3

CIRCUIT THEOREMS AND METHODS OF CIRCUIT ANALYSIS

3.1 INTRODUCTION

Circuit analysis is finding the current and voltage on every element of the circuit being analyzed. In previous chapters we addressed solving circuits using Ohm's and Kirchhoff's laws. This chapter will enhance your portfolio of circuit-solving techniques by introducing new circuit methods of analysis. The methods covered in this chapter are superposition, Thévenin's, Norton's, Mesh, and Nodal methods. But why do we need so many more methods? The answer is an issue of practicality. Solving a circuit becomes easier with more knowledge of different methods. This helps the person solving a circuit in several ways. Many times the number of variables in a circuit is too large, and thus difficult to solve by hand. If we have a computer to solve the circuit, why do I care about the number of variables? Well so far in our world, computers are faster but may not always generate the correct answer. It is important that we, as circuit analysis engineers, have at least a rough idea if the numerical answer that the computer will provide is within reasonable expectations. It is always important to be able to do a rough analysis to understand if computer findings are at least meaningful for the given circuit and within the expected range. All the following methods will ultimately allow us to use a *checks and balances* approach to circuit solving.

Electrical, Electronics, and Digital Hardware Essentials for Scientists and Engineers, First Edition. Ed Lipiansky.

3.2 THE SUPERPOSITION METHOD

The superposition method is applicable and valid for solving circuits if the circuit is linear. In a general sense, a function $f(x)$ is said to be linear if-and-only-if* the following conditions are met:

1. Function $f(x)$ domain and its range are linear spaces over the same scalar field.
2. *Homogeneity Property*: For all values of x in the function domain and every scalar α then

$$f(\alpha x) = \alpha f(x). \tag{3.1}$$

3. *Additivity Property:* For every pair of element domains x_1 and x_2, the following holds:

$$f(x_1 + x_2) = f(x_1) + f(x_2). \tag{3.2}$$

It can be observed that Equation (3.1) holds when $f(x)$ is a linear function.

In general a function $f(x)$, that has the form:

$$f(x) = ax + b, \tag{3.3}$$

where a is the slope of the line and b is its y-intercept, is said to be linear only if its y-intercept is zero. That is,

$$f(0) = 0. \tag{3.4}$$

Important Points:

$$f(\mathrm{x}) = ax + b, \tag{3.5}$$

for $b \neq 0$, *is NOT a linear function.* However, Equation (3.5) is still the equation of a straight line. The function

$$f(x) = ax \tag{3.6}$$

is linear for any value of a and x [1].

Graphically we state that a line that goes through the origin of coordinates is a linear function. However, a line, whose equation does not go through the

* The symbol "⇔" stands for "if-and-only-if," meaning that logically, it is a necessary and sufficient condition.

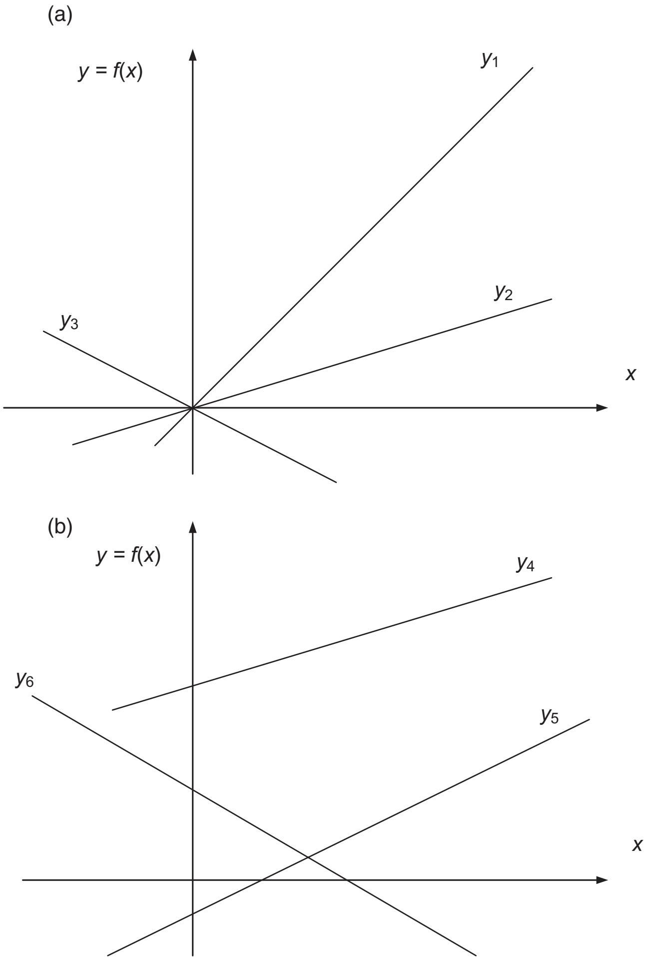

Figure 3.1 (a) Lines that are linear functions; (b) lines that are *not* linear functions.

origin of coordinates, is *not* a linear function. Figure 3.1a,b show lines that are linear functions and lines that are not linear functions.

Let us present an example of the homogeneity property given that

$$f(x) = 3x \tag{3.7}$$

and $\alpha = 4.5$, we need to verify that Equation (3.1) holds for all values of x when applied to Equation (3.7):

$$f(4.5x) = 3(4.5x). \tag{3.8}$$

To prove that the homogeneity property holds, let us present Table 3.1.

Table 3.1 Table used to exemplify numerically the homogeneity property

Col 1	Col 2	Col 3	Col 4	Col 5
x	$\boldsymbol{f(x) = 3x}$	α	$\boldsymbol{\alpha f(x)}$	$\boldsymbol{f(\alpha x) = 4.5 \times (3x)}$
0	$f(0) = 3 \times (0) = 0$	4.5	$4.5 \times (0) = 0$	$4.5 \times [3 \times (0)] = 0$
1	$f(1) = 3 \times (1) = 3$	4.5	$4.5 \times (3) = 13.5$	$4.5 \times [3 \times (1)] = 13.5$
2	$f(2) = 3 \times (2) = 6$	4.5	$4.5 \times (6) = 27$	$4.5 \times [3 \times (2)] = 27$
3	$f(3) = 3 \times (3) = 9$	4.5	$4.5 \times (9) = 40.5$	$4.5 \times [3 \times (3)] = 40.5$
n	$f(\mathrm{n}) = 3 \times (\mathrm{n}) = 3\ \mathrm{n}$	4.5	$4.5 \times (3\ \mathrm{n}) = 13.5\ \mathrm{n}$	$4.5 \times [3 \times (\mathrm{n})] = 13.5\ \mathrm{n}$

Note: Col stands for Column.

The entries of this table are:

Column 1: x
Column 2: $f(x) = 3x$
Column 3: α
Column 4: $\alpha f(x)$ and
Column 5: $f(\alpha x)$

By inspection of Table 3.1's columns 4 and 5, it is clear that Equation (3.1) holds. Without loss of generality we can state Equation (3.1) will hold for the infinitely many values of x for the given $f(x)$ and for any given value of α.

In reference to the additivity property we will prove that it is met by a linear function in a graphical way (Fig. 3.2).

By inspection of Figure 3.2 we can see that Equation (3.2) holds, repeated for the reader's convenience

$$f(x_1 + x_2) = f(x_1) + f(x_2).$$

Homogeneity and additivity properties together are completely equivalent to stating that a linear function complies with the *superposition property*:

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2). \tag{3.9}$$

So when a function complies with Equation (3.9), it is said to be linear. Conversely, when a function is linear, it complies with Equation (3.9).

Logically, the above is stated as follows:

Given $f(x)$, a function whose domain is x, and its range $f(x)$ is a

$$\textit{Linear function} \Leftrightarrow f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2).$$

Example 3.1 Prove that the equation of a straight line $f(x) = 4x + 7$, not passing through the origin of coordinates (i.e., $b \neq 0$), is not a linear function of x.

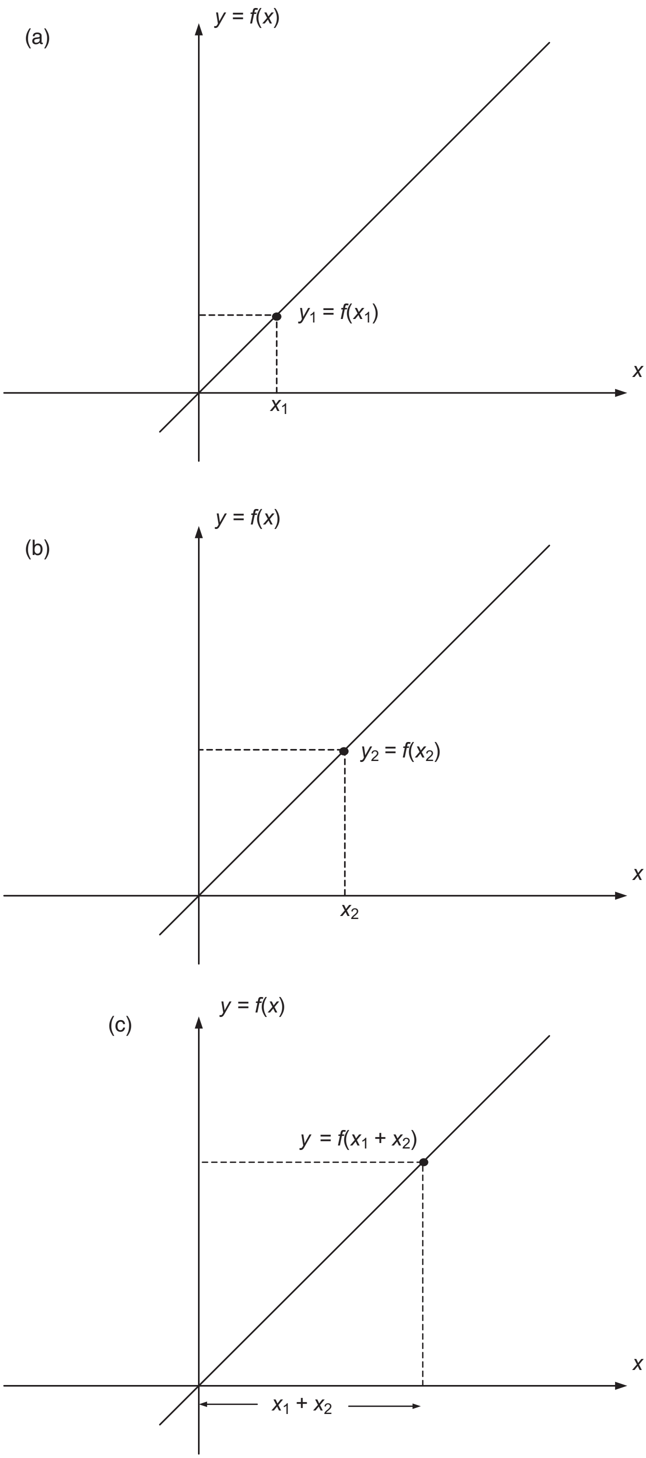

Figure 3.2 The validity of the additivity property for a linear function. (a) Linear function evaluated at x_1: $f(x_1)$; (b) linear function evaluated at x_2: $f(x_2)$; (c) linear function evaluated at $x_1 + x_2$: $f(x_1 + x_2) = f(x_1) + f(x_2)$.

Solution to Example 3.1

Simply using the homogeneity property, Equation (3.1), $f(\alpha x) = \alpha f(x)$.

It can be seen that $f(x) = 4x + 7$ is not a linear function because

$$f(\alpha x) = 4\alpha x + 7 \tag{3.10}$$

and

$$\alpha f(x) = \alpha(4x + 7) = 4\alpha x + 7\alpha. \tag{3.11}$$

From Equations (3.10) and (3.11) we see that

$$f(\alpha x) \neq \alpha f(x). \tag{3.12}$$

Thus, function $f(x) = 4x + 7$, the equation of a straight line, is *not* a linear function from the standpoint that it does not comply with Equation (3.9). Nonetheless, $f(x) = 4x + 7$ is the equation of a straight line.

3.2.1 Circuits Superposition

Let us now apply the superposition property to electric circuits. Assume that we are given an electrical circuit that can contain any number of resistors, in the black box represented in Figure 3.3. Two external voltage sources are applied to the circuit. We also refer to these two voltage sources as the circuit excitations. The output of the circuit is referred to as the circuit response.

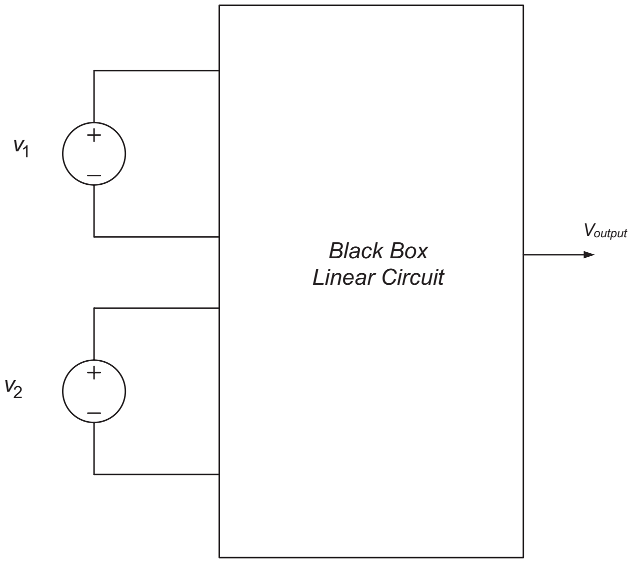

Figure 3.3 Electrical linear circuit with two external voltage sources: v_1 and v_2.

If we have a linear circuit where x is the excitation and $y = f(x)$ is its response, the superposition property tells us that

Given:

$y_1 = f(v_1)$, where y_1 is the response of the circuit due to excitation v_1 and
$y_2 = f(v_2)$, where y_2 is the response of the circuit due to excitation v_2.

The sum of the circuit responses $y_1 + y_2 = f(v_1) + f(v_2)$ equals the response of the sum of the circuit excitations $y_1 + y_2 = f(v_1 + v_2)$.

Moreover, thanks to the linearity of the circuit, we can also calculate the response of the circuit to excitation v_1 while excitation v_2 is inhibited. This yielding the response y_1 for $v_2 =$ inhibited. Similarly, we can calculate the response of the circuit y_2 when excitation v_1 is inhibited. Finally, adding the individually found responses we obtain

$$y_{1(\text{for } v2=\text{inhibited})} + y_{2(\text{for } v1=\text{inhibited})}. \tag{3.13}$$

Equation (3.13) provides the complete response of the circuit due to noninhibited excitations or the response of the circuit due to both excitations applied simultaneously.

When the excitation is a voltage source v, inhibiting the excitation means to replace the voltage source with a short circuit ($v = 0$). When the excitation is a current source i, inhibiting the excitation means to remove the current source from the circuit, or open-circuiting the current source.

To follow up with the circuit given in Figure 3.3, we can solve the circuit by superposition, which means by applying one excitation at a time, while inhibiting the other one. The complete response of the circuit is obtained by adding each of the individual responses as Figure 3.4a,b show. So the total response of the circuit is

$$V_{output} = V_{output\text{-}due\text{-}to\text{-}v1} + V_{output\text{-}due\text{-}to\text{-}v2}. \tag{3.14}$$

So now we may ask the question, why is it better to use superposition to solve a circuit, if it seems that the number of steps grows in the process? So let us address this question with an example.

Example 3.2 Given the circuit of Figure 3.5, find the current I_2 through resistor R_2 using superposition.

Now let us apply superposition. Calculate the current through R_2 but only due to the presence of the $V = 12$ V excitation, removing or open-circuiting current source I. We obtain the circuit shown in Figure 3.6, which clearly is simpler to solve than the original circuit of Figure 3.5. By inspection of circuit in Figure 3.6, the 12 V source is applied directly across R_1, thus the current through R_1 is easily calculated using Ohm's law:

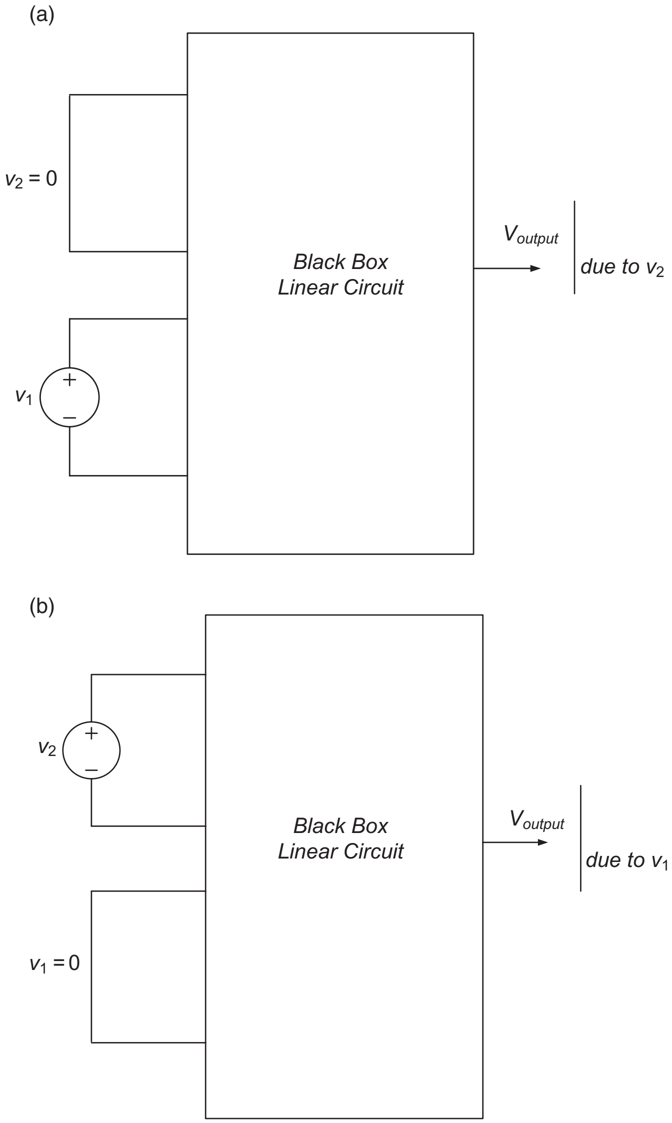

Figure 3.4 Application of superposition to the circuit of Figure 3.3: (a) circuit response due to v_1 when $v_2 = 0$; (b) circuit response due to v_2 when $v_1 = 0$.

$$I_1 = 12\ \mathrm{V}/6\ \Omega. \tag{3.15}$$

Now let us calculate the current through the series of R_2 and R_3. Thus, current I_{23} is

$$I_{23} = \mathrm{V}/(\mathrm{R}_2 + \mathrm{R}_3). \tag{3.16}$$

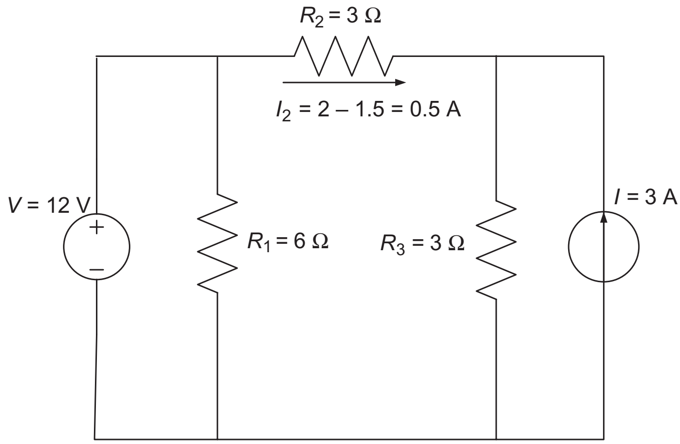

Figure 3.5 Circuit for Example 3.2.

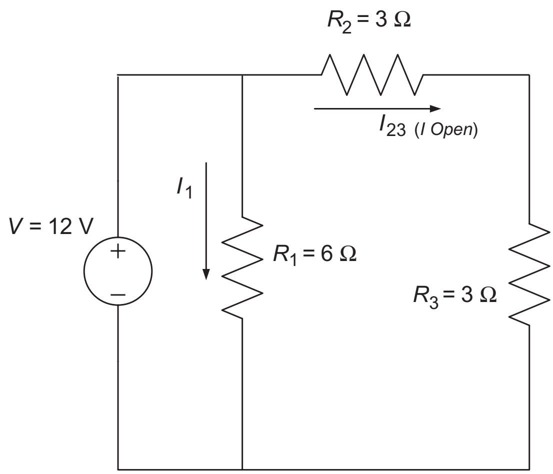

Figure 3.6 Example 3.2: Removing current source *I* and applying superposition under the effect of *V*.

Using the numerical values from Figure 3.6 leads to

$$I_{23} = 12\text{ V}/(3+3)\ \Omega = 2\text{ A}. \tag{3.17}$$

Now we need to calculate the current that flows through R_2 when excitation V = 12 V is replaced by a short circuit. We present this circuit in Figure 3.7.

By close examination we can see that R_1 is short-circuited, so basically only R_2 and R_3 are in parallel with the 3 A current source. This circuit is shown in Figure 3.8.

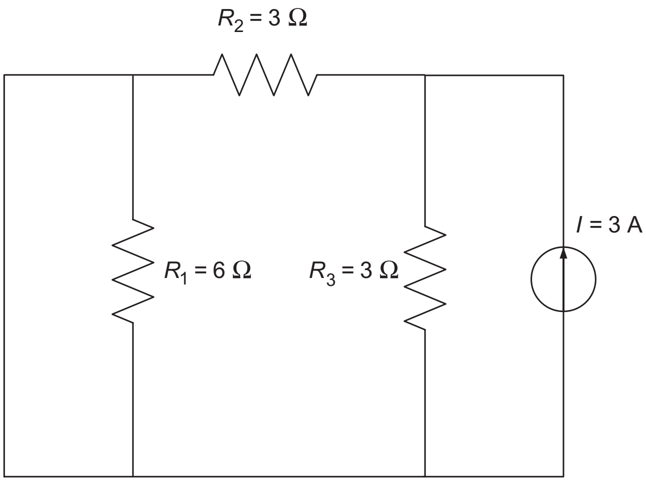

Figure 3.7 Removing voltage source V and applying superposition under the effect of current I.

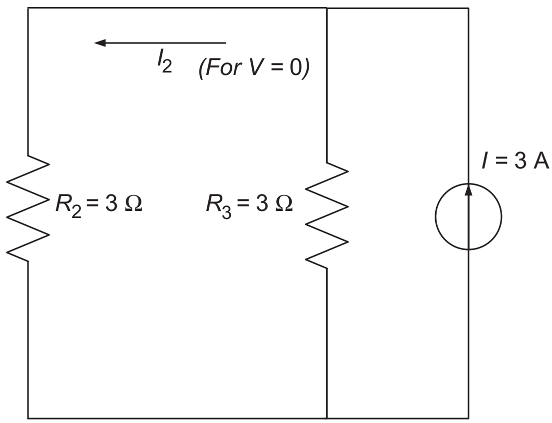

Figure 3.8 Example 3.2: Eliminating the short circuited resistor R_1.

By inspection of Figure 3.8 note that the circuit was redrawn eliminating the presence of R_1, because it was short-circuited (see Fig. 3.7). Clearly we see now that the 3-A current source I delivers current to two equal valued resistors in parallel R_2 and R_3. From Kirchoff's current law (KCL) we know that the current has to be divided equally between R_2 and R_3. Thus, the current flowing through R_2, after the elimination of the 12-V voltage source, is 1.5 A.

Finally, to complete the application of superposition for Example 3.2, we present the previously obtained currents that flow through R_2. When the 3-A current source was removed, the current through R_2 flowed from left to right, as shown in Figure 3.9, under $I_{due\ to\ V=12\ \mathrm{V}}$. When the 12-V voltage source was

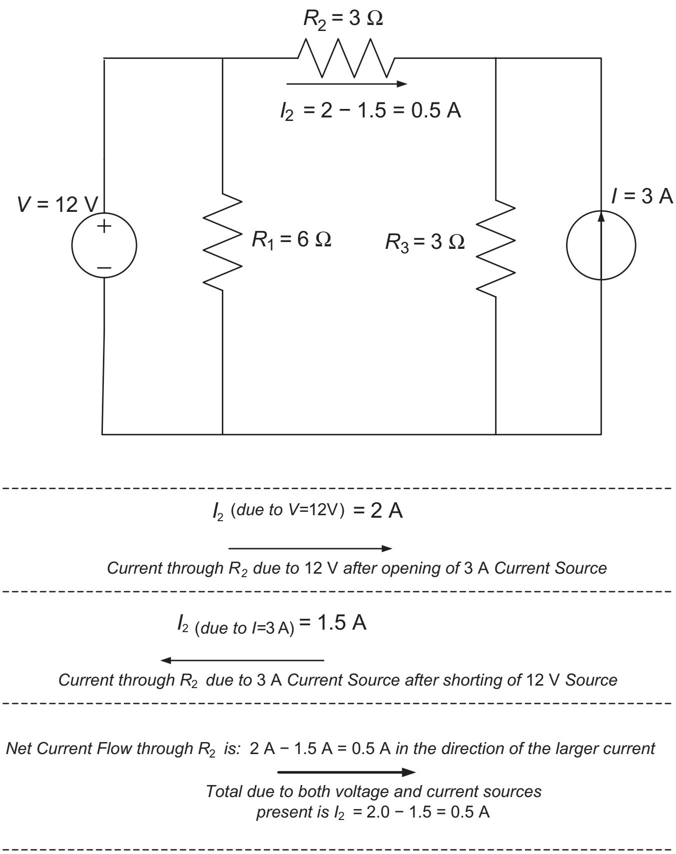

Figure 3.9 Net current flowing through resistor R_2.

removed, the current through R_2 flowed from right to left, as shown in Figure 3.9, under $I_{due\ to\ I=3\ \mathrm{A}}$.

The net resulting current of 0.5 A, flows through R_2 due to the simultaneous effect of both the voltage and the current sources in the direction on the larger current of 2 A.

3.3 THE THÉVENIN METHOD

The Thévenin method is very powerful since it allows one to replace a large linear circuit with a voltage source and a resistor in series. Such voltage is referred to as the Thévenin voltage and the resistor is called the Thévenin resistor. If we are dealing with an AC circuit, the term resistor is replaced with impedance. From Chapter 2, impedance is a combination of R, L, and C circuit elements.

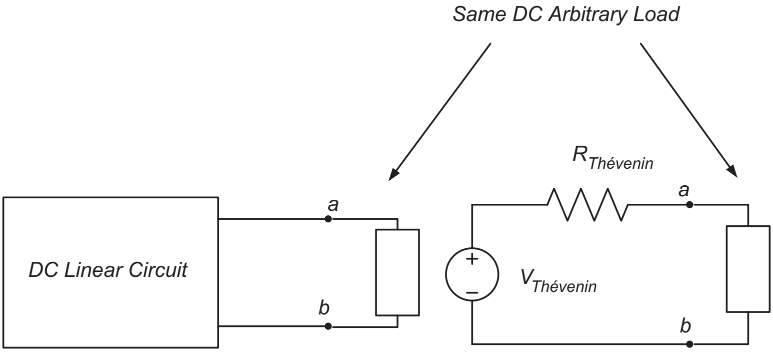

Figure 3.10 DC Thévenin equivalent of a DC circuit.

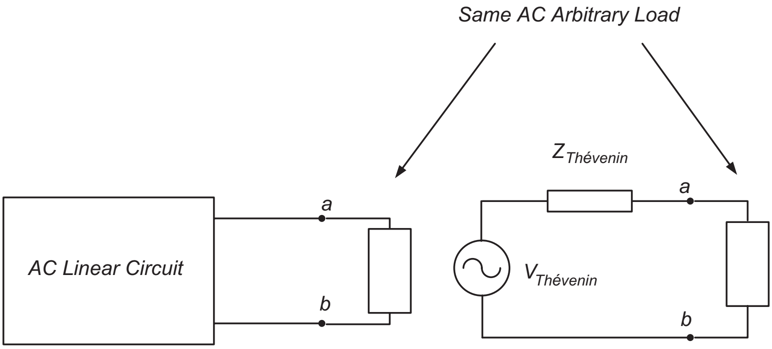

Figure 3.11 AC Thévenin equivalent of an AC circuit.

The linear circuit, to be replaced, can contain any number of circuit elements*, independent and/or dependent voltage, and current sources. It is important to note that the controlling voltage or current of the dependent sources need to reside within the same linear circuit that is to be replaced with the Thévenin equivalent circuit.

Figure 3.10 shows a DC linear circuit and its Thévenin equivalent circuit. The Thévenin equivalent voltage is a DC source for the DC case. Figure 3.11 shows an AC linear circuit and its Thévenin equivalent. The Thévenin equivalent voltage is an AC source for the AC case.

In both Figures 3.10 and 3.11, the load can even be a nonlinear load, it is not required for it to be linear, as the circuit that will be replaced with its Thévenin equivalent does. The arbitrary load may also have dependent voltage or current sources. Their controlling voltage or current shall be within the arbitrary load circuit itself.

* Circuit elements refer to resistors in DC circuits; but it refers to resistors, inductors, and capacitors in AC circuits.

3.3.1 Application of the Thévenin Method

Methodology to find the Thévenin equivalent circuit: (as it applies to Example 3.3, Fig. 3.12)

1. First slice the portion of the circuit that we want to find the Thévenin equivalent circuit at two nodes only. *This circuit has to be linear.*

   In our example we have drawn a dotted line *a-b*. The portion of the circuit that we want to *Thévenize* is on the left-hand side of the dotted line. Ensure that if dependent sources are present on the circuit to be *Thévenized*, the slicing of the circuit must not separate the dependent sources from their respective controlling variables.
2. The Thévenin voltage is calculated as follows: separate the circuit to be *Thévenized* at the terminals *a* and *b* from the associated circuit that is on the right-and side of the dotted line. The Thévenin voltage ($V_{Thévenin}$) is calculated as the *open-circuit voltage* across terminals *a* and *b*. In particular, this is the voltage across resistor R_3 for our example of Figure 3.12a.
3. To calculate the Thévenin resistance for DC circuits or the Thévenin impedance in AC circuits, inhibit in the linear circuit to be *Thévenized* all the independent voltage and current sources. To inhibit a voltage source, each source should be replaced with a short circuit. Open-circuiting or simply removing the current source from the circuit inhibits a current source. No action needs to be taken with dependent voltage or current sources. However, it is appropriate to remind the reader that any dependent sources in the linear circuit to be *Thévenized* must have their controlling variable within the same circuit to be *Thévenized*. Once all independent sources have been inhibited, calculate the resistance in the DC case (or the impedance in the AC case) seen across terminals *a* and *b*. The result is what we call the Thévenin resistance (or impedance) referred to as either R_{Th} (or Z_{Th}).
4. The entire circuit on the left of the dotted line can now be replaced by the series of the Thévenin voltage source and the Thévenin resistance (or impedance).

   Example 3.3, which follows, goes explicitly over a numerical application of Thévenin.

Example 3.3 Given the circuit of Figure 3.12a, find the Thévenin equivalent circuit at the left of the a-b dotted line. Use the Thévenin equivalent circuit found to calculate the current of the original circuit.

Referring to Figure 3.12a, we want to find the Thévenin equivalent of the circuit to the left of the a-b dotted line. Separating this circuit from the rest of the circuit on the right side of the a-b dotted line, we next find the open-circuit voltage across resistor R_3. This is the Thévenin voltage, and it is calculated as follows applying Kirchoff's voltage law (KVL):

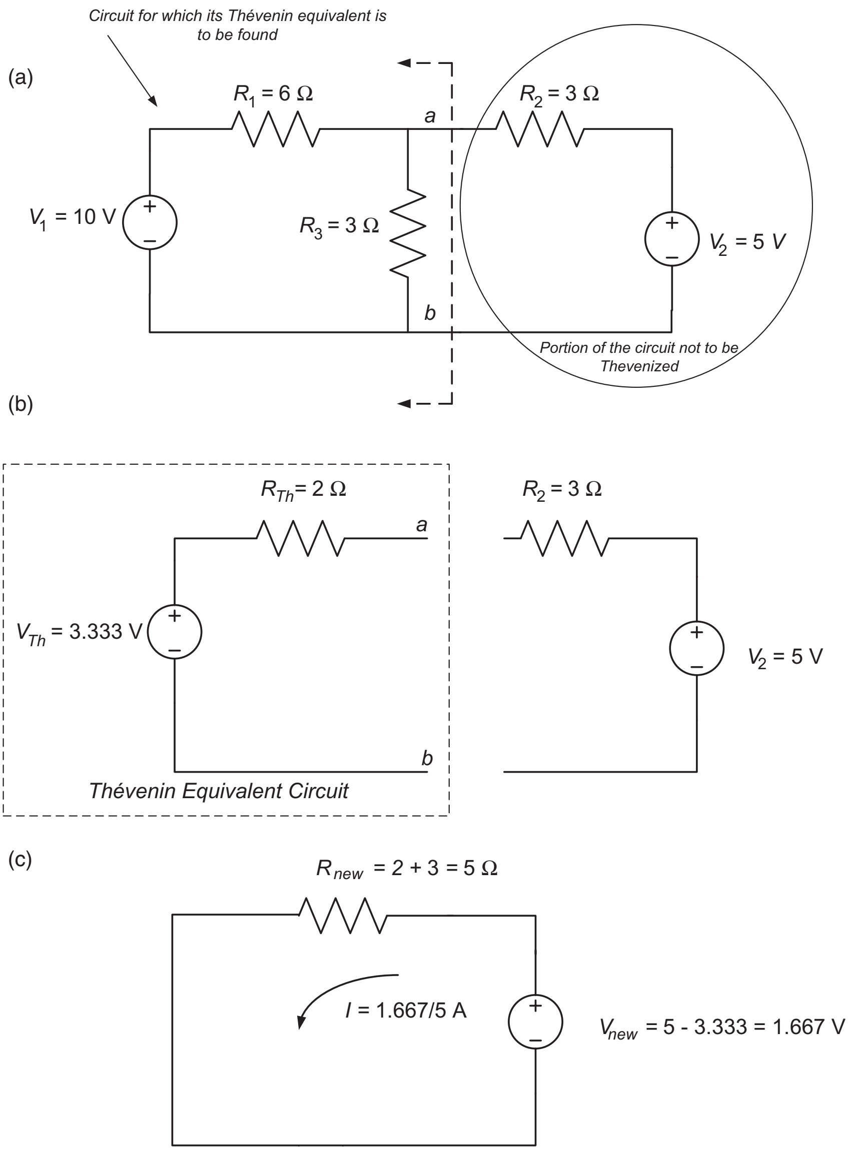

Figure 3.12 Circuit to apply the Thévenin method: (a) original circuit; (b) Thévenin equivalent on left of portion not to be Thévenized; (c) merging of the found Thévenin equivalent with the non-Thévenized portion of the circuit.

$$V_{Th} = V_1 R_3/(R_1 + R_3) \tag{3.18}$$

$$V_{Th} = 10 \times 3/(6+3) = 3.333 \text{ V}. \tag{3.19}$$

The Thévenin resistance R_{Th} is calculated by inhibiting (short-circuiting) the 10-V voltage source and calculating the resistance seen to the left of the *a-b* dotted line.

This yields the parallel of R_1 and R_3:

$$R_{Th} = (R_1 \times R_3)/(R_1 + R_3) = 2\ \Omega. \tag{3.20}$$

Next we substitute the circuit to the left of the *a-b* dotted line with its Thévenin equivalent, which is a 3.333-V DC source, from Equation (3.19) in series with $R_{Th} = 2\ \Omega$, from Equation (3.20). Finally, in Figure 3.12c we merged the Thévenin equivalent circuit with the rest of the untouched right-hand side original circuit. The resulting circuit is almost trivial and allows us to calculate the current in the circuit by Ohm's law in a straightforward fashion. We connect the found Thévenin equivalent circuit with the right-hand side circuit of Figure 3.12b. Combining the two voltage sources into V_{new}, we obtain Figure 3.12c:

$$I = (5 - 3.333)\text{ V}/5\ \Omega = 0.3334 \text{ A}. \tag{3.21}$$

Example 3.4 Given the circuit of Figure 3.13a, find the Thévenin equivalent circuit of the circuit to the left of the a-b dotted line. Reattach the equivalent circuit to resistor R_3 to calculate the current through R_3.

Figure 3.13a shows the originally given circuit, while Figure 3.13b shows the circuit to-be-Thévenized not connected to its load R_3. At this point we calculate the Thévenin voltage; the open-circuit voltage across terminals *a* and *b* needs to be determined. To proceed with this calculation, we will use the superposition method; refer to the circuit of Figure 3.13b. We will split the problem into two easier problems to solve. We will compute the voltage V_{ab} due to the effect of 8-V voltage source V, open-circuiting the current source I. This will yield $V_{Th\ due\ to\ 8\text{ V}}$. On the second step we calculate V_{ab} due to the effect of the 1-A current source I, short-circuiting the 8-V voltage source. This will yield $V_{Th\ due\ to\ 1\text{ A}}$. Upon obtaining those two partial voltages, the total voltage, which is the Thévenin voltage, is the algebraic sum of $V_{\text{Th due to 8 V}}$ and $V_{\text{Th due to 1 A}}$. An algebraic sum refers to performing a sum taking into account the signs or polarities of the voltages involved. In our particular case, both polarities are positive.

Let us refer to Figure 3.14a and b to see how we partition the originally given circuit (Fig. 3.13a) into two separate circuits, each of which will be driven by one of the sources while the other source is *inhibited*.

Referring to Figure 3.14a we calculate $V_{Th\ due\ to\ 8\text{ V}}$ as follows:

$$V_{Th\ due\ to\ 8\text{ V}} = VR_2/(R_1 + R_2). \tag{3.22}$$

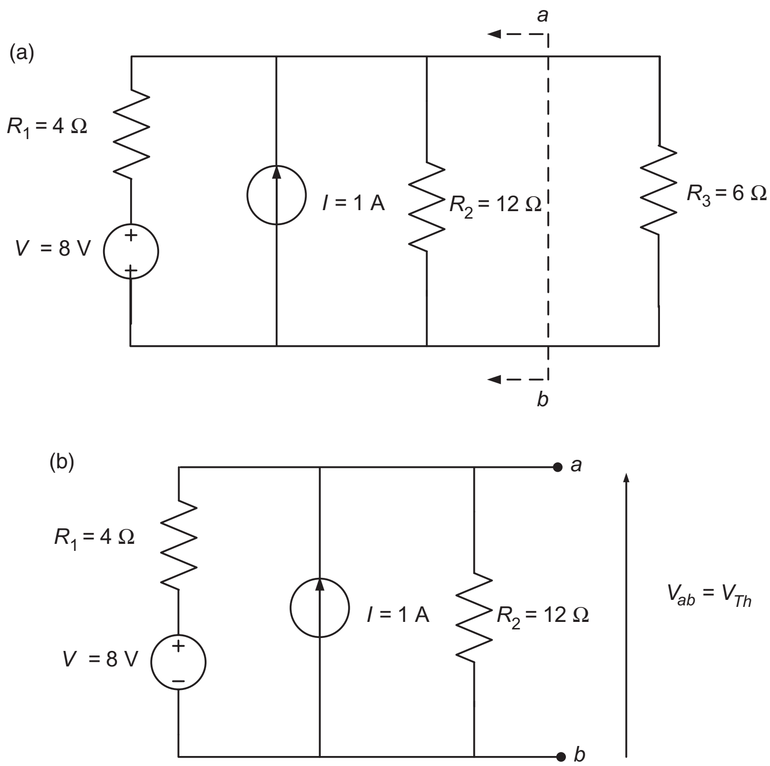

Figure 3.13 Circuit for Example 3.4: (a) original circuit; (b) sliced circuit to which Thévenin is applied.

Using the respective values in Equation (3.22) from Figure 3.14a, we obtain

$$V_{Th\ due\ to\ 8\ \mathrm{V}} = 8 \times 12/(4+12) = 6\ \mathrm{V}. \tag{3.23}$$

Now referring to Figure 3.13b, we calculate $V_{Th\ due\ to\ 1\ \mathrm{A}}$ as follows:

By KCL we can see by inspection that

$$1 = (V_{ab}/R_1) + (V_{ab}/R_2), \tag{3.24}$$

where: V_{ab}/R_1 and V_{ab}/R_2 are respectively the currents through resistors R_1 and R_2.

Using the component values from Figure 3.13b into Equation (3.24) yields

$$1 = V_{ab}(1/4 + 1/12). \tag{3.25}$$

$$V_{Th\ due\ to\ 1\ \mathrm{A}} = V_{ab} = 3\ \mathrm{V}. \tag{3.26}$$

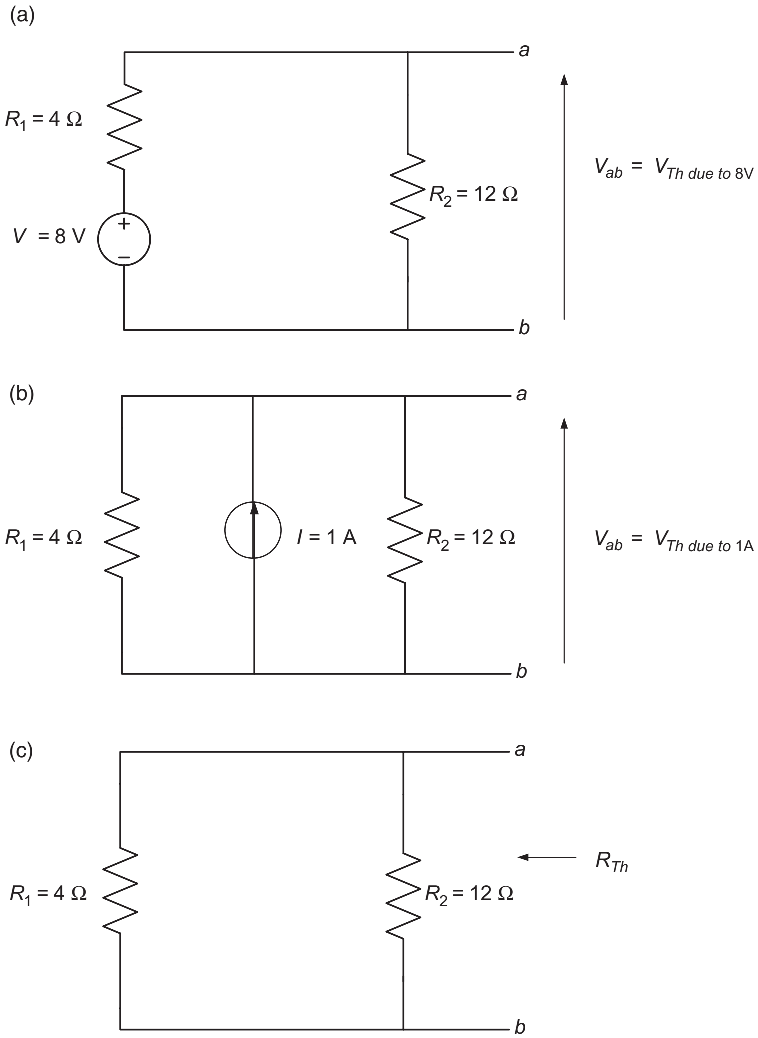

Figure 3.14 Superposition method for Thévenin Example 3.4: (a) effect of $V = 8$ V voltage source, 1 A current source open-circuited; (b) effect of $I = 1$ A current source, 8 V voltage source short-circuited; (c) elimination of all sources to compute the Thévenin resistance.

Now from the results of Equations (3.23) and (3.26) we add both voltages leading to

$$V_{Th\ due\ to\ 8\ \mathrm{V}} + V_{Th\ due\ to\ 1\ \mathrm{A}} = 6\ \mathrm{V} + 3\ \mathrm{V} = 9\ \mathrm{V}. \tag{3.27}$$

Now we need to calculate the Thévenin resistance (refer to Figure 3.14c). The voltage source is replaced by a short circuit and the current source is open-circuited. R_{Th} is simply the parallel of resistors R_1 and R_2. So now we have all

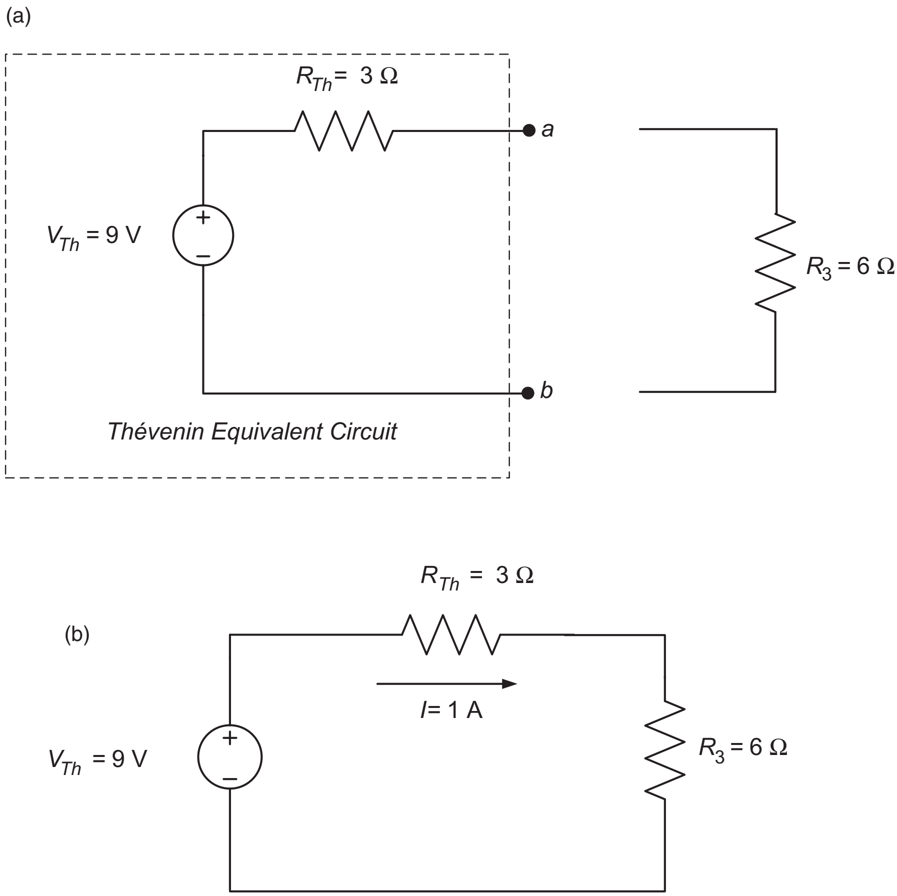

Figure 3.15 (a) Thévenin equivalent of Example 3.4; (b) Thévenin equivalent merged with the resistive load.

the elements of the Thévenin equivalent circuit. These are shown in Figure 3.15a. In Figure 3.15b, the Thévenin equivalent circuit is joined to resistor R_3. Finally, the current through R_3 is simply calculated using Ohm's law, leading to

$$I = V_{Th}(R_{Th} + R_3) = 9/(3+6) = 1 \text{ A}. \tag{3.28}$$

3.4 NORTON'S METHOD

Several decades after the invention of the Thévenin method, American engineer *Edward Norton* invented an analysis method which bears his name today. Norton's method of analysis is the dual of Thévenin's method. Duality, in

Table 3.2 Some dual pairs

Resistance	Conductance
Inductance	Capacitance
Voltage	Current
Voltage source	Current source
Node	Mesh
Open circuit	Short circuit
KVL	KCL
Thévenin	Norton
Elements in series	Elements in parallel

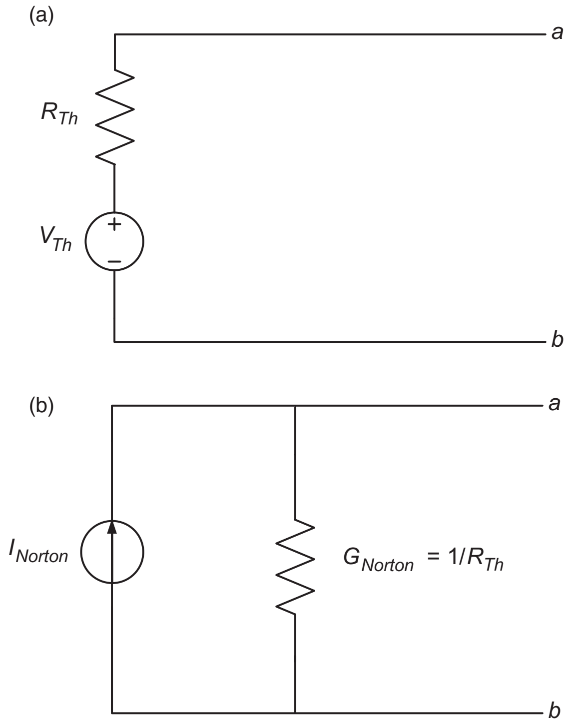

Figure 3.16 (a) Thévenin equivalent circuit; (b) Norton equivalent circuit.

circuit analysis, refers to circuits that can be described by the same set of equations and solutions, except that certain elements are interchanged.

Table 3.2 lists the dual-pair elements.

The dual of a Thévenin equivalent circuit is its Norton's equivalent.

Figure 3.16 shows the dual of Thévenin equivalent. Thévenin is a series of two elements; Norton is transformed by duality into a parallel of two elements. The Thévenin voltage source becomes a Norton current source. The Thévenin resistance becomes a Norton conductance. For more details on *duality in circuit theory*, the reader is referred to the Bibliography at the end of the chapter.

In Figure 3.16b the value of the Norton current source is given by

$$I_{Norton} = V_{Th}/R_{Th} \tag{3.29}$$

The Norton current is obtained short-circuiting the Thévenin equivalent circuit and calculating the current that flows through the short. So the Norton current source (I_{Norton}) is the short-circuit current in the Thévenin equivalent circuit.

And the Norton resistance is

$$R_{\mathrm{N}} = R_{Th}. \tag{3.30}$$

Note the Norton equivalent resistance is identical to the Thévenin equivalent resistance. For the sake of simplicity we will continue to use R_{Th} whether we use the Thévenin or the Norton equivalent circuits.

3.4.1 Source Transformations

Every voltage source in series with a resistance or impedance can be converted into a parallel equivalent of a current source in parallel with a conductance or admittance.

Equations (3.31) and (3.32) address the source transformations between Thévenin and Norton equivalent circuits. The equations below address the source transformation when we have an AC Thévenin source and in series with a Thévenin impedance.

$$\mathbf{I_{Norton} = V_{Th}/Z_{Th}} \tag{3.31}$$

and

$$\mathbf{Z_N = Z_{Th}}. \tag{3.32}$$

It is important to understand that the current, voltage, and impedance in Equations (3.31) and (3.32) are all phasors. Some textbooks define the Norton admittance $\mathrm{Y_N} = 1/\mathrm{Z_N}$, which is consistent with Equations (3.31) and (3.32).

Example 3.5 Given the Thévenin equivalent circuit of Figure 3.16a, find the Norton equivalent circuit. Assume that $V_{Th} = 24$ V and $R_{Th} = 4\ \Omega$.

Using the source transformation from Equations (3.31) and (3.32), the Norton current source is calculated as follows:

$$I_{Norton} = V_{Th}/R_{Th} = 24/4 = 6 \text{ A}. \tag{3.33}$$

The Norton equivalent resistance equals the Thévenin resistance:

$$R_{Th} = R_N = 4\ \Omega. \tag{3.34}$$

The source transformation method allows the conversion of a Thévenin equivalent circuit into a Norton equivalent circuit and vice versa. There is a direct way to obtain the Norton equivalent circuit from the given circuit, without previously finding its Thévenin equivalent. This is the topic of the following section.

3.4.2 Finding the Norton Equivalent Circuit Directly from the Given Circuit

The procedure follows:

1. Separate the circuit for which the Norton equivalent circuit is to be found from the rest of the circuit or its load. If there are any dependent voltage or current sources in the circuit for which the equivalent circuit is to be found, the dependent source and its control variable must reside within such circuit.
2. To find the Norton resistance, calculate it exactly as the Thévenin resistance was calculated. Inhibit all voltage and current sources in the circuit. That is, open-circuit all current sources, and short-circuit all voltage sources.
3. Calculate the Norton equivalent current source by shorting terminals a and b of the circuit whose Norton equivalent circuit is to be found. Refer to Figure 3.17. The Norton current source I_N is the calculated short-circuit current that flows through shorted terminals a and b.
4. The Norton equivalent of the original circuit is the parallel of I_N and R_{Th} (remember that R_{Th} and R_N are always identical).

Example 3.6 Find the Norton equivalent circuit of the circuit of Figure 3.17b. Use the found Norton equivalent circuit to calculate the load current I_L through resistor R_L, refer to Figure 3.17a.

By inspection of Figure 3.17b, inhibiting all voltage and current sources, we find that the Norton resistance equals the series of R_1 and R_2 in parallel with R_3. Thus,

$$R_N = R_{Th} = (R_1 + R_2)R_3/(R_1 + R_2 + R_3). \tag{3.35}$$

Using the given values for the resistors,

$$R_N = R_{Th} = (3+3)6/(3+3+6) = 3\ \Omega. \tag{3.36}$$

Let us calculate the Norton current source using Equation (3.33). Prior to this calculation, let us simplify the circuit from Figure 3.17b a little further.

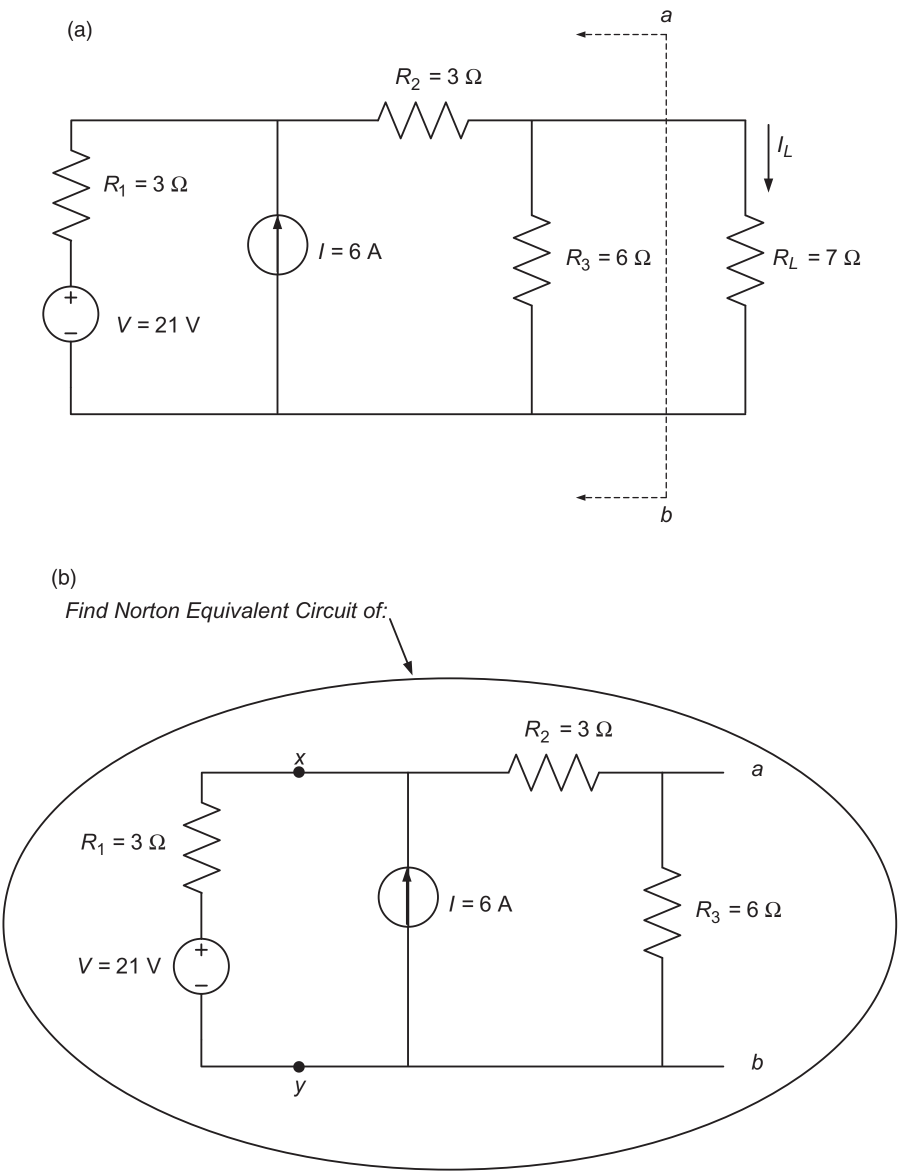

Figure 3.17 Finding the Norton equivalent circuit for Example 3.6: (a) original circuit; (b) circuit from which to find Norton's equivalent circuit.

By inspection of the circuit in Figure 3.17b note that V and R_1 are in series, on the left side of nodes x and y. We can do a source transformation of V and R_1 to convert them into a current source in parallel with a resistor.

$$I_{short\text{-}circuit} = I_{Norton} = V_{Th}/R_{Th} = V/R_1. \tag{3.37}$$

$$I_{short\text{-}circuit} = I_{Norton} = I_{ST} = 21/3 = 7\text{ A}. \tag{3.38}$$

The result of this source transformation is provided in Figure 3.18b. Note that after such source transformation, the 3 Ω resistor R_1 appears in Figure 3.18b renamed as R_{ST}. The Norton current calculated in the source transformation is presented as $I_{ST} = 7$ A. Note: The subscript ST stands for *source transformation*. Then, using the Norton equivalent circuit for the voltage source transformation and substituting it into the original circuit of Figure 3.17b, we obtain Figure 3.19a.

So let us combine the two current sources into one, thus 6 A + 7 A = 13 A and name this current I_{Combined} (refer to Figure 3.19b). Now let us short-circuit the terminals a and b and calculate the Norton short circuit current. This will be done in four steps, with Equations (3.39) through (3.42).

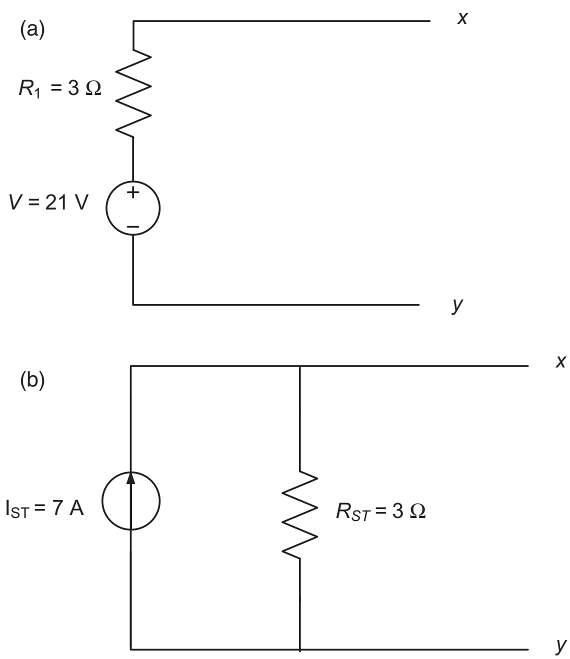

Figure 3.18 Source transformation as an interim step toward finding the Norton equivalent: (a) Thévenin equivalent; (b) Norton equivalent.

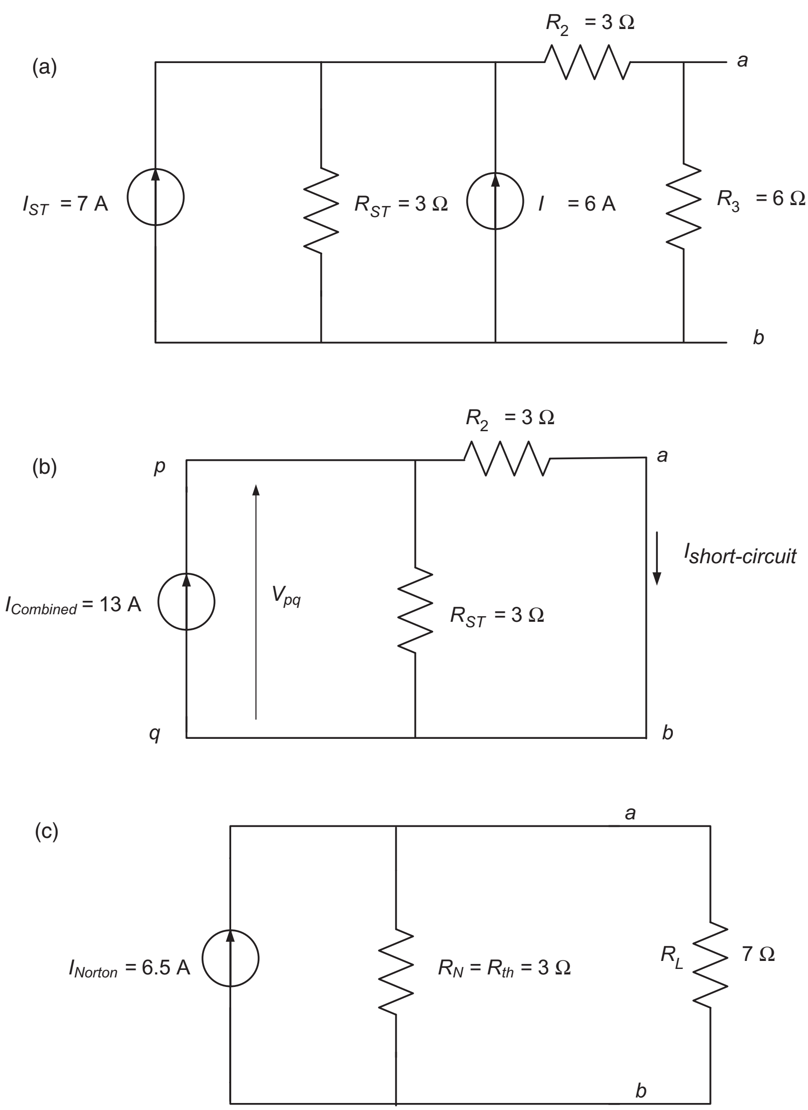

Figure 3.19 (a) Circuit of Example 3.6 after a source transformation; (b) circuit of Example 3.6 after combination of the two current sources; (c) final Norton equivalent circuit: Example 3.6.

From Figure 3.19b and since R_2 and R_{ST} are in parallel, we get

$$V_{pq} = I_{Combined} R_2 R_{ST} /(R_2 + R_{ST}), \tag{3.39}$$

where V_{pq} is the voltage across the terminals of the (13 A) $I_{Combined}$ current source.

Note: In Figure 3.19b, the shorting of terminals *a* and *b*, essentially eliminates resistor R_3 from the circuit seen in Figure 3.19a. Since this voltage V_{pq} is the same as the voltage across the parallel of R_{ST} and R_2, by Ohm's law:

$$I_{short\ circuit} = [I_{Combined} R_2 R_{ST} / (R_2 + R_{ST})] / R_2 \tag{3.40}$$

Eliminating R_2 from numerator and denominator:

$$I_{short\ circuit} = I_{Combined} R_{ST} / (R_2 + R_{ST}) \tag{3.41}$$

Now using the values for Equation (3.41) from Figure 3.19b:

$$I_{short\ circuit} = 13\ 3/(3+3) = 6.5\ \text{A}. \tag{3.42}$$

This short circuit current of 6.5 A will be the Norton equivalent current source of the originally provided circuit (Fig. 3.17b). The 3-Ω resistor R_{ST} from Equation (3.36) is the Thévenin resistor of the equivalent model, as it can be seen in Figure 3.19c.

The final step, Figure 3.19c, is to attach the 7-Ω load resistor R_L to the Norton equivalent circuit and calculate the current through R_L.

Thus, we obtain

$$V_{ab} = I_{Norton} [R_N R_L / (R_N + R_L)]. \tag{3.43}$$

Thus, the current through R_L is:

$$I_L = V_{ab} / R_L. \tag{3.44}$$

Plugging the value of V_{ab} from Equation (3.44) into Equation (3.43)

$$I_L = I_{Norton} [R_N / (R_N + R_L)]. \tag{3.45}$$

$$I_L = 1.95\ \text{A}. \tag{3.46}$$

3.5 THE MESH METHOD OF ANALYSIS

The mesh method of circuit analysis is based on KVL. It provides a more effective way of deriving circuit equations virtually by quick inspection of the circuit. The mesh method is more suitable and intuitive when the circuit contains independent voltage sources. The method is somewhat less intuitive when current sources are also included and probably the least intuitive when dependent voltage and current sources are also present. The mesh method solves for mesh currents as opposed to finding individual branch currents for every circuit branch. This is advantageous because the number of unknowns is somewhat reduced.

We will address the methodology of writing mesh equations for various circuits via examples that will grow in complexity.

The following assumptions when using the mesh method are made:

1. *All circuits that we will analyze are planar.* The mesh method does not work for non-planar circuits.
2. *A mesh is a closed loop that does not contain other loops within it.*

Planar circuits are those circuits that can be drawn on a plane without its branches crossing each other. Nonplanar circuits are those that cannot be drawn or redrawn without one or more branches crossing. Figure 3.20 presents examples of planar and nonplanar circuits. Nonplanar circuits are outside the scope of this book and are studied in advanced circuit analysis courses. Careful observation of Figure 3.20a reveals that the circuit is actually planar; however, at first sight it initially may appear nonplanar. Circuit Figure 3.20b is the exact same circuit as that in a. Finally, c is a true nonplanar circuit.

Figure 3.21 shows a simple circuit with two meshes: one of them is: a-b-c-d-a, the second one is b-e-f-c-b. It is important to observe that a-b-e-f-c-d-a is a loop and not a mesh, because it includes one previously defined mesh.

3.5.1 Establishing Mesh Equations. Circuits with Voltage Sources

Let us assume we have a two-mesh circuit as the one shown in Figure 3.22. Note that the circuit has three independent voltage sources, three resistors and two meshes. We also introduce the concept of mesh currents. Mesh current I_I is the current in the mesh formed by elements V_1, R_1, R_2, and V_2. Mesh current I_{II} is the current in the mesh formed by elements V_2, R_2, R_3, and V_3. The branch currents are I_{b1}, I_{b2}, and I_{b3}. It is important to see that mesh currents are not in general the same as branch currents. Note that mesh currents are named with Roman numeral subscripts in this example, whereas branch currents are named with regular number subscripts.

Branch current I_{b1} is the current that flows through the branch that contains voltage source V_1 and resistor R_1. Similarly, branch current I_{b2} is the current that flows through the branch that contains V_2 and R_2; and branch current I_{b3} is the branch current that flows through elements R_3 and V_3. So let us look into the relationship that exists between branch currents and mesh currents.

In particular for the circuit shown in Figure 3.22, the following are how the branch and mesh currents relate to each other:

$$I_I = I_{b1} \tag{3.47}$$

$$I_I - I_{II} = I_{b2} \tag{3.48}$$

$$I_{II} = I_{b3}, \tag{3.49}$$

where in Equations (3.47) through (3.49) the currents in the left-hand side of the equal signs are mesh currents. The currents on the right-hand side of the equal sign are branch currents. Once we establish the mesh equations for the

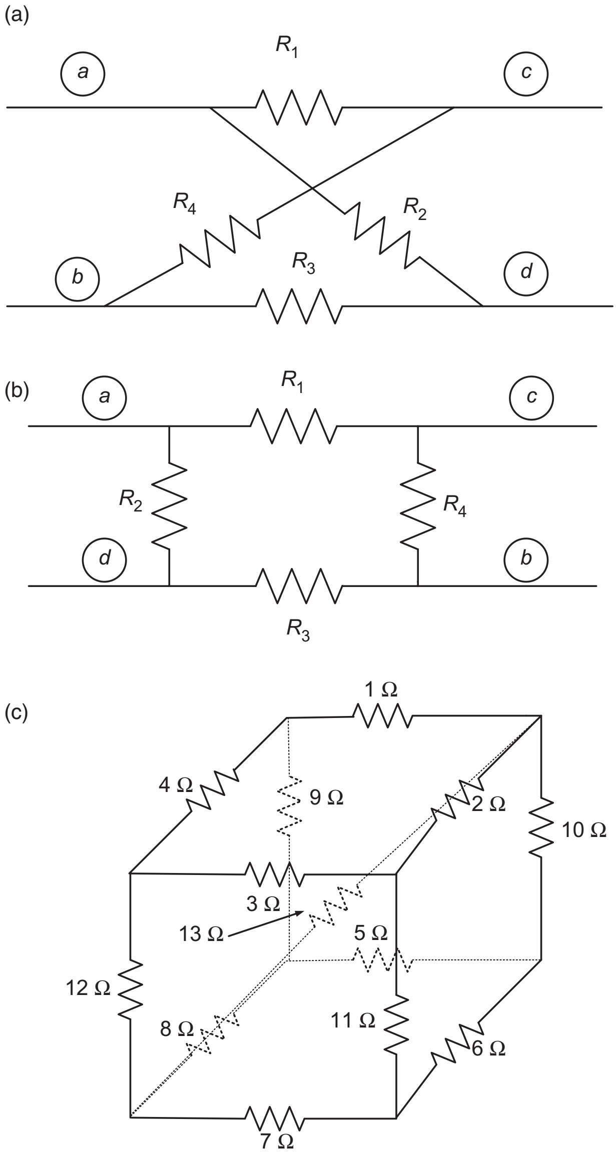

Figure 3.20 Planar and nonplanar circuits: (a) nonplanar circuit, that is, planar circuit in disguise; (b) same planar circuit redrawn; (c) true nonplanar circuit.

given circuit, the mesh currents are the unknowns in the mesh method of analysis. Since we have two meshes, we will be able to obtain two mesh equations and solve for the unknown mesh currents I_I and I_{II}. The branch currents in each specific circuit element are calculated using Equations (3.47) through (3.49) after the mesh currents are found.

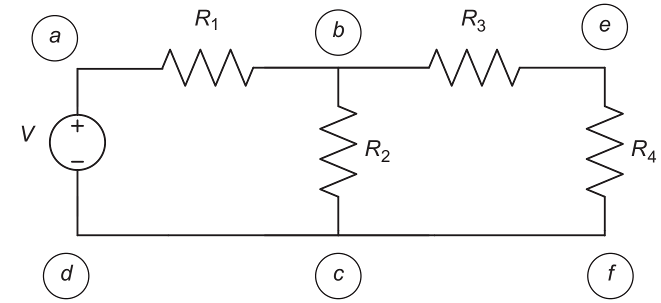

Figure 3.21 Meshes and loops in a circuit.

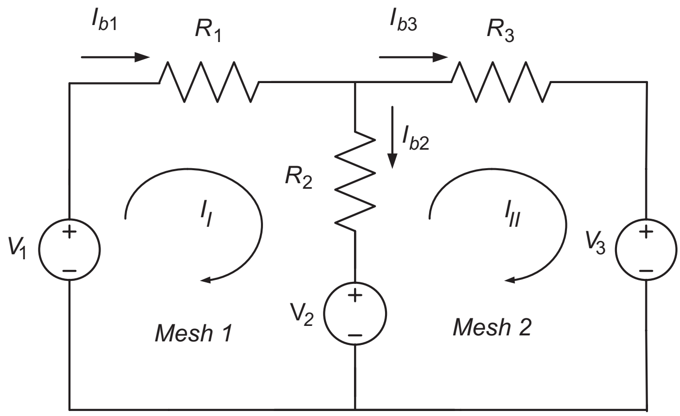

Figure 3.22 Two-mesh equations for Example 3.7.

Initially we will apply KVL to each mesh, but work with mesh currents instead of branch currents. So for

$$\text{Mesh 1: } V_1 - V_2 = I_I R_1 + (I_I - I_{II})R_2. \tag{3.50}$$

$$\text{Mesh 2: } V_2 - V_3 = (I_{II} - I_I)R_2 + I_{II}R_3. \tag{3.51}$$

Note that the direction of the mesh currents was arbitrarily chosen to be clockwise. When applying KVL to each mesh we travel each mesh in the clockwise direction too. It is usually a headache to the reader, understanding why is that currents directions and the direction of traveling the meshes are picked arbitrarily? The simple answer to this is that as long as the voltage rises and voltage drops signs are respected in a consistent manner, the numerical answer will provide a positive sign when such current direction was assigned in the way in which it was assumed; or it will provide a negative sign if the current actually flows in the direction opposite to the one assumed. Do not get hung up on this; solving problems will clarify these apparently confusing arbitrary choices.

So now let us regroup the terms of Equations (3.50) and (3.51) with respect to the mesh currents and obtain

$$\text{Mesh 1: } V_1 - V_2 = I_I(R_1 + R_2) - I_{II}R_2. \tag{3.52}$$

$$\text{Mesh 2: } V_2 - V_3 = -I_I R_2 + I_{II}(R_2 + R_3). \tag{3.53}$$

Equations (3.52) and (3.53) are a system of simultaneous linear equations that allows us to find the two unknown mesh currents I_I and I_{II}.

We can also rewrite the system of simultaneous equations in matrix form as follows:

$$\begin{vmatrix} V_1 - V_2 \\ V_2 - V_3 \end{vmatrix} = \begin{vmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 \end{vmatrix} \begin{vmatrix} I_I \\ I_{II} \end{vmatrix}. \tag{3.54}$$

Example 3.7 Now let us consider a numerical example using the circuit of Figure 3.22 and mesh equations in matrix form, from Equation (3.54), assuming the following component values:

$$R_1 = 3\,\Omega,\ R_2 = 1\,\Omega,\ R_3 = 3\,\Omega. \tag{3.55}$$

$$V_1 = 2\text{ V},\ V_2 = 1\text{ V},\ V_3 = 2\text{ V}. \tag{3.56}$$

Using the values from Equations (3.53) and (3.54) into the mesh equation obtained in Equation (3.52) we obtain

$$\begin{bmatrix} 2-1 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 3+1 & -1 \\ -1 & 3+1 \end{bmatrix} \begin{bmatrix} \mathrm{I_I} \\ \mathrm{I_{II}} \end{bmatrix}. \tag{3.57}$$

Solving matrix Equation (3.57) we obtain

$$I_I = 0.2\text{ A}. \tag{3.58}$$

$$I_{II} = -0.2\text{ A}. \tag{3.59}$$

Using the mesh to branch currents relationships from Equations (3.47), (3.48), and (3.49) we obtain

$$I_I = I_{b1} = 0.2\text{ A} \tag{3.60}$$

$$I_I - I_{II} = I_{b2} = 0.2\text{ A} - (-0.2\text{ A}) = 0.4\text{ A} \tag{3.61}$$

$$I_{II} = I_{\mathrm{b}3} = -0.2\text{ A}. \tag{3.62}$$

Refer to Figure 3.23 to see the original circuit from Figure 3.22 with the added mesh and branch currents values found in the above calculations.

To verify the correctness of the numerical results, work out the mesh equations of circuit of Figure 3.23, using the results of Equations (3.58) through (3.62). Make sure that KCL for all nodes and KVL for all meshes are met.

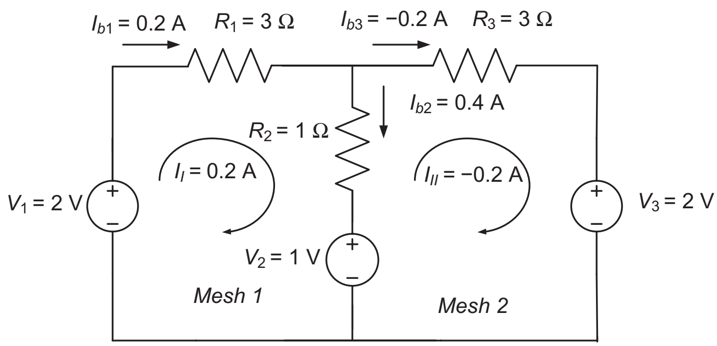

Figure 3.23 Two-mesh equations solutions for Example 3.7.

Example 3.8 Given the circuit of Figure 3.24, derive the mesh equations; find all the branch currents as functions of the mesh currents and the *voltage at node A with respect to ground.* Provide numerical answers for all of the currents and voltages requested.

By inspection of the circuit in Figure 3.24 we can write the mesh equations in the same way we did it for the previous problem, using KVL:

$$\text{Mesh 1: } V_1 - V_2 = (I_I - I_{III})R_1 + (I_I - I_{II})R_2. \tag{3.63}$$

$$\text{Mesh 2: } V_2 - V_3 = (I_{II} - I_I)R_2 + (I_{II} - I_{III})R_3. \tag{3.64}$$

$$\text{Mesh 3: } V_4 = I_{III}R_4 + (I_{III} - I_{II})R_3 + (I_{III} - I_I)R_1. \tag{3.65}$$

Regrouping Equations (3.63) through (3.65) based on each of the mesh currents, we obtain

$$\text{Mesh 1: } V_1 - V_2 = I_I(R_1 + R_2) - I_{II}R_2 - I_{III}R_1. \tag{3.66}$$

$$\text{Mesh 2: } V_2 - V_3 = -I_IR_2 + I_{II}(R_2 + R_3) - I_{III}R_3. \tag{3.67}$$

$$\text{Mesh 3: } V_2 - V_3 = -I_IR_1 - I_{II}R_3 + I_{III}(R_1 + R_3 + R_4). \tag{3.68}$$

We will come back to Equations (3.66) through (3.68) when we will cover finding out the mesh equations simply by inspection of the circuit; eliminating the steps where we applied KVL, Equations (3.63) through (3.65).

Now rewriting Equations (3.66) through (3.68) in matrix form we get

$$\begin{vmatrix} V_1 - V_2 \\ V_2 - V_3 \\ V_4 \end{vmatrix} = \begin{vmatrix} R_1 + R_2 & -R_2 & -R_1 \\ -R_2 & R_2 + R_3 & -R_3 \\ -R_1 & -R_3 & R_1 + R_3 + R_4 \end{vmatrix} \begin{vmatrix} I_I \\ I_{II} \\ I_{III} \end{vmatrix}. \tag{3.69}$$

Using the numerical values from Figure 3.24 into matrix Equation (3.69), we obtain

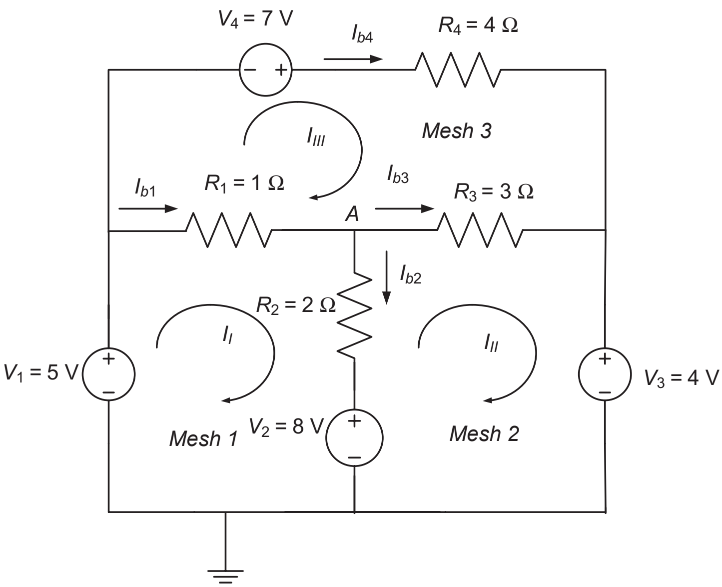

Figure 3.24 Mesh analysis for circuit for Example 3.8.

$$\begin{vmatrix} 5-8 \\ 8-4 \\ 7 \end{vmatrix} = \begin{vmatrix} 1+2 & -2 & -1 \\ -2 & 2+3 & -3 \\ -1 & -3 & 1+3+4 \end{vmatrix} \begin{vmatrix} I_I \\ I_{II} \\ I_{III} \end{vmatrix}. \tag{3.70}$$

Solving the matrix above, it yields:

$$I_I = 1.36364 \text{ A}. \tag{3.71}$$

$$I_{II} = 2.54545 \text{ A}. \tag{3.72}$$

$$I_{III} = 2 \text{ A}. \tag{3.73}$$

By inspection of the circuit of Figure 3.24 we see that the branch to mesh current relationships are

$$I_{b1} = I_I - I_{III}. \tag{3.74}$$

$$I_{b2} = I_I - I_{II}. \tag{3.75}$$

$$I_{b3} = I_{II} - I_{III}. \tag{3.76}$$

$$I_{b4} = I_{III}. \tag{3.77}$$

Plugging the values of I_I (Eq. 3.71) through I_{III} (Eq. 3.73) into Equations (3.74) through (3.77) yields:

$$I_{b1} = -0.63644 \text{ A.} \tag{3.78}$$

$$I_{b2} = -1.18182 \text{ A.} \tag{3.79}$$

$$I_{b3} = 0.54545 \text{ A.} \tag{3.80}$$

$$I_{b4} = 2 \text{ A.} \tag{3.81}$$

By inspection of the branch currents in Figure 3.24 and the results of Equations (3.78) through (3.81) we can see that results for currents I_{b1} and I_{b2} produced negative results. This means that if we go back to Figure 3.24, currents I_{b1} and I_{b2} actually flow in the opposite direction as that shown in the picture.

Finally, it is easy to see that voltage at node *A with respect to ground* (or V_A) equals

$$V_A = V_1 - I_{b1}R_1. \tag{3.82}$$

Plugging the given and the calculated values into Equation (3.77) we obtain

$$V_A = 5 - (-0.63644)1 = 5.63643 \text{ V.} \tag{3.83}$$

As an additional exercise to the reader, verify that all the branch currents, given by Equations (3.74) through (3.77), numerically comply with KCL. The reader should also verify numerically that all KVL Equations (3.66) through (3.68) hold. Hint: Use the calculated values and plug them into the appropriate circuit equations.

3.5.2 Establishing Mesh Equations by Inspection of the Circuit

From the example problems already addressed, we notice that we have been working with circuits that only have voltage sources. Because of this, it is more suitable and also straightforward to derive mesh equations using KVL around each mesh. What we will do in this section is to skip the writing of the circuit equations using KVL, as we did for Example 3.7; see Equations (3.52) and (3.53).

Example 3.9 Writing Mesh Equations by Inspection of the Circuit
Let us start first with the circuit of Example 3.7, Figure 3.22. We repeat this circuit for the reader's convenience in Figure 3.25.

Let us study the circuit diagram carefully. Mesh 1 contains voltage sources V_1 and V_2 and resistors R_1 and R_2. Mesh current I_I is defined to travel mesh 1 in the clockwise direction. Mesh 2 contains voltage sources V_2 and V_3 and resistors R_2 and R_3. Mesh current I_{II} is defined to travel mesh 2 also in a clockwise direction.

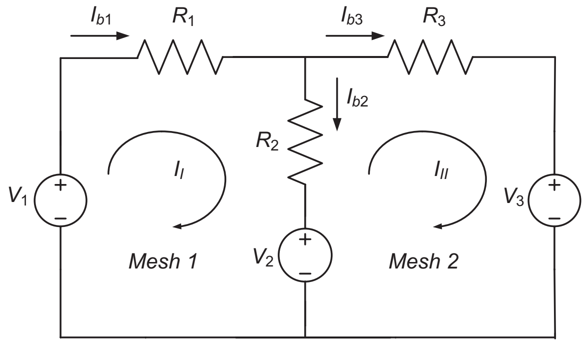

Figure 3.25 Circuit for Example 3.9 finding mesh equations by circuit inspection.

Referring to the previously obtained mesh equations in matrix form, refer to Equation (3.54), we repeat them here again for the reader's convenience:

$$\begin{vmatrix} V_1 - V_2 \\ V_2 - V_3 \end{vmatrix} = \begin{vmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 \end{vmatrix} \begin{vmatrix} I_I \\ I_{II} \end{vmatrix} \tag{3.84}$$

So now referring to matrix Equation (3.84), note that the vector column of voltages has 2×1 dimensions and elements:

$$v_1 = V_1 - V_2. \tag{3.85}$$

and

$$v_2 = V_2 - V_3. \tag{3.86}$$

The 2×2 resistance matrix has elements

$$a_{11} = R_1 + R_2 \quad a_{12} = -R_2 \tag{3.87}$$

$$a_{21} = -R_2 \quad a_{22} = R_2 + R_3 \tag{3.88}$$

The 2×1 vector column of currents contains mesh currents I_I and I_{II}. Usually the mesh currents are the unknowns to be found.

We can also express the mesh equations with Ohm's law in matrix form:

$$|V| = |R||I|, \tag{3.89}$$

where $|V|$ is a 2×1 voltage column, $|R|$ is a 2×2 resistance matrix, and $|I|$ is a 2×1 current column.

For the construction of the mesh equations in matrix form, we make the following observations:

The top element of the voltage column v_1 equals the algebraic sum (taking into account the sign of each source) of the voltage sources in mesh 1, traveling mesh 1 in the clockwise direction. That is $V_1 - V_2$ from Equation (3.85).

The bottom element of the voltage column v_2 equals the algebraic sum of the voltage sources of mesh 2 traveling mesh 2 in the clockwise direction. That is $V_2 - V_3$ from Equation (3.86)

Now for the resistive matrix, element a_{11} will always have the sum of all the resistive elements in mesh 1. Note that this sum will always be a sum of positive numbers.

For the resistive matrix, element a_{22} will always have the sum of all the resistive elements in mesh 2. Note that this sum will always be a sum of positive numbers.

Let us concentrate on the a_{12} term of the resistive matrix. We see (Fig. 3.25) that resistor R_2 is a common element between meshes I and II. And since mesh current I_I flows in the clockwise direction, while mesh current I_{II} flows through R_2 in the opposite direction; the contribution of the $I_{II}\ R_2$ term will have a negative sign. Note that if both mesh currents had been chosen such that they both flowed through the common element in the same direction, then the sign of term $I_{II}\ R_2$ would have been positive.

Finally, for the resistance matrix terms a_{21} is the term in mesh 2 that is common to mesh 1. For the same reason, since mesh current I_{II} flows in the opposite direction of mesh current I_I, the term a_{21} will have a negative sign.

The column of mesh currents simply contains the unknown mesh currents to be found which are I_I and I_{II}. Let us also observe that the resistance matrix will always have positive elements on its main diagonal: that is, elements a_{11} and a_{22}. The reciprocal terms (a_{12} and a_{21}) may be both positive and both negative depending on the directions chosen for the mesh currents, as explained earlier. Finally, if the circuit is passive, that means it does not contain any dependent sources, elements a_{12} and a_{21} are identical in sign and magnitude. This is to say that the resistance matrix is symmetrical.

Important Points: Deriving Mesh Equations by Circuit Inspection:
Resistance Matrix: Main diagonal always contains positive elements, none of which can be zero.

If the circuit is passive the resistance matrix is symmetrical (*i.e.*, $a_{12} = a_{21}$).

A passive circuit only contains resistors, inductors, or capacitors, but it cannot contain dependent voltage or dependent current sources. Some examples using dependent sources will be given in Chapter 6.

Drill Problem 3.10: Using the previously seen methodology, find out by inspection the mesh equations for the circuit of Figure 3.26 (this is the same circuit used for Example 3.8).

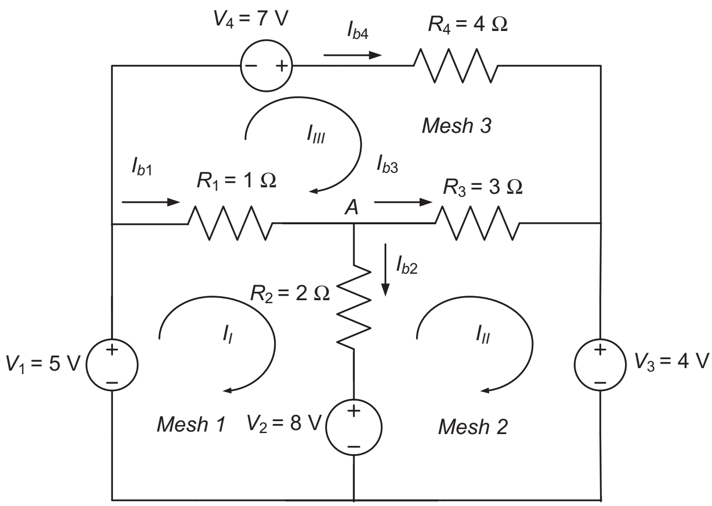

Figure 3.26 Mesh equations by circuit inspection for Drill Problem 3.10.

3.5.3 Establishing Mesh Equations When There Are also Current Sources

The mesh method of analysis is very straightforward when the circuit contains voltage sources. However, if the circuit in addition to containing voltage sources contains current sources, some changes will occur in the mesh equations. To better understand the differences, let us address this with Example 3.11.

Example 3.11 Refer to the circuit of Figure 3.27 to work on this example.

Some Important Notations Pertaining to Circuits:
Referring to the circuit of Figure 3.27: Note that the voltage across resistor R_1 is $V_A - V_{REF} = V_A - 0 = V_A$

The voltage across resistor R_2 assumes the highest voltage at node B with respect to node A is denoted V_{BA}, which is also equal to $V_B - V_A$. It also means that node B is more positive than node A. Note that if the voltage at node B is less positive than the voltage of node A, then $V_B - V_A$ is a negative number. For example, If $V_B = 4$ V and $V_A = 5$ V, then $V_B - V_A = -1$ V.

On the other hand, if we want to talk about the voltage across resistor R_2, where the higher voltage is assumed to be at node A, and the lower voltage is at node B, the voltage across R_2 is V_{AB} which is also equal to $V_A - V_B$. Also note that $V_{AB} = -V_{BA}$.

The voltage across current source I_{S1} is V_{DA}. The voltage at node D is assumed to be larger than the voltage at node A. That is: $V_{DA} = V_D - V_A$.

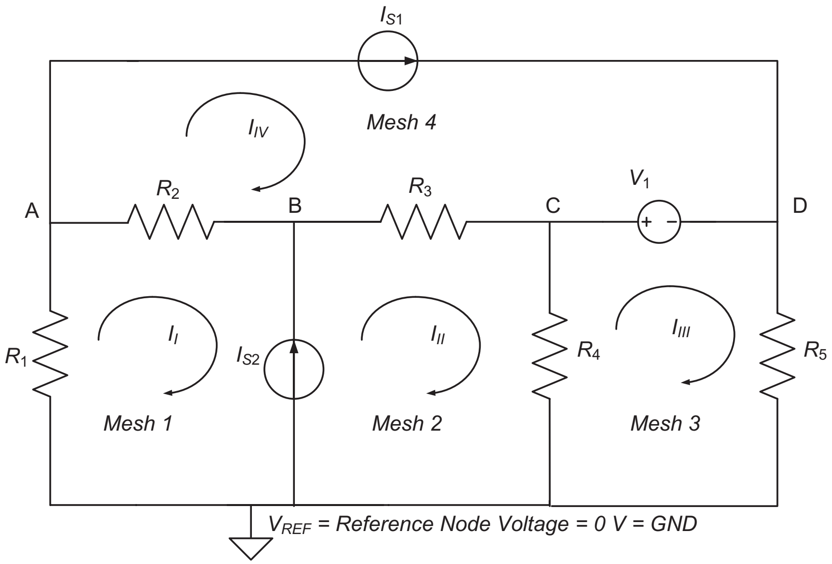

Figure 3.27 Mesh equations of circuits with voltage and current sources for Example 3.11.

Similarly, the voltage across current source I_{S2} is V_B. The voltage at node B is assumed to be higher than the voltage at node V_{REF}; note that V_{REF} was defined as our reference or zero-volt ground node.

So now by inspection of circuit of Figure 3.27 we see four meshes, that is, I, II, III, and IV. Starting with mesh I, if we want to establish the mesh equations for this mesh, we cannot use the value of the current source I_{S2} in the KVL equations since the voltage across current source I_{S2} is V_B. We can then write mesh I equation as follows:

$$Mesh\ I \quad V_B = (I_{IV} - I_I)R_2 - I_I R_1 \tag{3.90}$$

$$Mesh\ II \quad V_B = (I_{II} - I_{IV})R_3 + (I_{II} - I_{III})R_4 \tag{3.91}$$

$$Mesh\ III \quad -V_1 = I_{III}R_5 + (I_{III} - I_{II})R_4 \tag{3.92}$$

It is not necessary to write the equation for *Mesh IV* since mesh current IV (I_{IV}) is known numerically. That is,

$$I_{IV} = I_{S1}. \tag{3.93}$$

The last equation we need is current source I_{S2} which equals the differences between mesh currents I_{II} and I_I. That is,

$$I_{S2} = I_{II} - I_I. \tag{3.94}$$

Now if we subtract Equation (3.91) from Equation (3.90), the unknown voltage V_B is eliminated from the result and we obtain

$$(I_{IV} - I_I)R_2 - I_I R_1 + (I_{IV} - I_{II})R_3 + (I_{III} - I_{II})R_4 = 0. \tag{3.95}$$

Reordering Equation (3.95) grouping by mesh currents yields

$$-I_I(R_1 + R_2) - I_{II}(R_3 + R_4) + I_{III}R_4 + I_{IV}(R_2 + R_3) = 0. \tag{3.96}$$

The elimination of voltage V_B from Equations (3.95) and (3.96) is equivalent to thinking as merging meshes I and II; this new merged mesh is called a *super-mesh*. This super-mesh consists of elements R_1, R_2, R_3, and R_4 after the elimination of current source I_{S2}.

Refer to Figure 3.28, the super-mesh just described is shown after the physical removal of current source I_{S2}. Remember this step is justified by Equations (3.95) and (3.96).

Referring to Figure 3.28, note that independent current source I_{S1} is identical to the selected mesh current I_{IV}. In a typical circuit, like the one of Figure 3.27, the independent current and voltage sources are known; the same goes for the resistors. Generally, the mesh currents are unknown. But let us talk about how many mesh current equations we need and how many mesh currents are unknown. We have a total of four mesh currents I_I, I_{II}, I_{III}, and I_{IV}.

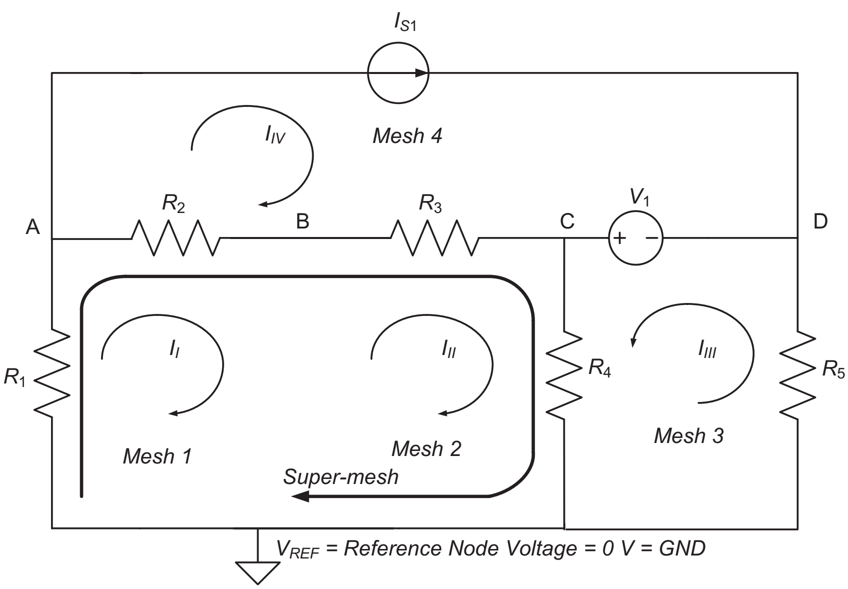

Figure 3.28 The creation of the super-mesh for Example 3.11.

We already mentioned that I_{IV} is numerically known because it is equal to the value of independent current source I_{S1}, see Figure 3.27. Thus, the only unknown mesh currents are: I_I, I_{II}, and I_{III}. To find the three unknown mesh currents we need three linearly independent equations. The first one is Equation (3.96) and the other two are Equations (3.92) and (3.94). We repeat these three key equations here for the reader's convenience. Since we also know from Equation (3.93) that mesh current I_{IV} is known and equals I_{S1}, we replace I_{IV} with I_{S1} in Equation (3.96) and obtain

$$-I_I(R_1+R_2)-I_{II}(R_3+R_4)+I_{III}R_4+I_{S1}(R_2+R_3)=0. \tag{3.97}$$

$$-V_1=I_{III}R_5+(I_{III}-I_{II})R_4. \tag{3.98}$$

where $I_{S2}=I_{II}-I_I$

In a typical problem all resistors R_1 through R_5, the two current sources I_{S1} and I_{S2}, and voltage source V_1 are numerically known.

What we just did mathematically with Equation (3.97) is the following:

The voltage V_B across current source I_{S2} is not initially known and Equation (3.96) eliminates V_B. This merges or creates a so called *super-mesh* with meshes *I* and *II*. Two meshes that share a current source are referred to as an essential mesh. So we re-draw the circuit of Figure 3.27 showing the newly formed super-mesh and it is shown in Figure 3.28. On the other hand, mesh *IV* is nonessential because its current source I_{S1} is not shared with any other mesh. Thus, we eliminate (or open circuit) current source I_{S1}. These steps along with the super-mesh are both shown in the circuit of Figure 3.29.

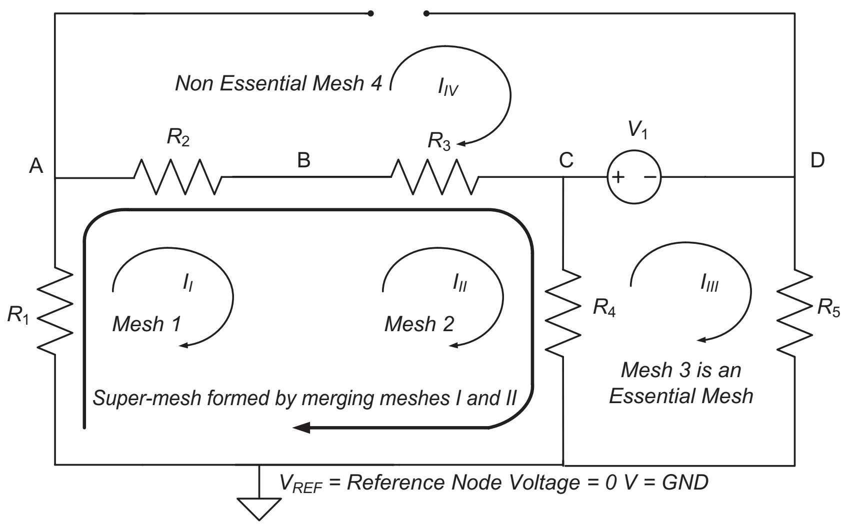

Figure 3.29 Circuit for Example 3.11 after the elimination of all independent current sources.

It is important to state that a super-mesh does not have a current of its own. Note that the original mesh currents I_I through I_{IV} continue to flow through the elements of the newly formed *super-mesh* in Figure 3.29. This is certainly a requirement which was derived by Equation (3.96). We will show next a simpler method using the super-mesh concept of deriving Equation (3.96), without having to write the individual equations for meshes *I* and *II* as we did previously.

Following the *super-mesh* of Figure 3.29, travel the super-mesh in the direction indicated by the heavy arrow accounting for all voltage drops and rises. In our example, the super-mesh does not have any voltage rises (i.e., voltage sources) of its own; however, it may have them in other examples.

$$I_I R_1 + (I_I - I_{IV})R_2 + (I_{II} - I_{IV})R_3 + (I_{II} - I_{III})\mathrm{R}_4 = 0. \tag{3.99}$$

Since mesh current I_{IV} equals the current I_{S1} (Fig. 3.27),

$$I_{IV} = I_{S1}. \tag{3.100}$$

After regrouping terms in Equation (3.99) around, the mesh current becomes:

$$-I_I(R_1 + R_2) - I_{II}(R_3 + R_4) + I_{III}R_4 + I_{S1}(R_2 + R_3) = 0. \tag{3.101}$$

Note that Equation (3.101) is identical to Equation (3.97).

The fourth and last equation is for essential mesh *III*. This is probably the simplest equation to write since it contains only a voltage source and we need to write the mesh equation using KVL. Note: An essential mesh is one that has current sources, and it is not a super mesh.

$$Mesh\ III - V_1 = (I_{III} - I_{II})R_4 + I_{III}R_5. \tag{3.102}$$

Solving the four equations, which we rewrite below for the reader's convenience, all mesh currents, are numerically obtained.

$$-I_I(R_1 + R_2) - I_{II}(R_3 + R_4) + I_{III}R_4 + I_{S1}(R_2 + R_3) = 0. \tag{3.103}$$

$$-V_1 = I_{III}R_5 + (I_{III} - I_{II})R_4. \tag{3.104}$$

$$I_{S2} = I_{II} - I_I. \tag{3.105}$$

A system of three simultaneous linear equations with three unknowns; Equations (3.103) through (3.105), is solved to obtain mesh currents I_I, I_{II}, and I_{III}.

Remember that I_{IV} is already known by inspection of the circuit of Figure 3.27, Equation (3.100).

By inspection of the circuit of Figure 3.30 we can find the branch currents on every resistor as a function of their mesh currents.

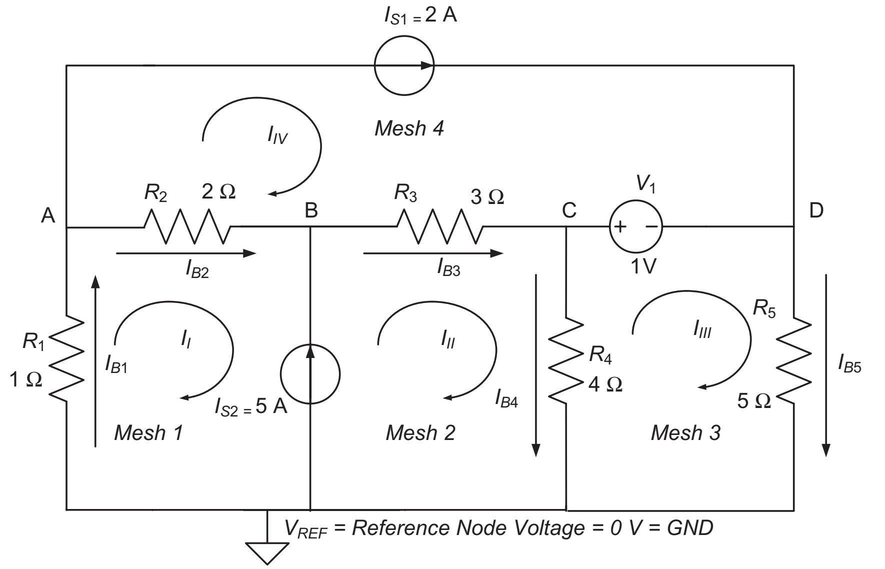

Figure 3.30 Circuit to solve by mesh analysis method for Example 3.12.

$$\text{Branch current through resistor } R_1 = I_{B1} = I_I. \tag{3.106}$$

$$\text{Branch current through resistor } R_2 = I_{B2} = I_\text{I} - I_{IV}. \tag{3.107}$$

$$\text{Branch current through resistor } R_3 = I_{B3} = I_{II} - I_{IV}. \tag{3.108}$$

$$\text{Branch current through resistor } R_4 = I_{B4} = I_{II} - I_{III}. \tag{3.109}$$

$$\text{Branch current through resistor } R_5 = I_{B5} = I_{III}. \tag{3.110}$$

Example 3.12 Using the circuit of Figure 3.27, and assuming the element values given by Equations (3.111) through (3.118), calculate the values of all four mesh currents. Hint: Use Equations (3.103) through (3.105).

Once the mesh currents are obtained, calculate the branch currents through resistors R_1 through R_5. Hint: Use Equations (3.106) through (3.110). Then, calculate the voltages at nodes A, B, C, and D with respect to ground.

$$R_1 = 1\,\Omega, \tag{3.111}$$

$$R_2 = 2\,\Omega, \tag{3.112}$$

$$R_3 = 3\,\Omega, \tag{3.113}$$

$$R_4 = 4\,\Omega, \tag{3.114}$$

$$R_5 = 5\,\Omega, \tag{3.115}$$

$$I_{S1} = 2\text{ A}, \tag{3.116}$$

$$I_{S2} = 5\text{ A}, \tag{3.117}$$

$$V_1 = 1\text{ V}. \tag{3.118}$$

The circuit is presented again for the reader's convenience in Figure 3.30.

Using the circuit values given by Equations (3.111) through (3.118), plugging them into mesh Equations (3.103) through (3.105), we obtain the following mesh currents:

The author assumes that the reader can solve a system of linear simultaneous equations. Bear in mind that one of this book's goals is to learn circuit analysis; however, it is not the main goal of this book to walk the reader through solving algebraic equations.

Then,

$$I_I = -2.01351\text{ A}. \tag{3.119}$$

$$I_{II} = 2.98649\text{ A}. \tag{3.120}$$

$$I_{III} = 1.21622\text{ A}. \tag{3.121}$$

$$I_{IV} = 2\text{ A}\ (\textit{found by circuit inspection}). \tag{3.122}$$

Now, using Equations (3.106) through (3.110) to calculate the branch currents, we obtain

$$I_{B1} = I_I = -2.01351\text{ A}. \tag{3.123}$$

$$I_{B2} = I_I - I_{IV} = -4.01351\text{ A}. \tag{3.124}$$

$$I_{B3} = I_{II} - I_{IV} = 0.98649\text{ A}. \tag{3.125}$$

$$I_{B4} = I_{II} - I_{III} = 1.77027\text{ A}. \tag{3.126}$$

$$I_{B5} = I_{III} = 1.21622\text{ A}. \tag{3.127}$$

Refer one more time to Figure 3.30 to see the branch current directions and compare them with the signs of Equations (3.123) through (3.127).

Note that branch current I_{B1} was defined in the same direction as mesh current I_I; however, the numerical result of $I_{B1} = -2.01351$ A means branch current I_{B1} actually flows from node A to the reference node. It is also true that branch current I_{B2} defined to flow from node B into node C, because of the negative sign of its result, actually flows from C to B.

By inspection of Figure 3.30 we can easily find the corresponding nodal voltages as function of their branch currents and their respective branch resistors.

$$V_A = -I_{B1}R_1. \tag{3.128}$$

$$V_A - V_B = I_{B2}R_2. \tag{3.129}$$

$$V_B - V_C = I_{B3}R_3. \tag{3.130}$$

$$V_C = I_{B4}R_4. \tag{3.131}$$

$$V_D = I_{B5}R_5. \tag{3.132}$$

Now plugging the values of branch current and resistors into (we find the nodal voltages)

$$V_A = 2.013521\text{ V}. \tag{3.133}$$

$$V_B = 10.0405\text{ V}. \tag{3.134}$$

$$V_C = 7.08108\text{ V}. \tag{3.135}$$

$$V_D = 6.08108\text{ V}. \tag{3.136}$$

Let us note that from Equation (3.133), V_A is a positive voltage with respect to ground, which means that $V_{GND} - V_A = 0 - V_A = -V_A = I_{B1} \times R_1$, which is consistent with the direction which branch current I_{B1} has in Figure 3.30 and its negative result given by Equation (3.123).

Similarly note that nodal voltage V_A is positive but smaller than the nodal voltage at V_B (i.e., $V_A < V_B$ or 2.01351 V < 10.0405 V). That explains why based on the direction defined for branch current I_{B2} (Fig. 3.30), the numerical result is negative; that is, from Equation (3.124)

$$I_{B2} = -4.01351\text{ A}.$$

On a final note on this example, mesh currents are defined currents just for the mesh method of analysis. Mesh currents are not currents that can be directly measured, like branch currents can.

3.5.4 Establishing Mesh Equations When There Are also Dependent Sources

In this section we will address a circuit with an independent voltage source and also a dependent current source. We will see that the mesh equations can be stated simply starting with the circuit KVL equations. The fact that there is a dependent source does not change significantly how KVL equations need to be written. *We will find then that a constraint equation links the dependant source output (a current $4I_A$ in our next example) and its independent variable (I_A).* Finally, we will see that the matrix mesh equations lead to a *nonsymmetrical* matrix, because a dependent source represents an active device. More on dependent sources will be covered on the chapter on transistors.

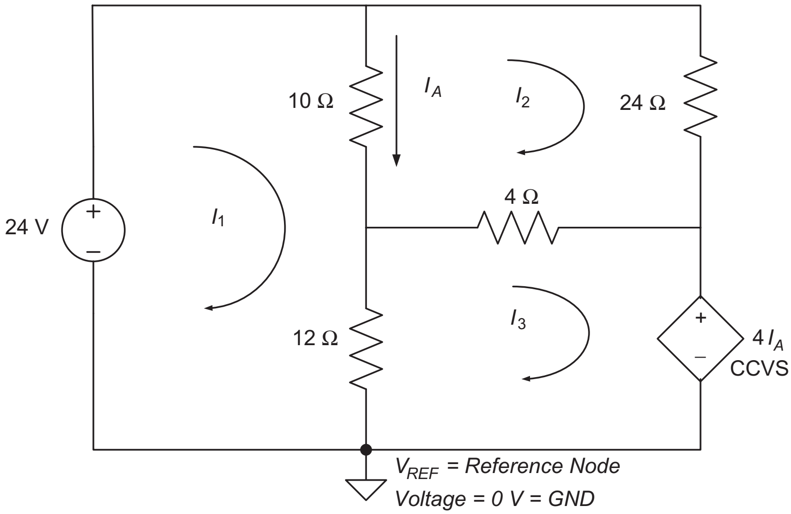

Figure 3.31 Establishing mesh equations for circuits with a dependent source.

Example 3.13 Establish the mesh equations starting with *KVL:* Let us refer to the circuit of Figure 3.31. There is a dependent voltage source, whose output voltage is, $4I_A$, which is referred to as the dependent variable or the voltage source output. The current I_A is the control variable of our dependent source. This current I_A is defined to be the current that flows through the 10-Ω resistor, with the direction shown in Figure 3.31. Keep in mind that dependent sources (current of voltage types) are mathematical models to represent devices that have gain. You cannot buy a dependent source in a battery store or anywhere else; a dependent source is a circuit-modeling concept. We will address the meaning of gain when studying operational amplifiers and transistorized circuits. Finally, we add that in cases that have dependent sources, the author prefers not to address a *by-inspection* method, because its rules are more complex than those for the straightforward cases of mesh equations with just independent voltage sources.

Using the already predefined mesh current of Figure 3.31, we can write for each mesh their respective mesh equations using KVL around each mesh:

$$\textit{Mesh 1:}\ 24 = 10(I_1 - I_2) + 12(I_1 - I_3). \tag{3.137}$$

$$\textit{Mesh 2:}\ 0 = 10(I_2 - I_1) + 24I_2 + 4(I_2 - I_3). \tag{3.138}$$

$$\textit{Mesh 3:}\ -4I_A = 12(I_3 - I_1) + 4(I_3 - I_2). \tag{3.139}$$

Note that in Mesh 3 (3.139), term $-4I_A$ is a voltage not a current; refer again to Figure 3.31.

By inspection of the circuit in Figure 3.31, it is easy to see that

$$I_A = I_1 - I_2. \tag{3.140}$$

Equation (3.139) shows branch current I_A expressed as a function of the circuit mesh currents. Now regrouping terms in Equations (3.137) through (3.139) and using Equation (3.140) to eliminate the use of I_A, we obtain the following mesh equations:

$$\textit{Mesh 1: } 24 = 22I_1 - 10I_2 - 12I_3. \tag{3.141}$$

$$\textit{Mesh 2: } 0 = -10I_1 + 38I_2 - 4I_3. \tag{3.142}$$

$$\textit{Mesh 3: } 0 = -8I_1 - 8I_2 + 16I_3. \tag{3.143}$$

Dividing by two on both sides of the equal sign Equations (3.141) and (3.142), dividing (3.143) by eight, and rewriting them in their matrix form yields:

$$\begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \tag{3.144}$$

Solving the matrix system, one obtains that

$$I_1 = 2.25 \text{ A}. \tag{3.145}$$

$$I_2 = 0.75 \text{ A}. \tag{3.146}$$

$$I_3 = 1.5 \text{ A}. \tag{3.147}$$

And using Equation (3.140) for I_A,

$$I_A = 2.25 - 0.75 = 1.5 \text{ A}. \tag{3.148}$$

The actual matrix solving is left as an exercise to the reader.

Notice that as predicted, the resistance matrix in Equation (3.144) is not symmetrical, that is, because there was a dependent source in the circuit. That is, $a_{23} = -2$ is not equal to $a_{32} = -1$. Question to the reader: Which other elements of matrix (3.144) prove that the matrix is not symmetrical?

3.5.4.1 Commentary on Mesh Analysis Note that given a circuit with only voltage sources and "n" meshes, there are n mesh currents that can be defined. This yields a system of n independent equations with n unknowns.

However, if there are any current sources in a mesh, each current source reduces the number of linearly independent equations by one per current source per mesh. Finally, if the circuit contains at least one dependent source, the resistance matrix will not be symmetrical like it is in the case of a passive circuit. A passive circuit only contains resistors (additionally capacitors and inductors if it is an AC circuit) and independent voltage and/or current sources.

3.6 THE NODAL METHOD OF ANALYSIS

The nodal method of circuit analysis is based on KCL. It provides a more effective way of deriving circuit equations virtually by quick inspection of the circuit. The nodal method is more suitable and intuitive when the circuit contains independent current sources. The method is somewhat less intuitive when voltage sources are also included and probably the least intuitive when dependent voltage and current sources are present. For a circuit that contains n nodes, one of the nodes is arbitrarily chosen as the reference node or ground, and the remaining "$n - 1$"nodal voltages of the circuits are typically the unknowns. We will address the methodology of writing KCL equations for various circuits via examples that will grow in complexity.

Unlike the mesh method, the nodal method works for *planar* and *nonplanar* circuits. It is commonly the method of choice of some electric and electronic circuit simulation programs.

3.6.1 Establishing Nodal Equations: Circuits with Independent Current Sources

Let us assume that we have a circuit such as the one presented in Figure 3.32. By inspection we see that the circuit has four nodes. The reference node is usually chosen to be at the bottom of the circuit. Additionally, the nonreference nodes are: A, B, and C.

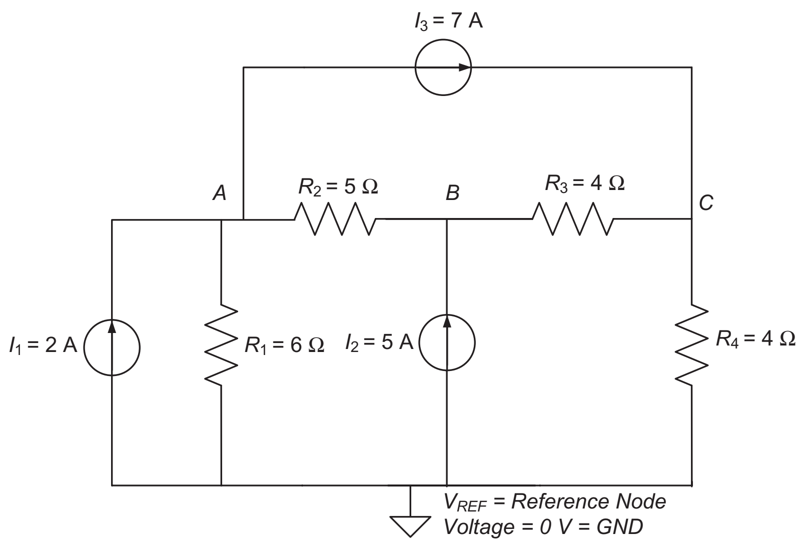

Figure 3.32 Circuit with current sources to establish nodal equations.

Using KCL at each of the nodes we obtain

$$\textit{Node A: } I_1 - I_3 = V_A/R_1 + (V_A - V_B)/R_2. \tag{3.149}$$

$$\textit{Node B: } I_2 = (V_B - V_A)/R_2 + (V_B - V_C)/R_3. \tag{3.150}$$

$$\textit{Node C: } I_3 = V_C/R_4 + (V_C - V_B)/R_3. \tag{3.151}$$

Regrouping Equations (3.149) through (3.151) around their nodal voltages we obtain

$$\textit{Node A: } I_1 - I_3 = V_A(1/R_1 + 1/R_2) - V_B(1/R_2). \tag{3.152}$$

$$\textit{Node B: } I_2 = -V_A(1/R_2) + V_B(1/R_2 + 1/R_3) - V_C(1/R_3). \tag{3.153}$$

$$\textit{Node C: } I_3 = -V_B(1/R_3) + V_C(1/R_3 + 1/R_4) \tag{3.154}$$

In the above three equations we have three unknowns, the nodal voltages V_A, V_B, and V_C. Once those voltages are found, the branch currents in every branch element can easily be calculated using Ohm's law.

Let us make a notation simplification, remembering that the inverse of a resistance R is its conductance G, where $G = 1/R$.

We can re-write Equations (3.152) through (3.154) in matrix form and they become:

$$\begin{bmatrix} I_1 - I_3 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} G_1 + G_2 & -G_2 & 0 \\ -G_2 & G_2 + G_3 & -G_3 \\ 0 & -G_3 & G_3 + G_4 \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix}, \tag{3.155}$$

where [I] is a 3×1 column of current sources. The 3×3 matrix in Equation (3.155) is referred to as the conductance matrix [G]. The vector of nodal voltages contains all nonreference node voltages, V_A, V_B, and V_C. Equation (3.155) can be written in a more compact form and that is

$$[I] = [G][V]. \tag{3.156}$$

Equation (3.156) is another matrix form of Ohm's law using the *conductance* matrix G.

Example 3.14 Find the nodal voltages V_A, V_B, and V_C and currents on resistors R_1 through R_4 in the circuit of Figure 3.32. Hint: Use Equations (3.152) through (3.154) or matrix system (Eq. 3.155).

Referring again to the circuit of Figure 3.32 and using the corresponding values for the independent current sources and resistors we rewrite Equation (3.155) as follows:

$$\begin{bmatrix} 2-7 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1/6+1/5 & -1/5 & 0 \\ -1/5 & 1/5+1/4 & -1/4 \\ 0 & -1/4 & 1/4+1/4 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}. \tag{3.157}$$

Solving the linear system of three equations with three unknowns we obtain for the nodal voltages:

$$V_A = 0.947368 \text{ V}. \tag{3.158}$$

$$V_B = 26.7368 \text{ V}. \tag{3.159}$$

$$V_C = 27.3684 \text{ V}. \tag{3.160}$$

Having obtained the nodal voltages we can use them to calculate the branch currents for resistors R_1 through R_5. Thus:

$$I_{B1} = V_A / R_1 = 0.947368/6 = 0.157895 \text{ A} \tag{3.161}$$

$$I_{B2} = (V_B - V_A)/R_2 = (26.7368 - 0.947368)/5 = 5.15789 \text{ A} \tag{3.162}$$

$$I_{B3} = (V_C - V_B)/R_3 = (27.3684 - 26.7368)/4 = 0.157895 \text{ A} \tag{3.163}$$

$$I_{B4} = V_C / R_4 = 27.3684/4 = 6.84211 \text{ A}, \tag{3.164}$$

where, in Equation (3.161), current I_{B1} flows from node A to ground; in Equation (3.162) I_{B2} flows from node B into node A; in (3.163) I_{B3} flows from node C into node B. Finally, in (3.164) I_{B4} flows from node C to ground.

Drill Problem 3.15: Using the circuit of Figure 3.32 and the found branch currents given by Equations (3.161) through (3.163), check that KCL is met at nodes A, B, C, and V_{REF} (the grounded node).

3.6.2 Establishing Nodal Equations by Inspection: Circuits with Current Sources

The nodal equations of circuits with current sources and a relatively small number of nodes are quite easy to determine. Let us start working on the circuit of the next example.

Example 3.16 Refer to the circuit of Figure 3.33. On first sight the circuit appears to be complicated; however, after looking at it for some time and understanding its topology, it is not so.

Although the circuit contains six resistors and six independent current sources, its topology is not that much different from that of the circuit of Figure 3.32.

Note that our circuit, Figure 3.33, still has three nonreference nodes and a reference node (ground). For simplicity and convenience, the nonreference nodes are named A, B, and C.

So although this circuit has many more resistors and current sources than the one of Figure 3.32, it is still a circuit whose nodal equations matrix is 3×3, that is, three rows by three columns. Let us assume that the elements of the

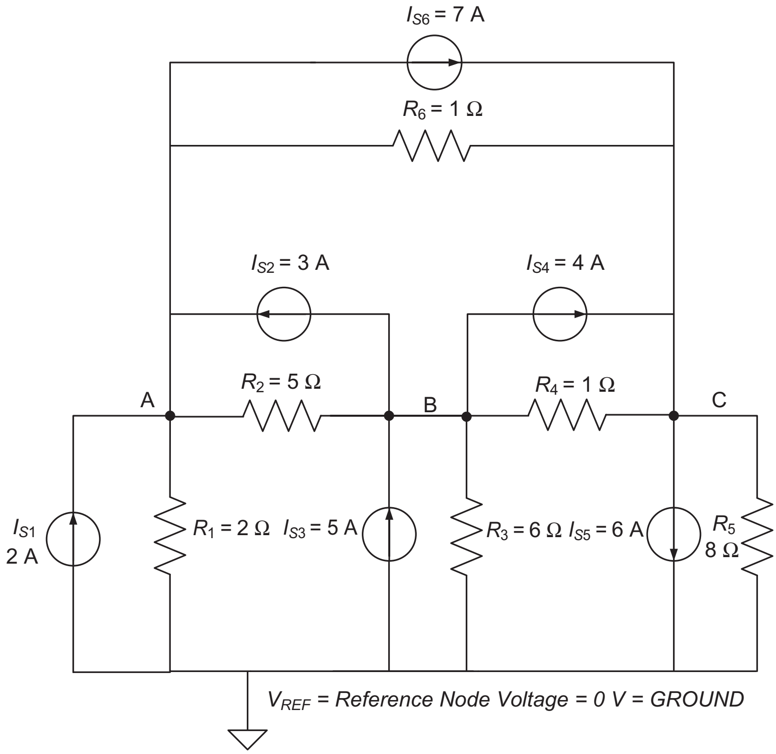

Figure 3.33 Circuit for Example 3.14: establishing nodal equations by inspection.

first row of our matrix 3 × 3 are: a_{11}, a_{12}, a_{13}. Second row elements are: a_{21}, a_{22}, a_{23}; and the third row elements are: a_{31}, a_{32}, a_{33}. The system of three nodal equations and three unknowns is provided first. Then we will look at every equation term and identify how it is associated to the given circuit diagram.

Again referring to Figure 3.33 by inspection of the circuit we have

$$\textit{Node A: } I_{S1} + I_{S2} - I_{S6} = (G_1 + G_2 + G_6)V_A - G_2V_B - G_6V_C. \tag{3.165}$$

$$\textit{Node B: } I_{S3} - I_{S2} - I_{S4} = -G_2V_A + (G_2 + G_3 + G_4)V_B - G_4V_C. \tag{3.166}$$

$$\textit{Node C: } I_{S4} + I_{S6} - I_{S5} = -G_6V_A - G_4V_B + (G_4 + G_5 + G_6)V_C. \tag{3.167}$$

A general inspection of Equations (3.165) through (3.167) should make it clear that each equation is written for every nonreference node and the effects of the other nonreference node over the node in question. Equation (3.165) corresponds to node A, Equation (3.166) corresponds to node B, and Equation (3.167) corresponds to node C. In particular in Equation (3.165), we observe that it has a current term on the left-hand side of the equal sign. The term $(G_1 + G_2 + G_6)V_A$ expresses the effect of all conductances connected to node

A; the term: $-G_2V_B$ expressed the effect of adjacent node B over node A. Finally, the last term: $-G_6V_C$ expresses the effect of adjacent node C over node A.

Referring to Figure 3.33 let us go over every term of Equation (3.165) one more time. The left hand term of Equation (3.165) is the algebraic sum of the currents at node A. Currents flowing into the node are positive, while currents leaving the node are negative. Thus, the term: $I_{S1} + I_{S2} - I_{S6}$. The first term to the right of the equal sign: $(G_1 + G_2 + G_6)V_A$ consists of the sum of all the conductances directly connected to node A, that is, conductances G_1, G_2, and G_6. Because they are all connected to node A, they need to be multiplied by V_A. The next term of Equation (3.165), that is, $-G_2V_B$, contains a term that is minus the conductance between nodes A and B times nodal voltage V_B. G_2 is also referred to as the *shared* conductance between nodes A and B. Finally, the last term of Equation (3.165), that is, $-G_6V_C$, is minus the shared conductance between nodes A and C $(-G_6)$ times the nodal voltage V_C. Similarly, we can go over Equation (3.166). In the current term $I_{S3} - I_{S2} - I_{S4}$, note that I_{S3} enters node B, that is why I_{S3} has a positive sign, while currents I_{S2} and I_{S4} leave node B, thus their negative sign. Continuing with Equation (3.166), the term: $-G_2V_A$ denotes the influence of node A over node B. The next term, $(G_2 + G_3 + G_4)V_B$ shows the effect of the actual node in question, that is, node B, and the sum of all conductances connecting to such node times its nodal voltage V_B. Finally, term: $-G_4V_C$ expresses the influence of node C over node B.

Last, let us briefly describe Equation (3.167). The current term: $I_{S4} - I_{S5} + I_{S6}$ is the sum of the currents entering node C minus the currents leaving node C. Term: $-G_6V_A$ shows the impact of node A over node C through their shared conductance G_6. Term: $-G_4V_B$ shows the impact of node B over node C through their shared conductance G_4. Finally, term: $(G_4 + G_5 + G_6)V_C$ is the effect of all conductances connecting to node C, times its nodal voltage V_C.

Now that we have gone over the terms of Equations (3.165) through (3.167), it is easy to convert such equation into their matrix form:

$$\begin{bmatrix} I_{S1}+I_{S2}-I_{S6} \\ I_{S3}-I_{S2}-I_{S4} \\ I_{S4}-I_{S5}+I_{S6} \end{bmatrix} = \begin{bmatrix} G_1+G_2+G_6 & -G_2 & -G_6 \\ -G_2 & G_2+G_3+G_4 & -G_4 \\ -G_6 & -G_4 & G_4+G_5+G_6 \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} \tag{3.168}$$

In the above matrix equations we have that

$$a_{11} = G_1 + G_2 + G_6;\ a_{12} = -G_2;\ a_{13} = -G_6. \tag{3.169}$$

$$a_{21} = -G_2;\ a_{22} = G_2 + G_3 + G_4;\ a_{23} = -G_4. \tag{3.170}$$

$$a_{31} = -G_6;\ a_{32} = -G_4;\ a_{33} = G_4 + G_5 + G_6. \tag{3.171}$$

Note all the main diagonal elements of the $[G]$ matrix, a_{11}, a_{22}, and a_{33} are always nonzero and positive. Additionally, if the circuit matrix has no dependent sources, that is to say, the circuit is passive and thus its matrix is symmetrical.

A symmetrical matrix is that whose elements that are mirrored around its main diagonal are identical. In general $a_{ij} = a_{ji}$ for *i and j* from 1 to 3, but $i \neq j$. In particular for our example this means that

$$a_{12} = a_{21}. \tag{3.172}$$

$$a_{13} = a_{31}. \tag{3.173}$$

$$a_{23} = a_{32}. \tag{3.174}$$

We will provide the method without applying KCL, as it was done in the previous section and stating directly the procedural steps:

1. For arithmetic convenience, convert every resistor into its equivalent conductance. That is, $G = 1/R$.
2. Identify all nonreference nodes and the reference node. This step is already given in our example.
3. Determine the conductance matrix $[G]$ which will have dimensions of (*number of nodes* – 1) × (*number of nodes* – 1), in our example: 3 × 3.
4. Determine the independent current sources column vector *[I]*.
5. Write the matrices in the form: $[I] = [G][V]$,

where $[V]$ is the column of nodal voltages or the unknowns. In our example:

$$[V] = \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix}$$

Example 3.17 Again, using the circuit of Figure 3.33, calculate the numerical values of nodal voltages V_A, V_B, and V_C. After you find the three nodal voltages, find the currents through every resistor. Hint: Use Equations (3.165) through (3.167) or solve the matrix system given by Equation (3.168).

For this example, all we have to do is to use the values given for the independent current sources and the resistors and solve Equation (3.168).

So plugging in the appropriate values into matrix Equation (3.168) we obtain

$$\begin{bmatrix} 2+3-7 \\ 5-3-4 \\ 4-6+7 \end{bmatrix} = \begin{bmatrix} 0.5+0.2+1 & -0.2 & -1 \\ -0.2 & 0.2+0.1667+1 & -1 \\ -1 & -1 & 1+0.125+1 \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix}. \tag{3.175}$$

Remember that the 3×3 matrix coefficients are formed by sum of the conductances (not the resistances) given by the circuit of Figure 3.33.

Solving Equation 3.175 we obtain the following nodal voltages:

$$V_A = 0.84812\ \text{V}. \tag{3.176}$$

$$V_B = 1.02857\ \text{V}. \tag{3.177}$$

$$V_C = 3.23609\ \text{V}. \tag{3.178}$$

The current for every resistor are simply given by Ohm's law:

$I_{R1} = V_A/R_1 = 0.84812\ \text{V}/2\ \Omega = 0.42406\ \text{A}$ (I_{R1} flows from node A to ground)

$I_{R2} = (V_B - V_A)/R_2 = (1.02857 - 0.84812)\ \text{V}/5\ \Omega$
$= 0.0360902\ \text{A}$ (I_{R2} flows from B to A)

$I_{R3} = V_B/R_3 = 1.02857\ \text{V}/6\ \Omega = 0.171429\ \text{A}$ (I_{R3} flows from B to ground)

$I_{R4} = (V_C - V_B)/R_4 = (3.23609 - 1.02857)\ \text{V}/1\ \Omega$
$= 2.20752\ \text{A}$ (I_{R4} flows from C to B)

$I_{R5} = V_C/R_5 = 3.23609\ \text{V}/8\ \Omega = 0.404511\ \text{A}$ (I_{R5} flows from C to ground)

$I_{R6} = (V_C - V_A)/R_6 = (3.23609 - 0.84812)\ \text{V}/1\ \Omega$
$= 2.38797\ \text{A}$ (I_{R6} flows from C to A)

3.6.3 Establishing Nodal Equations When There Are also Voltage Sources

The nodal method is more suitable and straightforward, when the circuit only contains current sources, because the method is derived using KCL at each nonreference node. It is also possible to apply the nodal method when, in addition to current sources, voltage sources are present. This, even though it is somewhat less intuitive, it simplifies the nodal equations by one equation per voltage source that constitutes a super node.

Example 3.18 Refer to Figure 3.34 to solve the three-node circuit that contains two independent current sources and two independent voltage sources. The nonreference nodes are clearly labeled: A, B, and C. We cannot apply nodal analysis as usual because of the presence of the voltage sources. It is not possible to know the currents through the independent voltage sources, before making any calculations. Inspecting the circuit of Figure 3.34 carefully, we can see that even though there are three nodes in the circuit, the voltage at node A is already known, it actually is $V_1 = 4$ V. We can also see by inspection that the difference of nodal voltages V_B and V_C equals the value of the voltage source $V_2 = 3$ V. That is, $V_2 = V_B - V_C = 3$ V.

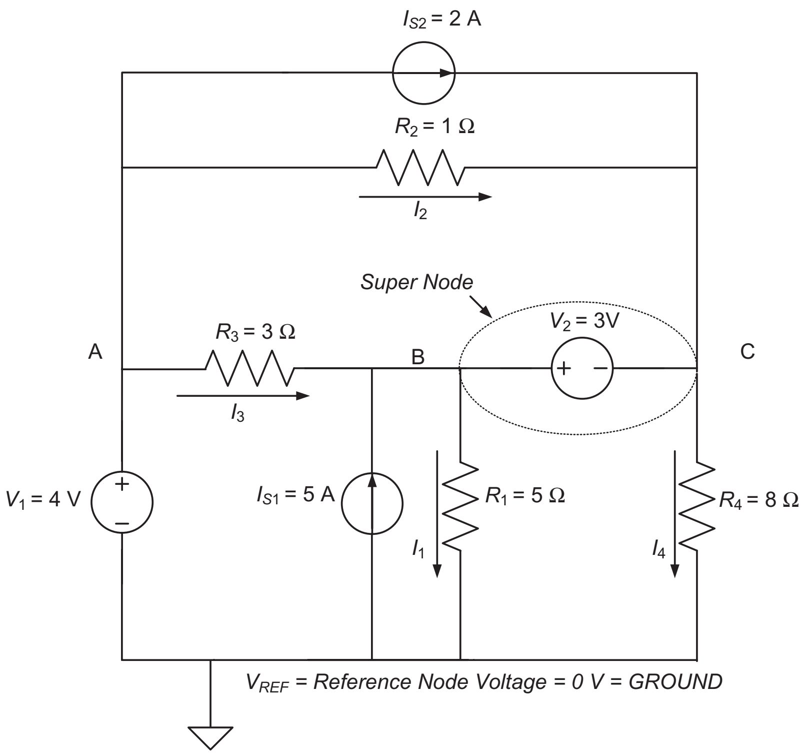

Figure 3.34 Circuit to be solved by the nodal method.

So there are two constraint equations:

$$V_2 = V_B - V_C = 3\text{ V}. \tag{3.179}$$

$$V_1 = V_A = 4\text{ V}. \tag{3.180}$$

Nodes B and C are encircled in Figure 3.34, and they are defined as a *super-node*.

Note that at the indicated *super-node*, KCL also applies. So the sum of all *super-node* entering current equals the sum of all super node leaving currents, that is,

$$I_{S2} + I_2 + I_3 + I_{S1} = I_1 + I_4. \tag{3.181}$$

Since $I_{S1} = 5$ A and $I_{S2} = 2$ A, then Equation (3.181) becomes

$$I_2 + I_3 + 7 = I_1 + I_4. \tag{3.182}$$

Additionally, by inspection of the circuit of Figure 3.34, we see that

$$I_1 = V_B/R_1 = V_B/5. \tag{3.183}$$

$$I_2 = (V_A - V_C)/R_2 = (V_A - V_C)/1. \tag{3.184}$$

$$I_3 = (V_A - V_B)/R_3 = (V_A - V_B)/3. \tag{3.185}$$

$$I_4 = V_C/R_4 = V_C/8. \tag{3.186}$$

Using Equations (3.183) through (3.186) in Equation (3.182) we obtain

$$(V_A - V_C)/1 + (V_A - V_B)/3 + 7 = V_B/5 + V_C/8. \tag{3.187}$$

Solving Equation (3.187) with constraint Equations (3.181) and (3.182), we obtain the values for V_B and V_C.

$$V_B = 9.47236 \text{ V}. \tag{3.188}$$

$$V_C = 6.47236 \text{ V}. \tag{3.189}$$

Now since all the nodal voltages are known, we can easily find the branch currents through all the resistors using Equations (3.183) through (3.186).

$$I_1 = V_B/R_1 = V_B/5 = 1.89447 \text{ A}. \tag{3.190}$$

$$I_2 = (V_A - V_C)/R_2 = (V_A - V_C)/1 = -2.47236 \text{ A}. \tag{3.191}$$

$$I_3 = (V_A - V_B)/R_3 = (V_A - V_B)/3 = -1.82412 \text{ A}. \tag{3.192}$$

$$I_4 = V_C/R_4 = V_C/8 = 0.809045 \text{ A}. \tag{3.193}$$

To determine the currents of each of the independent voltage sources, it is a simple application of KCL. This is left as an exercise to the reader.

3.6.4 Establishing Nodal Equations When There Are Dependent Sources

Let us analyze a circuit that contains two independent current sources, an independent voltage source and a dependent voltage source. An example of this nature is likely as complex as it can be, to solve by hand.

Example 3.19 Using the circuit of Figure 3.35 establish the nodal equations for the circuit using KCL.

By inspection of the circuit of Figure 3.35 we can see that the 10-V dependent voltage source has a control voltage V, which is the voltage developed across the independent 10-A current source, with the shown polarity. Nodes C and D are super-nodes. The voltage at node D with respect to ground is

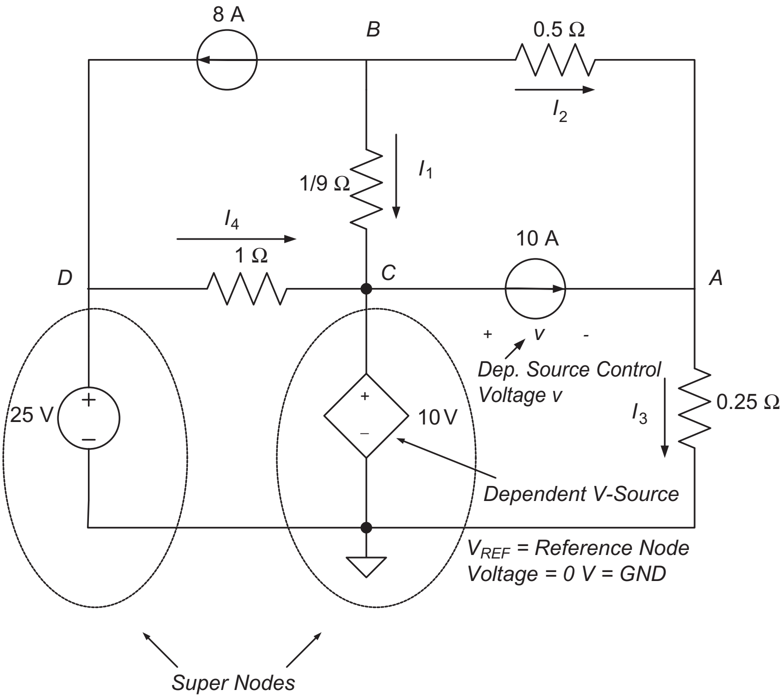

Figure 3.35 Circuit for Example 3.19: nodal method containing dependent sources.

25 V. The only nodal equations that we need to establish are those at nodes A and B.

One more time referring to the circuit in Figure 3.35, let us note that the currents entering node A are 10 A (an independent current source), and branch current I_2, while branch current I_3 leaves node A. Additionally, we observe that for node B, the 8 A-independent current source and both branch currents I_1 and I_2 leave node B.

Since that branch current I_1 flows from node B to node C, thus

$$I_1 = (V_B - V_C)/(1/9). \tag{3.194}$$

Branch current I_2 flows from node B to node A. Thus,

$$I_2 = (V_B - V_A)/0.5. \tag{3.195}$$

Branch current I_3 flows from node A to ground, and

$$I_3 = V_A/0.25, \tag{3.196}$$

we can establish the two nodal equations:

at Node A:

$$10 = V_A/0.25 - (V_B - V_A)/0.5 \tag{3.197}$$

and

at Node B:

$$-8 = (V_B - V_C)/1/9 + (V_B - V_A)/0.5. \tag{3.198}$$

Now, regrouping equations and finding the inverses of the resistances in Equations (3.197) and (3.198), we obtain the nodal equations for the circuit:

$$\textit{Node A:}\ 10 = (4+2)V_A - 2V_B. \tag{3.199}$$

$$\textit{Node B:}\ -8 = -2V_A + (9+2)V_B - 9V_C \tag{3.200}$$

Note that Equations (3.199) and (3.200) are two equations with three unknown nodal voltages: V_A, V_B, and V_C. Again by inspection of the circuit of Figure 3.35 we can write one constraint equation that relates nodal voltage V_C to the dependent source voltage V:

By inspection of the circuit we can see that

$$10\ \mathrm{V} = V_C. \tag{3.201}$$

$$V_A = 10\ \mathrm{V} - \mathrm{V} = 9\ \mathrm{V}. \tag{3.202}$$

And

$$V = V_C - V_A. \tag{3.203}$$

Using Equations (3.201) and (3.202) into Equations (3.199) and (3.200) and solving for the nodal voltages we get

$$V_A = 2.2381\ \mathrm{V}. \tag{3.204}$$

$$V_B = 1.71429\ \mathrm{V}. \tag{3.205}$$

$$V_C = 2.4867\ \mathrm{V}. \tag{3.206}$$

And since $V = V_C - V_A$ from Equation (3.203),

$$\mathrm{V} = 2.48677 - 2.2381 = 0.24867\ \mathrm{V}. \tag{3.207}$$

Now that we have all the nodal voltages, the calculation of the branch currents easily follows:

$$I_1 = -6.95238 \text{ A.} \tag{3.208}$$

$$I_2 = -1.04762 \text{ A.} \tag{3.209}$$

$$I_3 = 8.95238 \text{ A.} \tag{3.210}$$

$$I_4 = 22.5132 \text{ A.} \tag{3.211}$$

The current through the 25-V independent voltage source is: 14.5132 A *and the current through the voltage-controlled voltage source* 10 V *is* 5.56085 A.

The reader is strongly encouraged to apply KCL to each node of the circuit of Figure 3.35 to validate that the calculated currents are correct.

3.6.4.1 Commentary on Nodal Analysis

Note that given a circuit with only current sources and "n" nodes, there are "$n - 1$" necessary and sufficient linearly independent nodal equations required to find the n nodal voltages. However, if there are any voltage sources in the circuit, this creates so-called super-nodes. Each super-node voltage source reduces the number of nodal voltages by one per source. Finally, if the circuit contains at least one dependent source, the resistance matrix will not be symmetrical like it is in the case of a passive circuit. A passive circuit only contains resistors (additionally capacitors and inductors if it is an AC circuit) and independent sources.

3.7 WHICH ONE IS THE BEST METHOD?

We looked at the following circuit theorems and methods of analysis:

1. Superposition theorem
2. Thévenin theorem
3. Norton theorem
4. Source transformations
5. Mesh method of analysis
6. Nodal method of analysis

It is strictly the user, you, who has to make a decision of what method to use. Not everyone will necessarily agree that one method is better than another one. However, it is true, if one is comfortable using any of the methods presented, equally well, sometimes using one method instead of another one can really speed up the circuit solving process, particularly when this is done by hand.

3.7.1 Superposition Theorem Highlights

It is a divide-and-conquer approach. Given a circuit with multiple independent current or voltage sources, one is able to calculate the effect of all sources by enabling one source at a time while inhibiting all other ones. This is repeated until every source was enabled in a mutually exclusive fashion. The algebraic sum of the individual effects leads to the composite result, that is, as if all the sources were applied simultaneously. This method only works with linear circuits, usually the majority of circuits that we will be dealing with, but be careful it is not all of them (e.g., a diode is a nonlinear device). Superposition of power quantities *does not* apply even if the circuit is linear. The applicability of superposition applies to the currents and voltages of linear circuits. The most interesting feature of superposition is that one solves a larger number of problems, where each problem is easier to solve. Or at least that should be the idea when applying this method.

It works with independent and dependent sources, AC and DC voltages and currents. However, it is important to note that when applying superposition, the dependent sources *must not* be inhibited like the independent sources are (i.e., one at a time); the dependent sources must be left alone. Some Problems at the end of the chapter will allow you to practice solving circuits by superposition with dependent and independent sources.

3.7.2 Thévenin Theorem Highlights

Thévenin theorem allows one to replace a piece of a circuit that we choose, with a Thévenin voltage source in series with a Thévenin resistance (or impedance). Many times this simplifies solving the circuit. Thévenin applies to linear circuits with independent and dependent voltage and current sources and AC and DC voltages and currents. When dealing with dependent sources, just like when applying superposition, we do not touch (or inhibit) the dependent sources. The reason is that dependent sources have their own control variable. Finally, upon slicing a circuit to find its Thévenin equivalent, if the sliced circuit contains a dependent voltage or current source, you must make sure that the control variables of such sources *do not* get separated from the circuit being Thévenized.

3.7.3 Norton's Theorem Highlights

Norton's theorem allows one to replace a piece of a circuit that we choose, with a Norton current source in parallel with a Norton resistance (or impedance). The Norton resistance is calculated in exactly the same way as the Thévenin resistance. Norton only applies to linear circuits with independent and dependent voltage and current sources and AC and DC voltages and currents. When dealing with dependent sources, just like when applying superposition, we do not touch (or inhibit) the dependent sources. The reason is that dependent sources have their own control variable. Finally, upon slicing a circuit to find its Norton equivalent, if the sliced circuit contains a dependent voltage or current source, you must make sure that the control variables of such sources *do not* get separated from the circuit whose Norton equivalent is being sought.

3.7.4 Source Transformations Highlights

This method basically derives from the convertibility that exists between independent voltage into independent current sources, by applying Thévenin and Norton Theorems. An independent voltage source in series with a resistor (or impedance) can be converted into an independent current source in parallel with the same resistor (or impedance).

So upon solving a circuit with a large number of mixed current and voltage sources, it may become convenient to either transform all the current sources into voltage sources or vice-versa in order to apply the by-inspection mesh or nodal methods. So whether we apply Thévenin, Norton, or source transformations, it is important to note that the Thévenin resistance (or impedance) is identical to the Norton equivalent resistance (or impedance).

3.7.5 Mesh Method of Analysis Highlights

The mesh method applies to circuits that are planar. Recall that a planar circuit is one that can be drawn without any branches crossing any other. Beware that there are circuits that may appear to be nonplanar; however, redrawing them reveals that they are actually planar. This method is more appealing when we have voltage sources in the meshes, because it uses KVL as its main method of analysis.

This method seems more intuitive and easier to approach with the "*by inspection method"* earlier described, but only when voltage sources are present. When current sources are present, it reduces the number of mesh equations by one per current source. The mesh method with both voltage and current sources also has a by inspection method, but we do not cover this on this text. The by-inspection method with voltage and current sources is somewhat more complicated to memorize. The mesh method applies when there are dependent sources as well. Finally, it can be said that the mesh method is usually an attractive choice when the number of meshes is small.

3.7.6 Nodal Method of Analysis Highlights

The nodal method is a more general method than the mesh method. The nodal method applies to planar and nonplanar circuits. This method is more appealing to use when the circuit contains current sources because it fundamentally uses KCL for the analysis. The method, however, is also applicable when there are voltage sources. This creates a reduction in the number of nodal equations of one by voltage source. There is also a *by inspection nodal method* which is intuitive and easier to apply when only current sources are present. The *by*

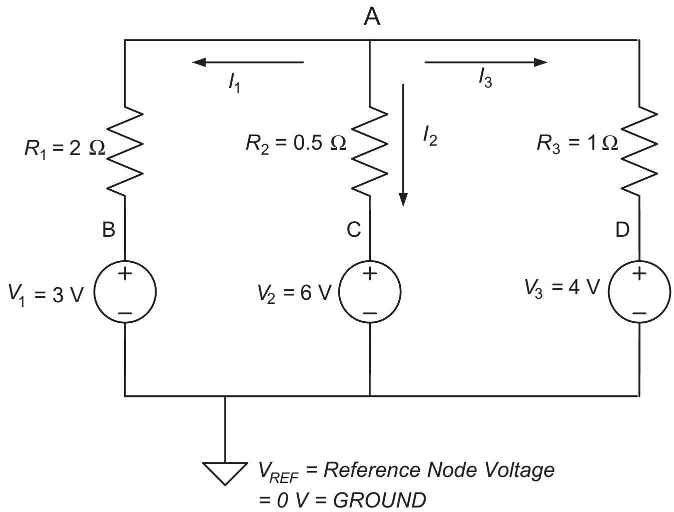

Figure 3.36 Circuit example to be solved by six different methods.

inspection nodal method also exists for current and voltage sources but it is less intuitive to apply, thus we do not cover it.

3.8 USING ALL THE METHODS

To help determine which method is more effective, the circuit of Figure 3.36 will be solved by all six methods covered throughout this chapter. These are: superposition, Thévenin, Norton, source transformations, mesh, and nodal.

3.8.1 Solving Using Superposition

Let us present Example 3.20 to appreciate the different approaches to solving circuits using the different methods presented in this chapter. Refer to the circuit of Figure 3.36. The problem to solve in this example is to find the value of the nodal voltage *A* with respect to ground.

Example 3.20 Solving the Circuit of Figure 3.36 by Superposition
This is a circuit with three independent voltage sources and three resistors. Solving this problem by superposition takes three steps to disable one source at a time and calculating the value of voltage *A*. In a fourth and last step, we calculate the composite solution by obtaining the algebraic sum of the three previous results.

Proceeding with this problem, we break this circuit down into three simpler circuits. Refer to Figure 3.37 parts *a*, *b*, and *c*.

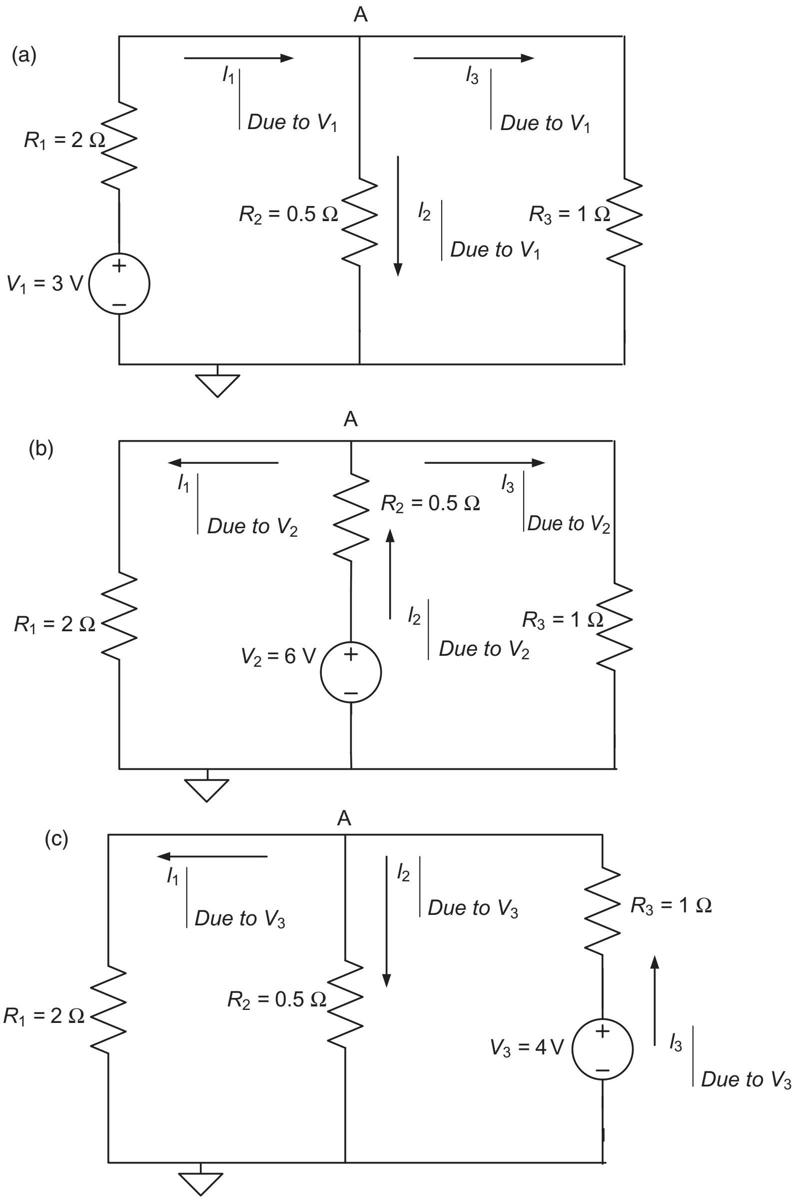

Figure 3.37 Solving circuit of Example 3.20 by superposition: (a) only source V_1 is present; (b) only source V_2 is present; (c) only source V_3 is present.

Circuit of Figure 3.37a is the original circuit powered by source V_1, while sources V_2 and V_3 are inhibited (replaced with short circuits). The circuit of b is powered by source V_2 while V_1 and V_3 are inhibited. Finally, the circuit of c is powered by source V_3, while V_1 and V_2 are inhibited.

So for the circuit of Figure 3.37a we have

$$V_{A|due\ to\ V1} = [V_1/(R_1 + R_2//R_3)](R_2//R_3). \tag{3.212}$$

Equation (3.212) tells us that voltage V_A is the voltage across the parallel of resistors R_2 and R_3. The current through the circuit is the voltage V_1 divided by the series combination of R_1 and ($R_2//R_3$), thus we arrive at Equation (3.212). $R_2//R_3$ is short-hand notation for the calculation of R_2 in parallel with R_3. Plugging in the component values from Figure 3.37a into Equation (3.212) we get

$$V_{A|due\ to\ V1} = [3/(2 + 0.5//1)](0.5//1) = 3/7\ \text{V}. \tag{3.213}$$

For the circuit of Figure 3.37b, the reasoning is similar as the case before. Thus,

$$V_{A|due\ to\ V2} = [V_2/(R_2 + R_1//R_3)](R_1//R_3) \tag{3.214}$$

Plugging in the component values from Figure 3.37*b* into Equation (3.214) we get:

$$V_{A|due\ to\ V2} = [6/(0.5 + 2//1)](2//1) = 24/7\ \text{V}. \tag{3.215}$$

For the circuit of Figure 3.37c, the reasoning is similar as the case before. Thus,

$$V_{A|due\ to\ V3} = [V_3/(R_3 + R_1//R_2)](R_1//R_2). \tag{3.216}$$

Plugging in the component values from Figure 3.37c into Equation (3.216) we get

$$V_{A|due\ to\ V3} = [4/(1 + 2//0.5)](2//0.5) = 8/7\ \text{V}. \tag{3.217}$$

$$V_A = 3/7\ \text{V, due to excitation } V_1 \text{ when } V_2 = 0 \text{ and } V_3 = 0. \tag{3.218}$$

$$V_A = 24/7\ \text{V, due to excitation } V_2 \text{ when } V_1 = 0 \text{ and } V_3 = 0. \tag{3.219}$$

$$V_A = 8/7\ \text{V, due to excitation } V_3 \text{ when } V_1 = 0 \text{ and } V_2 = 0, \tag{3.220}$$

Adding all three voltages at node A, due to voltage excitations V_1, V_2, and V_3 we obtain

$$V_{A|due\ to\ V1} + V_{A|due\ to\ V2} + V_{A|due\ to\ V3} = 3/7 + 24/7 + 8/7 = 5\ \text{V}. \tag{3.221}$$

Note: In this particular case, because all the voltages are positive, there is no difference between their sum and their algebraic sum.

Since V_A is known, it is now easy to find branch currents I_1, I_2. and I_3.

$$I_1 = (V_1 - V_A)/R_1 = (3-5)/2 = -1 \text{ A}. \tag{3.222}$$

$$I_2 = (V_2 - V_A)/R_2 = (6-5)/0.5 = 2 \text{ A}. \tag{3.223}$$

$$I_3 = (V_3 - V_A)/R_3 = (4-5)/1 = -1 \text{ A}. \tag{3.224}$$

Referring back to the circuit of Figure 3.36 note that current I_2 is positive; thus, it flows in the same direction assumed in the circuit. Currents I_1 and I_3 flow in the opposite direction than the one originally assumed. This is consistent with the fact that V_A = 5 V, which is smaller than V_2 but it is higher than V_1 and V_3. Remember, current flows from high voltages to lower voltages, and its flow is considered positive.

3.8.2 Example 3.21: Solving the Circuit of Figure 3.36 by Thévenin

Now let us analyze the same circuit of Figure 3.36 using Thévenin's theorem. We refer to the step-by-step procedure following the circuits in Figures 3.38 and 3.39. So the first thing we do is to slice the original circuit such that one piece is to be Thévenized while the remaining circuit will not be touched, initially.

We do exactly that if Figure 3.38a–c, respectively, show the circuits to calculate the Thévenin voltage and resistance.

Figure 3.38a shows the calculated Thévenin voltage and resistance, of 16/3 V and 1/3 Ω, respectively.

The calculation of V_{Th} simply is the open-circuit voltage at terminals A and B of the circuit of Figure 3.38a. Note that this circuit is a single mesh circuit and applying KVL to it we obtain

$$V_2 - V_3 = I_{mesh}(R_2 + R_3) \tag{3.225}$$

and

$$V_{Th} = V_2 - I_{mesh}R_2. \tag{3.226}$$

Solving Equations (3.225) and (3.226) we obtain that

$$I_{mesh} = (6-4)/(0.5+1) = 4/3 \text{ A}. \tag{3.227}$$

Thus, from Equation (3.226),

$$V_{Th} = 6 - (4/3)(1/2) = 6 - 2/3 = 16/3 \text{ V}. \tag{3.228}$$

From Figure 3.38c we see that R_{Th} is the parallel of R_2 and R_3; that is, R_{Th} = 1/3 Ω.

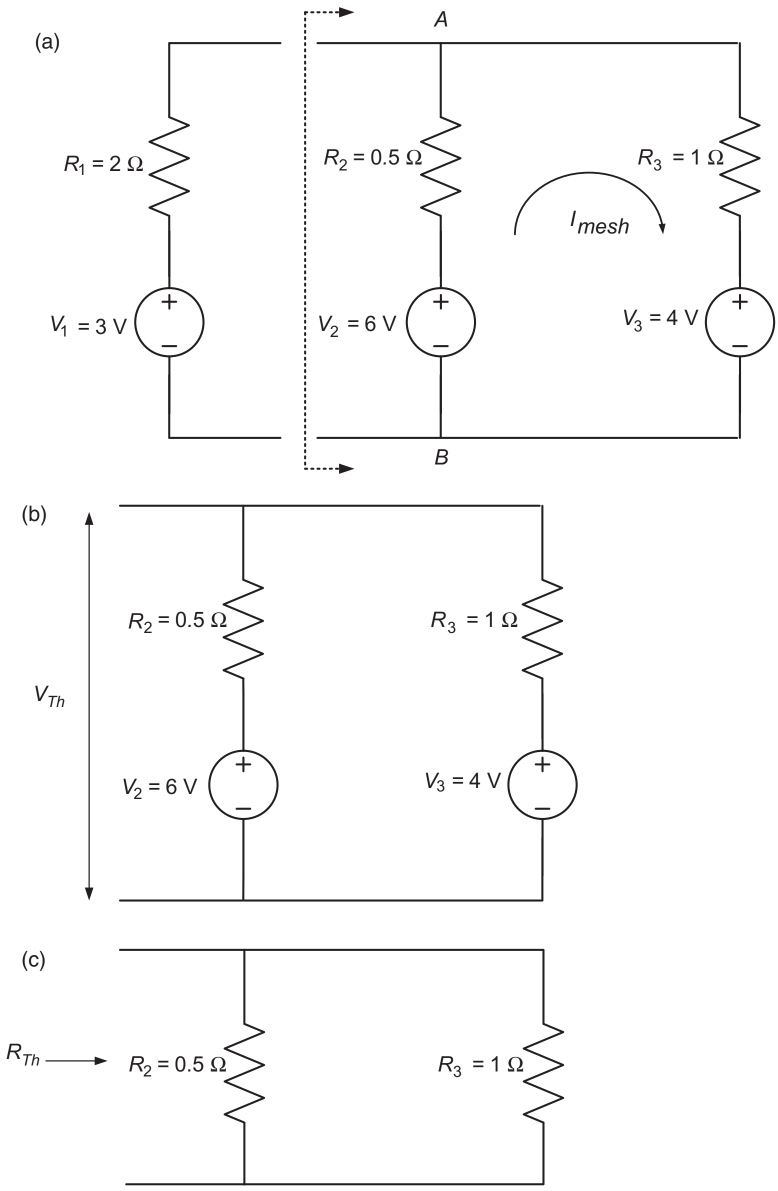

Figure 3.38 Example 3.21: (a) circuit to be Thévenized; (b) circuit used to find V_{Th}; (c) circuit used to find R_{Th}.

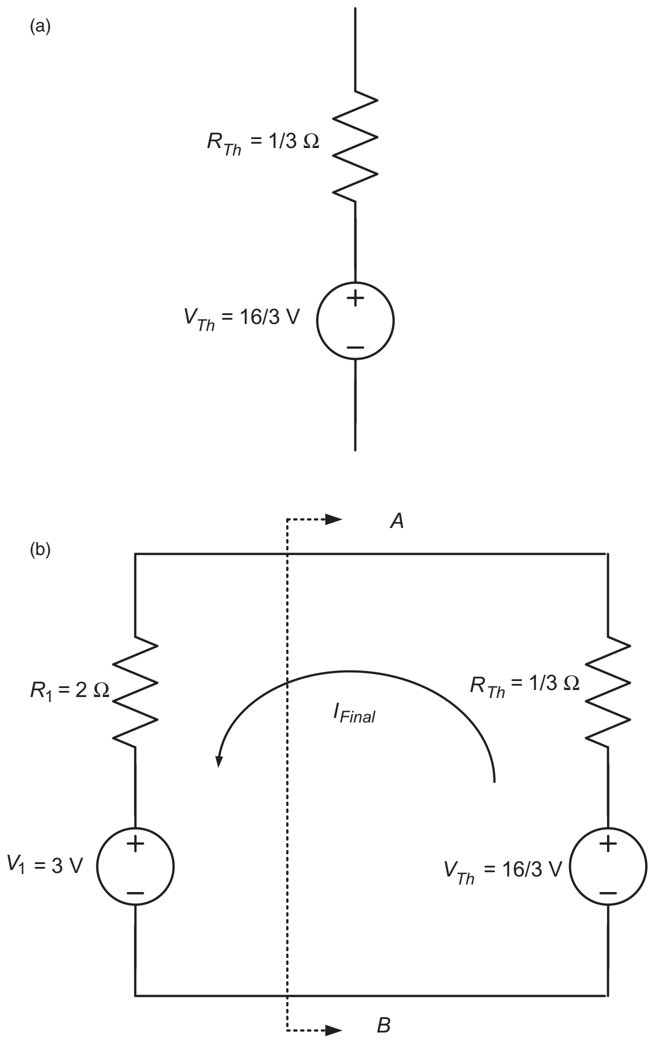

Figure 3.39 (a) The Thévenin equivalent; (b) the Thévenin equivalent circuit reattached to the left side of the sliced circuit, which was not Thévenized.

Figure 3.39 shows the final steps, the Thévenin equivalent circuit, and the attachment of the Thévenized circuit to the portion of the circuit that was not Thévenized.

The only things left are to apply KVL to the circuit of Figure 3.39b which is

$$V_{Th} - V_1 = I_{Final}(R_{Th} + R_1), \tag{3.229}$$

and calculate the voltage across nodes A and B.

$$V_{AB} = V_{Th} - I_{Final} R_{Th}. \tag{3.230}$$

Now using the numerical values for the terms of Equation (3.230) inspecting Figure 3.39b it yields

$$I_{Final} = 1 \text{ A}. \tag{3.231}$$

and

$$V_{AB} = 5 \text{ V}. \tag{3.232}$$

3.8.3 Example 3.22: Solving the Circuit of Figure 3.36 by Norton

In a similar fashion, the original circuit of Figure 3.36 can be solved using Norton's theorem. We use Figures 3.40 and 3.41 to follow the step-by-step process. We slice the circuit to which we will apply Norton in Figure 3.40a. The circuit to which Norton will be applied is on the right-hand side of the dotted line. Next we compute the short-circuit current, that is, the Norton current I_N seen on the circuit of Figure 3.40b, and calculate the Norton resistance with the circuit of Figure 3.40c. The Norton current of Figure 3.40b can be calculated using any other method of choice. You may choose to use KCL and state that the Norton or short-circuit current I_N of Figure 3.40b is

$$I_N = V_2/R_2 + V_3/R_3, \tag{3.233}$$

which numerically yields

$$I_N = 16 \text{ A}. \tag{3.234}$$

Note that the above takes into account that the voltage at node A is identical to the voltage at Ground or 0 V. Why? Because nodes A and *Ground* are tied together by a wire of zero resistance, thus both nodes are really the same node and they are at the same voltage level. Now looking at Figure 3.40c, the Norton resistance is the parallel equivalent of R_2 and R_3 and it is $R_N = 1/3\ \Omega$. Recall that to calculate the Norton resistance, we short-circuit voltage sources and open-circuit current sources. Our example only has two voltage sources V_1 and V_2, which are short-circuited to calculate the Norton resistance, Figure 3.40c.

Moving now to Figure 3.41 we draw the Norton equivalent circuit attached to the left-hand side of the original circuit that was left alone at slicing time, see Figure 3.40a. We use KCL at node A and state that

$$(V_1 - V_A)/R_1 + 16 = V_A/R_N. \tag{3.235}$$

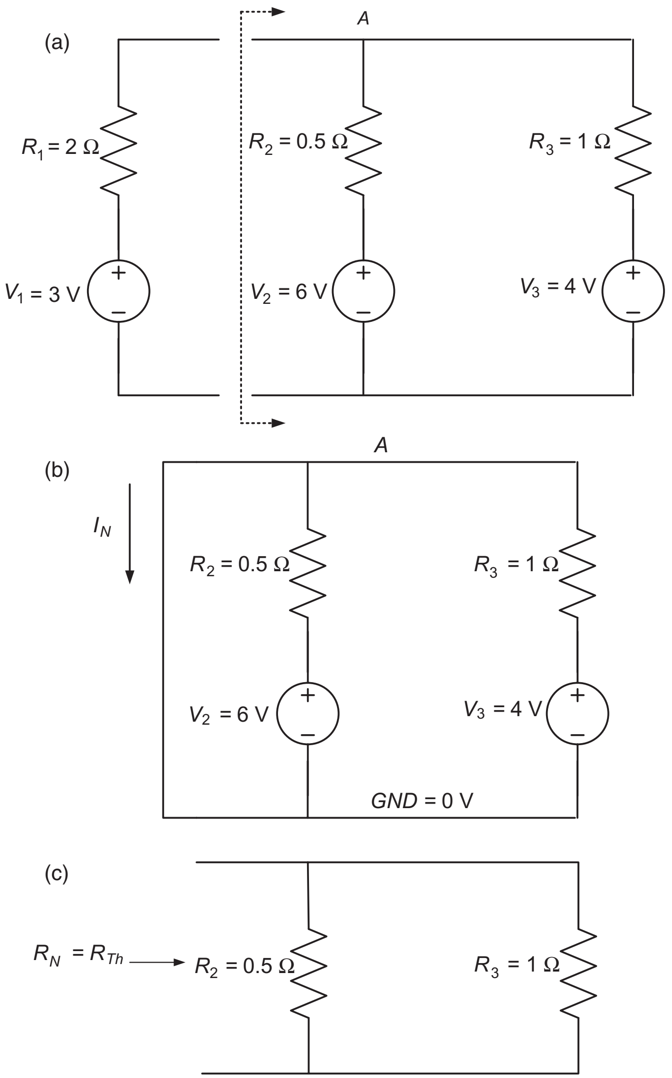

Figure 3.40 Example 3.22: (a) slicing the circuit; (b) calculating the Norton current; (c) calculating the Norton resistance.

Plugging numerical values into Equation (3.235) it yields that

$$V_A = 5\ \text{V}. \tag{3.236}$$

Note that in the circuit of Figure 3.40, V_A is now zero because node A is grounded; refer to Figure 3.40b.

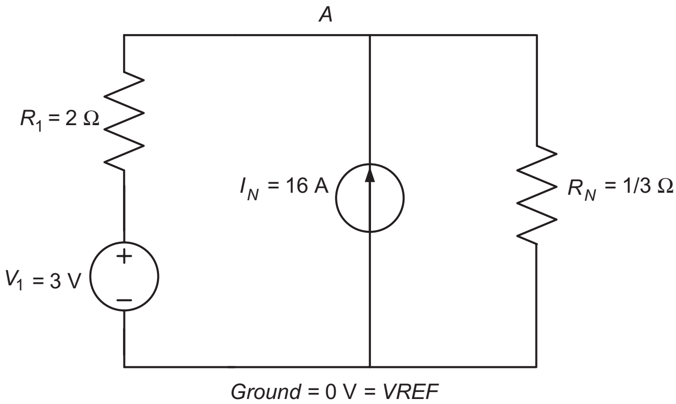

Figure 3.41 Example 3.22: Norton equivalent circuit attached to original circuit left out.

As a final step, we can apply a source transformation to the series circuit formed by V_1 and resistor R_1 of Figure 3.41. This last step, not shown in Figure 3.40, will convert the entire circuit into a single current source in parallel with a single resistor.

Drill Problem: Derive the above final step in Example 3.22, using Figure 3.41.

3.8.4 Example 3.23: Solving the Circuit of Figure 3.36 Using Source Transformations

Referring again to the original circuit of Figure 3.36, it is easy to see that we have three voltage sources in series with a resistor. Applying Thévenin's to Norton's source transformation to each source and its resistor in series, we convert them into a current source in parallel with a resistor.

The equations to do this are:

$$I_{N1} = V_1/R_1 = 3/2 = 1.5 \text{ A}. \tag{3.237}$$

$$I_{N2} = V_2/R_2 = 6/0.5 = 12 \text{ A}. \tag{3.238}$$

$$I_{N3} = V_3/R_3 = 4/1 = 4 \text{ A}. \tag{3.239}$$

The resistances in parallel with each one of the Norton current sources are the same resistors that were in series with each voltage source.

From Figure 3.42a we see that the parallel independent current sources equal to their algebraic sum of currents. The resistors in parallel are combined into a single parallel equivalent resistor, shown in Figure 3.42b

Thus, the total resulting current is obtained adding Equations (3.237) through (3.239), and this is

$$I_{N1} + I_{N2} + I_{N3} = I_{Ntotal} = 17.5 \text{ A}. \tag{3.240}$$

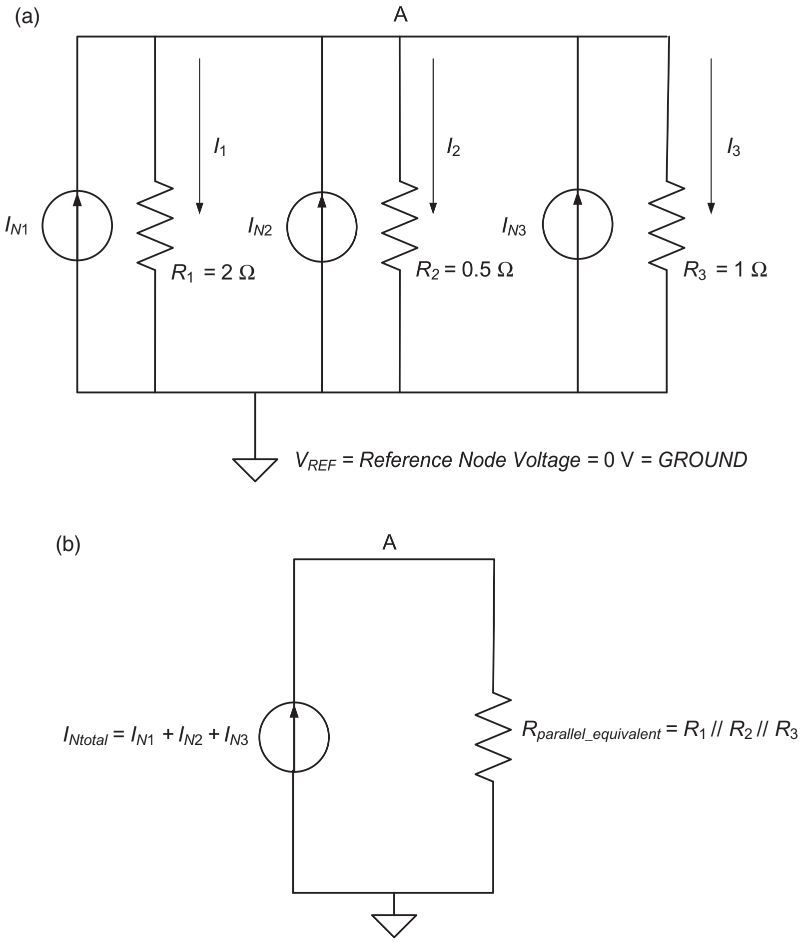

Figure 3.42 (a) Voltage source to current source transformation of circuit of Figure 3.36; (b) Norton resistance.

The total equivalent parallel resistance, that is, 2 Ω in parallel with 0.5 Ω and in parallel with 1 Ω equals

$$R_{par\text{-}equiv} = R_1//R_2//R_3 = 2//0.5//1 = 2/7\ \Omega \tag{3.241}$$

and the nodal voltage,

$$V_A = I_{Thtotal} R_{par\text{-}equiv} = 5\ \text{V}. \tag{3.242}$$

Note: The operator "//" stands for parallel equivalent resistor: so that $a//b$ is equal to: $(ab)/(a + b)$.

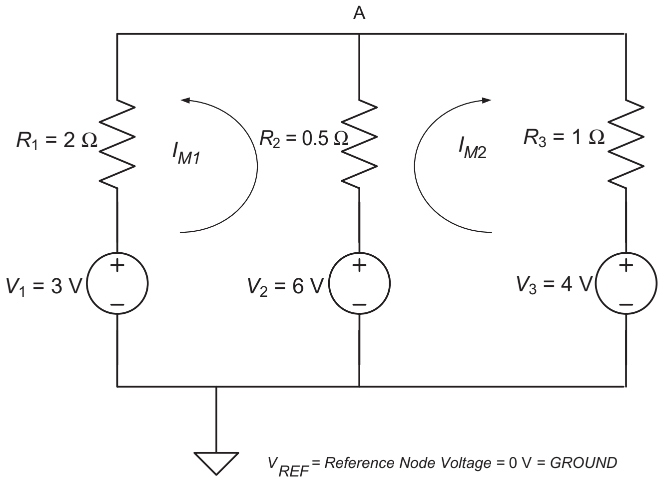

Figure 3.43 Circuit of Figure 3.36 solved by the mesh method.

3.8.5 Example 3.24: Solving the Circuit of Figure 3.36 Using the Mesh Method

The circuit of Figure 3.36 is repeated in Figure 3.43. In addition to the original information two mesh currents I_{M1} and I_{M2} have been arbitrarily defined. Note that I_{M1} was chosen to flow in the counter clockwise direction in mesh 1. I_{M2} was chosen to flow in the clockwise direction in mesh 2. Now we can start writing the mesh equations for this circuit.

$$\text{Mesh 1:}\quad V_2 - V_1 = (I_{M1} + I_{M2})R_2 + I_{M1}R_1 \tag{3.243}$$

$$\text{Mesh 2:}\quad V_2 - V_3 = (I_{M2} + I_{M1})R_2 + I_{M2}R_3. \tag{3.244}$$

Distributing and regrouping Equations (3.243) and (3.244) by mesh currents we obtain

$$\text{Mesh 1:}\quad V_2 - V_1 = I_{M1}(R_1 + R_2) + I_{M2}R_2 \tag{3.245}$$

$$\text{Mesh 2:}\quad V_2 - V_3 = I_{M1}R_2 + I_{M2}(R_2 + R_3). \tag{3.246}$$

Important observation: In Equations (3.245) and (3.246) the $I_{M2}R_2$ and $I_{M1}R_2$ terms have a positive sign because both mesh currents I_{M1} and I_{M2} flow through R_2, the common element between both meshes, in the same direction. We could have also obtained Equations (3.245) and (3.246) directly by inspection of the circuit in Figure 3.43.

We plug into Equations (3.245) and (3.246) the values from Figure 3.43 and get

$$6-3 = I_{M1}(2+0.5) + I_{M2}0.5. \tag{3.247}$$

$$6-4 = I_{M1}0.5 + I_{M2}1.5. \tag{3.248}$$

Solving Equations (3.247) and (3.248) for I_{M1} and I_{M2} we obtain

$$I_{M1} = 1 \text{ A}. \tag{3.249}$$

$$I_{M2} = 1 \text{ A}. \tag{3.250}$$

Finally, by inspection of the circuit of Figure 3.43 we see that

$$V_A = V_2 - (I_{M1} + I_{M2})R_2. \tag{3.251}$$

And again plugging the values from Figure 3.43 and from Equations (3.249) and (3.250) we obtain

$$V_A = 5 \text{ V}. \tag{3.252}$$

3.8.6 Example 3.25: Solving the Circuit of Figure 3.36 Using the Nodal Method

Figure 3.44 addresses the solving of this circuit by the nodal method. Note that the circuit of Figure 3.44 has actually four nodes (A, B, C, D, and V_{REF}). But fortunately, the nodal voltages at nodes B, C, D with respect to V_{REF} are known. That is,

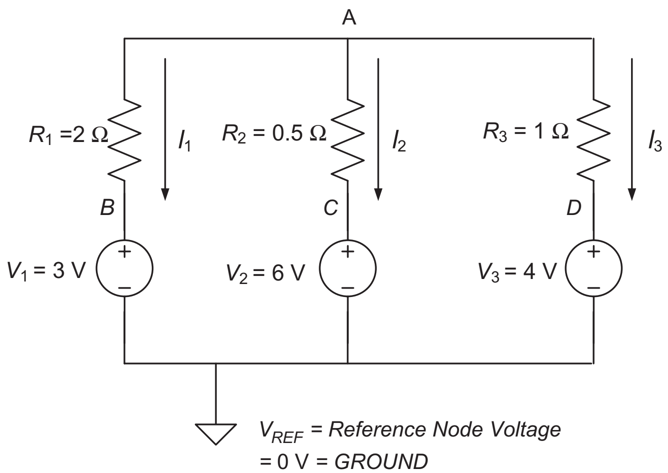

Figure 3.44 Solving the circuit of Figure 3.36 using the nodal method.

$$V_B = V_1 = 3\text{ V}. \tag{3.253}$$

$$V_C = V_2 = 6\text{ V}. \tag{3.254}$$

$$V_D = V_3 = 4\text{ V}. \tag{3.255}$$

So in actuality there is one nonreference node and an unknown nodal voltage V_A.

Nodes B, C, D are super nodes, and we can state the single nodal equation needed to solve V_A. By inspection of the circuit of Figure 3.44, the single nodal equation is

$$(V_A - V_1)/R_1 + (V_A - V_2)/R_2 + (V_A - V_3)/R_3 = 0. \tag{3.256}$$

Now plugging the values from Figure 3.44 into Equation (3.256) it yields

$$(V_A - 3)/2 + (V_A - 6)/0.5 + (V_A - 4)/1 = 0. \tag{3.257}$$

Solving for V_A we obtain that

$$V_A = 5\text{ V}. \tag{3.258}$$

3.9 SUMMARY AND CONCLUSIONS

After all six methods have been used, the reader is encouraged to go over them at least one more time to understand each of the techniques used. Leaving personal preferences aside, the nodal and the source transformation methods are quite brief and powerful. For example, look at the single Equation (3.257) used with the nodal method. The nodal method allows one to solve the problem with a single nodal equation because of the presence of super nodes. The source transformation method allows solving the problem with simple arithmetic. Now whether one can state that these two are the easiest methods is a different story. Clearly, superposition breaks down a single problem into three simpler ones. In some ways this introduces more opportunity to make an arithmetic error. Thévenin and Norton are not so bad. Finally, the mesh method allows us to solve for voltage V_A writing two equations. Ultimately, it is the reader who can clearly state which is the easiest and fastest method to apply for him or her, this becoming more of a personal preference and not an absolute fact.

FURTHER READING

1. Charles A. Desoer and Ernest S. Kuh, *Basic Electric Circuit Theory*, McGraw-Hill, New York, 1969.

2. Thomas Kailath, *Linear Systems*, Prentice Hall, Upper Saddle River, NJ, 1980.
3. Mahmood Nahvi and Joseph Edminister, *Electric Circuits*, 4th ed., McGraw-Hill, New York, 2003.
4. Charles K. Alexander and Matthew N. O. Sadiku, *Fundamentals of Electric Circuits*, 2nd ed., McGraw-Hill, New York, 2004.
5. Davis E. Johnson, Johnny R. Johnson, John L. Hilburn, and Peter D. Scott, *Electric Circuit Analysis*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 1997.

PROBLEMS

3.1 For the circuit in Figure 3.45 determine using the superposition method: (1) Current delivered by the current source I, (2) voltage across resistor R, (3) current through resistor R, (4) voltage across current source I, (5) power delivered by voltage source V, (6) power delivered by current source I, (7) power consumed by resistor R.

3.2 For the circuit in Figure 3.46 determine using the superposition method the power consumed by resistors R_1, R_2, and R_3, and the power delivered by voltage sources V_1, V_2, and V_3.

3.3 For the circuit given in Figure 3.47, find the voltage value at node A using superposition. Note that the circuit has two independent voltage sources and one voltage-controlled voltage source (VCVS). Hint: When you apply superposition, eliminate the independent sources one at a time. Never eliminate the dependent source and its control voltage.

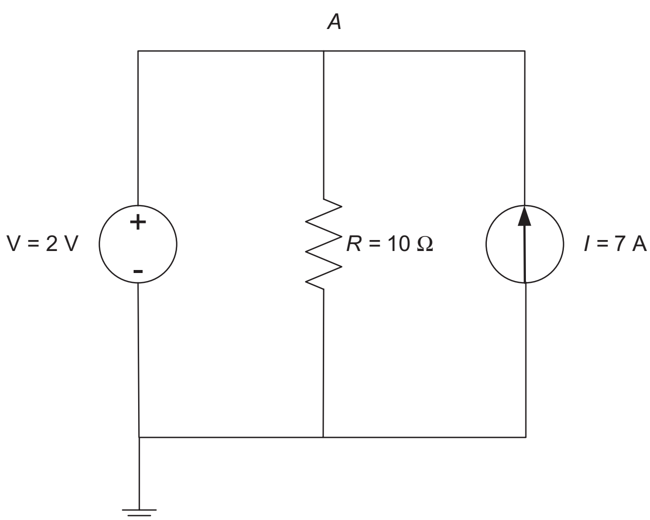

Figure 3.45 Circuit for Problem 3.1.

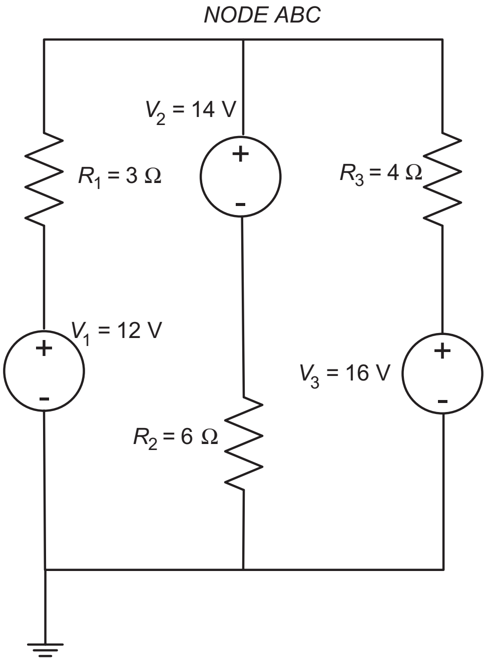

Figure 3.46 Circuit for Problem 3.2.

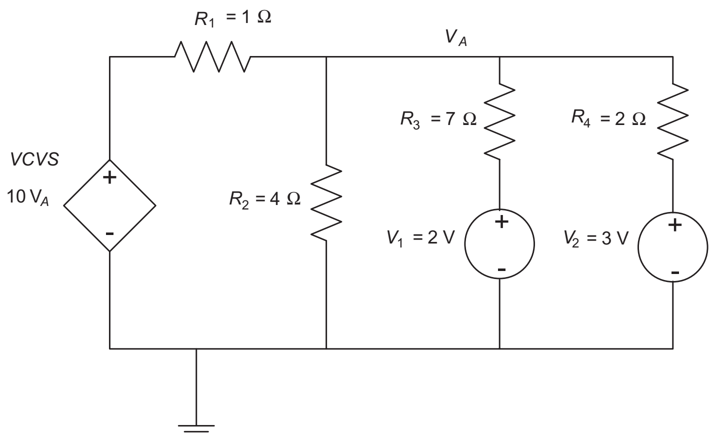

Figure 3.47 Circuit for Problem 3.3.

3.4 For the circuit given in Figure 3.48, find the voltage at node A using superposition. Note that the circuit has two independent sources and one VCVS. Hint: When you apply superposition, eliminate the independent sources one at a time. Never eliminate the dependent source and its control voltage or current.

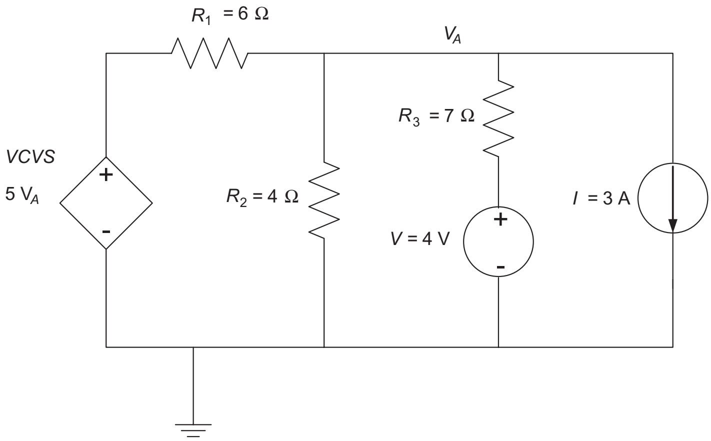

Figure 3.48 Circuit for Problem 3.4.

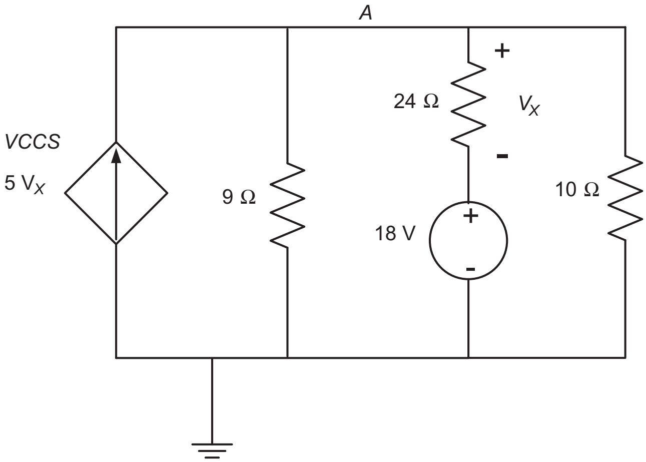

Figure 3.49 Circuit for Problem 3.5.

3.5 For the circuit given in Figure 3.49, find the voltage between node A and ground. Apply any circuit solving method of your preference.

3.6 For the circuit given in Figure 3.50, calculate the power consumed by load resistor $R_L = 2\ \Omega$.

3.7 Using the circuit of Figure 3.46 and assuming that $V_2 = 0$ V, recalculate using any circuit analysis method the power consumed by resistors R_1, R_2, and R_3, and the power delivered by voltage sources V_1, and V_3.

3.8 Using the circuit of Figure 3.51, apply any circuit analysis method to determine current I_3 through resistor R_3. Hint: This problem is much simpler than what it appears to be.

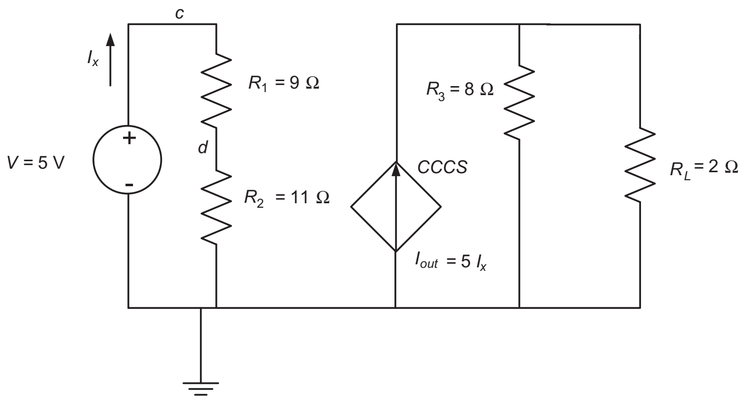

Figure 3.50 Circuit for Problem 3.6.

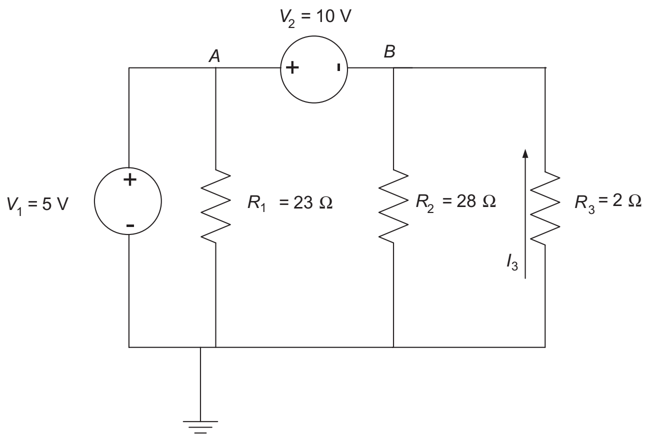

Figure 3.51 Circuit for Problem 3.8.

3.9 Using the circuit of Figure 3.52, apply Thévenin's method as many times as needed, to find current I_4 through resistor R_4.

3.10 Using the circuit of Figure 3.52, find current I_4 through resistor R_4 transforming all voltage sources to current sources and applying KCL.

3.11 Using the circuit of Figure 3.52, find current I_4 through resistor R_4 using superposition.

3.12 Using the circuit of Figure 3.52, find current I_4 through resistor R_4 using source transformations. Hint: Convert voltage sources into current sources and apply KCL.

3.13 Using the circuit of Figure 3.53, find the voltage across and current through every resistor, that is, R_1 through R_4.

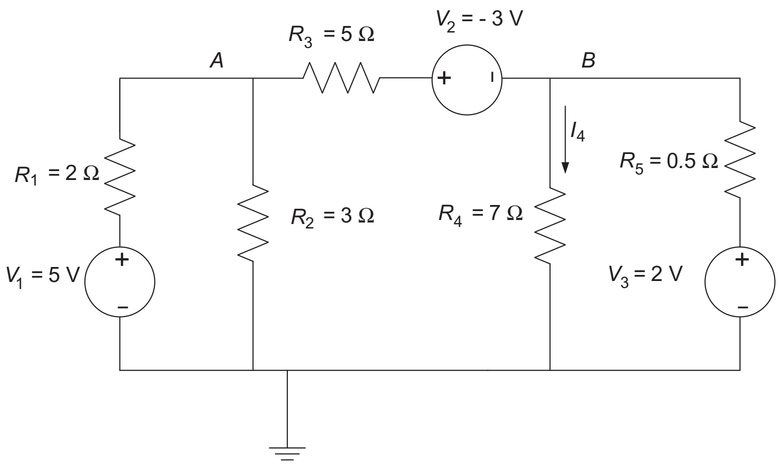

Figure 3.52 Circuit for Problems 3.9 through 3.12.

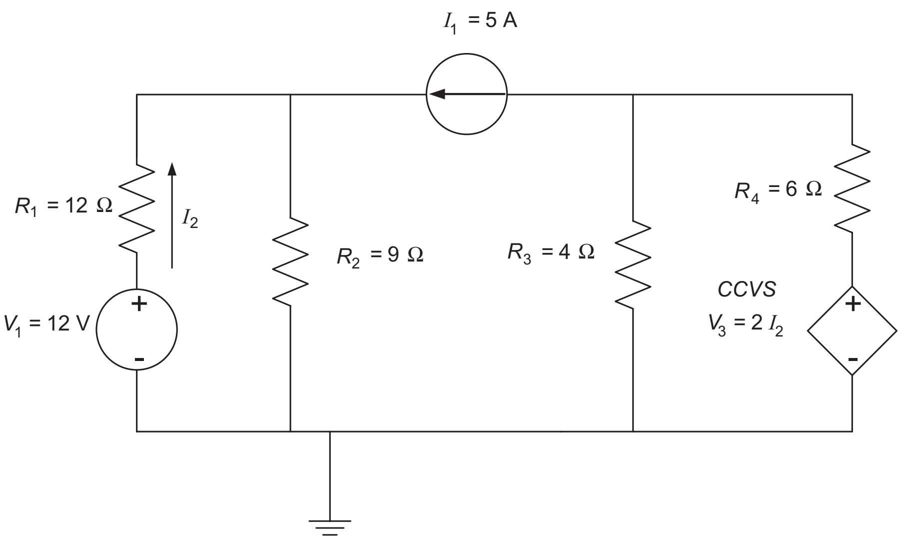

Figure 3.53 Circuit for Problem 3.13.

3.14 Calculate the total power delivered by all sources and consumed by all resistors in the circuit of Figure 3.54. Hint: There may be situations when a source does not deliver power, because it is being *charged* by some other source in the circuit. Question: Is that the case for this example? Justify your answer.

3.15 Find branch currents I_1, I_2, and I_3 of the circuit of Figure 3.55.

3.16 Apply mesh analysis to calculate branch currents I_1, I_2, and I_3 of the circuit of Figure 3.56.

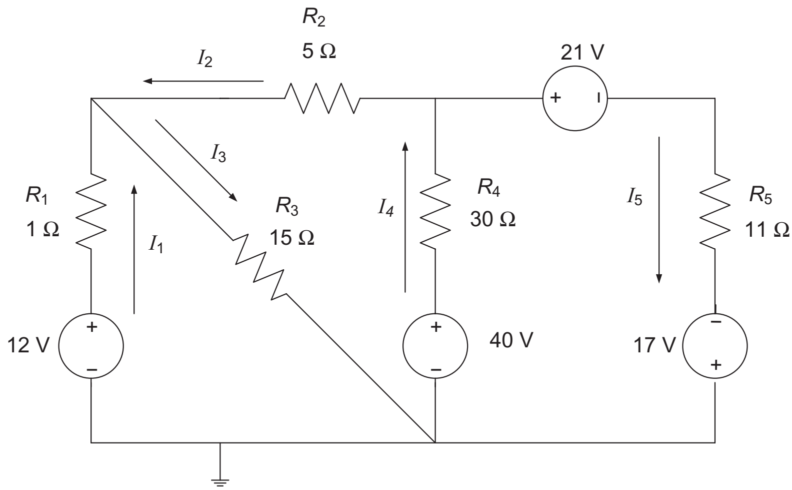

Figure 3.54 Circuit for Problem 3.14.

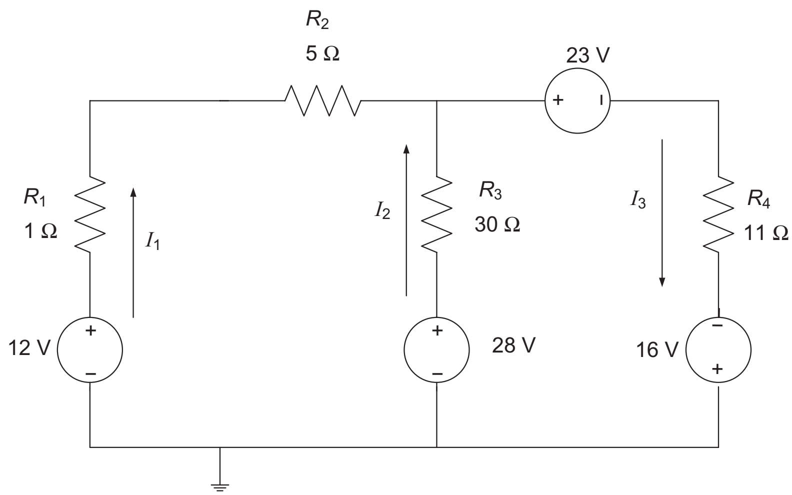

Figure 3.55 Circuit for Problem 3.15.

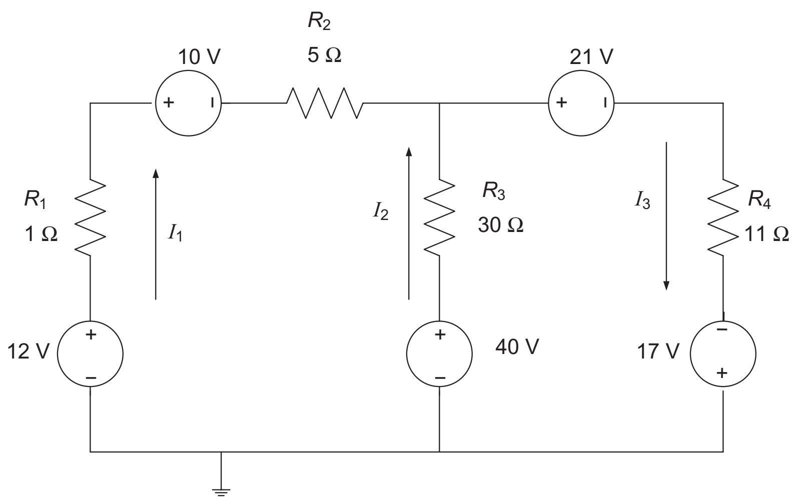

Figure 3.56 Circuit for Problem 3.16.

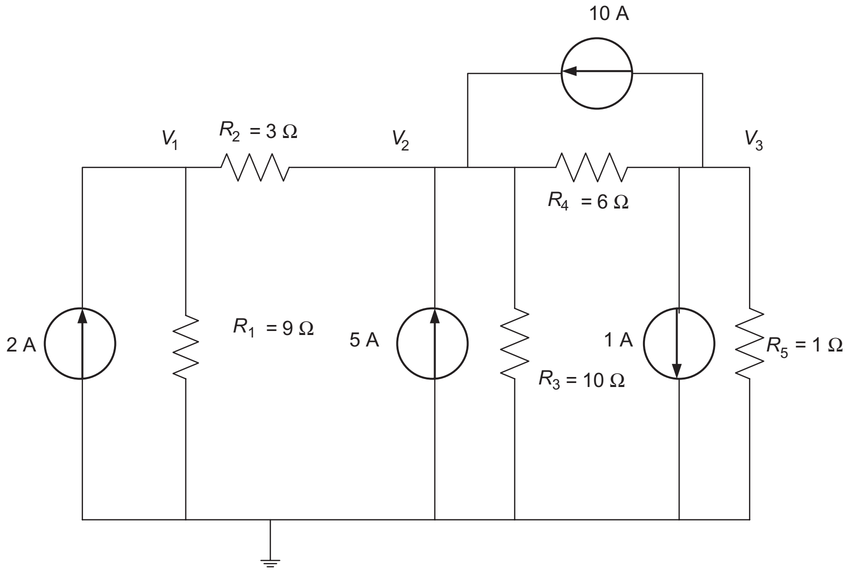

Figure 3.57 Circuit for Problem 3.17.

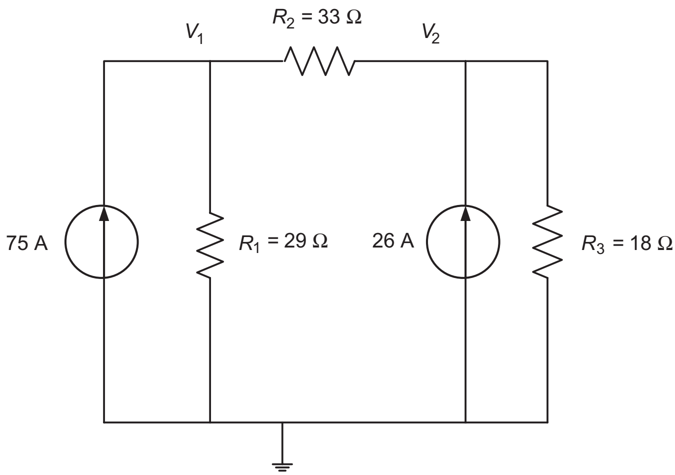

Figure 3.58 Circuit for Problems 3.18, 3.19, and 3.20.

3.17 Apply nodal analysis to calculate node voltages V_1, V_2, and V_3 of the circuit of Figure 3.57.

3.18 Apply nodal analysis to calculate node voltages V_1 and V_2 of the circuit of Figure 3.58.

3.19 Apply Norton's analysis to calculate node voltages V_1 and V_2 of the circuit of Figure 3.58.

3.20 Applying superposition calculate node voltages V_1 and V_2 of the circuit of Figure 3.58.