within $\frac{1}{8}$ cm and 125 m is precise to within 0.5, so $18\frac{1}{4}$ is the more precise measurement. The relative error of $18\frac{1}{4}$ is $\frac{0.125}{18.25}$ or about 0.7%, while the relative error of 125 is $\frac{0.5}{125}$ or 0.4%, so 125 is more accurate.

20. Suppose a person is 5.5 m tall. If this height is measured to the nearest meter, the relative error would be $\frac{0.5\text{ m}}{5.5\text{ m}}$ or about 9%. If measured to the nearest centimeter, the relative error would be $\frac{0.5\text{ cm}}{5.5\text{ m}} = \frac{0.5\text{ cm}}{65\text{ cm}}$ or about 0.8%. If measured to the nearest $\frac{1}{16}$ cm, the relative error would be $\frac{0.03125\text{ cm}}{65\text{ cm}}$ or about 0.05%. While measuring to the nearest $\frac{1}{16}$ cm is a more accurate measure, this level of accuracy is not necessary, about a 1% level of accuracy is sufficient, so measuring to the nearest centimeter is sufficient.

21. 1 m; Suppose a mountain is 4000 m tall. If this height is measured to the nearest meter, the relative error would be $\frac{0.5 \text{ m}}{4000 \text{ m}}$ or about 0.01%. If measured to the nearest centimeter, the relative error would be $\frac{0.5 \text{ cm}}{4000 \text{ m}} = \frac{0.5 \text{ cm}}{48{,}000 \text{ cm}}$ or about 0.001%. If measured to the nearest $\frac{1}{16}$ cm, the relative error would be $\frac{0.03125 \text{ cm}}{48{,}000 \text{ cm}}$ or about 0.0007%. While measuring to the nearest $\frac{1}{16}$ cm is a more accurate measure, this level of accuracy is not necessary; about a 1% level of accuracy is sufficient, so measuring to the nearest meter is sufficient.

22. 13.5 cm; each measurement is accurate within $\frac{1}{8}$ of an centimeter, so the least perimeter is $2\left(2\frac{1}{8}\right)$ cm + $2\left(4\frac{5}{8}\right)$ cm; 14.5 cm; Each measurement is accurate within $\frac{1}{8}$ of an centimeter , so the greatest perimeter is $2\left(2\frac{3}{8}\right)$ cm + $2\left(4\frac{7}{8}\right)$ cm.

Lesson 10-4

5. Given: $\overline{AB} \cong \overline{CD}$

Prove: $\overline{CD} \cong \overline{AB}$

Proof:

Statements (Reasons)

1. $\overline{AB} \cong \overline{CD}$ (Given)
2. $AB = CD$ (Def. of ≅ segs.)
3. $CD = AB$ (Symm. Prop.)
4. $\overline{CD} \cong \overline{AB}$ (Def. of ≅ segs.)

6. Given: $\overline{AB}$

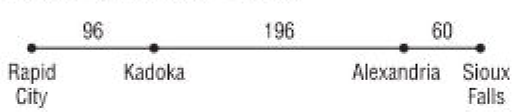
Prove: $\overline{AB} \cong \overline{AB}$

Proof:

Statements (Reasons)

1. $\overline{AB}$ (Given)
2. $AB = AB$ (Refl. Prop.)
3. $\overline{AB} \cong \overline{AB}$ (Def. of ≅ segs.)

7a.



7b. We are given that all of the points are collinear. Since Kadoka is 96 kilometers from Rapid City and Sioux Falls is 352 kilometers from Rapid City, Kadoka is between Rapid City and Sioux Falls. Since Alexandria is 292 kilometers from Rapid City, and Kadoka is 96 kilometers from Rapid City, Kadoka is between Alexandria and Rapid City. Since Sioux Falls is 352 kilometers from Rapid City and Alexandria is 292 kilometers from Rapid City, Alexandria is between Kadoka and Sioux Falls. Therefore, from west to east, the cities are Rapid City, Kadoka, Alexandria, and Sioux Falls.

8. Given: $\overline{XW} \cong \overline{YZ}$ and $\overline{YZ} \cong \overline{ZX}$

Prove: $\overline{XW} \cong \overline{ZX}$

Proof:

Statements (Reasons)

1. $\overline{XW} \cong \overline{YZ}$ and $\overline{YZ} \cong \overline{ZX}$ (Given)
2. $XW = YZ$ and $YZ = ZX$ (Definition of Congruence)
3. $XW = ZX$ (Substitution)
4. $\overline{XW} \cong \overline{ZX}$ (Definition of Congruence)

9. Given: $\overline{AC} \cong \overline{AD}$ and $\overline{ED} \cong \overline{BC}$

Prove: $\overline{AE} \cong \overline{AB}$

Proof:

Statements (Reasons)

1. $\overline{AC} \cong \overline{AD}$ and $\overline{ED} \cong \overline{BC}$ (Given)
2. $AC = AD$, $ED = BC$ (Definition of Congruence)
3. $AE + ED = AD$, $AB + BC = AC$ (Segment Addition Postulate)
4. $AE + ED = AB + BC$ (Substitution)
5. $AE = AB$ (Subtraction Property of Equality)
6. $\overline{AE} = \overline{AB}$ (Definition of Congruence)

10. Given: R is the midpoint of $\overline{QS}$ and $\overline{PQ} \cong \overline{ST}$.

Prove: $\overline{PR} \cong \overline{RT}$

Proof:

Statements (Reasons)

1. R is the midpoint of $\overline{QS}$, $\overline{PQ} \cong \overline{ST}$. (Given)
2. $QR = RS$ (Definition of Midpoint)
3. $PQ = ST$ (Definition of Congruence)
4. $PQ + QR = RS + ST$ (Addition Property of Equality)
5. $PR = PQ + QR$, $RT = RS + ST$ (Segment Addition Postulate)
6. $PR = RT$ (Substitution)
7. $\overline{PR} \cong \overline{RT}$ (Definition of Congruence)

11. Given: Q is the midpoint of $\overline{PR}$, S is the midpoint of $\overline{RT}$, and $\overline{QR} \cong \overline{RS}$.

Prove: $PT = 4QR$

Proof:

Statements (Reasons)

1. Q is the midpoint of $\overline{PR}$, S is the midpoint of $\overline{RT}$, and $\overline{QR} \cong \overline{RS}$. (Given)
2. $PQ = QR$ and $RS = ST$ (Definition of Midpoint)
3. $QR = RS$ (Definition of Congruence)
4. $PT = PQ + QR + RS + ST$ (Segment Addition Postulate)
5. $QR = ST$ (Transitive Property)
6. $PT = QR + QR + QR + QR$ (Substitution)
7. $PT = 4QR$ (Simplify)

12. Given: $\overline{AB} \cong \overline{EF}$, $\overline{ED} \cong \overline{HK}$, $AB + BE + ED = EF + EH + HK$

Prove: $\overline{BE} \cong \overline{EH}$

Proof:

Statements (Reasons)

1. $\overline{AB} \cong \overline{EF}$, $\overline{ED} \cong \overline{HK}$, $AB + BE + ED = EF + EH + HK$ (Given)
2. $AB = EF$, $ED = HK$ (Definition of Congruence)
3. $AB + BE + ED = AB + EH + HK$ (Substitution)
4. $AB - AB + BE + ED = AB - AB + EH + HK$ (Subtraction Property of Equality)
5. $BE + ED = EH + HK$ (Simplification)
6. $BE + ED = EH + ED$ (Substitution)
7. $BE + ED - ED = EH + ED - ED$ (Subtraction Property of Equality)

Chapter 10 Answer Appendix

8. $BE = EH$ (Simplification)
9. $\overline{BE} \cong \overline{EH}$ (Definition of Congruence)

13. 

Sample answer: I placed an initial point A on a line ℓ and constructed a point B on the line so that AB is equal to PQ. Using point B as an initial point, I marked point C on the line so that BC is also equal to PQ. The length of the whole segment AC is $AB + BC$ according to the Additional Postulate and $AB = BC = PQ$. Using substitution $AC = PQ + PQ$, or $AC = 2PQ$, so $\overline{AC}$ is twice as long as $\overline{PQ}$.

14a. 

14b. $8PC = PQ$

14c. 

I can measure $\overline{PC}$ and mark off segments of that length along $\overline{PQ}$, and count how many segments were formed.

14d. 

$8PC = PQ$

14e. Given: A is the midpoint of $\overline{PQ}$, B is the midpoint of $\overline{PA}$, and C is the midpoint of $\overline{PB}$.

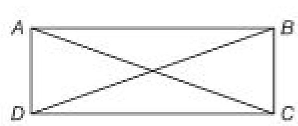
Prove: $8PC = PQ$

__Statements (Reasons)__

1. A is the midpoint of $\overline{PQ}$, B is the midpoint of $\overline{PA}$, and C is the midpoint of $\overline{PB}$. (Given)
2. $PA = AQ$, $PB = BA$, $PC = CB$ (Def. of Midpoint)
3. $PC + CB = PB$ (Seg. Add. Post.)
4. $PC + PC = PB$ (Subs.)
5. $2PC = PB$ (Subs.)
6. $PB + BA = PA$ (Seg. Add. Post.)
7. $PB + PB = PA$ (Subs.)
8. $2PB = PA$ (Add. Prop.)
9. $2(2PC) = PA$ (Subs.)
10. $4PC = PA$ (Subs.)
11. $PA + AQ = PQ$ (Seg. Add. Post.)
12. $PA + PA = PQ$ (Subs.)
13. $2PA = PQ$ (Subs.)
14. $2(4PC) = PQ$ (Subs.)
15. $8PC = PQ$ (Subs.)

16. Given: $ABCD$ is a rectangle.

Prove: $\overline{AC} \cong \overline{BD}$



__Statements (Reasons)__

1. $ABCD$ is a rectangle (Given)
2. $AB = CD$, $AD = BC$ (Definition of a Rectangle)
3. $(AC)^2 = (AD)^2 + (DC)^2$, $(BD)^2 = (BC)^2 + (DC)^2$ (Pythagorean Theorem)
4. $(AC)^2 = (BC)^2 + (DC)^2$ (Substitution)
5. $(AC)^2 = (BD)^2$ (Substitution or Transitive)
6. $AC = \pm\sqrt{(BD)^2}$ (Square Root Property)
7. $AC = \sqrt{(BD)^2}$ (By definition, length must be positive.)
8. $AC = BD$ (Definition of Square Root)
9. $\overline{AC} \cong \overline{BD}$ (Definition of Congruence)

18. True

19. 

20. Paragraph proofs and two-column proofs both use deductive reasoning presented in a logical order along with the postulates, theorems, and definitions used to support the steps of the proofs. Paragraph proofs are written as a paragraph with the reasons for each step incorporated into the sentences. Two-column proofs are numbered and itemized. Each step of the proof is provided on a separate line with the support for that step in the column beside the step.

Student Handbook

This **Student Handbook** can help you answer these questions.

What if I Forget a Vocabulary Word?

What if I Forget a Formula?

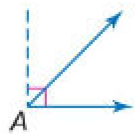
Glossary/القاموس

English

A

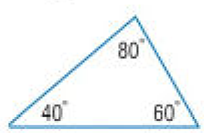
absolute value The distance a number is from zero on the number line.

acute angle An angle with a degree measure less than 90.



$0 < m\angle A < 90$

acute triangle A triangle in which all of the angles are acute angles.



three acute angles

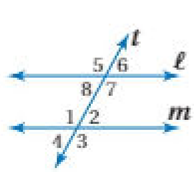
additive identity For any number a, a + 0 = 0 + a = a.

additive inverse Two integers, x and -x, are called additive inverses. The sum of any number and its additive inverse is zero.

adjacent angles Two angles that lie in the same plane, have a common vertex and a common side, but no common interior points.

algebraic expression An expression consisting of one or more numbers and variables along with one or more arithmetic operations.

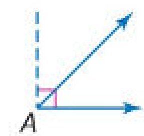
alternate exterior angles In the figure, transversal t intersects lines ℓ and m. $\angle 5$ and $\angle 3$, and $\angle 6$ and $\angle 4$ are alternate exterior angles.



العربية

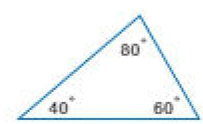
قيمة مطلقة المسافة الفاصلة بين العدد والصفر على خط الأعداد.

زاوية حادة زاوية قياس درجتها أقل من 90.



$0 < m\angle A < 90$

مثلث حاد الزوايا مثلث كل زواياه حادة.



ثلاث زوايا حادة

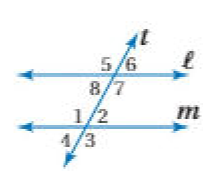
محايد جمعي لأي عدد a، $a + 0 = 0 + a = a$.

نظير جمعي يُطلق على العددين الصحيحين x و$-x$ النظيرين الجمعيين. وحاصل جمع أي عدد ونظيره الجمعي يساوي صفرًا.

الزوايا المتجاورة زاويتان تقعان في نفس المستوى، لهما رأس مشترك وضلع مشترك، لكن لا توجد لديهما نقاط داخلية مشتركة

تعبير جبري أي تعبير يتكون من عدد واحد أو أكثر ومتغير واحد أو أكثر، فضلًا عن عملية حسابية واحدة أو أكثر.

الزوايا الخارجية المتبادلة في الشكل، يتقاطع القاطع t مع المستقيمين ℓ و m. $\angle 5$ و$\angle 3$ و$\angle 6$ و$\angle 4$ زوايا خارجية متبادلة.



alternate interior angles In the figure at the bottom of page R115, transversal t intersects lines ℓ and m. $\angle 1$ and $\angle 7$, and $\angle 2$ and $\angle 8$ are alternate interior angles.

الزوايا الداخلية المتبادلة في الشكل في الجزء السفلي من صفحة R115، يتقاطع القاطع t مع المستقيمين ℓ و m. $\angle 1$ و$\angle 7$ و$\angle 2$ و$\angle 8$ زوايا داخلية متبادلة.

altitude **1.** In a triangle, a segment from a vertex of the triangle to the line containing the opposite side and perpendicular to that side. **2.** In a prism or cylinder, a segment perpendicular to the bases with an endpoint in each plane. **3.** In a pyramid or cone, the segment that has the vertex as one endpoint and is perpendicular to the base.

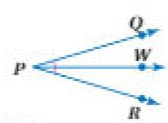
ارتفاع 1. في المثلث، قطعة مستقيمة تمتد من أحد رؤوس المثلث إلى الضلع المقابل وتكون متعامدة على هذا الضلع. 2. في المنشور أو الأسطوانة، الارتفاع هو قطعة مستقيمة متعامدة على القاعدتين ذات نقطة نهاية في كل مستوى. 3. في الهرم أو المخروط، الارتفاع هو الجزء الذي يتضمن الرأس بوصفها نقطة نهاية واحدة ويكون متعامدًا على القاعدة.

angle The intersection of two noncollinear rays at a common endpoint. The rays are called *sides* and the common endpoint is called the *vertex*.

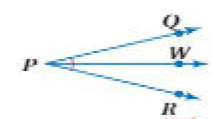
زاوية تقاطع شعاعين ليسا على خط مستقيم واحد عند نقطة نهاية مشتركة. تسمى الأشعة بالأضلاع وتسمى نقطة النهاية المشتركة بالرأس.

angle bisector A ray that divides an angle into two congruent angles.

مُنصِّف الزاوية شعاع يُقسِّم الزاوية إلى زاويتين متطابقتين.



$\overrightarrow{PW}$ is the bisector of $\angle P$.



$\overrightarrow{PW}$ هو منصف الزاوية $\angle P$

angle of rotation The angle through which a preimage is rotated to form the image.

زاوية الدوران الزاوية التي تدور الصورة الأصلية من خلالها لتكوّن الصورة.

area **1.** The measure of the surface enclosed by a geometric figure. **2.** The number of square units needed to cover a surface.

المساحة 1. قياس لمنطقة محصورة في نطاق معين على سطح شكل هندسي. 2. عدد الوحدات المربعة اللازمة لتغطية سطحٍ ما.

arithmetic sequence A numerical pattern that increases or decreases at a constant rate or value. The difference between successive terms of the sequence is constant.

متتالية حسابية نمط عددي يزيد أو يقل بمعدل أو قيمة ثابتة. ويكون الفرق بين الحدود المتتابعة للمتتالية ثابتًا.

asymptote A line that a graph approaches.

خط مقارب خط يقترب منه الرسم البياني.

augmented matrix A coefficient matrix with an extra column containing the constant terms.

مصفوفة موسعة مصفوفة معاملات بها عمود إضافي يتضمّن الحدود الثابتة.

auxiliary line An extra line or segment drawn in a figure to help complete a proof.

خط مساعد خط إضافي أو قطعة مستقيمة يتم رسمها في شكل للمساعدة على استكمال البرها

B

bar graph A graphic form using bars to make comparisons of statistics.

رسم بياني خطي رسم بياني يستخدم الأعمدة لإجراء المقارنات الإحصائية.

base In an expression of the form x^n, the base is x.

قاعدة في أي تعبير صيغته x^n، تكون القاعدة هي x.

base angle of an isosceles triangle See isosceles triangle and isosceles trapezoid.

زاوية القاعدة في مثلث متساوي الساقين راجع مثلث متساوي الساقين وشبه المنحرف متساوي الساقين.

base of parallelogram Any side of a parallelogram.

قاعدة متوازي الأضلاع أي ضلع في متوازي الأضلاع.

best-fit line The line that most closely approximates the data in a scatter plot.

خط المواءمة الأفضل الخط الذي يعد الأقرب تمثيلاً للبيانات في التمثيل البياني بالنقاط المبعثرة.

between For any two points A and B on a line, there is another point C between A and B if and only if A, B, and C are collinear and $AC + CB = AB$.

بينية بالنسبة إلى أي نقطتين، A و B في أي خط مستقيم، تكون هناك النقطة الأخرى C بين A و B إذا وفقط إذا كانت النقاط A و B و C تقع على خط مستقيم واحد، فإن $AC + CB = AB$

betweenness of points See between.

بينية النقاط راجع بينية.

bivariate data Data with two variables.

البيانات ذات المتغيرين بيانات تحتوي على متغيرين.

boundary A line or curve that separates the coordinate plane into regions.

حد الخط أو المنحنى الذي يقسم المستوى الإحداثي إلى مناطق.

box-and-whisker plot A diagram that divides a set of data into four parts using the median and quartiles.
A box is drawn around the quartile values and whiskers extend from each quartile to the extreme data points.

مخطط الصندوق ذو العارضين مخطط بياني يقسم مجموعة من البيانات إلى أربعة أجزاء باستخدام الوسيط والربيعيات.
وَيُرسم الصندوق حول قيم الربيعيات ويمتد العارضان من كل رُبَيع إلى نقاط البيانات القصوى.

C

center The given point from which all points on the circle are the same distance.

المركز النقطة المعلومة التي تبعد عنها كل النقاط على الدائرة نفس المسافة.

center of circle The central point where radii form a locus of points called a circle.

مركز الدائرة النقطة المركزية حيث تكوّن أنصاف الأقطار موضعًا هندسيًا للنقاط يُسمى دائرة.

center of dilation The center point from which dilations are performed.

مركز تغيير الأبعاد نقطة المركز الذي يتم منه تغيير الأبعاد بمقياس.

center of rotation A fixed point around which shapes move in a circular motion to a new position.

مركز الدوران نقطة ثابتة تتحرك حولها أشكال حركة دائرية إلى مؤقع جديد.

center of symmetry See point of symmetry.

مركز التناظر راجع نقطة التناظر.

chord 1. For a given circle, a segment with endpoints that are on the circle. 2. For a given sphere, a segment with endpoints that are on the sphere.

وتر 1. بالنسبة إلى دائرة معينة، قطعة مستقيمة توجد نقطتا نهايتيها على الدائرة. 2. بالنسبة إلى كرة معينة، قطعة مستقيمة توجد نقطتا نهايتيها على الكرة.

circle The locus of all points in a plane equidistant from a given point called the center of the circle.

دائرة موضع كل النقاط في مستوى متساو من نقطة محددة تُسمى مركز الدائرة.

P is the center of the circle.

P هي مركز الدائرة

circle The set of all points in a plane that are the same distance from a given point called the center.

دائرة مجموعة النقاط في المستوى والتي تبعد نفس المسافة عن نقطة معلومة تُسمى المركز.

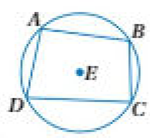
circle graph A type of statistical graph used to compare parts of a whole.

رسم بياني دائري نوع من الرسوم البيانية الإحصائية المستخدمة في مقارنة أجزاءٍ من كل.

circumference The distance around a circle.

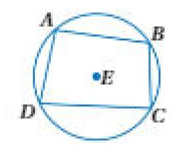
محيط الدائرة المسافة حول الدائرة.

circumscribed A circle is circumscribed about a polygon if the circle contains all the vertices of the polygon.



⊙*E* is circumscribed about quadrilateral *ABCD*.

مُحاطة تكون الدائرة مُحاطة بمضلع إذا كانت تحوي كل رؤوس المضلع.



⊙*E* محيطة برباعي الأضلاع *ABCD*

closed A set is closed under an operation if for any numbers in the set, the result of the operation is also in the set.

مغلقة تكون المجموعة مغلقة في ظل إحدى العمليات إذا تم إجراء عملية بين أي من أعدادها ونتج عنصر ينتمي لنفس المجموعة.

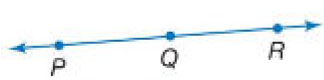
closed half-plane The solution of a linear inequality that includes the boundary line.

نصف مستو مغلق حل المتباينة الخطية التي تتضمن مستقيمًا حدِّيًا.

coefficient The numerical factor of a term.

معامل العامل العددي للحد.

collinear Points that lie on the same line.



P, *Q*, and *R* are collinear.

على خط مستقيم واحد أن تقع النقاط على نفس الخط.

P، *Q* و*R* على الخط نفسه.

common difference The difference between the terms in an arithmetic sequence.

فرق مشترك الفرق بين الحدود في المتتالية الحسابية.

common ratio The ratio of successive terms of a geometric sequence.

نسبة مشتركة نسبة الحدود المتتالية للتسلسل الهندسي.

common tangent A line or segment that is tangent to two circles in the same plane.

مماس مشترك خط مستقيم أو قطعة مستقيمة تلامس دائرتين في نفس المستوى.

complementary angles Two angles with measures that have a sum of 90.

زاويتان متكاملتان زاويتان مجموع قياسهما يساوي 90.

complements One of two parts of a probability making a whole.

متممات جزء أو جزءان من احتمال متمم للكل.

composition of transformations The resulting transformation when a transformation is applied to a figure and then another transformation is applied to its image.

تركيب التحويلات التحويل الناتج عند تطبيق تحويل على شكل ما ثم تطبيق تحويل آخر على صورة هذا الشكل.

compound inequality Two or more inequalities that are connected by the words *and* or *or*.

متباينة مركبة متباينتان أو أكثر متصلتان بالكلمتين و أو أو.

compound interest A special application of exponential growth.

فائدة مركبة تطبيق خاص للنمو الأسّي.

concave polygon A polygon for which there is a line containing a side of the polygon that also contains a point in the interior of the polygon.

مضلع مقعر مضلع به خط مستقيم يحتوي على أحد أضلاع المضلع ونقطة في المناطق الداخلية بالمضلع.

concentric circles Coplanar circles with the same center.

دوائر متحدة المركز دوائر متحدة المستوى لها نفس المركز.

congruence transformations A mapping for which a geometric figure and its image are congruent.

تحويلات التطابق رسم يكون فيه الشكل الهندسي وصورته متطابقين.

congruent Having the same measure.

التطابق وجود نفس القياس.

congruent polygons Polygons in which all matching parts are congruent.

مضلعات متطابقة المضلعات التي تتطابق فيها كل الأجزاء المماثلة.

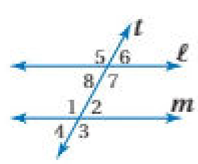
conjugates Binomials of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$.

مترافقات ثنائيات ذات حدين صيغتها $a\sqrt{b} + c\sqrt{d}$ و $a\sqrt{b} - c\sqrt{d}$.

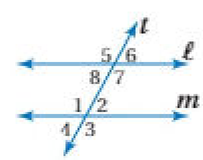
consecutive integers Integers in counting order.

أعداد صحيحة متتالية الأعداد الصحيحة حسب ترتيب العد.

consecutive interior angles In the figure, transversal t intersects lines ℓ and m. There are two pairs of consecutive interior angles: $\angle 8$ and $\angle 1$, and $\angle 7$ and $\angle 2$.



زوايا داخلية متتالية في الشكل، يتقاطع القاطع t مع المستقيمين ℓ و m ؛ هناك زوجان من الزوايا الداخلية المتتالية: $\angle 8$ و $\angle 1$ و $\angle 7$ و $\angle 2$.



consistent A system of equations that has at least one ordered pair that satisfies both equations.

متوافق نظام من المعادلات التي لها زوج مرتب واحد على الأقل يحقق كلتا المعادلتين.

constant A monomial that is a real number.

ثابت أحادي الحد يمثل عددًا حقيقيًا.

constant function A linear function of the form $y = b$.

دالة ثابتة دالة خطية بالصيغة $y = b$.

constant of variation The number k in equations of the form $y = kx$.

ثابت التغير العدد k في المعادلات بالصيغة $y = kx$.

construction A method of creating geometric figures without the benefit of measuring tools. Generally, only a pencil, straightedge, and compass are used.

تركيب إحدى طرق إنشاء أشكال هندسية دون الاستعانة بأدوات القياس. بشكل عام، يتم استخدام قلم رصاص ومسطرة عدلة وفرجار فقط.

continuous function A function that can be graphed with a line or a smooth curve.

دالة متصلة دالة يمكن تمثيلها بيانيًا بخط مستقيم أو بمنحنى منتظم.

convex polygon A polygon for which there is no line that contains both a side of the polygon and a point in the interior of the polygon.

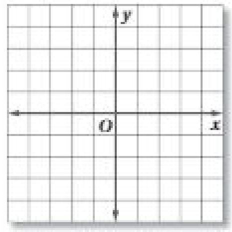
مضلع محدب مضلع لا يوجد له خط مستقيم يحوي أحد أضلاع المضلع ونقطة في المناطق الداخلية بالمضلع.

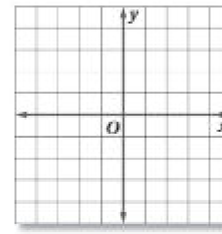
coordinate The number that corresponds to a point on a number line.

إحداثي العدد الذي يتوافق مع نقطة على خط الأعداد.

coordinate plane The plane containing the x- and y-axes.

مستوى إحداثي المستوى الذي يحوي المحورx والمحور y.





coordinate proofs Proofs that use figures in the coordinate plane and algebra to prove geometric concepts.

براهين إحداثية البراهين التي تستخدم الأشكال في المستوى الإحداثي والجبر لإثبات مفاهيم هندسية.

coordinate system The grid formed by the intersection of two number lines, the horizontal axis and the vertical axis.

نظام إحداثي الشبكة التي تنتج عن تقاطع خطين من الأعداد، المحور الأفقي والمحور الرأسي.

coplanar Points that lie in the same plane.

متحدة المستوى النقاط التي تقع في نفس المستوى.

corollary A statement that can be easily proved using a theorem is called a corollary of that theorem.

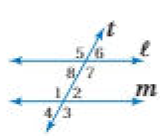
نتيجة عبارة يمكن إثباتها بسهولة باستخدام نظرية تسمى لازمة تلك النظرية.

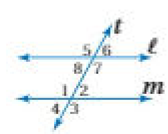
correlation coefficient A value that shows how close data points are to a line.

معامل الارتباط القيمة التي توضح مدى قرب نقاط البيانات من الخط المستقيم.

corresponding angles In the figure, transversal t intersects lines ℓ and m. There are four pairs of corresponding angles: $\angle 5$ and $\angle 1$, $\angle 8$ and $\angle 4$, $\angle 6$ and $\angle 2$, and $\angle 7$ and $\angle 3$.

زوايا متطابقة في الشكل، يتقاطع القاطع t مع المستقيمين ℓ و m . توجد أربعة أزواج من الزوايا المتطابقة: $\angle 5$ و $\angle 1$ و $\angle 8$ و $\angle 4$ و $\angle 6$ و $\angle 2$ و $\angle 7$ و $\angle 3$.





corresponding parts Matching parts of congruent polygons.

أجزاء متوافقة الأجزاء المتماثلة من المضلعات المتطابقة.

counterexample A specific case in which a statement is false.

مثال مضاد حالة خاصة تكون فيها العبارة خطأ.

cube root If $a^3 = b$, then a is the cube root of b.

جذر تكعيبي إذا كان $a^3 = b$، إذن a هو الجذر التكعيبي لـ b.

D

decreasing The graph of a function goes down on a portion of its domain when viewed from left to right.

متناقص التمثيل البياني لدالة تهبط على جزء من مجالها عند النظر إليها من اليسار إلى اليمين.

deductive reasoning The process of using facts, rules, definitions, or properties to reach a valid conclusion.

تبرير استنتاجي عملية استخدام الحقائق أو القواعد أو التعريفات أو الخواص للتوصل إلى استنتاج صحيح.

defining a variable Choosing a variable to represent one of the unspecified numbers in a problem and using it to write expressions for the other unspecified numbers in the problem.

تعيين متغير اختيار متغير لتمثيل أحد الأعداد غير المحددة في مسألة واستخدامه لكتابة التعبيرات للأعداد غير المحددة الأخرى في المسألة.

degree A unit of measure used in measuring angles and arcs. An arc of a circle with a measure of 1° is $\frac{1}{360}$ of the entire circle.

درجة وحدة القياس المستخدمة في قياس الزوايا والأقواس. قوس الدائرة الذي يبلغ قياسه °1 يكون $\frac{1}{360}$ من الدائرة بأكملها.

dependent A system of equations that has an infinite number of solutions.

غير مستقل نظام معادلات له عدد لا نهائي من الحلول.

dependent variable The variable in a relation with a value that depends on the value of the independent variable.

متغير تابع المتغير الذي تعتمد قيمته في علاقة على قيمة المتغير المستقل.

diameter 1. In a circle, a chord that passes through the center of the circle. 2. In a sphere, a segment that contains the center of the sphere, and has endpoints that are on the sphere. 3. The distance across a circle through its center.

قطر 1. في الدائرة، الوتر الذي يمر عبر مركز الدائرة. 2. في الكرة، القطعة المستقيمة التي تتضمن مركز الكرة ولها نقاط نهاية على الكرة. 3. المسافة المارة بالمركز داخل الدائرة.

dilation A transformation that enlarges or reduces the original figure proportionally. A dilation with center C and positive scale factor k, $k \neq 1$, is a function that maps a point P in a figure to its image such that

- if point P and C coincide, then the image and preimage are the same point, or
- if point P is not the center of dilation, then P′ lies on $\overrightarrow{CP}$ and $CP' = k(CP)$.

If $k < 0$, P′ is the point on the ray opposite $\overrightarrow{CP}$ such that $CP' = |k|(CP)$.

تغيير الأبعاد بمقياسo تحويل يكبر الشكل الأصلي أو يختزله بشكل متناسب. تغيير الأبعاد بمقياس عند المركز C ومعامل المقياس الإيجابي k، $k \neq 1$ ، هو الدالة التي ترسم النقطة P في شكل ما على صورتها بحيث

- إذا تطابقت النقطتان P وC ، تكون كل من الصورة والصورة الأصلية نفس النقطة
- أو إذا لم تكن النقطة P هي مركز تغيير الأبعاد، إذن P′ تقع على CP و$CP' = k(CP)$.

إذا كانت $k < 0$ ، P′ هي النقطة الموجودة على الشعاع المقابل CP فإن $CP' = |k|(CP)$.

dimension The number of rows, m, and the number of columns, n, of a matrix written as $m \times n$.

بُعد عدد الصفوف m، وعدد الأعمدة n للمصفوفة المكتوبة بالصورة $m \times n$.

dimensional analysis The process of carrying units throughout a computation.

تحليل بُعدي عملية نقل الوحدات طوال العملية الحسابية.

direct isometry An isometry in which the image of a figure is found by moving the figure intact within the plane.

تساوي القياس المباشر تساوي القياس الذي يمكن العثور فيه على صورة شكل عن طريق تحريك الشكل مع الحفاظ على المستوى.

direct variation An equation of the form $y = kx$, where $k \neq 0$.

تغير طردي معادلة بالصيغة $y = kx$، حيث $k \neq 0$.

discrete function A function of points that are not connected.

distance between two points The length of the segment between two points.

distribution A graph or table that shows the theoretical frequency of each possible data value.

domain The set of the first numbers of the ordered pairs in a relation.

E

element Each entry in a matrix.

elimination The use of addition or subtraction to eliminate one variable and solve a system of equations.

end behavior Describes how the values of a function behave at each end of the graph.

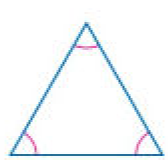
enlargement An image that is larger that the original figure.

equally likely The outcomes of an experiment are equally likely if there are n outcomes and the probability of each is $\frac{1}{n}$.

equation A mathematical sentence that contains an equals sign, =.

equiangular polygon A polygon with all congruent angles.

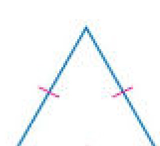
equiangular triangle A triangle with all angles congruent.



equidistant The distance between two lines measured along a perpendicular line is always the same.

equilateral polygon A polygon with all congruent sides.

equilateral triangle A triangle with all sides congruent.



دالة منفصلة دالة تتكون من نقاط غير متصلة.

مسافة بين نقطتين طول القطعة المستقيمة الواصلة بين نقطتين.

توزيع رسم بياني أو جدول يعرض التكرار النظري لكل قيمة بيانات محتملة.

مجال مجموعة المساقط الأولى للأزواج المرتبة في علاقة ما.

عنصر كل إدخال في مصفوفة.

حذف استخدام عملية الجمع أو الطرح لحذف متغير واحد وحل نظام المعادلات.

سلوك النهاية يصف سلوك قيم الدالة في كل طرف بالرسم البياني.

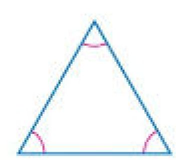
تكبير طريقة تصبح الصورة من خلالها أكبر من الشكل الأصلي.

مرجح بالتساوي تكون نتائج التجربة مرجحة بالتساوي إذا كانت هناك نتائج n وكان احتمال كل منها $\frac{1}{n}$.

معادلة عبارة رياضية تحتوي على علامة التساوي =.

مضلع متساوي الزوايا مضلع كل زواياه متطابقة.

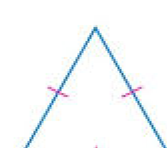
مثلث متساوي الزوايا مثلث كل زواياه متطابقة.



متساوي الأبعاد تكون المسافة بين الخطين المستقيمين عند قياسهما على طول المستقيم المتعامد عليهما متساوية دائمًا.

مضلع متساوي الأضلاع مضلع كل أضلاعه متطابقة.

مثلث متساوي الأضلاع مثلث كل أضلاعه متطابقة.



equivalent equations Equations that have the same solution.

معادلات متكافئة معادلات لها نفس الحل.

equivalent expressions Expressions that denote the same value for all values of the variable(s).

تعابير متكافئة تعابير لها نفس القيمة لكل قيم المتغير أو المتغيرات.

evaluate To find the value of an expression.

إيجاد القيمة إيجاد قيمة أي تعبير.

excluded values Any values of a variable that result in a denominator of 0 must be excluded from the domain of that variable.

قيم مستبعدة أي قيم ينتج عنها صفر في المقام لا بد من استبعادها من مجال ذلك المتغير.

exponent In an expression of the form x^n, the exponent is n. It indicates the number of times x is used as a factor.

أس في تعبير صيغته x^n، يكون الأس هو n. وهو يشير إلى عدد مرات x الذي يمثل العامل.

exponential decay When an initial amount decreases by the same percent over a given period of time.

تضاؤل أسي عندما تقل القيمة الأولية بالنسبة المئوية ذاتها على مدى فترة زمنية محددة.

exponential equation An equation in which the variables occur as exponents.

المعادلة الأسية معادلة تنطوي على المتغيرات كأسس.

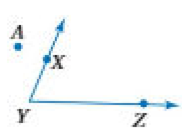
exponential function A function that can be described by an equation of the form $y = a^x$, where $a > 0$ and $a \neq 1$.

دالة أسية دالة يمكن وصفها بمعادلة بالصيغة $y = a^x$، حيث $a > 0$ و $a \neq 1$.

exponential growth When an initial amount increases by the same percent over a given period of time.

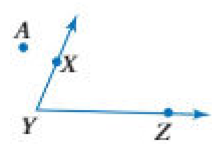
نمو أسي عندما تزيد القيمة الأولية بالنسبة المئوية ذاتها على مدى فترة زمنية محددة.

exterior A point is in the exterior of an angle if it is neither on the angle nor in the interior of the angle.



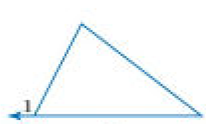
A is in the exterior of ∠XYZ.

نقطة خارجية النقطة التي تقع خارج الزاوية ولا تقع على الزاوية أو في داخلها.



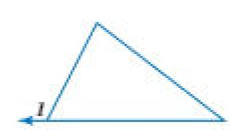
A تقع خارج ∠XYZ

exterior angles **1.** An angle that lies in the region that is not between two transversals that intersect the same line. **2.** An angle formed by one side of a triangle and the extension of another side.



∠1 is an exterior angle.

زوايا خارجية 1. الزاوية التي تقع في المنطقة خارج القاطعين اللذين يقطعان نفس الخط المستقيم. 2. الزاوية المكونة من أحد أضلاع المثلث وامتداد ضلع آخر.



∠1 هي زاوية خارجية

extraneous solutions Results that are not solutions to the original equation.

حلول دخيلة النتائج التي لا تمثل حلولاً للمعادلة الأصلية.

extremes In the ratio $\frac{a}{b} = \frac{c}{d}$, a and d are the extremes.

طرفا التناسب في النسبة $\frac{a}{b} = \frac{c}{d}$، a و d هما طرفا التناسب.

Glossary/القاموس

F

factors In an algebraic expression, the quantities being multiplied are called factors.

عوامل في التعبير الجبري، يطلق على الكميات المضروبة اسم العوامل.

family of graphs Graphs and equations of graphs that have at least one characteristic in common.

عائلة الرسوم البيانية الرسوم البيانية ومعادلات الرسوم البيانية التي تشترك في خاصية واحدة على الأقل.

finite plane A plane that has boundaries or does not extend indefinitely.

مستوى مُتناهٍ المستوى الذي له حدود أو الذي لا يمتد إلى ما لا نهاية.

flow proof A proof that organizes statements in logical order, starting with the given statements. Each statement is written in a box with the reason verifying the statement written below the box. Arrows are used to indicate the order of the statements.

برهان متسلسل البرهان الذي ينظم العبارات بترتيب منطقي، بدءا بعبارات المعطيات. ُتكتب كل عبارة في مربع مع كتابة السبب المبرر للعبارة أسفل المربع. وتستخدم الأسهم لتوضيح ترتيب العبارات.

formula An equation that states a rule for the relationship between certain quantities.

صيغة معادلة توضح قاعدة للعلاقة بين كميات معينة.

four-step problem-solving plan
Step 1 Explore the problem.
Step 2 Plan the solution.
Step 3 Solve the problem.
Step 4 Check the solution.

خطة حل المسألة ذات الخطوات الأربع
خطوة 1 فهم المسألة.
خطوة 2 تخطيط الحل.
خطوة 3 حل المسألة.
خطوة 4 التحقق من الحل.

frequency table A chart that indicates the number of values in each interval.

جدول تكرار جدول يوضح عدد القيم في كل فترة زمنية.

function A relation in which each element of the domain is paired with exactly one element of the range.

دالة علاقة يقترن فيها كل عنصر في المجال بعنصر واحد آخر في المدى.

function notation A way to name a function that is defined by an equation. In function notation, the equation $y = 3x - 8$ is written as $f(x) = 3x - 8$.

ترميز الدالة طريقة لتسمية الدالة المعرفة بمعادلة. في ترميز الدالة، المعادلة $y = 3x - 8$ تُكتب بالصيغة $f(x) = 3x - 8$.

Fundamental Counting Principle If an event M can occur in m ways and is followed by an event N that can occur in n ways, then the event M followed by the event N can occur in $m \times n$ ways.

مبدأ العد الأساسي إذا كان الحدث M يمكن أن يحدث بعدد m من الطرق ويتبعه العدد N الذي يمكن أن يحدث بعدد n من الطرق، إذن فالحدث M الذي يتبع الحدث N يمكن أن يحدث بعدد $m \times n$ من الطرق.

G

general equation for exponential decay
$y = C(1 - r)^t$, where y is the final amount, C is the initial amount, r is the rate of decay expressed as a decimal, and t is time.

معادلة عامة للتضاؤل الأسي $y = C(1 - r)^t$، حيث y هو القيمة النهائية، وC هو القيمة الأولية، وr هو معدل التضاؤل المتمثل في عدد عشري، وt هو الزمن.

general equation for exponential growth
$y = C(1 + r)^t$, where y is the final amount, C is the initial amount, r is the rate of change expressed as a decimal, and t is time.

معادلة عامة للنمو الأسي $y = C(1 + r)^t$، حيث y هو القيمة النهائية، وC هو القيمة الأولية، وr هو معدل التغير المتمثل في عدد عشري، وt هو الزمن.

geometric sequence A sequence in which each term after the first is found by multiplying the previous term by a constant r, called the common ratio.

متتالية هندسية متتالية يتم فيها إيجاد كل حد بعد الحد الأول عن طريق ضرب الحد السابق في ثابت r، ويُسمى النسبة المشتركة.

glide reflection The composition of a translation followed by a reflection in a line parallel to the translation vector.

انعكاس انحداري تركيب من الانسحاب متبوعًا بانعكاس في الخط المستقيم الموازي لمتجه الانسحاب.

graph To draw, or plot, the points named by certain numbers or ordered pairs on a number line or coordinate plane.

رسم بياني رسم أو تعيين نقاط ممثلة بأرقام محددة أو أزواج مرتبة على خط أعداد أو مستوى إحداثي.

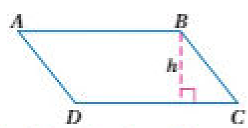
H

half-plane The region of the graph of an inequality on one side of a boundary.

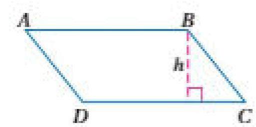
نصف المستوى منطقة الرسم البياني لمتباينة على جانب واحد من الحد.

height of a parallelogram The length of an altitude of a parallelogram.

ارتفاع متوازي الأضلاع طول أحد ارتفاعات متوازي الأضلاع.



h is the height of parallelogram *ABCD*.



h هي ارتفاع متوازي الأضلاع *ABCD*

histogram A graphical display that uses bars to display numerical data that have been organized into equal intervals.

مدرج تكراري عرض رسومي تُستخدم فيه الأعمدة لعرض بيانات عددية منظمة إلى فئات متساوية.

I

identity An equation that is true for every value of the variable.

متطابقة معادلة حقيقية لكل قيمة من قيم المتغير.

identity function The function $y = x$.

دالة محايدة الدالة $y = x$.

identity matrix A square matrix that, when multiplied by another matrix, equals that same matrix. If A is any $n \times n$ matrix and I is the $n \times n$ identity matrix, then $A \cdot I = A$ and $I \cdot A = A$.

مصفوفة متطابقة مصفوفة مربعة عندما يتم ضربها في مصفوفة أخرى، فإنها تساوي المصفوفة نفسها. إذا كان A هو أي مصفوفة $n \times n$، و I هو المصفوفة المتطابقة $n \times n$، إذن $A \cdot I = A$ و $I \cdot A = A$.

image A figure that results from the transformation of a geometric figure.

صورة شكل ينتج عن تحويل شكل هندسي.

included angle In a triangle, the angle formed by two sides is the included angle for those two sides.

زاوية محصورة في المثلث، الزاوية المكونة من التقاء ضلعي المثلث هي الزاوية المحصورة لهذين الضلعين.

included side The side of a polygon that is a side of each of two angles.

ضلع محصور أحد أضلاع المضلع الذي يمثل ضلعًا لكلتا الزاويتين.

inconsistent A system of equations with no ordered pair that satisfy both equations.

غير متوافق نظام من معادلتين لا يوجد به زوج مرتب يحقق كلتا المعادلتين.

increasing The graph of a function goes up on a portion of its domain when viewed from left to right.

متزايد الرسم البياني لدالة ترتفع على جزء من مجالها عند النظر إليها من اليسار إلى اليمين.

independent A system of equations with exactly one solution.

معادلة مستقلة نظام معادلات له حل واحد فقط.

independent variable The variable in a function with a value that is subject to choice.

متغير مستقل متغير في الدالة تخضع قيمته للاختيار.

indirect isometry An isometry that cannot be performed by maintaining the orientation of the points, as in a direct isometry.

تساوي القياس غير المباشر تساوي القياس الذي لا يمكن تنفيذه من خلال الحفاظ على اتجاه النقاط، كما يحدث في تساوي القياس المباشر.

inductive reasoning A conclusion based on a pattern of examples.

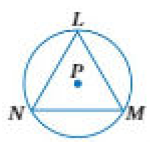
استدلال استقرائي استنتاج قائم على نمط من الأمثلة.

inequality An open sentence that contains the symbol $<$, $\leq$, $>$, or $\geq$.

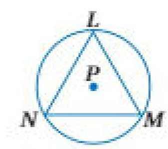
متباينة جملة مفتوحة تحتوي على الرمز $<$ أو $\geq$ أو $>$ أو $\leq$.

inscribed A polygon is inscribed in a circle if each of its vertices lie on the circle.

شكل محصور يكون المضلع محصورًا داخل دائرة إذا كانت كل رأس من رؤوسه تقع على الدائرة.



nLMN is inscribed in⊙P.



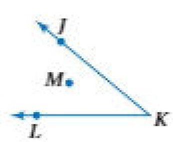
$\triangle LMN$ محصور في $\odot P$

integers The set {..., −2, −1, 0, 1, 2, ...}.

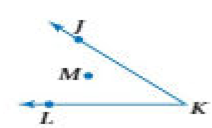
أعداد صحيحة مجموعة الأعداد {...،-2، -1، 0، 1، 2، ...}.

interior A point is in the interior of an angle if it does not lie on the angle itself and it lies on a segment with endpoints that are on the sides of the angle.

نقطة داخلية قطة داخل الزاوية ما لم تكن تقع على الزاوية نفسها، وتقع على قطعة مستقيمة ذات نقاط نهاية واقعة على ضلعي الزاوية.



M is in the interior of ∠*JKL*.



M تقع داخل ∠*JKL*.

interior angles Angles that lie between two transversals that intersect the same line.

زوايا داخلية الزوايا الواقعة بين القاطعين اللذين يقطعان نفس الخط المستقيم.

interquartile range The range of the middle half of a set of data. It is the difference between the upper quartile and the lower quartile.

مدى رُبعي مدى النصف الأوسط من مجموعة بيانات. وهو الفرق بين الرّبيع الأعلى والرّبيع الأدنى.

intersection **1.** The graph of a compound inequality containing and; the solution is the set of elements common to both inequalities. **2.** A set of points common to two or more geometric figures.

تقاطع 1. تمثيل لمتباينة مركبة تحتوي على العناصر المشتركة لكلتا المتباينتين والحل هو مجموعة تلك العناصر. 2. مجموعة من النقاط المشتركة بين شكلين هندسيين أو أكثر.

inverse variation An equation of the form $xy = k$, where $k \neq 0$.

تغير عكسي معادلة تأخذ الصورة $xy - k$، حيث $k \neq 0$.

irrational number A number that cannot be expressed as a terminating or repeating decimal.

عدد غير نسبي عدد لا يمكن التعبير عنه بكسور عشرية منتهية أو دورية.

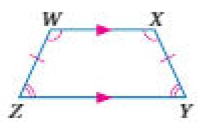
irregular figure A polygon with sides and angles that are not all congruent.

شكل غير منتظم مضلع ذو أضلاع وزوايا غير متطابقة.

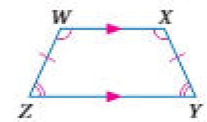
isometry A mapping for which the original figure and its image are congruent.

تساوي القياس رسم يكون فيه الشكل الأصلي وصورته متطابقين.

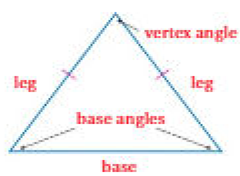
isosceles trapezoid A trapezoid in which the legs are congruent, both pairs of base angles are congruent, and the diagonals are congruent.



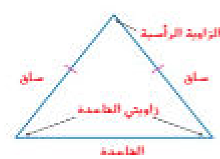
شبه منحرف متساوي الساقين هو شبه المنحرف الذي يتطابق فيه الساقان ويتطابق زوجا زوايا القاعدة والأقطار.



isosceles triangle A triangle with at least two sides congruent. The congruent sides are called legs. The angles opposite the legs are base angles. The angle formed by the two legs is the vertex angle. The side opposite the vertex angle is the base.

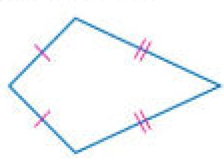


مثلث متساوي الساقين المثلث الذي يتطابق ضلعان على الأقل من أضلاعه. ويطلق على الأضلاع المتطابقة الساقين. كما يطلق على الزوايا المقابلة للأضلاع زوايا القاعدة. يطلق على الزاوية المكونة من التقاء ضلعي المثلث الزاوية الرأسية. وُيطلق على الضلع المقابل للزاوية الرأسية اسم القاعدة.

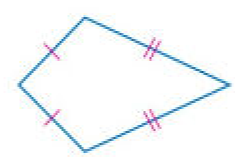


K

kite A quadrilateral with exactly two distinct pairs of adjacent congruent sides.



شكل رباعي محدب شكل رباعي مكون من زوجين مختلفين من الأضلاع المتطابقة المجاورة.



L

legs of a right triangle The shorter sides of a right triangle.

ساقا المثلث القائم الزاوية الضلعان الأقصر في المثلث القائم الزاوية.

legs of a trapezoid The nonparallel sides of a trapezoid.

ساقا شبه المنحرف الضلعان غير المتوازيين في شبه المنحرف.

legs of an isosceles triangle The two congruent sides of an isosceles triangle.

ساقا المثلث متساوي الساقين الضلعان المتطابقان في المثلث متساوي الساقين.

like terms Terms that contain the same variables, with corresponding variables having the same exponent.

حدود متشابهة حدود تتكون من المتغيرات نفسها مع متغيرات متناظرة لها الأس ذاته.

line A basic undefined term of geometry. A line is made up of points and has no thickness or width. In a figure, a line is shown with an arrowhead at each end. Lines are usually named by lowercase script letters or by writing capital letters for two points on the line, with a double arrow over the pair of letters.

نقطة مصطلح أساسي غير محدد للهندسة. يتكون الخط المستقيم من نقاط وليس له سُمك أو عرض. في الشكل، ُيوضح الخط المستقيم بسهمين عند نهايتيه. عادةً ما تسمى الخطوط المستقيمة بأحرف صغيرة أو بكتابة أحرف كبيرة تمثل نقطتين على الخط المستقيم، مع سهمين على الحرفين.

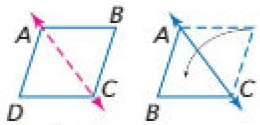
line of fit A line that describes the trend of the data in a scatter plot.

خط المواءمة خط مستقيم يعبر عن اتجاه البيانات في التمثيل البياني بالنقاط المبعثرة.

line of reflection A line in which each point on the preimage and its corresponding point on the image are the same distance from this line.

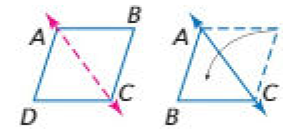
خط الانعكاس الخط الذي تكون فيه كل نقطة على الصورة الأصلية والنقطة المقابلة لها على الصورة على نفس المسافة من هذا الخط.

line of symmetry A line that can be drawn through a plane figure so that the figure on one side is the reflection image of the figure on the opposite side.



$\overleftrightarrow{AC}$ is a line of symmetry

خط التناظر الخط الذي يمكن رسمه عبر شكل المستوى بحيث يمثل الشكل في أحد الجانبين صورة منعكسة للشكل في الجانب المقابل.



$\overleftrightarrow{AC}$ هو خط التماثل

line segment A measurable part of a line that consists of two points, called endpoints, and all of the points between them.

قطعة مستقيمة الجزء القابل للقياس من الخط المستقيم الذي يتكون من نقطتين، يطلق عليهما اسم نقاط النهاية، وكل النقاط التي تقع بينهما.

linear equation An equation in the form $Ax + By = C$, with a graph that is a straight line.

معادلة خطية معادلة تأخذ الصورة $Ax + By = C$، ويتم تمثيلها بخط مستقيم.

linear extrapolation The use of a linear equation to predict values that are outside the range of data.

استكمال خارجي خطي استخدام معادلة خطية في توقع القيم التي تقع خارج مدى البيانات.

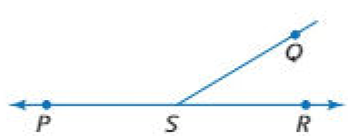
linear function A function with ordered pairs that satisfy a linear equation.

دالة خطية دالة تتكون من زوج مرتب يحقق المعادلة الخطية.

linear interpolation The use of a linear equation to predict values that are inside of the data range.

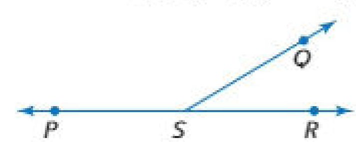
استكمال داخلي خطي استخدام معادلة خطية في توقع قيم تقع ضمن مدى البيانات.

linear pair A pair of adjacent angles whose non-common sides are opposite rays.



$\angle PSQ$ and $\angle QSR$ are a linear pair

زوج خطي زوج من الزوايا المجاورة التي يمثل ضلعاها غير المشتركين شعاعين معكوسين.



$\angle PSQ$ و $\angle QSR$ هما زوج خطي

linear regression An algorithm to find a precise line of fit for a set of data.

انحدار خطي خوارزمية لإيجاد خط المواءمة الدقيق لمجموعة من البيانات.

linear transformation One or more operations performed on a set of data that can be written as a linear function.

تحويل خطي عملية أو أكثر يتم إجراؤها على مجموعة من البيانات يمكن قراءتها باعتبارها دالة خطية.

literal equation A formula or equation with several variables.

معادلة حرفية صيغة أو معادلة متعددة المتغيرات.

locus The set of points that satisfy a given condition.

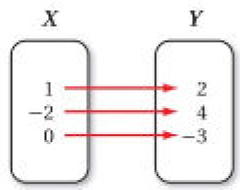
محل هندسي مجموعة النقاط تستوفي شرطًا محددًا.

lower quartile Divides the lower half of the data into two equal parts.

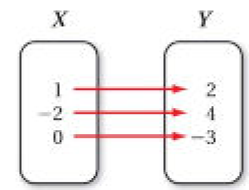
رُبيع أدنى يقسم النصف السفلي من البيانات إلى جزأين متساويين.

M

mapping Illustrates how each element of the domain is paired with an element in the range.



رسم الخرائط يوضح كيفية اقتران كل عنصر في المجال بعنصر آخر في المدى.



matrix Any rectangular arrangement of numbers in rows and columns.

مصفوفة أي مجموعة مستطيلة من الأعداد في الصفوف والأعمدة.

maximum The highest point on the graph of a curve.

قيمة عظمى أعلى نقطة في الرسم البياني للمنحنى.

mean The sum of numbers in a set of data divided by the number of items in the data set.

المتوسط الحسابي مجموع الأعداد في مجموعة البيانات مقسومًا على عدد العناصر في مجموعة البيانات.

mean absolute deviation The average of the absolute values of differences between the mean and each value in a data set. It is used to predict errors and to judge equality.

متوسط الانحراف المطلق متوسط القيم المطلقة للفروق بين المتوسط الحسابي وكل قيمة في مجموعة البيانات. ويُستخدم للتنبؤ بالأخطاء والحكم على التساوي.

means The middle terms of the proportion.

متوسطات حسابية الحدود المتوسطة للتناسب.

measures of central tendency Numbers or pieces of data that can represent the whole set of data.

مقاييس النزعة المركزية الأعداد أو أجزاء البيانات التي يمكنها تمثيل مجموعة البيانات ككل.

measures of position Measures that compare the position of a value relative to other values in a set.

مقاييس الموقع مقاييس تُستخدم لمقارنة موضع قيمة منسوبة إلى قيم أخرى في المجموعة.

measures of variation Used to describe the distribution of statistical data.

مقاييس التشتت تُستخدم لوصف توزيع البيانات الإحصائية.

median The middle number in a set of data when the data are arranged in numerical order. If the data set has an even number, the median is the mean of the two middle numbers.

وسيط العدد الأوسط في مجموعة البيانات عند ترتيب البيانات ترتيبًا عدديًا. وإذا كانت مجموعة البيانات تحتوي على عدد زوجي، فالوسيط هو متوسط العددين الأوسطين.

median fit line A type of best-fit line that is calculated using the medians of the coordinates of the data points.

وسيط خط المواءمة نوع من خط المواءمة الأفضل والذي يتم حسابه باستخدام وسيطات إحداثيات نقاط البيانات.

metric A rule for assigning a number to some characteristic or attribute.

قياس قاعدة لتعيين العدد إلى خاصية أو صفة ما.

midpoint The point on a segment exactly halfway between the endpoints of the segment.

نقطة المنتصف النقطة الواقعة في منتصف القطعة المستقيمة بين نقطتي نهايتها.

midsegment A segment with endpoints that are the midpoints of two sides of a triangle.

منصف الساقين القطعة المستقيمة التي لها نقطتا نهاية تمثلان نقطتي منتصف على ضلعي المثلث.

midsegment of trapezoid A segment that connects the midpoints of the legs of a trapezoid.

منصف ساقي شبه المنحرف القطعة المستقيمة التي تربط بين نقطتي المنتصف على ساقي شبه المنحرف.

midsegment of triangle A segment with endpoints that are the midpoints of two sides of a triangle.

منصف ساقي المثلث القطعة المستقيمة التي لها نقطتا نهاية تمثلان نقطتي منتصف على ضلعي المثلث.

minimum The lowest point on the graph of a curve.

قيمة صغرى أدنى نقطة في الرسم البياني للمنحنى.

mixture problems Problems in which two or more parts are combined into a whole.

مسائل مختلطة مسائل يجتمع فيها جزءان أو أكثر لتكوين المسائل ككل.

mode The number(s) that appear most often in a set of data.

منوال العدد (الأعداد) الأكثر تكرارًا في مجموعة من البيانات.

monomial A number, a variable, or a product of a number and one or more variables.

أحادي الحد عدد أو متغير أو حاصل ضرب عدد ومتغير واحد أو أكثر.

multiplicative identity For any number a, $a \cdot 1 = 1 \cdot a = a$.

محايد ضربي لأي عدد a، $a \cdot 1 = 1 \cdot a = a$.

multiplicative inverses Two numbers with a product of 1.

النظير الضربي عددان حاصل ضربهما يساوي 1.

multi-step equation Equations with more than one operation.

معادلة متعددة الخطوات معادلات تتكون من أكثر من عملية واحدة.

N

nth root If $a^n = b$ for a positive integer n, then a is an nth root of b.

جذر نوني n th إذا كان $a^n = b$ لعدد صحيح n، إذن a هو الجذر النوني nth لـ b.

n-gon A polygon with n sides.

n-gon مضلع له أضلاع n.

natural numbers The set {1, 2, 3, ...}.

أعداد طبيعية مجموعة الأعداد {1، 2، 3، ...}.

negative A function is negative on a portion of its domain where its graph lies below the x-axis.

دالة سالبة تكون الدالة سالبة في الجزء من مجالها الذي يقع رسمه البياني أدنى المحور الأفقي x.

negative correlation In a scatter plot, as x increases, y decreases.

ارتباط سالب في التمثيل البياني بالنقاط المبعثرة، كلما زادت قيمة x، قلت قيمة y.

negative exponent For any real number $a \neq 0$ and any integer n, $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.

أس سالب لأي عدد حقيقي $a \neq 0$ وأي عدد صحيح n، $\frac{1}{a^n}$ و $\frac{1}{a^{-n}} = a^n$

negative number Any value less than zero.

عدد سالب أي قيمة أقل من صفر.

nonlinear function A function with a graph that is not a straight line.

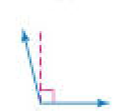
دالة غير خطية دالة تمثيلها البياني ليس خطًا مستقيمًا.

number theory The study of numbers and the relationships between them.

نظرية الأعداد دراسة الأعداد والعلاقات بينها.

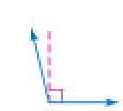
O

obtuse angle An angle with degree measure greater than 90 and less than 180.



$90 < m\angle A < 180$

زاوية منفرجة زاوية قياسها أكبر من 90 وأصغر من 180.



$90 < m\angle A < 180$

obtuse triangle A triangle with an obtuse angle.

one obtuse angle

odds The ratio of the probability of the success of an event to the probability of its complement.

open half-plane The solution of a linear inequality that does not include the boundary line.

open sentence A mathematical statement with one or more variables.

opposite rays Two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ such that B is between A and C.

opposites Two numbers with the same absolute value but different signs.

ordered pair A set of numbers or coordinates used to locate any point on a coordinate plane, written in the form (x, y).

order of magnitude The order of magnitude of a quantity is the number rounded to the nearest power of 10.

order of operations

1. Evaluate expressions inside grouping symbols.
2. Evaluate all powers.
3. Do all multiplications and/or divisions from left to right.
4. Do all additions and/or subtractions from left to right.

order of symmetry The number of times a figure can map onto itself as it rotates from 0° to 360°.

origin The point where the two axes intersect at their zero points.

outliers Data that are more than 1.5 times the interquartile range beyond the quartiles.

مثلث منفرج الزاوية مثلث إحدى زواياه منفرجة.

زاوية منفرجة واحدة

فرص نسبة احتمال نجاح الحدث إلى احتمال إتمامه.

نصف مستوٍ مفتوح حل المتباينة الخطية الذي لا يتضمن مستقيمًا حديًا.

عبارة مفتوحة عبارة رياضية ذات متغير واحد أو أكثر.

شعاعان معكوسان الشعاعان $\overrightarrow{BA}$ و $\overrightarrow{BC}$ بحيث تقع B بين A و C.

معكوسات عددان يتطابقان في القيمة المطلقة ويختلفان في العلامة.

زوج مرتب مجموعة من الأعداد أو الإحداثيات المستخدمة لتعيين أي نقطة على مستوى إحداثي، وتُكتب بالصيغة (x, y).

القيمة الأسية المقدار الأسي للكمية هو العدد التقريبي لأقرب قيمة أسية من 10.

ترتيب العمليات

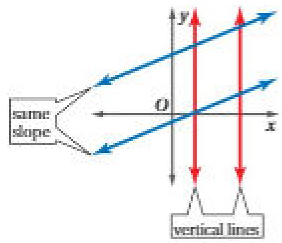
1. إيجاد قيم التعبيرات داخل رموز المجموعات.
2. إيجاد قيم كل القوى.
3. إجراء كل عمليات الضرب و/أو القسمة من اليسار إلى اليمين.
4. إجراء كل عمليات الجمع و/أو الطرح من اليسار إلى اليمين.

ترتيب التناظر عدد مرات انطباق الشكل على نفسه بينما يدور من 0° إلى 360°.

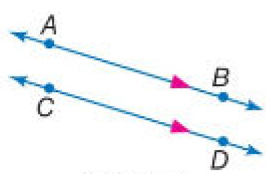
نقطة الأصل النقطة التي يتقاطع عندها المحوران في نقطتي الصفر لهما.

قيم متطرفة بيانات تكون أكبر بمقدار مرة ونصف من المدى بين الربيعين بما يتجاوز الربيعيات.

parallel lines 1. Lines in the same plane that do not intersect and either have the same slope or are vertical lines.



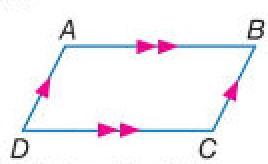
2. Coplanar lines that do not intersect.



$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

parallel planes Planes that do not intersect.

parallelogram A quadrilateral with parallel opposite sides. Any side of a parallelogram may be called a *base*.



$\overline{AB} \parallel \overline{DC} : \overline{AD} \parallel \overline{BC}$

parameter A measure that describes a characteristic of the population as a whole.

parent function The simplest of functions in a family.

parent graph The simplest of the graphs in a family of graphs.

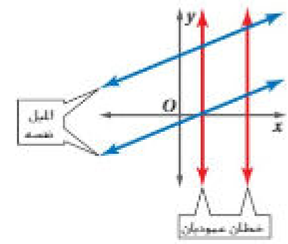
percent A ratio that compares a number to 100.

percent of change When an increase or decrease is expressed as a percent.

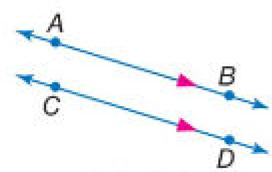
percent of decrease The ratio of an amount of decrease to the previous amount, expressed as a percent.

percent of increase The ratio of an amount of increase to the previous amount, expressed as a percent.

خطوط مستقيمة متوازية 1.. مستقيمات متوازية في نفس المستوى لا تتقاطع وإما أن يكون لهما نفس الميل أو أنهما مستقيمان عموديان.



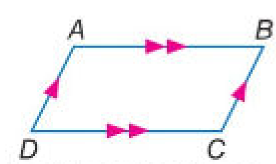
2. المستقيمات المتحدة المستوى التي لا تتقاطع.



$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

مستويات متوازية المستويات التي لا تتقاطع.

متوازي الأضلاع شكل رباعي الأضلاع له أربعة أضلاع متقابلة متوازية. يمكن تسمية أي ضلع من متوازي الأضلاع قاعدة.



$\overline{AB} \parallel \overline{DC} : \overline{AD} \parallel \overline{BC}$

معلمة قياس يصف خاصية المجتمع الإحصائي ككل.

دالة أصلية أبسط دالة في عائلة الدوال.

رسم بياني أصلي أبسط رسم بياني في عائلة الرسومات البيانية.

نسبة مئوية نسبة تقارن العدد بالعدد 100.

النسبة المئوية للتغيير التعبير عن التزايد أو التناقص بنسبة مئوية.

النسبة المئوية للتناقص نسبة كمية التناقص إلى الكمية السابقة معبرًا عنها بالنسبة المئوية.

النسبة المئوية للتزايد هي نسبة كمية التزايد إلى الكمية السابقة معبرًا عنها بالنسبة المئوية.

percent proportion
$$\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100} \text{ or } \frac{a}{b} = \frac{p}{100}$$

النسبة المئوية
$$\frac{\text{الجزء}}{\text{الكل}} = \frac{\text{النسبة المئوية}}{100} \text{ أو } \frac{a}{b} = \frac{P}{100}$$

perfect square A number with a square root that is a rational number.

مربع تام عدد له جذر تربيعي عبارة عن عدد نسبي.

perimeter The distance around a geometric figure.

محيط المسافة حول شكل هندسي.

perimeter The sum of the lengths of the sides of a polygon.

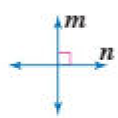
محيط مجموع أطوال أضلاع المضلع.

perpendicular lines Lines that intersect to form a right angle.

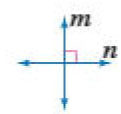
مستقيمات متعامدة الخطوط المستقيمة التي تتقاطع لتكون زاوية قائمة.

perpendicular lines Lines that form right angles.

مستقيمات متعامدة الخطوط المستقيمة التي تكون زوايا قائمة.



line m ⊥ line n



المستقيم m ⊥ المستقيم n

pi (π) An irrational number represented by the ratio of the circumference of a circle to the diameter of the circle.

باي (π) عدد غير نسبي تمثله نسبة محيط الدائرة إلى قطر الدائرة.

plane A basic undefined term of geometry. A plane is a flat surface made up of points that has no depth and extends indefinitely in all directions. In a figure, a plane is often represented by a shaded, slanted four-sided figure. Planes are usually named by a capital script letter or by three noncollinear points on the plane.

مستوى مصطلح هندسي أساسي غير محدد. مستوى السطح المستوي الذي يتكون من نقاط ليس لها عمق وتمتد إلى ما لا نهاية في كل الاتجاهات. في أي شكل، غالبًا ما يتم تمثيل المستوى بشكل مظلل مائل ذي أربعة أضلاع. وتسمى المستويات عادةً بحرف مطبوع كبير أو بثلاثة نقاط لا تقع على خط مستقيم.

plane Euclidean geometry Geometry based on Euclid's axioms dealing with a system of points, lines, and planes.

هندسة إقليدية مستوية الهندسة المبنية على مسلّمات إقليدس والتي تتناول نظامًا من النقاط والخطوط والمستويات.

plane symmetry Symmetry in a three- dimensional figure that occurs if the figure can be mapped onto itself by a reflection in a plane.

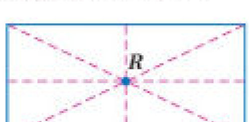
تماثل المستوى التماثل في أي شكل ثلاثي الأبعاد الذي يحدث إذا أمكن تخطيط الشكل على نفسه بالانعكاس في أي مستوى.

point A basic undefined term of geometry. A point is a location. In a figure, points are represented by a dot. Points are named by capital letters.

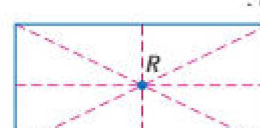
نقطة مصطلح هندسي أساسي غير محدد. وتعد النقطة موضعًا. في أي شكل، يتم تمثيل النقاط بنقطة. تتم تسمية النقاط بأحرف كبيرة.

point of symmetry A figure that can be mapped onto itself by a rotation of 180°.

نقطة التناظر الشكل الذي يمكن انطباقه على نفسه بزاوية دوران مقدارها 180°.



R is a point of symmetry.



R نقطة تماثل

Glossary/القاموس

point of tangency For a line that intersects a circle in only one point, the point at which they intersect.

نقطة التماس بالنسبة إلى الخط المستقيم الذي يقطع دائرة في نقطة واحدة فقط، هي النقطة التي يتقاطع فيها الخط المستقيم مع الدائرة.

point–slope form An equation of the form $y - y_1 = m(x - x_1)$, where (x_1, y_1) are the coordinates of any point on the line and m is the slope of the line.

نموذج نقطة الميل معادلة تأخذ الصورة $y - y_1 = m(x - x_1)$، حيث يكون (x_1, y_1) الإحداثيين لأي نقطة على المستقيم و m هو ميل المستقيم.

polygon A closed figure formed by a finite number of coplanar segments called *sides* such that the following conditions are met:

1. The sides that have a common endpoint are noncollinear.
2. Each side intersects exactly two other sides, but only at their endpoints, called the *vertices.*

مضلع شكل مغلق يتكون من عدد محدود من القطع متحدة المستوى تسمى الأضلاع مع توافر الشروط التالية:

1. الأضلاع التي لها نقطة نهاية واحدة لا تقع على خط
2. تقاطع كل ضلع مع ضلعين آخرين، ولكن في نقطتي نهايتيهما فقط وتسمى رؤوس الزوايا.

positive A function is positive on a portion of its domain where its graph lies above the x-axis.

موجبة تكون الدالة موجبة في جزء من مجالها عندما يقع تمثيلها البياني أعلى المحور x.

positive correlation In a scatter plot, as x increases, y increases.

ارتباط موجب في التمثيل البياني بالنقاط المبعثرة، كلما زادت القيمة x، زادت قيمة y.

positive number Any value that is greater than zero.

عدد موجب أي قيمة أكبر من الصفر.

power An expression of the form x^n, read *x to the nth power.*

قيمة أسية تعبير عن الصيغة x^n وتقرأ x مرفوعة إلى القيمة الأسية n.

preimage The graph of an object before a transformation.

صورة أصلية الرسم البياني لكائن ما قبل التحويل.

principal square root The nonnegative square root of a number.

جذر تربيعي أساسي الجذر التربيعي غير السالب للعدد.

principle of superposition Two figures are congruent if and only if there is a rigid motion or a series of rigid motions that maps one figure exactly onto the other.

مبدأ التراكب يتطابق الشكلان فقط في حالة وجود حركة غير مرنة أو سلسلة من الحركات غير المرنة التي تجعل أحد الشكلين ينطبق على الآخر تمامًا.

probability The ratio of the number of favorable equally likely outcomes to the number of possible equally likely outcomes.

احتمال نسبة عدد النتائج المفضلة المتساوية الاحتمال إلى عدد النتائج المحتملة المتساوية الاحتمال.

probability graph A way to give the probability distribution for a random variable and obtain other data.

التمثيل البياني للاحتمال طريقة لإيجاد توزيع الاحتمال لمتغير عشوائي والحصول على بيانات أخرى.

product In an algebraic expression, the result of quantities being multiplied is called the product.

حاصل ضرب في التعبير الجبري، يسمى ناتج ضرب الكميات حاصل الضرب.

product rule If (x_1, y_1) and (x_2, y_2) are solutions to an inverse variation, then $y_1x_1 = y_2x_2$.

قاعدة حاصل الضرب إذا كان (x_1, y_1) و(x_2, y_2) حلولاً للتباين العكسي، إذن $y_1x_1 = y_2x_2$.

proportion An equation of the form $\frac{a}{b} = \frac{c}{d}$, where b, d ≠ 0, stating that two ratios are equivalent.

نسبة معادلة تُكتب بالصيغة $\frac{a}{b} = \frac{c}{d}$، حيث b و $d \neq 0$، توضح أن النسبتين متكافئتان.

Glossary/ القاموس

Q

quartile The values that divide a set of data into four equal parts.

رُبيعيات قيم تقسم مجموعة من البيانات إلى أربعة أجزاء متساوية.

R

radical equations Equations that contain radicals with variables in the radicand.

معادلات جذرية المعادلات التي تحتوي على جذور بمتغيرات في مجذورات.

radical expression An expression that contains a square root.

تعبير جذري تعبير يحتوي على جذر تربيعي.

radical function A function that contains radicals with variables in the radicand.

دالة جذرية الدالة التي تحتوي على جذور بمتغيرات في المجذورات.

radicand The expression that is under the radical sign.

مجذور التعبير الذي يكون تحت علامة الجذر.

radius 1. In a circle, any segment with endpoints that are the center of the circle and a point on the circle.
2. In a sphere, any segment with endpoints that are the center and a point on the sphere.

نصف القطر 1. في الدائرة، هو أي قطعة مستقيمة لها نقطتي نهاية، إحداهما مركز الدائرة والأخرى نقطة على الدائرة.
2. في الكرة، هو أي قطعة مستقيمة لها نقطتي نهاية، إحداهما مركز الكرة والأخرى نقطة على الكرة.

range 1. The set of second numbers of the ordered pairs in a relation. 2. The difference between the greatest and least data values.

المدى 1. مجموعة الأعداد الثانية للأزواج المرتبة في العلاقة. 2. الفرق بين قيمتي البيانات العليا والدنيا.

rate The ratio of two measurements having different units of measure.

معدل النسبة بين قياسين لهما وحدات قياس مختلفة.

rate of change 1. How a quantity is changing with respect to a change in another quantity. 2. Describes how a quantity is changing over time.

معدل التغير 1. كيفية تغير كمية ما بالنسبة إلى تغير في كمية أخرى. 2. يصف كيفية تغير الكمية بمرور الوقت.

rate problems Problems in which an object moves at a certain speed, or rate.

مسائل المعدل المسائل التي يتحرك فيها جسم ما بسرعة معينة أو معدل معين.

ratio A comparison of two numbers by division.

نسبة مقارنة بين عددين من خلال القسمة.

rational exponent For any positive real number b and any integers m and $n > 1$, $b^{\frac{m}{n}} = (\sqrt[n]{b})^m$ or $\sqrt[n]{b^m}$. $\frac{m}{n}$ is a rational exponent.

الأس النسبي بالنسبة إلى أي عدد حقيقي موجب b وأي أعداد صحيحة $n > 1$, $b^{\frac{m}{n}} = (\sqrt[n]{b})^m$ أو $\sqrt[n]{b^m}$. $\frac{m}{n}$ هو أس نسبي.

rational expression An algebraic fraction with a numerator and denominator that are polynomials.

تعبير نسبي الكسر الجبري الذي له بسط ومقام كثيريّ الحدود.

rational function An equation of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions, and $q(x) \neq 0$.

دالة نسبية معادلة تأخذ الصورة $f(x) = \frac{p(x)}{q(x)}$، حيث $p(x)$ و $q(x)$ دوال كثيرة الحدود، و $q(x) \neq 0$.

rationalizing the denominator A method used to eliminate radicals from the denominator of a fraction.

إنطاق المقام طريقة تستخدم للتخلص من الجذور في مقام الكسر.

rational numbers The set of numbers expressed in the form of a fraction $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

أعداد نسبية مجموعة الأعداد التي يعبر عنها في شكل كسر $\frac{a}{b}$، حيث a و b أعداد صحيحة و$b \neq 0$.

ray $\overrightarrow{PQ}$ is a ray if it is the set of points consisting of $\overline{PQ}$ and all points S for which Q is between P and S.

شعاع $\overrightarrow{PQ}$ شعاع يتكون من مجموعة من النقاط تتكون من $\overline{PQ}$ وكل النقاط S التي تكون فيها Q بين P و S.

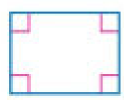
real numbers The set of rational numbers and the set of irrational numbers together.

أعداد حقيقية مجموعة الأعداد النسبية ومجموعة الأعداد غير النسبية.

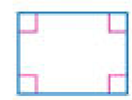
reciprocal The multiplicative inverse of a number.

معكوس ضربي النظير الضربي العدد، أو مقلوبه الضربي.

rectangle A quadrilateral with four right angles.



المستطيل رباعي أضلاع مستطيل زواياه الأربع قائمة.



recursive formula Each term is formulated from one or more previous terms.

صيغة تكرارية كل حد تتم صياغته من حد أو أكثر من الحدود السابقة.

reduction An image that is smaller than the original figure.

اختزال صورة أصغر من الشكل الأصلي.

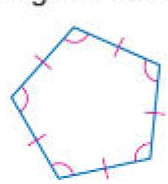
reflection A transformation representing the flip of a figure over a point, line or plane. A reflection in a line is a function that maps a point to its image such that

- if the point is on the line, then the image and preimage are the same point, or
- if the point is not on the line, the line is the perpendicular bisector of the segment joining the two points.

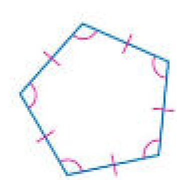
انعكاس تحويل يمثل قلب الشكل على نقطة أو مستقيم أو مستوى. الانعكاس في المستقيم هو دالة تخطط نقطة إلى صورتها بحيث

- إذا كانت النقطة على المستقيم، فإن الصورة والصورة الأصلية تكونان على نفس النقطة
- أو إذا كانت النقطة على المستقيم، فسيكون المستقيم متصفًا عموديًا للقطعة التي تصل بين النقطتين.

regular polygon A convex polygon in which all of the sides are congruent and all of the angles are congruent.



مضلع منتظم المضلع المحدب الذي تتطابق فيه كل الزوايا وكل الأضلاع.



regular tessellation A tessellation formed by only one type of regular polygon.

فسيفساء منتظمة اصطفاف فسيفسائي يتكوّن من نوع واحد فقط من المضلعات المنتظمة.

relation A set of ordered pairs.

علاقة أي مجموعة من الأزواج المرتبة.

remote interior angles The angles of a triangle that are not adjacent to a given exterior angle.

زوايا داخلية غير مجاورة زوايا المثلث التي تكون غير مجاورة للزاوية الخارجية المعطاة.

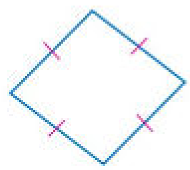
replacement set A set of numbers from which replacements for a variable may be chosen.

مجموعة التعويض مجموعة من الأعداد التي قد يتم منها اختيار بدائل الإحلال للمتغير.

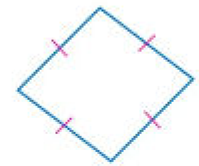
residual The difference between an observed y-value and its predicted y-value on a regression line.

قيمة متبقية الفرق بين قيمة y الملاحظة وقيمة y المتوقعة على خط الانحدار.

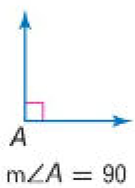
rhombus A quadrilateral with all four sides congruent.



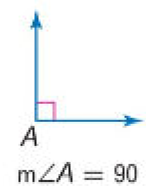
معين رباعي الأضلاع معيّن يتكون من أربعة أضلاع متطابقة.



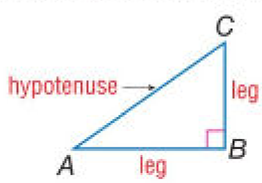
right angle An angle with a degree measure of 90.



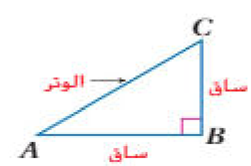
زاوية قائمة الزاوية التي قياسها 90.



right triangle A triangle with a right angle. The side opposite the right angle is called the *hypotenuse*. The other two sides are called *legs*.



مثلث قائم الزاوية مثلث إحدى زواياه قائمة. يُسمى الضلع المقابل للزاوية القائمة الوتر. ويطلق على الضلعين الآخرين الساقين.



root The solutions of a quadratic equation.

جذر حلول المعادلة التربيعية.

rotation A transformation that turns every point of a preimage through a specified angle and direction about a fixed point, called the *center of rotation*. A rotation about a fixed point through an angle of $x°$ is a function that maps a point to its image such that

- if the point is the center of rotation, then the image and preimage are the same point, or
- if the point is not the center of rotation, then the image and preimage are the same distance from the center of rotation and the measure of the angle of rotation formed by the preimage, center of rotation, and image points is x.

دوران التحويل الذي يدير كل نقطة في الصورة الأصلية بزاوية واتجاه محددين نحو نقطة ثابتة تُسمى مركز الدوران. الدوران حول نقطة ثابتة بزاوية $x°$ هو دالة ترسم نقطة على صورتها بحيث

- إذا كانت النقطة هي مركز الدوران، فإن الصورة والصورة الأصلية تكونان نفس النقطة،
- إذا لم تكن النقطة هي مركز الدوران، فستكون الصورة والصورة الأصلية على نفس المسافة من مركز الدوران وقياس زاوية الدوران التي كونتها الصورة الأصلية ومركز الدوران ونقاط الصورة هي x.

rotational symmetry If a figure can be rotated less than 360° about a point so that the image and the preimage are indistinguishable, the figure has rotational symmetry.

تناظر دوراني إذا أمكن تدوير شكل بزاوية أقل من 360° حول نقطة ما بحيث لا يمكن التمييز بين الصورة والصورة الأصلية، يكون للشكل تناظر دوراني.

row reduction The process of performing elementary row operations on an augmented matrix to solve a system.

اختزال صف عملية تنفيذ عمليات الصف الأولية على مصفوفة تم زيادتها لحل النظام.

Glossary/القاموس

S

sample space The list of all possible outcomes.

الفضاء العيني قائمة بكل النتائج المحتملة.

scale The relationship between the measurements on a drawing or model and the measurements of the real object.

مقياس العلاقة بين المقاييس في الرسم أو النموذج والمقاييس للكائن الحقيقي.

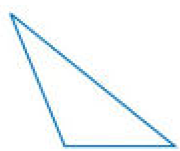
scale factor of dilation The ratio of a length on an image to a corresponding length on the preimage.

معامل مقياس تغيير الأبعاد نسبة الطول في صورة إلى الطول المناظر في الصورة الأصلية.

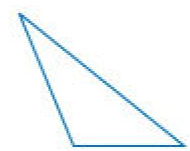
scale model A model used to represent an object that is too large or too small to be built at actual size.

نموذج مقياسي نموذج يستخدم لتمثيل شيء كبير جدًا أو صغير جدًا بحيث لا يمكن بناؤه في الحجم الفعلي.

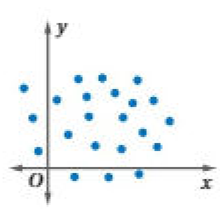
scalene triangle A triangle with no two sides congruent.



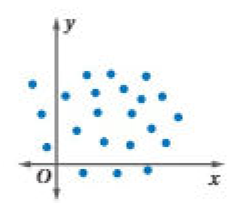
مثلث مختلف الأضلاع مثلث لا يوجد به ضلعان متطابقان.



scatter plot A scatter plot shows the relationship between a set of data with two variables, graphed as ordered pairs on a coordinate plane.



تمثيل بياني بالنقاط المبعثرة تمثيل بياني يعرض العلاقة بين مجموعة بيانات ذات متغيرين، ويمثلها في صورة أزواج مرتبة على المستوى الإحداثي.



scientific notation A number in scientific notation is expressed as $a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

ترميز علمي العدد في الترميز العلمي الذي يتم التعبير عنه بالصورة $a \times 10^n$، حيث $1 \leq a < 10$ وn رقم صحيح.

segment See *line segment*.

قطعة مستقيمة انظر قطعة مستقيمة.

segment bisector A segment, line, or plane that intersects a segment at its midpoint.

منصف القطعة المستقيمة القطعة المستقيمة أو المستقيم أو المستوى الذي يتقاطع مع القطعة المستقيمة في نقطة المنتصف.

semi-regular tessellation A uniform tessellation formed using two or more regular polygons.

فسيفساء شبه منتظمة اصطفاف فسيفسائي منتظمة مكون من مضلعين منتظمين أو أكثر.

sequence A set of numbers in a specific order.

متتالية مجموعة من الأعداد مرتبة بشكل محدد.

set-builder notation A concise way of writing a solution set. For example, $\{t \mid t < 17\}$ represents the set of all numbers t such that t is less than 17.

رمز بناء مجموعة الحل طريقة موجزة لكتابة مجموعة الحل. على سبيل المثال، $\{t \mid t < 17\}$ تمثل مجموعة جميع الأعداد t حيث t أقل من 17.

sides of an angle The rays of an angle.

ضلعا الزاوية شعاعا الزاوية.

similarity transformation When a figure and its transformation image are similar.

تحويل تشابهي عندما يكون الشكل وصورة تحويله متشابهين.

simplest form An expression is in simplest form when it is replaced by an equivalent expression having no like terms or parentheses.

أبسط صورة يكون التعبير في أبسط صوره عندما يتم استبداله بتعبير مكافئ ليس له حدود أو أقواس مشابهة.

skew lines Lines that do not intersect and are not coplanar.

مستقيمات متخالفة المستقيمات التي لا تتقاطع وغير متحدة المستوى.

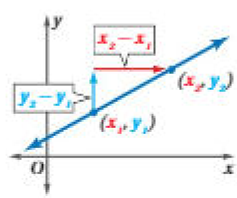
slope For a (nonvertical) line containing two points (x_1, y_1) and (x_2, y_2), the number m given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ where } x_2 \neq x_1.$$

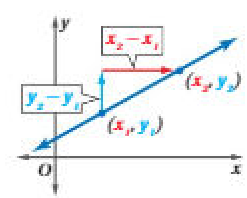
ميل بالنسبة إلى خط مستقيم (غير عمودي) يحتوي على نقطتين (x_1, y_1) و(x_2, y_2)، ويكون الرقم m من معطيات الصيغة

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ حيث } x_2 \neq x_1.$$

slope The ratio of the change in the y-coordinates (rise) to the corresponding change in the x-coordinates (run) as you move from one point to another along a line.



ميل نسبة التغير في الإحداثي y (التغير الرأسي) إلى التغير المناظر في الإحداثي x (التغير الأفقي) كلما تحركت من نقطة إلى أخرى بطول المستقيم.



slope–intercept form A linear equation of the form $y = mx + b$. The graph of such an equation has slope m and y-intercept b.

صيغة الميل والمقطع معادلة خطية تأخذ الصورة $y = mx + b$. التمثيل البياني لهذه المعادلة له ميل m ويتقاطع مع المحور الرأسي y في b.

slope–intercept form An equation of the form $y = mx + b$, where m is the slope and b is the y-intercept.

صيغة الميل والمقطع معادلة تأخذ الصورة $y = mx + b$، حيث m هو الميل و b هو التقاطع مع المحور y.

solid of revolution A three-dimensional figure obtained by rotating a plane figure about a line.

مجسم دوراني شكل ثلاثي الأبعاد نحصل عليه بدوران شكل مستو حول مستقيم.

solution A replacement value for the variable in an open sentence.

حل قيمة إحلال لمتغير في عبارة مفتوحة.

solution set The set of elements from the replacement set that make an open sentence true.

مجموعة الحل مجموعة العناصر من مجموعة الإحلال التي تكون عبارة مفتوحة صحيحة.

solve an equation The process of finding all values of the variable that make the equation a true statement.

حل معادلة عملية إيجاد كل القيم للمتغير الذي يجعل المعادلة عبارة صحيحة.

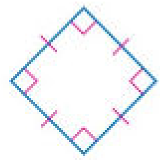
solving an open sentence Finding a replacement value for the variable that results in a true sentence or an ordered pair that results in a true statement when substituted into the equation.

حل عبارة مفتوحة إيجاد قيمة إحلال للمتغير الذي يؤدي إلى تكوين عبارة صحيحة أو الزوج المرتب الذي يؤدي إلى تكوين عبارة صحيحة عندما يتم إحلاله في المعادلة.

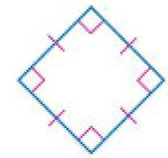
space A boundless three-dimensional set of all points.

فراغ مجموعة من كل النقاط غير المحدودة ثلاثية الأبعاد.

square A quadrilateral with four right angles and four congruent sides.



مربع شكل رباعي الأضلاع مكون من أربع زوايا قائمة وأربع أضلاع متطابقة.



square root One of two equal factors of a number.

جذر تربيعي أحد العاملين المتساويين للعدد.

square root function Function that contains the square root of a variable.

دالة الجذر التربيعي الدالة التي تحتوي على الجذر التربيعي لمتغير.

standard deviation The square root of the variance.

انحراف معياري الجذر التربيعي للتباين.

standard form The standard form of a linear equation is $Ax + By = C$, where $A \geq 0$, A and B are not both zero, and A, B, and C are integers with a greatest common factor of 1.

صيغة قياسية الصيغة القياسية للمعادلة الخطية هي $Ax + By = C$، حيث $A \geq 0$ وA وB لا تساويان صفر، وA وB وC أعداد صحيحة ذات قاسم مشترك أكبر يساوي 1.

statistic A quantity calculated from a sample.

إحصائي كمية يتم حسابها من عينة.

statistical inference The statistics of a sample are used to draw conclusions about the population.

استدلال إحصائي إحصائيات من عينة تستخدم للتوصل إلى استنتاجات عن المجتمع الإحصائي.

stem-and-leaf plot A system used to condense a set of data where the greatest place value of the data forms the stem and the next greatest place value forms the leaves.

مخطط الساق والأوراق نظام يستخدم لتكثيف مجموعة البيانات حيث تكوّن القيمة المنزلية الأكبر من البيانات الساق وتكوّن القيمة المنزلية الأكبر التالية الأوراق.

substitution Use algebraic methods to find an exact solution of a system of equations.

تعويض استخدام طرق جبرية لإيجاد حل دقيق لنظام من المعادلات.

supplementary angles Two angles with measures that have a sum of 180.

زاويتان متكاملتان زاويتان مجموع قياسهما يساوي 180.

surface area The sum of the areas of all the surfaces of a three-dimensional figure.

مساحة السطح مجموع مساحات أسطح الشكل ثلاثي الأبعاد.

symmetry 1. A geometric property of figures that can be folded and each half matches the other exactly. 2. A figure has symmetry if there exists a rigid motion—reflection, translation, rotation, or glide reflection—that maps the figure onto itself.

تناظر 1. خاصية هندسية للأشكال التي يمكن ثنيها وتطابق أحد نصفيها تماما على النصف الآخر. 2. يحتوي أحد الأشكال على تناظر في حالة وجود حركة قوية – انعكاس أو انسحاب أو دوران أو انعكاس انحداري – تتطابق مع الشكل نفسه.

system of equations A set of equations with the same variables.

نظام المعادلات مجموعة المعادلات التي تحتوي على نفس المتغيرات.

system of inequalities A set of two or more inequalities with the same variables.

نظام المتباينات مجموعة المتباينات المكونة من متباينتين أو أكثر والتي تحتوي على نفس المتغيرات.

term A number, a variable, or a product or quotient of numbers and variables.

حد العدد أو المتغير أو حاصل ضرب أو قسمة الأعداد والمتغيرات.

terms of a sequence The numbers in a sequence.

حدود المتسلسلة الأعداد المتتالية.

tessellation A pattern that covers a plane by transforming the same figure or set of figures so that there are no overlapping or empty spaces.

تغطية بالفسيفساء نمط ما يغطي المستوى من خلال تحويل نفس الشكل أو مجموعة الأشكال بحيث لا يوجد تداخل أو مساحات فارغة.

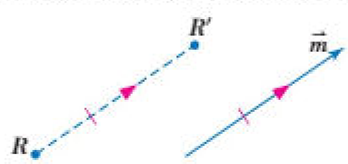
transformation In a plane, a mapping for which each point has exactly one image point and each image point has exactly one preimage point.

تحويل يشير في أحد المستويات إلى التخطيط الذي تحتوي كل نقطة فيه على نقطة صورة واحدة بالضبط وتحتوي كل نقطة صورة على نقطة صورة أصلية واحدة بالضبط.

translation A transformation that moves a figure the same distance in the same direction. A translation is a function that maps each point to its image along a vector such that each segment joining a point and its image has the same length as the vector, and this segment is also parallel to the vector.

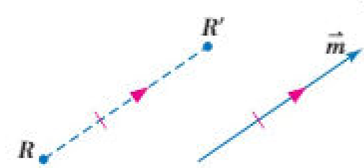
انسحاب تحويل يحرك أحد الأشكال في نفس البُعد وفي نفس الاتجاه. الانسحابo دالة تربط كل نقطة بصورتها على متجه ما بحيث تكون كل قطعة مستقيمة متصلة بنقطة وتكون صورة هذه النقطة بنفس طول المتجه، وتكون هذه القطعة المستقيمة أيضًا موازية للمتجه.

translation vector The vector in which a translation maps each point to its image.



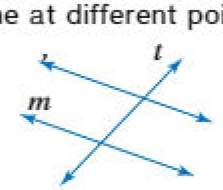
Point R', is a translation of point R along translation vector m.

متجه الانسحاب المتجه الذي يربط فيه الانسحاب كل نقطة بصورتها.



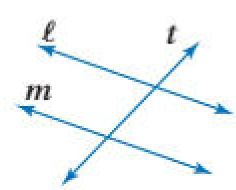
النقطة R'، هي انسحاب للنقطة
R على امتداد متجه الانسحاب m

transversal A line that intersects two or more lines in a plane at different points.



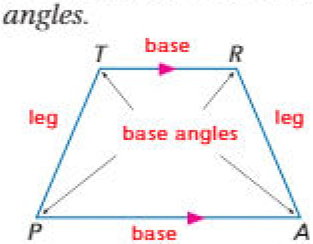
Line t is a transversal.

قاطع خط يقطع خطين أو أكثر في مستوى ما عند نقاط مختلفة.

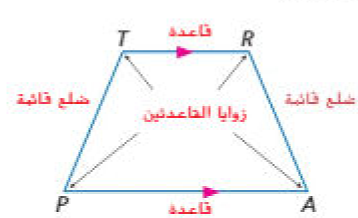


الخط t قاطع

trapezoid A quadrilateral with exactly one pair of parallel sides. The parallel sides of a trapezoid are called *bases*. The nonparallel sides are called *legs*. The pairs of angles with their vertices at the endpoints of the same base are called *base angles*.



شبه المنحرف رباعي أضلاع مكون من زوج من الأضلاع الموازية تماما. يُطلق على الأضلاع الموازية لشبه المنحرف القواعد. ويُطلق على الضلعان غير الموازيين ضلعا القائمة. ويُطلق على أزواج الزوايا مع رؤوسها عند نقاط نهاية نفس القاعدة زوايا القاعدة.



tree diagram A diagram used to show the total number of possible outcomes.

مخطط الشجرة مخطط مستخدم في عرض العدد الإجمالي للنتائج المحتملة.

two-column proof A formal proof that contains statements and reasons organized in two columns. Each step is called a statement, and the properties that justify each step are called reasons.

برهان ذو عمودين برهان شكلي يحتوي على عبارات ومبررات مرتبة في عمودين. يُطلق على كل خطوة عبارة ويطلق على الخصائص التي تبرر كل خطوة مبررات.

U

undefined term Words, usually readily understood, that are not formally explained by means of more basic words and concepts. The basic undefined terms of geometry are point, line, and plane.

مصطلح غير معرّف الكلمات التي عادة ما تفهم بسهولة ولا يتم شرحها بطريقة رسمية عن طريق المزيد من الكلمات والمفاهيم الأساسية. المصطلحات الهندسية الأساسية غير المعرّفة هي النقطة والخط المستقيم والمستوى.

uniform motion problems Problems in which an object moves at a certain speed, or rate.

مسائل الحركة المنتظمة المسائل التي يتحرك فيها جسم ما بسرعة أو معدل معين.

uniform tessellations Tessellations containing the same arrangement of shapes and angles at each vertex.

تغطيات منتظمة بالفسيفساء التغطيات بالفسيفساء التي تحتوي على نفس ترتيب الأشكال والزوايا عند كل رأس.

union The graph of a compound inequality containing or; the solution is a solution of either inequality, not necessarily both.

ربط تمثيل بياني لحل متباينة مركبة؛ بحيث يكون الحل لإحدى المتباينتين وليس بالضرورة لكلتيهما.

unit analysis The process of including units of measurement when computing.

تحليل الوحدة العملية التي تتضمن وحدات القياس عند الحساب.

unit rate A ratio of two quantities, the second of which is one unit.

معدل الوحدة نسبة كميتين، وتكون الكمية الثانية وحدة واحدة.

univariate data Data with one variable.

بيانات أحادية المتغير بيانات ذات متغير واحد.

upper quartile The median of the upper half of a set of data.

رُبيع أعلى وسيط النصف العلوي من مجموعة البيانات.

V

variable 1. Symbols used to represent unspecified numbers or values. 2. a characteristic of a group of people or objects that can assume different values

متغير 1. الرموز المستخدمة لتمثيل الأعداد أو القيم غير المحددة. 2. سمة مجموعة من الأفراد أو الأجسام التي يمكن أن تحمل قيمًا مختلفة

variance The mean of the squares of the deviations from the arithmetic mean.

تباين متوسط مربعات الانحرافات عن المتوسط الحسابي.

vertex angle of an isosceles triangle See isosceles triangle.

زاوية رأس مثلث متساوي الساقين انظر المثلث متساوي الساقين.

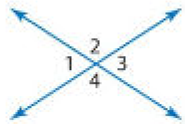
vertex of an angle The common endpoint of an angle.

رأس الزاوية نقطة النهاية المشتركة لزاوية ما.

vertex of a polygon The vertex of each angle of a polygon.

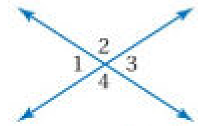
رأس المضلع رأس كل زاوية في المضلع.

vertical angles Two nonadjacent angles formed by two intersecting lines.



∠1 and ∠3 are vertical angles.
∠2 and ∠4 are vertical angles.

زوايا عمودية زاويتان غير متجاورتين يتم تشكيلهما عن طريق خطين متقاطعين.



∠1 و ∠3 زوايا عمودية
∠2 و ∠4 زوايا عمودية

vertical line test If any vertical line passes through no more than one point of the graph of a relation, then the relation is a function.

اختبار الخط الرأسي يكون في حالة مرور أي خط رأسي عبر ما لا يزيد عن نقطة واحدة من الرسم البياني للعلاقة، ومن ثم تصبح العلاقة دالة.

volume The measure of space occupied by a solid region.

حجم مقياس المساحة التي تشغلها منطقة المجسم.

W

weighted average The sum of the product of the number of units and the value per unit divided by the sum of the number of units, represented by M.

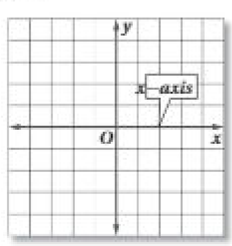
متوسط حسابي مرجح إجمالي حاصل ضرب عدد الوحدات والقيمة لكل وحدة مقسوم على مجموع عدد الوحدات التي يمثلها M.

whole numbers The set {0, 1, 2, 3, ...}.

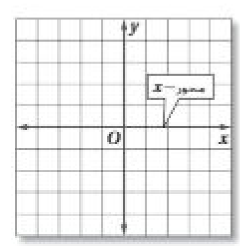
أعداد صحيحة مجموعة تضم {0، 1، 2، 3، ...}.

X

x-axis The horizontal number line on a coordinate plane.



المحور x خط الأعداد الأفقي على مستوى إحداثي.



x-coordinate The first number in an ordered pair.

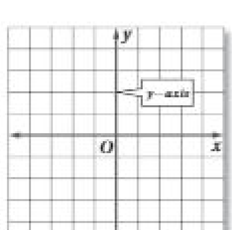
الإحداثيx العدد الأول في الزوج المرتب.

x-intercept The x-coordinate of a point where a graph crosses the x-axis.

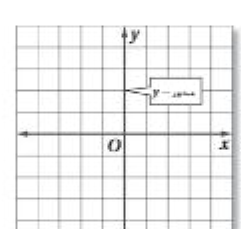
تقاطع مع المحور x الإحداثي x للنقطة التي يتقاطع عندها الرسم البياني مع المحور x.

Y

y-axis The vertical number line on a coordinate plane.



المحور y خط الأعداد الرأسي على مستوى إحداثي.



Glossary/القاموس

y-coordinate The second number in an ordered pair.

الإحداثي y العدد الثاني في الزوج المرتب.

y-intercept The y-coordinate of a point where a graph crosses the y-axis.

تقاطع مع المحور y الإحداثي y للنقطة التي يتعارض عندها الرسم البياني مع المحور y.

Z

zero The x-intercepts of the graph of a function; the points for which $f(x) = 0$.

صفر تقاطعات مع المحور x في التمثيل البياني لدالة ما، وتكون نقاطها $f(x) = 0$.

zero exponent For any nonzero number a, $a^0 = 1$. Any nonzero number raised to the zero power is equal to 1.

أس صفري بالنسبة إلى أي عدد غير الصفر a، $a^0 = 1$. وأي عدد غير الصفر مرفوع إلى الأس الصفري يساوي 1.

Symbols

$\neq$	is not equal to	AB	measure of $\overline{AB}$
$\approx$	is approximately equal to	$\angle$	angle
$\sim$	is similar to	$\triangle$	triangle
$>, \geq$	is greater than, is greater than or equal to	$^\circ$	degree
$<, \leq$	is less than, is less than or equal to	π	pi
$-a$	opposite or additive inverse of a	$\sin x$	sine of x
$\|a\|$	absolute value of a	$\cos x$	cosine of x
$\sqrt{a}$	principal square root of a	$\tan x$	tangent of x
$a : b$	ratio of a to b	!	factorial
(x, y)	ordered pair	$P(a)$	probability of a
$f(x)$	f of x, the value of f at x	$P(n, r)$	permutation of n objects taken r at a time
$\overline{AB}$	line segment AB	$C(n, r)$	combination of n objects taken r at a time

Algebraic Properties and Key Concepts

Identity	For any number a, $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
Substitution (=)	If $a = b$, then a may be replaced by b.
Reflexive (=)	$a = a$
Symmetric (=)	If $a = b$, then $b = a$.
Transitive (=)	If $a = b$ and $b = c$, then $a = c$.
Commutative	For any numbers a and b, $a + b = b + a$ and $a \cdot b = b \cdot a$.
Associative	For any numbers a, b, and c, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
Distributive	For any numbers a, b, and c, $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.
Additive Inverse	For any number a, there is exactly one number $-a$ such that $a + (-a) = 0$.
Multiplicative Inverse	For any number $\frac{a}{b}$, where $a, b \neq 0$, there is exactly one number $\frac{b}{a}$ such that $\frac{a}{b} \cdot \frac{b}{a} = 1$.
Multiplicative (0)	For any number a, $a \cdot 0 = 0 \cdot a = 0$.
Addition (=)	For any numbers a, b, and c, if $a = b$, then $a + c = b + c$.
Subtraction (=)	For any numbers a, b, and c, if $a = b$, then $a - c = b - c$.
Multiplication and Division (=)	For any numbers a, b, and c, with $c \neq 0$, if $a = b$, then $ac = bc$ and $\frac{a}{c} = \frac{b}{c}$.
Addition (>)*	For any numbers a, b, and c, if $a > b$, then $a + c > b + c$.
Subtraction (>)*	For any numbers a, b, and c, if $a > b$, then $a - c > b - c$.
Multiplication and Division (>)*	For any numbers a, b, and c, 1. if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$. 2. if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
Zero Product	For any real numbers a and b, if $ab = 0$, then $a = 0$, $b = 0$, or both a and b equal 0.
Square of a Sum	$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$
Square of a Difference	$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$
Product of a Sum and a Difference	$(a + b)(a - b) = (a - b)(a + b) = a^2 - b^2$

** These properties are also true for $<$, $\geq$, and $\leq$.*

Formulas

Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Distance on a coordinate plane	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Midpoint on a coordinate plane	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Pythagorean Theorem	$a^2 + b^2 = c^2$
Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Perimeter of a rectangle	$P = 2\ell + 2w$ or $P = 2(\ell + w)$
Circumference of a circle	$C = 2\pi r$ or $C = \pi d$

Area

rectangle	$A = \ell w$	**trapezoid**	$A = \frac{1}{2}h(b_1 + b_2)$
parallelogram	$A = bh$	**circle**	$A = \pi r^2$
triangle	$A = \frac{1}{2}bh$		

Surface Area

cube	$S = 6s^2$	**regular pyramid**	$S = \frac{1}{2}P\ell + B$
prism	$S = Ph + 2B$	**cone**	$S = \pi r\ell + \pi r^2$
cylinder	$S = 2\pi rh + 2\pi r^2$		

Volume

cube	$V = s^3$	**regular pyramid**	$V = \frac{1}{3}Bh$
prism	$V = Bh$	**cone**	$V = \frac{1}{3}\pi r^2 h$
cylinder	$V = \pi r^2 h$		

Measures

Metric	Customary
Length	
1 kilometer (km) = 1000 meters (m)	1 kilometer (km) = 1760 meters (m)
1 meter = 100 centimeters (cm)	1 kilometer = 5280 meter (m)
1 centimeter = 10 millimeters (mm)	1 meter = 3 meter
	1 meter = 12 centimeters (cm)
	1 meter = 36 centimeters
Volume and Capacity	
1 liter (L) = 1000 milliliters (mL)	1 milliliter (mls) = 4 quarts (qt)
1 kiloliter (kL) = 1000 liters	1 milliliter = 128 fluid ounces (fl oz)
	1 quart = 2 pints (pt)
	1 pint = 2 cups (c)
	1 cup = 8 fluid ounces
Weight and Mass	
1 kilogram (kg) = 1000 grams (g)	1 ton (T) = 2000 kilograms (kg)
1 gram = 1000 milligrams (mg)	1 pound = 16 ounces (oz)
1 metric ton (t) = 1000 kilograms	

Formulas

Coordinate Geometry

Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Distance on a number line:	$d = \lvert a - b \rvert$
Distance on a coordinate plane:	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Distance in space:	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
Distance arc length:	$\ell = \frac{x}{360} \cdot 2\pi r$
Midpoint on a number line:	$M = \frac{a + b}{2}$
Midpoint on a coordinate plane:	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Midpoint in space:	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

Perimeter and Circumference

square	$P = 4s$	**rectangle**	$P = 2\ell + 2w$	**circle**	$C = 2\pi r$ or $C = \pi d$

Area

square	$A = s^2$	**triangle**	$A = \frac{1}{2}bh$
rectangle	$A = \ell w$ or $A = bh$	**regular polygon**	$A = \frac{1}{2}Pa$
parallelogram	$A = bh$	**circle**	$A = \pi r^2$
trapezoid	$A = \frac{1}{2}h(b_1 + b_2)$	**sector of a circle**	$A = \frac{x}{360} \cdot \pi r^2$
rhombus	$A = \frac{1}{2}d_1 d_2$ or $A = bh$		

Lateral Surface Area

prism	$L = Ph$	**pyramid**	$L = \frac{1}{2}P\ell$
cylinder	$L = 2\pi rh$	**cone**	$L = \pi r\ell$

Total Surface Area

prism	$S = Ph + 2B$	**cone**	$S = \pi r\ell + \pi r^2$
cylinder	$S = 2\pi rh + 2\pi r^2$	**sphere**	$S = 4\pi r^2$
pyramid	$S = \frac{1}{2}P\ell + B$		

Volume

cube	$V = s^3$	**pyramid**	$V = \frac{1}{3}Bh$
rectangular prism	$V = \ell wh$	**cone**	$V = \frac{1}{3}\pi r^2 h$
prism	$V = Bh$	**sphere**	$V = \frac{4}{3}\pi r^3$
cylinder	$V = \pi r^2 h$		

Equations for Figures on a Coordinate Plane

slope-intercept form of a line	$y = mx + b$	**circle**	$(x - h)^2 + (y - k)^2 = r^2$
point-slope form of a line	$y - y_1 = m(x - x_1)$		

Trigonometry

Law of Sines	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	**Law of Cosines**	$a^2 = b^2 + c^2 - 2bc \cos A$ $b^2 = a^2 + c^2 - 2ac \cos B$ $c^2 = a^2 + b^2 - 2ab \cos C$
Pythagorean Theorem	$a^2 + b^2 = c^2$		

Symbols

$\neq$	is not equal to	$\parallel$	is parallel to	$\lvert\overrightarrow{AB}\rvert$	magnitude of the vector from A to B
$\approx$	is approximately equal to	$\nparallel$	is not parallel to	A'	the image of preimage A
$\cong$	is congruent to	$\perp$	is perpendicular to	$\rightarrow$	is mapped onto
$\sim$	is similar to	$\triangle$	triangle	$\odot A$	circle with center A
$\angle$, $\measuredangle$	angle, angles	$>, \geq$	is greater than, is greater than or equal to	π	pi
$m\angle A$	degree measure of $\angle A$	$<, \leq$	is less than, is less than or equal to	$\widehat{AB}$	minor arc with endpoints A and B
$^\circ$	degree	$\square$	parallelogram	$\widehat{ABC}$	major arc with endpoints A and C
$\overleftrightarrow{AB}$	line containing points A and B	n-gon	polygon with n sides	$m\widehat{AB}$	degree measure of arc AB
$\overline{AB}$	segment with endpoints A and B	$a:b$	ratio of a to b	$f(x)$	f of x, the value of f at x
$\overrightarrow{AB}$	ray with endpoint A containing B	(x, y)	ordered pair	!	factorial
AB	measure of $\overline{AB}$, distance between points A and B	(x, y, z)	ordered triple	${}_nP_r$	permutation of n objects taken r at a time
$\sim p$	negation of p, not p	$\sin x$	sine of x	${}_nC_r$	combination of n objects taken r at a time
$p \wedge q$	conjunction of p and q	$\cos x$	cosine of x	$P(A)$	probability of A
$p \vee q$	disjunction of p and q	$\tan x$	tangent of x	$P(A\|B)$	the probability of A given that B has already occurred
$p \longrightarrow q$	conditional statement, if p then q	$\vec{a}$	vector a		
$p \longleftrightarrow q$	biconditional statement, p if and only if q	$\overrightarrow{AB}$	vector from A to B		

Measures

Metric	Customary
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Volume and Capacity	
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Weight and Mass	
1 kilogram (kg) = 1000 grams (g) 1 gram = 1000 milligrams (mg) 1 metric ton (t) = 1000 kilograms	1 ton (T) = 2000 kilograms (km) 1 kilogram = 16 ounces (oz)

Formulas

Coordinate Geometry			
Midpoint	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	**Distance**	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
		Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}, x_2 \neq x_1$

Matrices			
Adding	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$	**Multiplying by a Scalar**	$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$
Subtracting	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$	**Multiplying**	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ab+bg & af-bh \\ ce+dg & cf-dh \end{bmatrix}$

Polynomials			
Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$	**Square of a Difference**	$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$
Square of a Sum	$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$	**Product of Sum and Difference**	$(a + b)(a - b) = (a - b)(a + b) = a^2 - b^2$

Logarithms			
Product Property	$\log_x ab = \log_x a + \log_x b$	**Power Property**	$\log_b m^p = p \log_b m$
Quotient Property	$\log_x \frac{a}{b} = \log_x a - \log_x b, b \neq 0$	**Change of Base**	$\log_a n = \frac{\log_b n}{\log_b a}$

Conic Sections			
Parabola	$y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$	**Ellipse**	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, a, b \neq 0$
Circle	$x^2 + y^2 = r^2$ or $(x - h)^2 + (y - k)^2 = r^2$	**Hyperbola**	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, a, b \neq 0$

Sequences and Series			
***n*th term, Arithmetic**	$a_n = a_1 + (n - 1)d$	***n*th term, Geometric**	$a_n = a_1 r^{n-1}$
Sum of Arithmetic Series	$S_n = n\left(\frac{a_1 + a_2}{2}\right)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$	**Sum of Geometric Series**	$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ or $S_n = \frac{a_1 - a_n r}{1 - r}, r \neq 1$

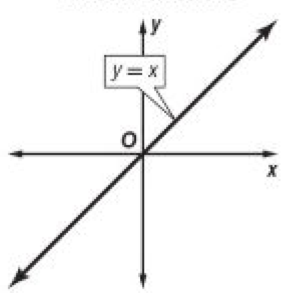
Trigonometry			
Law of Sines	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}, a, b, c \neq 0$		
Law of Cosines	$a^2 = b^2 + c^2 - 2bc \cos A$	$b^2 = a^2 + c^2 - 2ac \cos B$	$c^2 = a^2 + b^2 - 2ab \cos C$
Trigonometric Functions	$\sin \theta = \frac{\text{opp}}{\text{hyp}}$	$\cos \theta = \frac{\text{adj}}{\text{hyp}}$	$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{\sin \theta}{\cos \theta}$
	$\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin \theta}$	$\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos \theta}$	$\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{\cos \theta}{\sin \theta}$
Pythagorean Identities	$\cos^2 \theta + \sin^2 \theta = 1$	$\tan^2 \theta + 1 = \sec^2 \theta$	$\cot^2 \theta + 1 = \csc^2 \theta$

Symbols

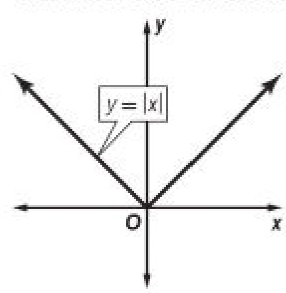
$f(x) = \{$	piecewise-defined function
$f(x) = \|x\|$	absolute value function
$f(x) = [\![x]\!]$	function of greatest integer not greater than a
$f(x, y)$	f of x and y, a function with two variables, x and y
$\overrightarrow{AB}$	vector AB
i	the imaginary unit
$[f \circ g](x)$	f of g of x, the composition of functions f and g
$f^{-1}(x)$	inverse of $f(x)$
$b^{\frac{1}{n}} = \sqrt[n]{b}$	nth root of b
$\log_b x$	logarithm base b of x
$\log x$	common logarithm of x
$\ln x$	natural logarithm of x
$\sum$	sigma, summation
$\bar{x}$	mean of a sample
μ	mean of a population
s	standard deviation of a sample
σ	standard deviation of a population
$P(B \| A)$	the probability of B given that A has already occurred
nPr	permutation of n objects taken r at a time
nCr	combination of n objects taken r at a time
$\text{Sin}^{-1} x$	Arcsin x
$\text{Cos}^{-1} x$	Arccos x
$\text{Tan}^{-1} x$	Arctan x

Parent Functions

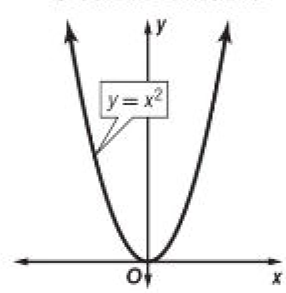
Linear Functions



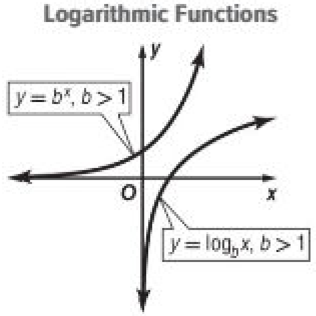
Absolute Value Functions



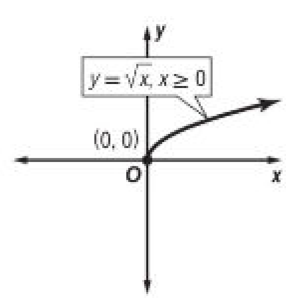
Quadratic Functions



Exponential and Logarithmic Functions



Square Root Functions



Reciprocal and Rational Functions