5

Magnetic Induction Phenomena

5.1 Fundamental Equations

The fundamental laws governing magnetic induction problems are those in (PIII.3)

$$\begin{cases} \text{curl } \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \text{div } \mathbf{B} = 0 \\ \text{curl } \mathbf{H} \approx \mathbf{J} \end{cases} \tag{5.1}$$

together with the constitutive relations concerning $\mathbf{B}(\mathbf{H})$ and $\mathbf{J}(\mathbf{E})$.

In order to spur on your interest in this new topic, we will point out just two very simple situations that cannot be justified in the framework of stationary fields, and that require the above equations for their correct interpretation. Here, you will find cases where closed circuits, containing no generators whatsoever, can have currents flowing in them. Conversely, you also will find cases where open circuits can display voltages across their terminals despite the absence of generators.

5.2 Gradient and Induction Electric Fields, Potential Vector

As you know, for purely stationary phenomena, the electric field vector is a gradient field originated by electric charges. From Part II, we found $\mathbf{E} = \mathbf{E}_g$, with

$$\begin{gathered} \text{curl } \mathbf{E}_g = 0 \rightarrow \mathbf{E}_g = -\text{grad } V \\ \text{div } \mathbf{E}_g = \rho/\varepsilon \end{gathered} \tag{5.2}$$

For time-varying phenomena we necessarily have $\mathbf{E} \neq \mathbf{E}_g$. Yet, it may be useful to consider the electric field vector to be broken down into two different contributions, the first of which is the gradient electric field, and the second is a new contribution, termed the electric induction field $\mathbf{E}_i$:

$$\mathbf{E} = \mathbf{E}_g + \mathbf{E}_i \tag{5.3}$$

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The properties of $\mathbf{E}_i$ can be obtained simply by substituting (5.3) into Maxwell's equations for homogeneous media:

$$\text{curl } \mathbf{E} = \text{curl } \mathbf{E}_g + \text{curl } \mathbf{E}_i = -\partial \mathbf{B}/\partial t$$

$$\text{div } \mathbf{E} = \text{div } \mathbf{E}_g + \text{div } \mathbf{E}_i = \rho/\varepsilon$$

By taking (5.2) into account we obtain for the electric induction field

$$\begin{cases} \text{curl } \mathbf{E}_i = -\partial \mathbf{B}/\partial t \\ \text{div } \mathbf{E}_i = 0 \end{cases} \tag{5.4}$$

from which we can see that the field lines of $\mathbf{E}_i$ are closed, embracing field lines of time-varying $\mathbf{B}$ – see Figure 5.1. If $\mathbf{B}$ happens to be a stationary field then $\mathbf{E}_i$ no longer exists.

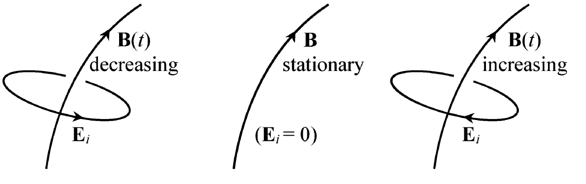

Figure 5.1 Time-varying **B** fields give rise to electric induction $\mathbf{E}_i$ fields

At this point, we recommend that you look again at div $\mathbf{B} = 0$. From vector calculus we know that div curl $\equiv 0$, and therefore we are allowed to define **B** as the *curl* of an auxiliary vector function **A** (with the same degree of arbitrariness mentioned earlier when the potential function V was introduced in Chapter 2):

$$\mathbf{B} = \text{curl } \mathbf{A} \tag{5.5}$$

Function **A**, just introduced, is known by the name of potential vector (units: Wb/m, weber per meter). Substituting (5.5) into (5.4), you readily get $\mathbf{E}_i = -\partial \mathbf{A}/\partial t$. This allows you to express the electric field vector as the sum of two contributions where both the old scalar potential and the new potential vector appear:

$$\mathbf{E} = -\left(\text{grad } V + \frac{\partial \mathbf{A}}{\partial t}\right) \tag{5.6}$$

5.3 Revisiting the Voltage Concept

The aspect we are now going to address is new and critical, because it goes against your intuition. So, please pay attention. If you have two ideal voltmeters V_1 and V_2 connected between the same pair of terminals a and b – see Figure 5.2 – and if you are asked about the relationship between their readings, what will your answer be?

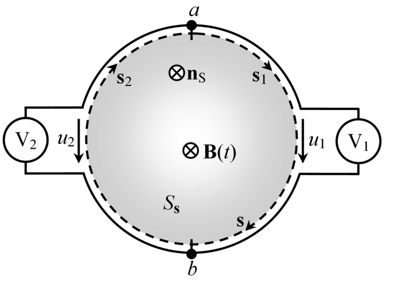

Figure 5.2 For time-varying regimes, $u_1(t) \neq u_2(t) \neq (V_a - V_b)$

Most of you will say that the readings are the same. But that may be well wrong!

Let us go back to the original definition of voltage in (1.7) and replace **E** with its new definition in (5.6):

$$u = \int_{\overrightarrow{ab}} \mathbf{E}\cdot d\mathbf{s} = -\int_{\overrightarrow{ab}} \mathrm{grad}\ V \cdot d\mathbf{s} - \frac{d}{dt}\int_{\overrightarrow{ab}} \mathbf{A}\cdot d\mathbf{s} = (V_a - V_b) - \frac{d}{dt}\int_{\overrightarrow{ab}} \mathbf{A}\cdot d\mathbf{s} \tag{5.7}$$

Since **A** is not a conservative field (curl $\mathbf{A} \neq 0$), the evaluation of

$$\int_{\overrightarrow{ab}} \mathbf{A}\cdot d\mathbf{s}$$

depends, in general, on the particular path going from a to b. This shows not only that voltage and potential difference are quite different things, $u \neq V_a - V_b$, but also that the evaluation of u requires a clear specification of the integration path between the two points a and b. Then, from (5.7), considering the voltages in Figure 5.2, we have

$$u_1 = (V_a - V_b) - \frac{d}{dt}\int_{\overrightarrow{(ab)_{\mathbf{s}_1}}} \mathbf{A}\cdot d\mathbf{s};\ \ u_2 = (V_a - V_b) + \frac{d}{dt}\int_{\overrightarrow{(ba)_{\mathbf{s}_2}}} \mathbf{A}\cdot d\mathbf{s}$$

By subtracting the preceding results we obtain the difference of the two voltmeters' readings:

$$u_1 - u_2 = -\frac{d}{dt}\left(\int_{\overrightarrow{(ab)_{\mathbf{s}_1}}} \mathbf{A}\cdot d\mathbf{s} + \int_{\overrightarrow{(ba)_{\mathbf{s}_2}}} \mathbf{A}\cdot d\mathbf{s}\right) = -\frac{d}{dt}\oint_{\mathbf{s}} \mathbf{A}\cdot d\mathbf{s} \tag{5.8}$$

where **s**, the reunion of the subpaths $\mathbf{s}_1$ and $\mathbf{s}_2$, is a closed clockwise-oriented path.

If **A** is null ($\mathbf{B} = 0$) or time invariant then we will get $u_1 = u_2$.

5.4 Induction Law

The cornerstone of magnetic induction phenomena, which allows us to evaluate voltages arising from time-varying **B**-fields, is the Maxwell–Faraday induction law.

From curl $\mathbf{E} = -\partial\mathbf{B}/\partial t$, using the already familiar Stokes theorem, we get

$$\oint_{\mathbf{s}} \mathbf{E}\cdot d\mathbf{s} = -\frac{d}{dt}\int_{S_s} \mathbf{B}\cdot\mathbf{n}_S \, dS \quad \text{or} \quad \oint_{\mathbf{s}} \mathbf{E}\cdot d\mathbf{s} = -\frac{d\psi_S}{dt} \tag{5.9}$$

where, as defined in (4.28), ψ_S is the magnetic flux linked with the circulation path **s**. In (5.9), the unit vector $\mathbf{n}_S$ normal to S_s is oriented according to the Stokes rule (right-hand screw rule).

The induction law in (5.9) simply states that the electromotive force induced along a closed path is equal to the negative derivative of the magnetic flux linked with that path.

The question posed in Figure 5.2, concerning the voltmeters' readings, can be reanswered (avoiding the use of the potential vector **A**) by making use of the induction law:

$$u_1 - u_2 = -\frac{d}{dt}\int_{S_s} \mathbf{B}\cdot\mathbf{n}_S \, dS = -\frac{d\psi_S}{dt}$$

As long as the open surface S_s having the path **s** as its bounding edge is traversed by a time-varying magnetic field, voltages u_1 and u_2 will be different.

5.5 Application Example (Magnetic Noise Effects)

Figure 5.3 shows an indoor electrical socket where a two-wire line is plugged in; the line is left open ($i = 0$) at the opposite end.

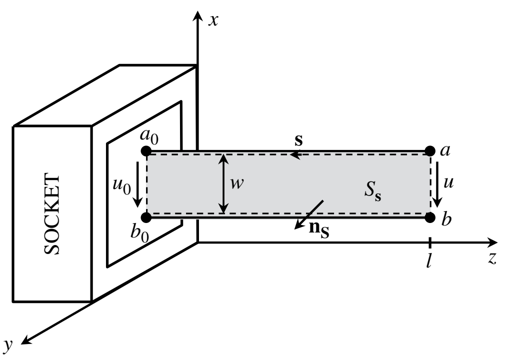

Figure 5.3 The relationship between voltages u_0 and u depends on the magnetic noise interference produced by externally produced **B** fields in the region under analysis

Assume that the two-wire line is longitudinally stretched so as to define a rectangular path of length l and width w, lying in the zx plane. Assume also that the socket voltage is $u_0(t) = U_0 \cos(\omega_0 t)$ and consider electronic equipment placed somewhere in the neighborhood that gives rise, in the region of the line, to a uniform magnetic field noise described by $B(t) = B_M \cos(\omega_\mathrm{B} t)$.

Questions

Q_1 Determine $u(t)$ assuming that $\mathbf{B} = B(t)\vec{e}_x$.

Q_2 Determine $u(t)$ assuming that $\mathbf{B} = B(t)\vec{e}_y$.

Q_3 Describe a simple solution to mitigate the magnetic noise effects.

Solutions

Q_1 By applying the induction law to the closed path $\overrightarrow{aa_0b_0ba}$ we find, noting that $\mathbf{B} \perp \mathbf{n}_\mathrm{S}$,

$$\underbrace{\oint_{\mathrm{S}} \mathbf{E} \cdot d\mathbf{s}}_{u_0 - u} = -\frac{d}{dt} \underbrace{\int_{S_\mathrm{s}} \mathbf{B} \cdot \vec{e}_y dS}_{0}$$

from which we obtain $u(t) = u_0(t) = U_0 \cos(\omega_0 t)$.

Q_2 By applying the induction law to the closed path $\overrightarrow{aa_0b_0ba}$ we find, noting that $\mathbf{B} \parallel \mathbf{n}_\mathrm{S}$,

$$\underbrace{\oint_{\mathrm{S}} \mathbf{E} \cdot d\mathbf{s}}_{u_0 - u} = -\frac{d}{dt} \underbrace{\int_{S_\mathrm{s}} \mathbf{B} \cdot \vec{e}_y dS}_{Bwl}$$

from which we obtain $u(t) = U_0 \cos(\omega_0 t) - wl\omega_B B_M \sin(\omega_B t)$.

Q_3 If the two-wire line is twisted at regular intervals the Stokes normal $\mathbf{n}_\mathrm{S}$ concomitantly switches from $+\vec{e}_y$ to $-\vec{e}_y$. If an odd number of twists are used then we will find $u \approx u_0$ provided that the perturbing magnetic field remains uniform in the region of the line.

5.6 Voltages and Currents in Magnetically Multicoupled Systems

We have already introduced in Section 4.12 the problem of magnetically coupled linear circuits, where self- and mutual inductances were defined. Here we elaborate on that problem considering time-varying currents.

In Figure 5.4 we show a ferromagnetic core provided with two coils. One is connected to a generator and the other is left open. Coil 1, with N_1 turns, has an internal resistance R_1; likewise coil 2, with N_2 turns, has an internal resistance R_2. Magnetic coupling between the coils is ensured through the magnetic field lines in the core that simultaneously embrace both coils. Current $i_1(t)$ flowing in coil 1 is given in the form of a triangular impulse (Figure 5.5(a)) of peak amplitude I_M.

To simplify matters let us assume that dispersion phenomena are negligible; that is, the magnetic flux in the core $\phi(t)$ is the same at every cross-section.

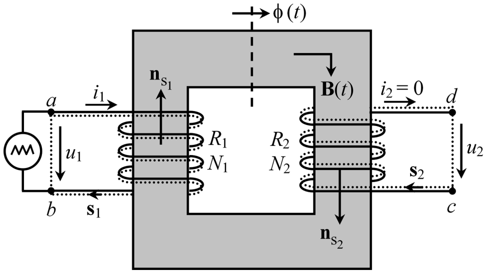

Figure 5.4 Circulation paths $\mathbf{s}_1$ and $\mathbf{s}_2$ used for the application of the induction law in order to evaluate the voltages u_1 and u_2 at the terminals of two magnetically coupled coils

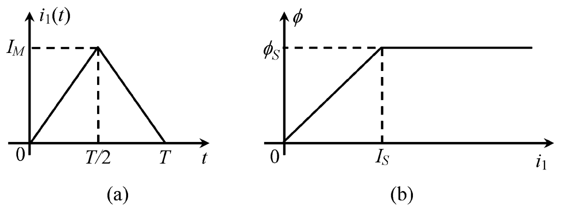

Figure 5.5 (a) Generator current (triangular pulse). (b) Stylized nonlinear magnetic characteristic $\phi(i_1)$ of the ferromagnetic core

Irrespective of the linear or nonlinear character of the core, the determination of ϕ as a function of i_1 is done by application of Ampère's law examined in Chapter 4.

Let us consider that the magnetic characteristic $\phi = \phi(i_1)$ is nonlinear, as stylized in Figure 5.5(b). The knee point $(I_S,\ \phi_S)$ defines an abrupt transition between the linear zone and the saturation zone.

In order to determine the generator voltage u_1 we apply the induction law to the closed circulation path $\mathbf{s}_1$, passing inside the coil conducting turns and through the generator. The path is oriented according to the prescribed reference direction of i_1.

For the sake of clarity, let us determine separately the left- and right-hand sides of the induction law,

$$\oint_{\mathbf{s}_1} \mathbf{E} \cdot d\mathbf{s} = \int_{\substack{\frown \\ ab \\ \text{coil}}} \frac{1}{\sigma_1} \mathbf{J}_1 \cdot d\mathbf{s} + \int_{\substack{\frown \\ ba \\ \text{(G)}}} \mathbf{E} \cdot d\mathbf{s} = +R_1 i_1 - u_1 \tag{5.10a}$$

$$-\frac{d}{dt} \int_{S_{s_1}} \mathbf{B} \cdot \mathbf{n}_{S_1}\ dS = -\frac{d}{dt} \psi_1 = -N_1 \frac{d\phi}{dt} \tag{5.10b}$$

Equating the results in (5.10), we find

$$u_1(t) = R_1 i_1(t) + N_1 \frac{d\phi(t)}{dt} \tag{5.11}$$

Now, let us turn our attention to coil 2 which, remember, carries no current. Nevertheless, as you will see, a voltage u_2 is going to appear across its terminals!

To determine voltage u_2 at the open terminals of coil 2 we reapply the induction law to the closed circulation path $\mathbf{s}_2$, passing inside the coil conducting turns and through the air. The path is oriented according to the prescribed reference direction of i_2. Following the same procedure as before, we find

$$\oint_{\mathbf{s}_2} \mathbf{E}\cdot d\mathbf{s} = \int_{\substack{\widehat{cd}\\ \text{coil}}} \frac{1}{\sigma_2}\mathbf{J}_2\cdot d\mathbf{s} + \int_{\substack{\widehat{dc}\\ \text{air}}} \mathbf{E}\cdot d\mathbf{s} = 0 + u_2 \tag{5.12a}$$

$$-\frac{d}{dt}\int_{S_{s_2}} \mathbf{B}\cdot\mathbf{n}_{S_2}\, dS = -\frac{d}{dt}\psi_2 = -N_2\frac{d\phi}{dt} \tag{5.12b}$$

Note that, in (5.12a), the integration $\widehat{cd}$ along the coil's conductor gives zero because we have assumed $i_2 = 0$, otherwise we would obtain $R_2 i_2$.

Equating the results in (5.12), we find

$$u_2(t) = -N_2\frac{d\phi(t)}{dt} \tag{5.13}$$

from which you can see that, although coil 2 carries no current, a voltage will appear at its terminals, as a result of induction phenomena.

Next, we analyze the results in (5.11) and (5.13), presenting some pertinent graphics and interpreting from a physical point of view the reason why u_2 exists (despite $i_2 = 0$).

To start with, consider the simple case where the core behaves linearly (that is, $i_1(t) \le I_S$). Take for instance $I_M = I_S$. Combining the information conveyed in Figure 5.5, you will obtain for $\phi(t)$ a triangular function, with peak value ϕ_S, similar to the one describing $i_1(t)$.

The derivative $d\phi/dt$ appearing in (5.11) and (5.13) is evaluated as

$$\frac{d\phi}{dt} = \begin{cases} +2\phi_S/T; & \text{for } 0 < t < T/2 \\ -2\phi_S/T; & \text{for } T/2 < t < T \end{cases}$$

The evolution of $u_1(t)$ is obtained by summing the resistive voltage component $u_R(t) = R_1 i_1(t)$ with the inductive voltage component $u_L(t) = N_1\, d\phi/dt$. The graphical construction leading to $u_1(t)$ is illustrated in Figure 5.6.

Things are a bit simpler for $u_2(t)$ since the resistive component is absent – see Figure 5.7.

Let us now interpret the results obtained for $u_2(t)$. For this purpose it is helpful to visualize a cross-section of the core leg where coil 2 is wound around (Figure 5.8).

In the time interval 0 to $T/2$ the field $B(t) = \phi(t)/S$ is time increasing. The associated field lines of the electric induction field $\mathbf{E}_i$ are circumferences oriented anticlockwise; this field actuates on the free charged particles of coil 2 ($\mathbf{F}_e = q\mathbf{E}_i$) giving rise to a distribution of positive charges at terminal c and negative charges at terminal d. A gradient electric field $\mathbf{E}_g$ oriented from c to d appears and a negative pulse voltage is revealed in $u_2(t)$.

A similar rationale applies to the time interval from $T/2$ to T during which the field $\mathbf{B}(t)$ is time decreasing. The associated field lines of the electric induction field $\mathbf{E}_i$ are

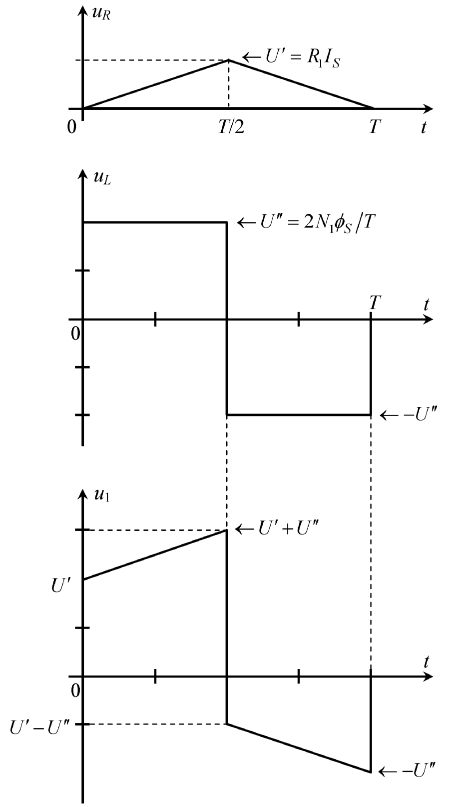

Figure 5.6 Graphical representation of the generator voltage against time, $u_1(t)$, showing its decomposition into resistive and inductive components u_R and u_L, respectively

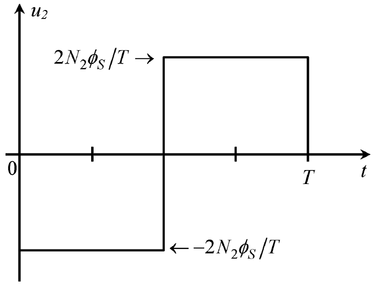

Figure 5.7 Graphical representation of coil 2 voltage against time, $u_2(t)$

circumferences oriented clockwise; this field actuates on the free charged particles of coil 2 giving rise to a distribution of positive charges at terminal d and negative charges at terminal c. A gradient electric field $\mathbf{E}_g$ oriented from d to c appears and a positive pulse voltage is revealed in $u_2(t)$.

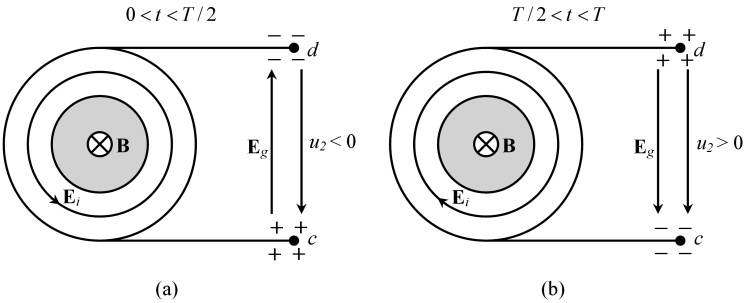

Figure 5.8 Interpretation of the positive and negative pulses appearing in $u_2(t)$. (a) Time-increasing **B** originates an induction electric field $\mathbf{E}_i$ that drives the positive charges to terminal c and negative charges to terminal d. (b) Time-decreasing **B** reverses the orientation of the induction electric field $\mathbf{E}_i$; positive charges are driven to terminal d and negative charges to terminal c

Let us now complicate things a little, allowing the ferromagnetic core to saturate due to imposed higher current intensity.

Make $I_M = 2I_S$. Combining the information conveyed in Figure 5.5, we obtain for $\phi(t)$ a trapezoidal function as shown in Figure 5.9.

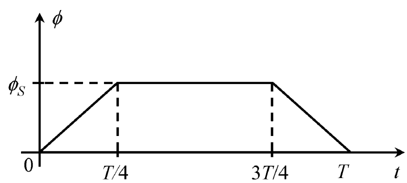

Figure 5.9 Trapezoidal function $\phi(t)$ originated by an intense triangular current pulse $i_1(t)$ that brings the core into saturation

The derivative $d\phi/dt$ shown in (5.11) and (5.13) is evaluated as

$$\frac{d\phi}{dt} = \begin{cases} +4\phi_S/T; & \text{for } 0 < t < T/4 \\ 0; & \text{for } T/4 < t < 3T/4 \\ -4\phi_S/T; & \text{for } 3T/4 < t < T \end{cases}$$

We strongly recommend that you try to redraw new graphs for

$$u_t(t) = R_1 i_1(t) + N_1 \frac{d\phi(t)}{dt} \quad \text{and} \quad u_2(t) = -N_2 \frac{d\phi(t)}{dt}$$

Omitting any details, we present the final results in Figure 5.10. Note that in the time interval from $T/4$ to $3T/4$ the **B** field remains constant with time, induction phenomena are absent, and therefore you get $u_2 = 0$.

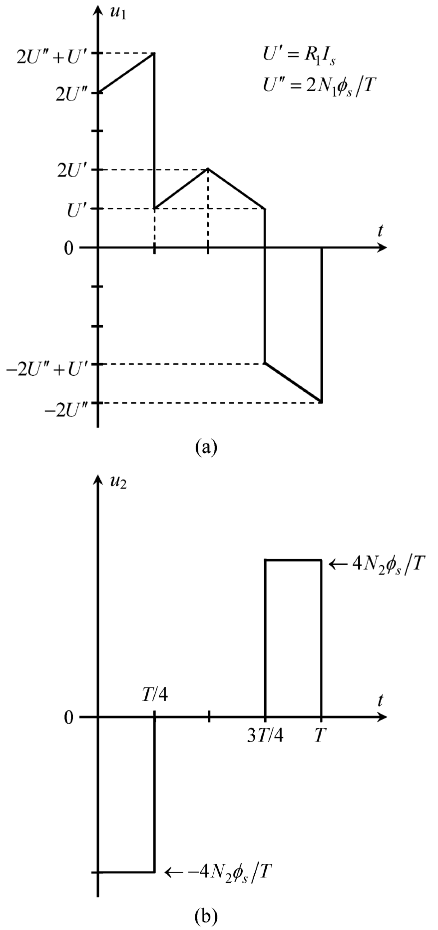

Figure 5.10 Voltage plots against time for the case of a saturated core. (a) Generator voltage (compare Figure 5.6). (b) Voltage of coil 2 (compare Figure 5.7)

The first part of the problem we have just finished solving (linear case) could have been handled more easily if the inductance coefficients characterizing the two-coupled coils were known beforehand. In fact, by using the results of Chapter 4, we could have put

$$\psi_1 = L_{11} i_1 + L_M i_2 \quad \text{and} \quad \psi_2 = L_M i_1 + L_{22} i_2$$

Substituting ψ_1 and ψ_2 above into (5.10b) and (5.12b), and making $i_2 = 0$, we would obtain, as an alternative, the following results:

$$u_1 = R_1 i_1 + L_{11} \frac{di_1}{dt} \quad \text{and} \quad u_2 = -L_M \frac{di_1}{dt} \tag{5.14}$$

Note, however, that you cannot use this alternative when dealing with nonlinear problems (where the concept of inductance does not apply).

In the more general case of a system of N-coupled coils, characterized by their internal resistances R_k (with $k = 1$ to N) and inductance matrix $[L]$, (4.41), the application of the induction law to every coil (using circulation paths coinciding with the reference directions of coil currents) permits coil voltages to be compactly determined from coil currents by making use of the following matrix equation:

$$[u(t)] = [R][i(t)] + [L]\frac{d}{dt}[i(t)] \tag{5.15}$$

where the resistance matrix $[R]$ is diagonal, the column matrix $[i(t)]$ gathers the coil currents, and the column matrix $[u(t)]$ gathers the coil voltages.

A word of caution: depending on how you choose the arbitrary reference directions for the coil voltages, $u_1, \ldots, u_k, \ldots, u_N$, both positive and negative algebraic signs need to be included in the elements of the column matrix $[u(t)]$ in (5.15). If all coil voltages u_k are marked so as to oppose the orientation of the circulation paths $\mathbf{s}_k$ then only positive algebraic signs will appear.

5.7 Application Example (Magnetic Coupling in Printed Circuit Boards)

A printed circuit board consisting of three conducting lands on the surface of a dielectric board above a reference conducting ground plane is shown in Figure 5.11. To simplify things, let's assume that all conductor resistances are negligibly small.

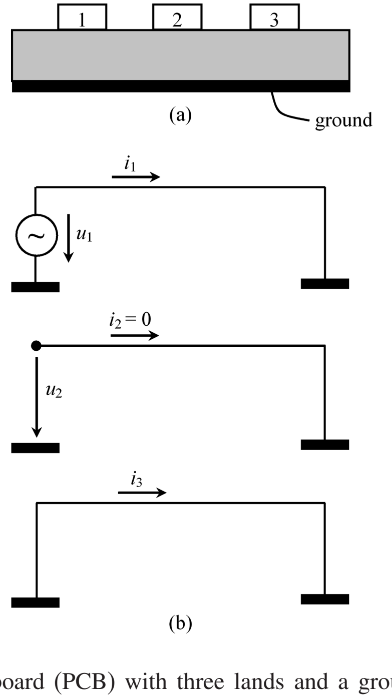

Figure 5.11 Printed circuit board (PCB) with three lands and a ground plane. (a) Cross-sectional view. (b) Enforced boundary conditions at the near and far ends of the PCB lands

All lands are short-circuited to ground at the far end. As for the near end, the situation is described as follows: land 1 is driven by a voltage u_1, land 2 is left open ($i_2 = 0$) and land 3 is short-circuited to ground ($u_3 = 0$). The conductor system is characterized by a symmetric inductance matrix

$$[L] = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

Voltage u_1 is a low-frequency sinusoidal function, $u_1(t) = U_1 \cos(\omega t)$.

Questions

Q_1 By application of the induction law obtain the governing equations of the system.

Q_2 Determine $i_1(t)$, $i_3(t)$ and $u_2(t)$.

Solutions

Q_1 Taking into account that $i_2 = 0$, from (5.15), we write

$$\begin{cases} u_1 = L_{11}\dfrac{di_1}{dt} + L_{13}\dfrac{di_3}{dt} \\ u_2 = L_{21}\dfrac{di_1}{dt} + L_{23}\dfrac{di_3}{dt} \\ 0 = L_{31}\dfrac{di_1}{dt} + L_{33}\dfrac{di_3}{dt} \end{cases}$$

Q_2 From the third equation we get

$$i_3 = -\frac{L_{31}}{L_{33}} i_1$$

Substituting this information into the first equation we find

$$u_1 = \underbrace{\left(L_{11} - \frac{L_{13}^2}{L_{33}}\right)}_{L_1} \frac{di_1}{dt} \rightarrow i_1(t) = \frac{1}{L_1}\int u_1(t)dt = \frac{U_1}{\omega L_1}\cos(\omega t - \pi/2)$$

Using the already established relationship between i_3 and i_1 we find

$$i_3(t) = \frac{L_{31}U_1}{\omega L_{33}L_1}\cos(\omega t + \pi/2)$$

From the second equation in Q_1 we obtain

$$u_2(t) = U_1\left(\frac{L_{21}}{L_1} - \frac{L_{23}L_{31}}{L_1 L_{33}}\right)\cos(\omega t)$$

5.8 Eddy Currents

We have already mentioned in Section 4.11 that ferromagnetic cores, subjected to periodic magnetization–demagnetization processes, dissipate energy because of hysteresis phenomena. There is another physical mechanism that can also cause additional core losses (heating).

You may still remember that in the preceding section (Figure 5.8) voltage u_2 was created by the action of an electric induction field $\mathbf{E}_i$ with closed field lines embracing lines of $\mathbf{B}(t)$. Such a field $\mathbf{E}_i$ exists not only outside the core, but also in its interior. Since ferromagnetic materials are also conductors (conductivity σ_{Fe}) this implies necessarily that electric currents with density $\mathbf{J}_i = \sigma_{\mathrm{Fe}}\mathbf{E}_i$ are allowed to circulate in planes transversal to $\mathbf{B}$, that is in the core cross-sections (see Figure 5.12). These currents are called eddy currents, or Foucault currents.

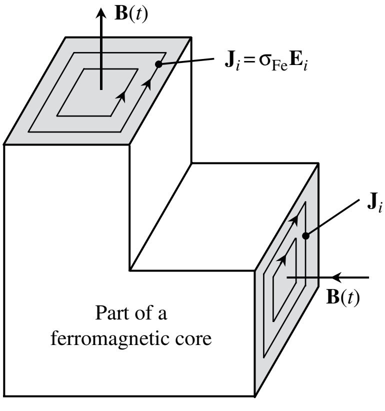

Figure 5.12 Eddy currents in conducting ferromagnetic materials originated by time-varying **B** fields

From (3.14) in Chapter 3, the power losses (Joule effect) associated with these currents are evaluated through

$$P_{\mathrm{Foucault}} = \int_{\mathrm{Core}} \hat{p}_{\mathrm{J}}\, dV; \;\; \hat{p}_{\mathrm{J}} = \mathbf{J}_i \cdot \mathbf{E}_i = \sigma_{\mathrm{Fe}} E_i^2 \propto \left(\frac{dB}{dt}\right)^2 \tag{5.16}$$

This shows that Foucault losses depend on the squared intensity of the electric induction field, which, in turn, depends on the time derivative of $B(t)$. In conclusion, the faster the variation of **B**, the more important the losses.

Another problem created by eddy currents is their own magnetic field which can significantly perturb the original field existing in the ferromagnetic core.

There are two known techniques that can be employed to mitigate eddy current effects. One consists of using laminated cores where each ferromagnetic sheet is electrically insulated from the others using a non-conducting varnish. This technique is relatively effective for time-varying **B** fields up to 20 kHz. For higher frequencies, ferrites should rather be used. Ferrites are electrically non-conductive ceramic compound materials consisting of a mix of iron, zinc and manganese oxides.

5.9 Generalization of the Induction Law to Moving Circuit Systems

This important topic deals with induction phenomena not originated by time-varying **B** fields, but by circuits (subjected to stationary **B** fields) whose geometrical configuration is time dependent.

Even if the velocity of the moving circuits is much smaller than the speed of light, for a rigorous and sound interpretation of the problem a contribution from the theory of relativity would be required. The introductory nature of this textbook prevents us from following such an approach here.

In any case, imagine the following *Gedanken* (thought) experiment. In a certain region of space an electromagnetic field exists. An observer O′ at rest characterizes the electromagnetic field in the region by a pair of vectors $\mathbf{E}'$ and $\mathbf{B}'$. O′ also observes that a particle with electric charge Q moves across the region with velocity $\mathbf{v}$, its trajectory changing according to the exerted Lorentz force $\mathbf{F} = Q\mathbf{E}' + Q\mathbf{v}\times\mathbf{B}'$. A second observer O seated on the particle looks at the region where the particle travels and characterizes the *same* electromagnetic field by a different pair of vectors **E** and **B**. Since the charged particle is at rest ($\mathbf{v} = 0$) with respect to O, this observer interprets its trajectory change as the result of a purely electric force $\mathbf{F} = Q\mathbf{E}$.

In order to make both observations agree we have to conclude that

$$\mathbf{E} = \mathbf{E}' + \underbrace{\mathbf{v}\times\mathbf{B}'}_{\mathbf{E}_\mathrm{v}} \tag{5.17}$$

where $\mathbf{E}_\mathrm{v}$ is the so-called dynamic electric field.

Although we are not going to prove it, **B** could be obtained as $\mathbf{B} = \mathbf{B}' - \mathbf{v}\times\mathbf{E}'/c^2$. Therefore, for typical applications ($v \ll c$), $\mathbf{B} = \mathbf{B}'$.

The induction law, for bodies at rest subjected to time-varying magnetic fields, has been formulated in (5.9) as

$$\oint_{\mathbf{S}} (\mathbf{E}_g + \mathbf{E}_i)\cdot d\mathbf{s} = -\frac{d}{dt}\int_{S_\mathbf{s}} \mathbf{B}(t)\cdot\mathbf{n}_\mathrm{S}\; dS$$

For moving bodies subjected to stationary magnetic fields, the above equation should be modified to

$$\oint_{\mathbf{S}(t)} (\mathbf{E}_g + \mathbf{E}_\mathrm{v})\cdot d\mathbf{s} = -\frac{d}{dt}\int_{S_{\mathbf{s}(t)}} \mathbf{B}\cdot\mathbf{n}_\mathrm{S}(t)\; dS \tag{5.18}$$

where the circulation path moves with the moving parts of the circuit, $\mathbf{s} = \mathbf{s}(t)$.

In the most general case of moving bodies subjected to time-varying magnetic fields, the generalization of the Maxwell–Faraday induction law takes the form

$$\oint_{\mathbf{S}(t)} \underbrace{(\mathbf{E}_g + \mathbf{E}_i + \mathbf{E}_\mathrm{v})}_{\mathbf{E}}\cdot d\mathbf{s} = -\frac{d}{dt}\underbrace{\int_{S_{\mathbf{s}(t)}} \mathbf{B}(t)\cdot\mathbf{n}_\mathrm{S}(t)\; dS}_{\psi_{\mathbf{s}(t)}} \tag{5.19}$$

where, it should be stressed, the **E** field on the left-hand side refers to the electric field as observed in the moving reference frame.

5.10 Application Example (Electromechanical Energy Conversion)

We now present an example that illustrates the principle of conversion of mechanical energy into electric energy and the conversion of electric energy into mechanical energy.

Take the situation depicted in Figure 5.13 where a moving bar, of mass M and internal resistance R, slides (frictionless) with velocity $\mathbf{v} = v(t)\vec{e}_x$ over two conducting rails. Perpendicular to the plane defined by the two rails, a uniform time-invariant $\mathbf{B}$ field is enforced. Neglect the internal resistances of the rails as well as the magnetic field produced by the circulating current.

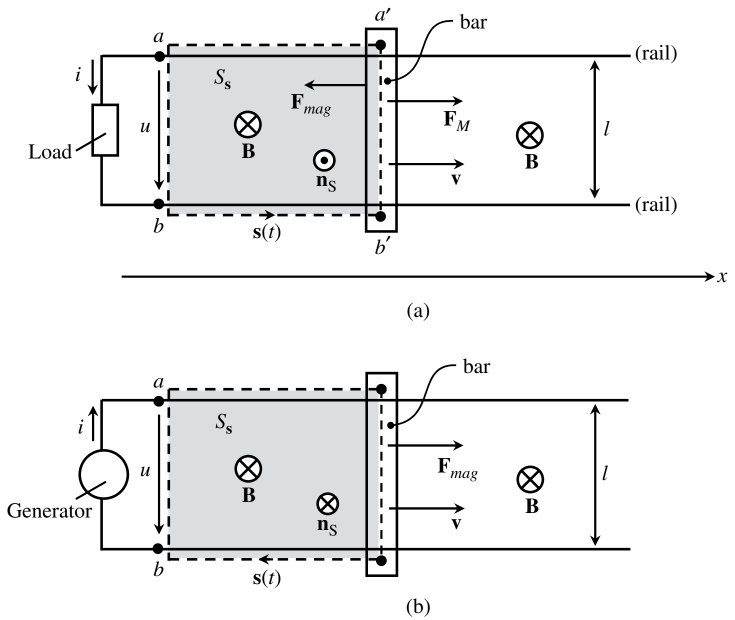

Figure 5.13 Illustration of the principles of electromechanical energy conversion using, as an example, a bar–rail system. (a) An external mechanical force $\mathbf{F}_\mathrm{M}$ drives the bar into movement giving rise to an induced emf, electric energy being delivered to the load. (b) An electric power supply (generator) produces a current flow that in conjunction with the $\mathbf{B}$ field originates a magnetic force $\mathbf{F}_\mathrm{mag}$, the latter driving the bar into movement (production of mechanical kinetic energy)

Questions

Q_1 In Figure 5.13(a) a passive electric load is connected between terminals a and b where a voltage u appears as a consequence of induction phenomena. The bar is driven by an externally applied mechanical force $\mathbf{F}_\mathrm{M}$ which causes the bar to move with velocity $\mathbf{v}$. Determine the voltage u and establish the energy balance equation of the system.

Q_2 Now consider the reverse problem in Figure 5.13(b), where an electric generator provides a voltage u between terminals a and b. The bar is free from any mechanical external force; however, due to the interaction between the $\mathbf{B}$ field and the current i in the bar, the latter undergoes the action of a magnetic force $\mathbf{F}_{mag}$ which drives the bar into movement with velocity $\mathbf{v}$. Determine the voltage u and establish the energy balance equation of the system.

Solutions

Q_1 In addition to the external mechanical force $\mathbf{F}_M$ the bar is also actuated by an opposite magnetic force $\mathbf{F}_{mag}$ originating from the interaction of the $\mathbf{B}$ field with the current i flowing in the bar.

Let us first apply the induction law to the closed path $\mathbf{s}(t)$ passing by the load, rails and moving bar. The path $\mathbf{s}$ is oriented according to the reference direction assigned to i:

$$\oint_{\mathbf{s}(t)} \mathbf{E} \cdot d\mathbf{s} = u + Ri \tag{5.20a}$$

$$-\frac{d}{dt}\int_{S_{\mathbf{s}(t)}} \mathbf{B} \cdot \mathbf{n}_S \; dS = +B\frac{d}{dt}S_{\mathbf{s}(t)} = B\frac{d}{dt}(lx) = Bl\frac{dx}{dt} = Blv \tag{5.20b}$$

Equating the results in (5.20) we obtain

$$u = Blv - Ri \tag{5.21}$$

The equation describing the movement of the bar along x is

$$\mathbf{F}_M + \mathbf{F}_{mag} = M\frac{d\mathbf{v}}{dt} \tag{5.22}$$

Taking into account that the magnetic force (Lorentz force) exerted along the length of the bar is

$$\mathbf{F}_{mag} = \int_{\overrightarrow{b'a'}} i \; d\mathbf{s} \times \mathbf{B} = -ilB\vec{e}_x$$

we can write from (5.22)

$$\mathbf{F}_M = M\frac{d\mathbf{v}}{dt} + iBl\vec{e}_x$$

The mechanical power $P_M = \mathbf{F}_M \cdot \mathbf{v}$ associated with the external driving force $\mathbf{F}_M$ is next determined as

$$P_M = M\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + iBlv = \frac{d}{dt}\left(\frac{1}{2}Mv^2\right) + i(u + Ri) = \frac{d}{dt}\left(\frac{1}{2}Mv^2\right) + ui + Ri^2 \tag{5.23}$$

The term $Mv^2/2$ is the kinetic energy of the bar W_K, the term ui is the electric power available to the load P_L, and Ri^2 is the power losses (Joule effect) in the bar P_J. Thus, we conclude for the power balance

$$P_M = dW_K/dt + P_L + P_J$$

Integration over time gives $W_M = W_K + W_L + W_J$.

This result illustrates the conversion of external mechanical energy into electric energy.

Q_2 Reapplying the induction law to path $\mathbf{s}(t)$ oriented according to the reference direction assigned to i, we obtain (see Figure 5.13(b))

$$\oint_{\mathbf{s}(t)} \mathbf{E} \cdot d\mathbf{s} = -u + Ri \tag{5.24a}$$

$$-\frac{d}{dt}\int_{S_{\mathbf{s}(t)}} \mathbf{B} \cdot \mathbf{n}_S \; dS = -B\frac{d}{dt}S_{\mathbf{s}(t)} = -Blv \tag{5.24b}$$

Equating the results in (5.24) we obtain $u = Blv + Ri$.

The equation describing the movement of the bar along x is

$$\mathbf{F}_{\text{mag}} = M\frac{d\mathbf{v}}{dt} \quad \text{or} \quad iBl\vec{e}_x = M\frac{d\mathbf{v}}{dt}$$

The inner product with $\mathbf{v}$ on both sides of the above equation yields

$$iBlv = \frac{d}{dt}\left(\frac{1}{2}Mv^2\right) \quad \text{or} \quad i(u - Ri) = \frac{d}{dt}\left(\frac{1}{2}Mv^2\right)$$

Rearranging terms

$$ui = \frac{d}{dt}\left(\frac{1}{2}Mv^2\right) + Ri^2 \quad \text{or} \quad P_G = dW_K/dt + P_J$$

where P_G is the electric power delivered by the generator.

Integrating over time we get the energy balance, $W_G = W_K + W_J$.

This result illustrates the conversion of electric energy into mechanical kinetic energy.

5.11 DC Voltage Generation

This section deals only with the functioning principle of the simplest DC generator.

Consider, as shown in Figure 5.14, a rotating disk (Faraday's disk) of radius R illuminated perpendicularly by a uniform time-invariant $\mathbf{B}$ field. Brush-type conducting contacts are established with the disk shaft and with the disk periphery, allowing a voltage U to be collected at the accessible open terminals a and b. The angular velocity of the rotating disk is ω.

Consider the application of the induction law to the closed path $\mathbf{s}$ (accompanying the disk movement). Since currents are absent we find

$$\oint_{\mathbf{s}} \mathbf{E} \cdot d\mathbf{s} = \int_{\substack{\overrightarrow{ab} \\ \text{air}}} \mathbf{E} \cdot d\mathbf{s} = U \tag{5.25a}$$

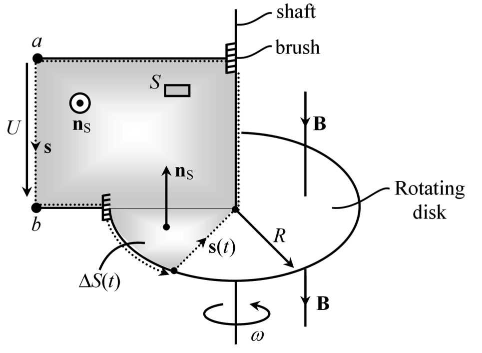

Figure 5.14 Illustration of DC voltage generation principles using, as an example, Faraday's rotating disk. Voltage U is proportional to both B and ω

As for the right-hand side of the induction law equation, concerning the time derivative of the linkage flux, you may evaluate it as

$$-\frac{d\psi_{\mathbf{s}(t)}}{dt} = -\frac{d}{dt}\left(\int_{S_\square} \mathbf{B}\cdot\mathbf{n}_S\, dS + \int_{\Delta S(t)} \mathbf{B}\cdot\mathbf{n}_S\, dS\right) \tag{5.25b}$$

Noting that the flux across the rectangular surface $S_\square$ is null and time invariant, the first contribution on the right-hand side of (5.25b) is zero. As for the second contribution, noting that $\mathbf{B} \uparrow\downarrow \mathbf{n}_S$ and that the area $\Delta S(t)$ steadily increases with time, $\Delta S(t) = \frac{1}{2}R^2\omega t$, we get

$$\psi_{\mathbf{s}(t)} = -B\Delta S = -\tfrac{1}{2}BR^2\omega t \quad \text{and} \quad -d\psi_{\mathbf{s}}/dt = \tfrac{1}{2}BR^2\omega$$

Finally, equating the results from (5.25), we obtain a DC voltage given by

$$U = \frac{BR^2\omega}{2} \tag{5.26}$$

Taking into account that $\omega = 2\pi N_{\text{rps}}$ (where N_{rps} is the number of rotations per second), and that the magnetic flux through the whole disk is $\phi = B\pi R^2$, the result in (5.26) can be rewritten in the more insightful form $U = N_{\text{rps}}\phi$.

5.12 AC Voltage Generation

Similar to the above, this section deals only with the functioning principle of the simplest AC generator.

Consider, as shown in Figure 5.15, a rectangular single-turn coil of area A rotating with angular velocity ω around its own axis in a region where a uniform time-invariant $\mathbf{B}$ field

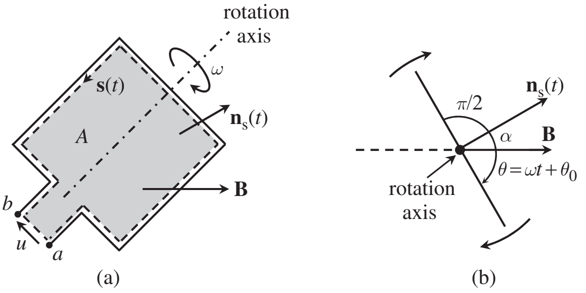

Figure 5.15 Illustration of AC voltage generation principles using, as an example, a rotating rectangular coil immersed in a uniform **B** field. (a) Perspective view and circulation path for the application of the induction law. (b) Side view, showing that the angle between **B** and the Stokes normal changes with time

exists perpendicular to the axis of rotation. A voltage $u(t)$ is collected at the coil's accessible terminals a and b which are left open.

Consider the application of the induction law to the closed path $\mathbf{s}(t)$ (accompanying the coil rotation). Since currents are absent we find

$$\oint_{\mathbf{S}} \mathbf{E} \cdot d\mathbf{s} = \int_{\substack{\overrightarrow{ba} \\ \text{air}}} \mathbf{E} \cdot d\mathbf{s} = -u \tag{5.27a}$$

As for the right-hand side of the induction law equation, concerning the time derivative of the linkage flux, we may evaluate it as

$$-\frac{d\psi_s}{dt} = -\frac{d}{dt}\int_A \mathbf{B} \cdot \mathbf{n}_S(t)\ dS$$

Noting that the angle α between $\mathbf{B}$ and $\mathbf{n}_S$ changes with time, $\alpha = \pi/2 - \theta(t)$, where $\theta(t) = \omega t + \theta_0$ (with arbitrary θ_0), we have

$$\mathbf{B} \cdot \mathbf{n}_S(t) = B\cos\alpha = B\sin(\omega t + \theta_0)$$

and therefore

$$-\frac{d\psi_s}{dt} = -\frac{d}{dt}\int_A \mathbf{B} \cdot \mathbf{n}_S(t) dS = -AB\omega\cos(\omega t + \theta_0) \tag{5.27b}$$

Equating the results from (5.27), we obtain an AC voltage given by $u(t) = AB\omega\cos(\omega t + \theta_0)$.

If the coil contains N turns tightly packed, the above result is modified to

$$u(t) = U_M\cos(\omega t + \theta_0), \text{ with } U_M = AB\omega N \tag{5.28}$$

5.13 Proposed Homework Problems

Problem 5.13.1

An overhead line conductor at height h_L above ground carries a sinusoidal current $i_L(t) = I\sin(\omega t)$. As shown in Figure 5.16, a fence of length l is situated just beneath the line and parallel to it. The fence has two horizontal supporting conducting wires, one at the soil level and the other at height h_F. Both wires are connected at one end of the fence by a conducting post, whereas at the opposite end a wooden post is used.

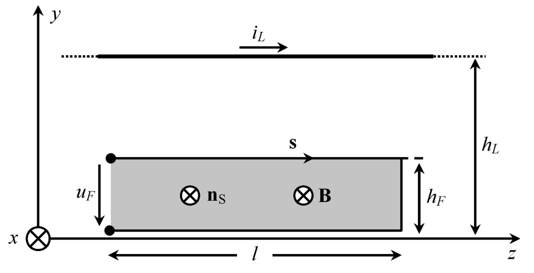

Figure 5.16 Induced voltage in a wire fence placed near to an overhead line

Assume that the return current in the poorly conducting soil contributes negligibly to the evaluation of the **B** field in air.

Data: $I = 5\,\text{kA},\ \omega = 2\pi f,\ f = 50\,\text{Hz},\ h_L = 12\,\text{m},\ h_F = 3\,\text{m},\ l = 200\,\text{m}.$

Q_1 From Chapter 4, determine the **B** field originated by the overhead line conductor in the fence region ($x = 0$).

Q_2 Determine the fence voltage $u_F(t)$ between the two supporting wires at the end where the wooden post has been placed.

Answers

Q_1

$$\mathbf{B}(y, t) = \mu_0 \frac{i_L(t)}{2\pi(h_L - y)} \vec{e}_x, \ \text{for } 0 < y < h_F$$

Q_2 Application of the induction law to the closed path **s** along the fence wires and wooden post gives

$$\underbrace{\oint_{\mathbf{s}} \mathbf{E} \cdot d\mathbf{s}}_{-u_F} = -\frac{d\psi_F}{dt}, \ \text{with } \psi_F = \int_{S_s} \mathbf{B} \cdot \mathbf{n}_S \, dS = \int_{y=0}^{y=h_F} B(y, t)\ l\ dy$$

$$\psi_F(t) = \underbrace{\left(\frac{\mu_0 l}{2\pi} \ln \frac{h_L}{h_L - h_F}\right)}_{L_M} i_L(t); \;\; L_M = 11.5\,\mu\text{H}$$

$$u_F(t) = L_M \frac{di_L(t)}{dt} = \underbrace{L_M I\omega}_{U} \cos(\omega t); \;\; U = 18.1\,\text{V}$$

Problem 5.13.2

A coil with N_1 turns and a conducting ring are placed in a transformer core – see Figure 5.17. The coil, with resistance R_1, is driven by a voltage generator. The ring can be left open or short-circuited depending on the switch position. The transformer core has a uniform cross-section S. The self-inductance of the coil is L_{11}. Neglect dispersion phenomena and the resistance of the ring.

Data: $N_1 = 10,\; R_1 = 10\;\Omega,\; S = 1\,\text{cm}^2,\; L_{11} = 10\,\text{mH}.$

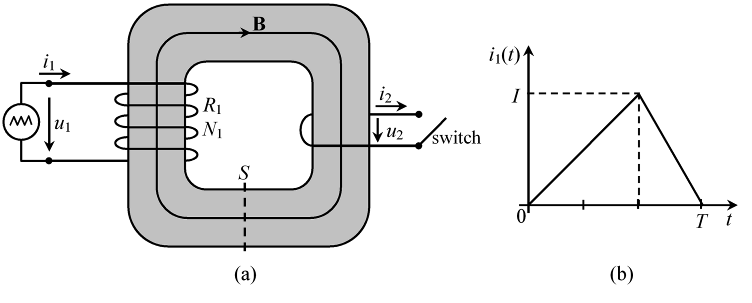

Figure 5.17 A transformer core with a coil of N_1 turns and one ring. (a) General view. (b) Generator current

Q_1 Using your knowledge from Chapter 4, determine the self-inductance of the ring L_{22} as well as the mutual inductance L_M.

Q_2 With the switch open ($i_2 = 0$) the generator current $i_1(t)$ is described by an asymmetrical triangular pulse of duration $T = 3\,\text{ms}$ and peak value $I = 0.1\,\text{A}$.
Using the induction law, determine the voltages $u_1(t)$ and $u_2(t)$.
Using your knowledge from Chapter 4, determine the time evolution of $B(t)$ in the core.

Q_3 Now consider that the switch is closed ($u_2 = 0$) and that the generator voltage u_1 is the same as you determined in Q_2.
Evaluate $i_1(t)$ and $i_2(t)$.
Determine the magnetic induction field in the transformer core. Comment on the result.

Answers

Q_1

$$L_{11} = \frac{N_1^2}{R_m} \rightarrow R_m = 10^4\,\mathrm{H}^{-1}, \quad L_{22} = \frac{N_2^2}{R_m} = 0.1\,\mathrm{mH}, \quad L_M = \frac{N_1 N_2}{R_m} = 1.0\,\mathrm{mH}$$

Q_2

$$u_1 = R_1 i_1 + L_{11}\frac{di_1}{dt}; \quad u_2 = -L_M \frac{di_1}{dt}$$

(see the illustrative plots in Figure 5.18)

$$\psi_1 = \begin{cases} L_{11} i_1 \\ N_1 \phi = N_1 BS \end{cases} \rightarrow B(t) = \frac{L_{11}}{N_1 S} i_1(t)$$

$B(t)$ is a triangular pulse as $i_1(t)$, with peak value of 1 T.

Q_3

$$u_2 = 0 = -\frac{d\psi_2}{dt} \rightarrow \psi_2(t) = 0 \rightarrow B(t) = 0 \rightarrow \psi_1(t) = 0$$

$$u_1 = R_1 i_1 + \frac{d\psi_1}{dt} = R_1 i_1 \rightarrow i_1(t) = \frac{u_1(t)}{R_1}$$

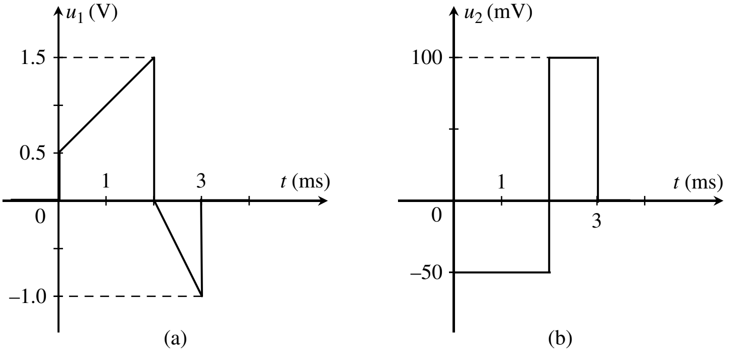

Figure 5.18 Voltage plots against time when the switch is open. (a) Coil voltage. (b) Ring voltage

Current $i_1(t)$ has the shape of the voltage $u_1(t)$ established in Figure 5.18 apart from a scale factor determined by the value of R_1.

Since the magnetic flux in the core is zero, then, from Ampère's law (Chapter 4), you will have $R_m\phi = 0 = N_1 i_1 + N_2 i_2$ (with $N_2 = 1$), and, consequently,

$$i_2(t) = -N_1 i_1(t) = -N_1 \frac{u_1(t)}{R_1}$$

Current $i_2(t)$ has the shape of $i_1(t)$, but with opposite sign, and scaled by a factor of ten ($N_1 = 10$).

When the switch is closed, the new current i_2 creates its own magnetic field which opposes the original one existing in the core (this is known in the literature as Lenz's 'law'). In our case, because the ring is a perfect conductor, the ring's magnetic field has exactly the same magnitude as the one previously existing in the core due to the excitation coil. Therefore, their sum is zero for the resultant field, $\mathbf{B}_1 + \mathbf{B}_2 = 0$.

Firstly, and most important to note, the idea that current $i_2(t)$ always reacts against the induction action of $i_1(t)$ – something equivalent to the action/reaction principle you have heard about in physics – is a false concept.

The reaction of $i_2(t)$ is critically dependent on the type of load connected to the second coil (ring). If, for instance, a capacitor is connected to the terminals of the second coil it may well happen that the total **B** field in the core may actually increase as compared to $\mathbf{B}_1$ (we will come back to this topic in Section 7.3.2).

Secondly, the problem treated here exemplifies a technological application of induction phenomena, namely induction heating. You saw that the current in the ring is N_1 times bigger than the one in the excitation coil. In reality, the ring has an internal resistance and because of that, energy dissipation due to the Joule effect will take place there:

$$W_{\mathrm{J}} = \int_0^T R_{\mathrm{ring}}\, i_2^2(t)\; dt$$

Depending on the system's design parameters, the structure in Figure 5.17 may be engineered so as to ensure that the heat generated in the metallic ring melts it down.

Problem 5.13.3

A ferromagnetic core (see Figure 5.19) is excited by a sinusoidal current flowing in an inductor (not shown) which gives rise to a magnetic flux $\phi(t) = \phi_M\ \sin(\omega t)$ circulating in the core. As shown in the figure, a single-turn coil is wound around the core. The coil is left open ($i = 0$), but due to induction phenomena a voltage $u(t)$ appears across its terminals a and b. In order to visualize the coil voltage, two oscilloscopes O_1 and O_2 are connected between a and b.

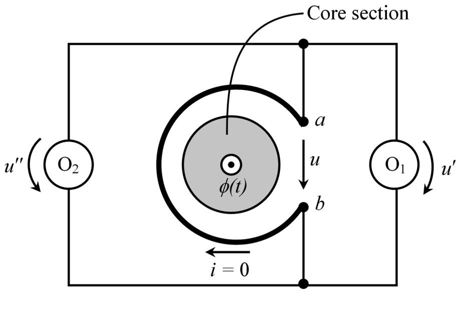

Figure 5.19 The single-turn coil voltage $u(t)$ is read differently by the oscilloscopes O_1 and O_2, both connected between a and b

Q_1 Determine $u(t)$.

Q_2 Determine the voltages $u'(t)$ and $u''(t)$ retrieved by both oscilloscopes.

Q_3 Repeat the problem for the case of a two-turn coil.

Q_4 Generalize the above result for an N-turn coil.

Answers

Q_1

$$u(t) = \frac{d\phi}{dt} = U\cos(\omega t), \text{ with } U = \omega\phi_M$$

Q_2

$$u'(t) = u(t); \; u''(t) = 0$$

Q_3

$$u'(t) = u(t) = 2\frac{d\phi}{dt} = 2U\cos(\omega t); \quad u''(t) = \frac{d\phi}{dt} = U\cos(\omega t)$$

Q_4

$$u'(t) = u(t) = N\frac{d\phi}{dt} = NU\cos(\omega t); \quad u''(t) = (N-1)\frac{d\phi}{dt} = (N-1)\; U\cos(\omega t)$$

Only for $N \gg 1$ will you have similar oscilloscope readings, $u' \approx u''$.

Problem 5.13.4

Two inductors are connected in series as shown in Figure 5.20. Inductors are characterized by internal resistances R_1 and R_2 and self-inductances L_{11} and L_{22}. The magnetic coupling factor between the inductors is $k = 0.75$.

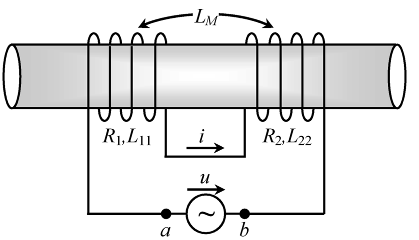

Figure 5.20 Two series-connected magnetically coupled inductors

The current common to both inductors is $i(t) = I\cos(\omega t)$.

Q_1 Determine the mutual inductance L_M between the inductors.

Q_2 Write an analytical expression for the applied voltage $u(t)$ between terminals a and b.

Q_3 Considering that $R_1 = 60\ \Omega,\ R_2 = 140\ \Omega,\ L_{11} = 0.1\,\text{H},\ L_{22} = 0.4\,\text{H},\ I = 0.25\,\text{A}$, and $\omega = 1\,\text{krad/s}$, determine $u(t)$ numerically.

Answers

Q_1 $L_M = -150\,\text{mH}$

Q_2

$$\begin{cases} \oint_{\mathbf{s}} \mathbf{E}\cdot d\mathbf{s} = -u + R_1 i + R_2 i \\ -\dfrac{d\psi_S}{dt} = -\dfrac{d}{dt}(\psi_1 + \psi_2) = -\dfrac{d}{dt}((L_{11}+L_M)i + (L_M + L_{22})i) \end{cases}$$

$$u = Ri + L\frac{di}{dt},\ \text{where } R = R_1 + R_2 \text{ and } L = L_{11} + L_{22} + 2L_M$$

$$u(t) = RI\cos(\omega t) - \omega LI\sin(\omega t)$$

Q_3 $R = 200\ \Omega,\ L = 200\,\text{mH},\ \omega L = 200\ \Omega$:

$$u(t) = 50\cos(\omega t) - 50\sin(\omega t)\ \text{V}$$

$$u(t) = 50\sqrt{2}\ \cos(\omega t + \pi/4)\ \text{V}$$

Problem 5.13.5

Consider the three-legged transformer shown in Figure 5.21, where the three vertical legs share the same geometrical and magnetic properties. For simplification purposes assume that the reluctances of the upper and lower yokes are negligibly small. The inductor placed on the left leg, with N_1 turns and self-inductance L_{11}, is driven by a sinusoidal voltage generator. The inductor placed on the right leg, with N_2 turns, is left open. Around the central leg a perfectly conducting ring can be switched from open state to short-circuit state depending on the switch position.

Assume that the internal resistance of inductor 1 is negligible, and that dispersion phenomena are absent.

Data: $L_{11} = 1\,\text{H},\ u_1(t) = U_1\cos(\omega t + \pi/2),\ U_1 = 325\,\text{V},\ \omega = 100\pi\,\text{rad/s},\ N_1 = 100,$ $N_2 = 400$.

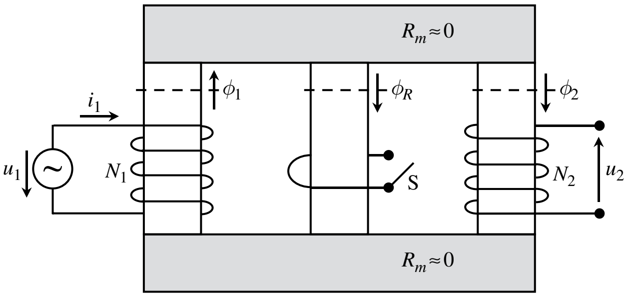

Figure 5.21 A three-legged magnetic circuit containing a ring in the center leg which can be switched on or off

Q_1 Find the magnetic flux $\phi_1(t)$ in the transformer's left leg.

Q_2 Consider that the ring is in the open state. Determine $\phi_2(t)$ and $u_2(t)$.

Q_3 Consider that the ring is in the short-circuit state. Determine $\phi_2(t)$ and $u_2(t)$.

Answers

Q_1

$$u_1(t) = N_1 \frac{d\phi_1(t)}{dt} \rightarrow \phi_1(t) = \frac{1}{N_1} \int u_1(t)\ dt = \Phi_1\ \cos(\omega t), \quad \Phi_1 = \frac{U_1}{\omega N_1} = 10.35\,\text{mWb}$$

Q_2 $\phi_1 = \phi_R + \phi_2$; $\phi_R = \phi_2$ (due to symmetry reasons); $\phi_2 = \phi_1/2$.

$$u_2(t) = N_2 \frac{d\phi_2(t)}{dt} = \frac{N_2}{2} \frac{d\phi_1(t)}{dt} = \frac{N_2}{2N_1} u_1(t) = U_2 \cos(\omega t + \pi/2), \text{ with } U_2 = 650\ \text{V}$$

Q_3 Because the ring is short-circuited, $\phi_R = 0 \rightarrow \phi_2 = \phi_1$:

$$u_2(t) = N_2 \frac{d\phi_2(t)}{dt} = N_2 \frac{d\phi_1(t)}{dt} = \frac{N_2}{N_1} u_1(t) = U_2\ \cos(\omega t + \pi/2), \text{ with } U_2 = 1.3\,\text{kV}$$

Problem 5.13.6

Consider the situation analyzed in Section 5.12 (AC voltage generation) but where, instead of a single rotating rectangular coil, you have three rotating rectangular coils making angles of $2\pi/3$ to each other – see Figure 5.22 (three-phase AC generator).

Q_1 Write the equations for the voltages $u_1(t)$, $u_2(t)$ and $u_3(t)$ at the coil terminals.

Q_2 Determine $u_1(t) + u_2(t) + u_3(t)$.

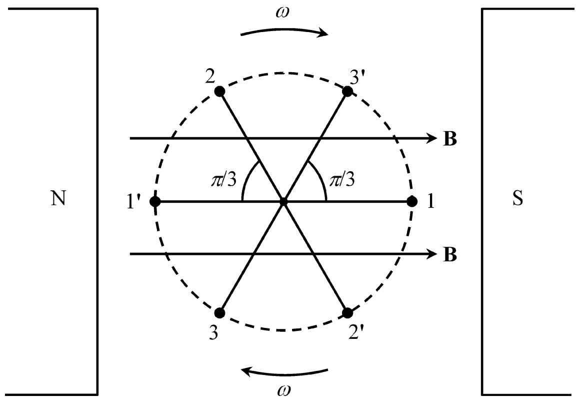

Figure 5.22 Operating principle of a three-phase AC generator. Three identical coils synchronously rotate in the space between the north and south poles of a magnet

Answers

Q_1

$$u_1(t) = U_M \ \cos(\omega t + \theta_0), \text{ with } U_M = AB\omega N$$
$$u_2(t) = U_M \ \cos(\omega t + \theta_0 - 2\pi/3)$$
$$u_3(t) = U_M \ \cos(\omega t + \theta_0 - 4\pi/3)$$

Q_2

$$u_1(t) + u_2(t) + u_3(t) = 0$$

Problem 5.13.7

Figure 5.23 illustrates the functioning principle of the moving-coil microphone. The coil is attached to a diaphragm (not shown) on which sound waves impinge. When the diaphragm vibrates, the coil is set in motion. The coil moves in a region where a radial stationary magnetic field, produced by a magnet, exists. The terminals of the coil are left open ($i = 0$). Due to magnetic induction, a voltage signal at the moving-coil terminals appears.

To simplify matters, assume that the coil is a single-turn coil and that the magnetic induction field in the (very small) air gap between the north and south poles of the magnet is approximately uniform, $B = 0.5\,\text{T}$.

Also assume that the coil movement is described by $x(t) = l/2 + X\sin(\omega t)$, with $X < l/2$.

Q_1 Find the magnetic flux $\phi(x)$ across the single-turn coil positioned at x.

Q_2 By applying the generalized induction law, determine the voltage $u(t)$ between the terminals a and b of the moving coil.

Q_3 Consider that the coil contains N turns tightly packed.
Take $N = 5,\ X = 10\,\mu\text{m},\ R = 5\,\text{mm},\ \omega = 2\pi f$ and $f = 3\,\text{kHz}$.
Determine $u(t)$ numerically.

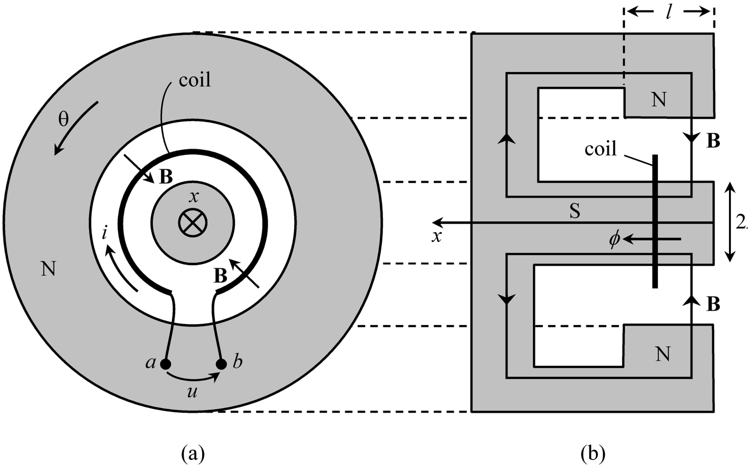

Figure 5.23 Moving-coil microphone. (a) Front view. (b) Transverse cross-section

Answers

Q_1 From div $\mathbf{B} = 0$, and noting that the radial $\mathbf{B}$ field in the air gap is redirected and transformed into an x-oriented field along the south pole piece, you find $\phi(x) = 2\pi RBx$.

Q_2

$$\begin{cases} \oint\limits_{\mathbf{S}(t)} \mathbf{E}\cdot d\mathbf{s} = -u \\ -\dfrac{d}{dt}\int\limits_{S_{\mathbf{s}(t)}} \mathbf{B}\cdot\mathbf{n}_S \, dS = -\dfrac{d\phi}{dt} = -\dfrac{d\phi}{dx}\dfrac{dx}{dt} = -2\pi RB\, v = -2\pi RBX\omega\cos(\omega t) \end{cases}$$

where $v = dx/dt$ is the coil's instantaneous velocity along x,

$$u(t) = 2\pi RBX\omega\cos(\omega t)$$

Q_3

$$u(t) = U\cos(\omega t), \text{ with } U = (2\pi)^2 RNBXf = 14.8\,\text{mV}$$

Problem 5.13.8

Figure 5.23 illustrating the functioning principle of the moving-coil microphone can also be used to explain the functioning principle of the loudspeaker.

Assume that a sinusoidal current $i(t) = I\cos(\omega t)$ is made to flow in the coil, with $I = 100\,\text{mA}$.

Q_1 Determine the magnetic Lorentz force $\mathbf{F} = F\vec{e}_x$ that puts the loudspeaker membrane into motion (producing sound waves).

Q_2 Evaluate the force numerically, considering the data specified in Problem 5.13.7.

Answers

Q_1

$$\mathbf{F} = \oint_{\mathbf{S}} i\ d\mathbf{s} \times \mathbf{B} = i \int_0^{2\pi N} \left(-R\ d\theta\ \vec{e}_\theta\right) \times \left(-B\ \vec{e}_r\right) = \underbrace{2\pi NBRI\cos(\omega t)}_{F(t)}\,\vec{e}_x$$

Q_2

$$F(t) = F_{\max}\cos(\omega t),\ \ F_{\max} = 2\pi NBRI = 7.85\,\text{mN}$$