7

Lumped Parameters Circuit Analysis

7.1 Introduction

This chapter is probably the easiest one for you. Vector calculus, differential operators, the Stokes and Gauss theorems, all of which have been harsh tools in previous chapters, will be almost absent here. Nonetheless, new, but softer, difficulties may now arise with the handling of complex algebra (see Appendix C).

In this new chapter we will make extensive use of the results derived in Chapters 5 and 6 concerning slow time-varying field phenomena (quasi-stationary regimes), which is the standard framework for circuit analysis. Magnetic induction phenomena and electric induction phenomena are considered separately; while the former are taken into account when lumped inductors are analyzed, the latter are taken into account when lumped capacitors are analyzed. In the case of lumped resistors, neither induction phenomena are considered.

Again, bear in mind that this lumped parameters approach is only valid when the length of the circuit structure under analysis is much shorter than the lowest wavelength characterizing the time evolution of the field.

To make things clearer, consider the typical *RLC* series circuit in Figure 7.1.

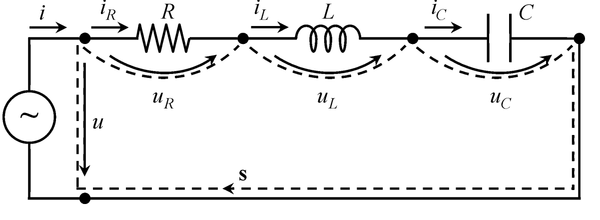

Figure 7.1 *RLC* series circuit

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Assuming that the term $\partial\mathbf{B}/\partial t$ created by the time-varying currents in the connecting wires is negligibly small, the application of the induction law gives

$$\oint_{\mathbf{S}} \mathbf{E}\cdot d\mathbf{s} = -\int_{s_s} \frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{n}_s \; dS \approx 0 \rightarrow u(t) = u_R(t) + u_L(t) + u_C(t) \tag{7.1}$$

Assuming that the displacement currents $\partial\mathbf{D}/\partial t$ created by the time-varying voltages between the connecting wires is negligibly small, application of the generalized Ampère's law gives

$$\int_{S_V} \mathbf{J}\cdot \mathbf{n}_o \; dS = -\int_{S_V} \frac{\partial \mathbf{D}}{\partial t}\cdot \mathbf{n}_o \; dS \approx 0 \rightarrow i(t) = i_R(t) = i_L(t) = i_C(t) \tag{7.2}$$

Assuming that both terms $\partial\mathbf{B}/\partial t$ and $\partial\mathbf{D}/\partial t$ are negligible for the resistor analysis, we have from Ohm's law (Chapter 3)

$$u_R(t) = R\, i_R(t) \tag{7.3}$$

Neglecting $\partial\mathbf{D}/\partial t$, but taking into account the important magnetic induction phenomena in the inductor, we have, from the induction law (Chapter 5),

$$u_L = L\frac{di_L(t)}{dt} \tag{7.4}$$

Finally, neglecting $\partial\mathbf{B}/\partial t$, but taking into account the important electric induction phenomena in the capacitor, we have, from the generalized Ampère's law (Chapter 6),

$$i_C = C\frac{du_C(t)}{dt} \rightarrow u_C(t) = \frac{1}{C}\int i_C(t)\, dt \tag{7.5}$$

By using the results from (7.1) to (7.5), valid for quasi-stationary regimes, we find the time-domain equation that governs the lumped parameters circuit in Figure 7.1:

$$u(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\, dt \tag{7.6}$$

7.2 Steady-State Harmonic Regimes

In this section we particularize circuit analysis for the special case of steady-state harmonic regimes, where fields, voltages, magnetic fluxes, electric charges and current intensities are described by time-varying sinusoidal functions. Moreover, it is assumed that the generator driving the circuit under analysis was turned on a long time ago (transient phenomena discarded).

In order to ensure that all the quantities referred to above have a sinusoidal description, an additional condition ought to be fulfilled: all the lumped components of the circuit must exhibit linear behavior. Note for instance that if you apply a sinusoidal voltage to a diode (nonlinear component) its current will be non-sinusoidal.

Before we proceed to the analysis of harmonic regimes a necessary comment is in order to justify the need for this type of analysis. Why are sinusoidal functions so important?

First of all, the transmission and distribution of electric power is made using AC (Alternating Current) – that is, using sinusoidal currents. In residential applications, the accessible voltage at the sockets that you use every day at home to plug in your appliances is a sinusoidal voltage (its frequency can be 50 Hz or 60 Hz depending on the country you live in).

At this stage you may be wondering about communication signals. They are certainly not sinusoidal functions!

Well, you are right. But you are missing the point.

Any regular well-behaved time-varying signal $s(t)$ can be expanded into a discrete or continuous sum of sinusoids of different frequencies, which constitute the so-called signal spectrum as follows:

$$\text{Periodic signals of period } T:\ s(t) = s_{av} + \sum_{k=1}^{\infty} S_k \cos(\omega_k t - \phi_k), \text{ with } \omega_k = 2\pi k/T$$

$$\text{Non-periodic signals}:\ s(t) = \int_0^{\infty} S(\omega)\cos(\omega t - \phi(\omega))\, d\omega$$

(See the results on Fourier series and transforms, in Appendix D.)

So, what you have to do is to analyze the time response of the circuit to each and every sinusoidal component of the signal spectrum and, at the end, superpose the results obtained. We emphasize again that this procedure with sinusoidal functions is only valid for linear circuits!

7.2.1 Characterization of Sinusoidal Quantities

In this subsection we introduce the standard terminology used to deal with sinusoidal functions.

Let $u_1(t)$ be a sinusoidal voltage given by

$$u_1(t) = U_1 \cos \underbrace{(\omega t + \alpha_1)}_{\phi_1(t)} \tag{7.7}$$

where U_1 denotes the maximum value of the voltage or amplitude, $\phi_1(t)$ denotes the time-varying phase, α_1 denotes the initial phase (for $t = 0$) and ω denotes the angular frequency $\omega = 2\pi f$, where f is the frequency in hertz.

If $\phi_1(t)$ is plotted against time you will obtain a tilted straight line (Figure 7.2). Since the cosine function repeats itself upon an angle shift of 2π, you can see that the sinusoidal time period is such that $T = 2\pi/\omega = 1/f$.

If two sinusoidal functions of the same frequency are compared, $u_1(t)$ given above and $u_2(t)$ given by

$$u_2(t) = U_2 \cos \underbrace{(\omega t + \alpha_2)}_{\phi_2(t)}$$

we will say that they are out of phase, because $\varphi = \phi_1(t) - \phi_2(t) = \alpha_1 - \alpha_2 \neq 0$.

Note that, in the specification of phase shifts, the interval $-\pi < \varphi < +\pi$ is commonly used.

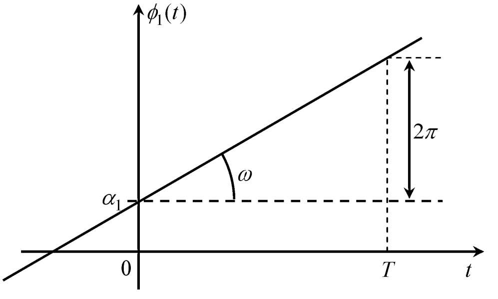

Figure 7.2 Phase against time; α_1 denotes the initial phase and ω denotes the angular frequency

Let us analyze a few particular cases:

- If $\varphi = 0$ the sinusoidal functions are said to be in phase.
- If $\varphi > 0$ we say that u_1 leads u_2; on the contrary, if $\varphi < 0$ we say that u_1 lags u_2.
- If $\varphi = \pm\pi$ the sinusoidal functions are said to be in phase opposition, which is equivalent to a $T/2$ time shift.
- If $\varphi = \pm\pi/2$ the sinusoidal functions are said to be in phase quadrature, which is equivalent to a $T/4$ time shift.

Figure 7.3 illustrates the above particular cases for you.

The specification of the average value of sinusoidal functions is absolutely useless. In fact, from (7.7) you can immediately recognize that any sinusoidal function has zero as its average value. A really important piece of information related to the amplitude of a sinusoidal function is its root-mean-square (rms) value. In general, for a time-periodic function $u(t)$, its rms value is defined as

$$U_{rms} = \sqrt{(u^2(t))_{av}} \tag{7.8}$$

The importance of this concept is linked to the evaluation of the average value of time-varying energetic quantities, like power and energy, as illustrated in the following cases.

Joule power losses in a resistor:

$$p_J(t) = Ri^2(t) \rightarrow P_J = (p_J(t))_{av} = R\left(i^2(t)\right)_{av} = RI^2_{rms} \tag{7.9a}$$

Magnetic energy in an inductor:

$$W_m(t) = \tfrac{1}{2}Li^2(t) \rightarrow (W_m)_{av} = \tfrac{1}{2}L\left(i^2(t)\right)_{av} = \tfrac{1}{2}LI^2_{rms} \tag{7.9b}$$

Electric energy in a capacitor:

$$W_e(t) = \tfrac{1}{2}Cu^2(t) \rightarrow (W_e)_{av} = \tfrac{1}{2}C\left(u^2(t)\right)_{av} = \tfrac{1}{2}CU^2_{rms} \tag{7.9c}$$

In the particular situation of sinusoidal quantities, where $\left(\cos^2(\phi(t))\right)_{av} = 1/2$, the result in (7.8) simplifies to

$$U_{rms} = \sqrt{(u^2(t))_{av}} = U/\sqrt{2} \tag{7.10}$$

where U is the maximum value of $u(t)$.

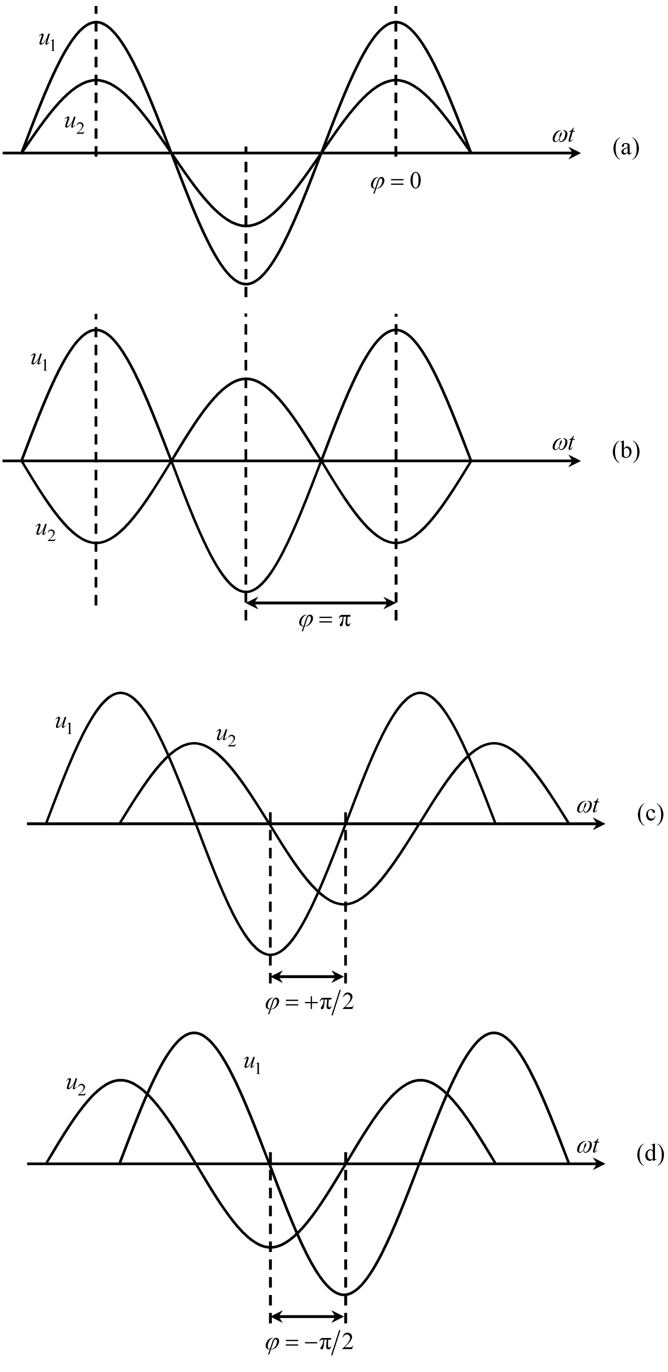

Figure 7.3 (a) Voltages u_1 and u_2 are in phase. (b) Voltages u_1 and u_2 are in phase opposition. (c) Voltages u_1 and u_2 are in phase quadrature with u_1 leading u_2. (d) Voltages u_1 and u_2 are in phase quadrature with u_1 lagging u_2

7.2.2 Complex Amplitudes or Phasors

Time-varying sinusoidal quantities of a given frequency can be represented by complex constants conveying information not only on the amplitude, but also on the initial phase of the sinusoid. These complex constants are termed complex amplitudes or phasors.

Consider, for instance, the voltage $u_1(t)$ in (7.7),

$$u_1(t) = U_1 \cos \underbrace{(\omega t + \alpha_1)}_{\phi_1(t)}$$

By using the Euler identity

$$e^{j\phi_1(t)} = \cos(\phi_1(t)) + j\sin(\phi_1(t)) \rightarrow \Re\left(e^{j\phi_1(t)}\right) = \cos(\phi_1(t))$$

we immediately recognize that $u_1(t)$ can be rewritten in the form

$$u_1(t) = \Re\left(\left(U_1 e^{j\alpha_1}\right) e^{j\omega t}\right) = \Re\left(\overline{U}_1 \, e^{j\omega t}\right) \tag{7.11}$$

The time-invariant quantity $\overline{U}_1 = U_1 e^{j\alpha_1}$ is the complex amplitude of the sinusoidal voltage $u_1(t)$.

The simplicity of the preceding formulation should certainly not pose any doubt. However, some of you may be thinking what the purpose of this is. What is the usefulness of substituting complex amplitudes by the time functions they represent?

This a simple question, with a simple answer. First of all, you should note that, by this means, we are using a time-invariant quantity to represent a time-varying function. Secondly, as will show next, operations (like sum, differentiation and integration) involving sinusoidal functions can be much more easily performed in the complex domain than in the time domain.

Consider the following example.

You want to determine the sum, $u_3 = u_1 + u_2$, of two sinusoidal voltages with the same frequency, $u_1(t) = U_1 \cos(\omega t + \alpha_1)$ and $u_2(t) = U_2 \cos(\omega t + \alpha_2)$. You can do that by resorting to standard trigonometry but you will waste a lot a time. One thing that you should know is that the sum of two sinusoids of the same frequency will yield a resultant sinusoid with the same frequency – that is, you are expecting a result in the form $u_3(t) = U_3 \cos(\omega t + \alpha_3)$. So, let us then use the phasor technique to find U_3 and α_3:

$$U_3 \cos(\omega t + \alpha_3) = U_1 \cos(\omega t + \alpha_1) + U_2 \cos(\omega t + \alpha_2)$$

$$\Re\left(\overline{U}_3 \, e^{j\omega t}\right) = \Re\left(\overline{U}_1 \, e^{j\omega t}\right) + \Re\left(\overline{U}_2 \, e^{j\omega t}\right) = \Re\left(\left(\overline{U}_1 + \overline{U}_2\right) \, e^{j\omega t}\right)$$

From the above result you immediately obtain $\overline{U}_3 = U_3 \; e^{j\alpha_3} = \overline{U}_1 + \overline{U}_2$ (see Figure 7.4). Now that's simple, isn't it?

Consider another example.

Let $q(t)$ be a given sinusoidal function, $q(t) = Q\cos(\omega t + \alpha_q)$. We wish to determine its time derivative, $i(t) = dq/dt = I\cos(\omega t + \alpha_i)$. Let us then use the phasor technique to find I and α_i:

$$i(t) = \Re\left(\overline{I} \, e^{j\omega t}\right) = \frac{d}{dt} q(t) = \frac{d}{dt} \Re\left(\overline{Q} \, e^{j\omega t}\right) = \Re\left(\left(j\omega \overline{Q}\right) \, e^{j\omega t}\right)$$

from which you can see that $\overline{I} = j\omega\overline{Q}$ or, equivalently, $I \; e^{j\alpha_i} = \omega Q \; e^{j(\alpha_q + \pi/2)}$.

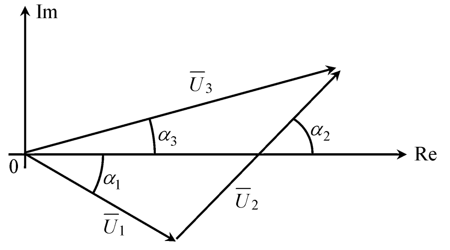

Figure 7.4 Illustration of the sum operation involving complex amplitudes

Results for time integration can be obtained from the above:

$$q(t) = \int i(t)\, dt \rightarrow \overline{Q} = \frac{1}{j\omega}\overline{I}$$

Table 7.1 summarizes the principal conclusions concerning the equivalence between time-domain operations and their corresponding phasor-domain operations.

Table 7.1 Time- and phasor-domain operations

Time domain	Phasor domain
$u_3(t) = u_1(t) + u_2(t)$	$\overline{U}_3 = \overline{U}_1 + \overline{U}_2$
$i(t) = \frac{d}{dt}q(t)$	$\overline{I} = j\omega\overline{Q}$
$q(t) = \int i(t)\ dt$	$\overline{Q} = \frac{1}{j\omega}\overline{I}$

7.2.3 Application Example (RLC Circuit)

Consider the time-domain equations for the *RLC* circuit obtained in (7.1)–(7.6). Assume that all voltages and currents are sinusoidal functions of angular frequency ω.

Questions

Q_1 Obtain the phasor-domain equations for this circuit (Figure 7.1).

Q_2 Comment on the phase relationships between voltages and currents at the resistor, inductor and capacitor terminals.

Q_3 Draw an illustrative phasor diagram showing the phasor's positions in the complex plane. Consider the following data: $i(t) = I\cos(\omega t)$, $I = 1\,\mathrm{A}$, $\omega = 1\,\mathrm{krad/s}$.
Take $R = 100\ \Omega$, $L = 0.2\,\mathrm{H}$, $C = 0.1\,\mu\mathrm{F}$.

Q_4 Several voltmeters are connected to the circuit as shown in Figure 7.5; voltmeter readings are rms values. Determine the readings of V_1, V_2, V_3, V_4 and V_5. Comment on the results.

Q_5 Write the expression for $u(t)$ and plot it against time.

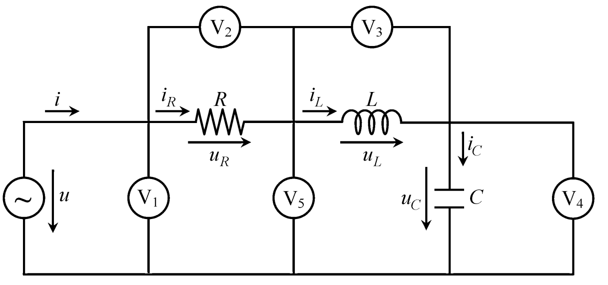

Figure 7.5 *RLC* series circuit provided with a set of voltmeters for measuring voltage rms values

Solutions

Q_1

$$u(t) = u_R(t) + u_L(t) + u_C(t) \rightarrow \overline{U} = \overline{U}_R + \overline{U}_L + \overline{U}_C \tag{7.12}$$

$$i(t) = i_R(t) = i_L(t) = i_C(t) \rightarrow \overline{I} = \overline{I}_R = \overline{I}_L = \overline{I}_C$$

$$u_R(t) = R\, i_R(t) \rightarrow \overline{U}_R = R\,\overline{I}_R \tag{7.13}$$

$$u_L = L\frac{di_L(t)}{dt} \rightarrow \overline{U}_L = j\omega L\,\overline{I}_L \tag{7.14}$$

$$u_C(t) = \frac{1}{C}\int i_C(t)\,dt \rightarrow \overline{U}_C = \frac{1}{j\omega C}\overline{I}_C \tag{7.15}$$

$$u(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt \rightarrow \overline{U} = \left(R + j\omega L + \frac{1}{j\omega C}\right)\overline{I} \tag{7.16}$$

Q_2 From (7.13) you can see that, in the complex plane, the phasors $\overline{U}_R$ and $\overline{I}_R$ are parallel vectors. This means that $u_R(t)$ and $i_R(t)$ are in phase.

From (7.14) you can see that the phasors $\overline{U}_L$ and $\overline{I}_L$ are orthogonal vectors. This means that $u_L(t)$ and $i_L(t)$ are in phase quadrature, with u_L leading i_L.

From (7.15) you can see that the phasors $\overline{U}_C$ and $\overline{I}_C$ are again orthogonal. This means that $u_C(t)$ and $i_C(t)$ are in phase quadrature, with u_C lagging i_C.

Q_3 $\overline{I} = 1\,\text{A}$, $\overline{U}_R = 100\,\text{V}$, $\overline{U}_L = 200\; e^{j\pi/2}\,\text{V}$, $\overline{U}_C = 100\; e^{-j\pi/2}\,\text{V}$, $\overline{U} = \sqrt{2}\;100\,e^{j\pi/4}\,\text{V}$.

See the corresponding phasor diagram in Figure 7.6.

You may be commenting to yourself that the phasor diagram is not correct, because the size of the phasor $\overline{I}$ appears to be bigger than the size of the voltage phasors. Well, you are wrong!

In the same way that you cannot compare kg to km/s, you cannot compare ampere with volt. In no case at all can you establish inequality relations between quantities of different nature. This means that, in order to draw the phasor diagram in Figure 7.6, different scales have to be adopted, one for currents and another for voltages!

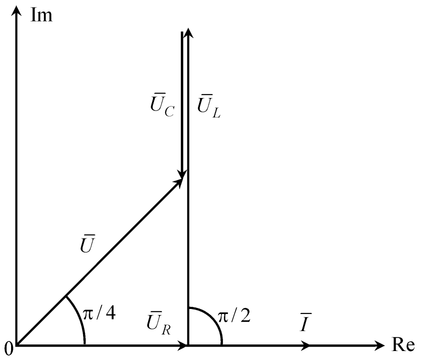

Figure 7.6 Illustrative phasor diagram for the *RLC* series circuit

Q_4

$$V_1 \equiv \left|\overline{U}\right|/\sqrt{2} = 100.0 \text{ V}$$

$$V_2 \equiv \left|\overline{U}_R\right|/\sqrt{2} = 70.7 \text{ V},\ V_3 \equiv \left|\overline{U}_L\right|/\sqrt{2} = 141.4 \text{ V},\ V_4 \equiv \left|\overline{U}_C\right|/\sqrt{2} = 70.7 \text{ V}$$

$$V_5 \equiv \left|\overline{U}_L + \overline{U}_L\right|/\sqrt{2} = 70.7 \text{ V}$$

It should be emphatically noted that $V_1 \neq V_2 + V_3 + V_4$!
Although the sum of the voltage phasors concerning the R, L and C circuit components is equal to the generator voltage phasor, see (7.12), the same is not true for the sum of the corresponding amplitudes or for the sum of the corresponding rms values, that is

$$\overline{U} = \overline{U}_R + \overline{U}_L + \overline{U}_C, \text{ but } \begin{cases} U \neq U_R + U_L + U_C \\ U_{rms} \neq U_{R_{rms}} + U_{L_{rms}} + U_{C_{rms}} \end{cases}$$

Summing vectors is not the same thing as summing scalars. Take care with this issue because it is a source of frequent mistakes.

Q_5 $u(t) = \Re\left(\overline{U}\ e^{j\omega t}\right) = \Re\left(\left(U\ e^{j\pi/4}\right)\ e^{j\omega t}\right) = U\ \cos(\omega t + \pi/4)$.
$u(t) = \sqrt{2}\ 100\ \cos(\omega t + \pi/4)$ V. See Figure 7.7.

A word of caution: you might be tempted to write $u(t)$ as $u(t) = U\ \cos(\omega t + 45°)$, but don't do it! That too is a common mistake. The problem is that the units for ωt are radians, and as you should know, radians and degrees cannot be mixed.

7.2.4 Instantaneous Power, Active Power, Power Balance Equation

As shown in Figure 7.8, a sinusoidal voltage $u(t) = U\ \cos(\omega t + \alpha_u)$ is applied across the terminals of a given linear passive circuit, that is a circuit containing resistors, inductors, capacitors, but no energy sources. Since the circuit behaves linearly, the generator current is also sinusoidal, $i(t) = I\ \cos(\omega t + \alpha_i)$.

The instantaneous power delivered by the generator, $p(t) = u(t)i(t)$, can be evaluated as

$$p(t) = u(t)i(t) = UI\cos(\omega t + \alpha_u)\cos(\omega t + \alpha_i)$$

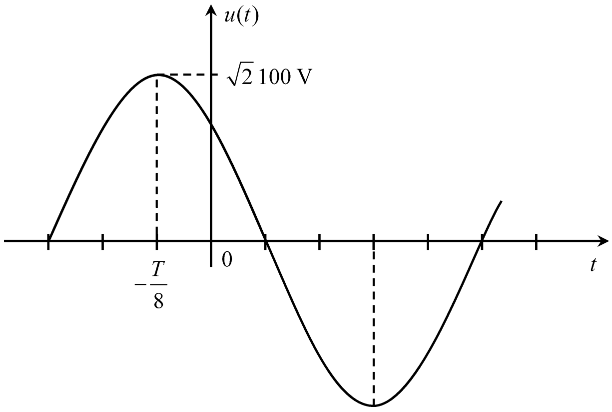

Figure 7.7 Graphical plot of generator voltage against time

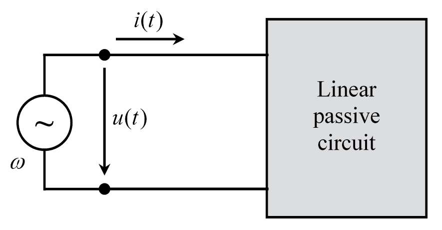

Figure 7.8 In a linear passive circuit, if $u(t)$ is sinusoidal then $i(t)$ will also be sinusoidal

From trigonometry we find

$$p(t) = \frac{UI}{2}\cos(\alpha_u - \alpha_i) + \frac{UI}{2}\cos(2\omega t + \alpha_u + \alpha_i) \tag{7.17a}$$

As an alternative, using the rms definition in (7.10), we can write

$$p(t) = U_{rms}I_{rms}\cos\underbrace{(\alpha_u - \alpha_i)}_{\varphi} + U_{rms}I_{rms}\cos(2\omega t + \alpha_u + \alpha_i) \tag{7.17b}$$

where $\varphi = \alpha_u - \alpha_i$ denotes the phase shift between the sinusoidal voltage and current.

A typical plot of the instantaneous power $p(t)$ is depicted in Figure 7.9.

From (7.17) and from Figure 7.9 we observe that the instantaneous power is not a sinusoidal function of time. Further, we see that $p(t)$ contains a time-invariant term plus a sinusoidal function of angular frequency 2ω. The constant term, representing the averaged power over time, is the so-called active power (units: W, watt)

$$\text{Active power}: \; P = (p(t))_{av} = U_{rms}I_{rms}\cos\varphi \tag{7.18}$$

From Figure 7.9 you can also see that when u and i are out of phase ($\varphi \neq 0$) the power delivered by the generator is negative during certain time intervals. The physical

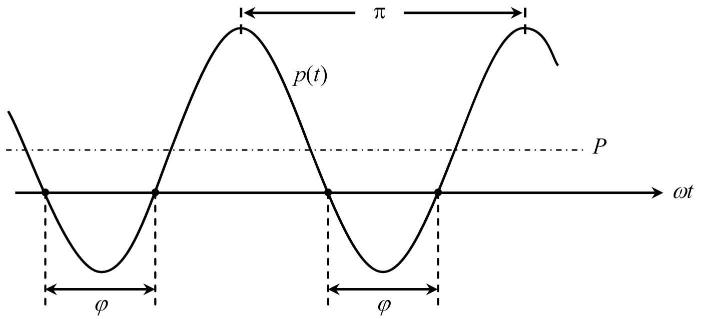

Figure 7.9 Typical plot of the instantaneous power against time. The time average of the instantaneous power, denoted by P, is the so-called active power. The angle φ indicates the difference between the phases of the voltage and current

interpretation for $p < 0$ is rather simple: during those intervals, inductors and/or capacitors discharge their stored energy towards the generator, thus reversing the normal flow of energy.

Moreover, if the circuit contains no resistors at all, the time intervals during which $p < 0$ will have the same duration as the time intervals during which $p > 0$. In this case u and i are in phase quadrature, $\varphi = \pm\pi/2$, and the active power is zero.

The preceding remarks, together with the result in (7.18), allows you to reach an important conclusion concerning the phase shift φ between $u(t)$ and $i(t)$. Since the average power delivered to a passive circuit cannot be negative, you must always have $\cos\varphi \geq 0$. This is tantamount to saying that, for passive circuits, you can only find values for φ in the range $-\pi/2 \leq \varphi \leq \pi/2$.

Next we introduce, in an intuitive way, the power balance equation in the time domain. Using the energy conservation principle we write

$$W(t) = W_J(t) + W_m(t) + W_e(t) \tag{7.19}$$

where W, W_J, W_m and W_e respectively represent the energy brought into by the generator, the energy dissipated by the Joule effect in resistors, the magnetic energy stored in inductors, and the electric energy stored in capacitors. Taking the time derivative of (7.19), we obtain the corresponding powers

$$p(t) = p_J(t) + p_m(t) + p_e(t) \;\leftrightarrow\; p(t) = p_J(t) + \frac{d}{dt}\left(W_m(t) + W_e(t)\right) \tag{7.20}$$

This last result is called the Poynting theorem. A rigorous proof of this theorem, based directly on Maxwell's equations, will be given later in Chapter 8.

The Poynting theorem can be immediately used to show that, for time-harmonic regimes, the active power is to be physically identified with Joule losses averaged over time. In fact, from (7.20), you can obtain $(p(t))_{av} = (p_J(t))_{av} + (p_m(t))_{av} + (p_e(t))_{av}$. Since the powers $p_m(t)$ and $p_e(t)$ associated to inductors and capacitors are purely sinusoidal functions, their average values are zero (do not forget that voltages and currents across inductors and capacitors are in phase quadrature), hence

$$P = (p_J(t))_{av} = P_J \tag{7.21}$$

7.2.5 Complex Power, Complex Poynting Theorem

In the framework of time-harmonic regimes, we introduced in Section 7.2.2 the phasor representation of sinusoidal voltages and sinusoidal currents. Can we do the same with the instantaneous power $p(t)$? The answer is no! Writing $p(t) = \Re\left(\overline{P}\, e^{j\omega t}\right)$ would be complete nonsense.

Phasors are only defined for sinusoidally varying quantities and, as you can see in Figure 7.9, the instantaneous power $p(t)$ is not in general a sinusoid. However, at this stage, it is customary to introduce a helpful auxiliary entity, the so-called complex power, whose definition is

$$\overline{P} = \frac{\overline{U}\,\overline{I}^*}{2} = P + jP_{\mathrm{Q}} = P_S\, e^{j\varphi} \tag{7.22a}$$

which (we emphasize) it is not a phasor. In (7.22a), the asterisk on $\overline{I}$ denotes complex conjugation.

One of the main advantages of introducing this new entity is that the evaluation of its real part directly yields the active power, $P = \Re\left(\overline{P}\right)$. Another advantage is that the angle of $\overline{P}$ provides information on the existing phase shift between u and i:

$$\measuredangle \overline{P} = \varphi = \alpha_u - \alpha_i.$$

To see that this is true let us substitute the expressions for $\overline{U}$ and $\overline{I}$ into (7.22a):

$$\overline{P} = \frac{\left(\sqrt{2}U_{rms}e^{j\alpha_u}\right)\left(\sqrt{2}I_{rms}e^{-j\alpha_i}\right)}{2} = U_{rms}I_{rms}\, e^{j(\alpha_u-\alpha_i)} = P_S\, e^{j\varphi} \tag{7.22b}$$

$$\overline{P} = \underbrace{U_{rms}I_{rms}\cos\varphi}_{P} + j\,\underbrace{U_{rms}I_{rms}\sin\varphi}_{P_{\mathrm{Q}}} \tag{7.22c}$$

By doing this, two new auxiliary quantities show up, the apparent power and the reactive power:

$$\text{Apparent power}: \; P_S = \left|\overline{P}\right| = U_{rms}I_{rms} \;\text{(units: VA, volt ampere)}$$

$$\text{Reactive power}: \; P_Q = \Im\left(\overline{P}\right) = P_S\,\sin\varphi \quad \text{(units: VAr, volt ampere reactive)} \tag{7.23}$$

Several physical interpretations for the apparent power can be given. From (7.17b) you will see that P_S represents the amplitude of the sinusoidal term of frequency 2ω belonging to $p(t)$. In addition, P_S represents the averaged power over time for circuits that behave as pure resistors ($\varphi = 0$).

The relationship among the diverse powers $\overline{P}$, P, P_S and P_Q is illustrated through the triangle representation shown in Figure 7.10.

The best way to learn about the physical significance of the reactive power P_Q is through the complex Poynting theorem. This theorem is an absolutely general theorem that can be deduced directly from Maxwell's equations for time-harmonic regimes. Here, we will skip a general demonstration of the theorem, and limit ourselves to arriving to it with the help of a particular example.

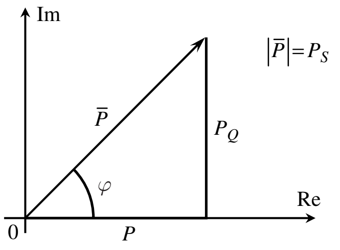

Figure 7.10 Triangle representation of powers

Take the phasor equation governing the *RLC* circuit obtained in (7.16):

$$\overline{U} = \left(R + j\omega L - j\frac{1}{\omega C}\right)\overline{I}$$

Let us multiply both sides of the above equation by $\overline{I}^*/2$. This yields on the left-hand side the complex power $\overline{P}$ introduced in (7.22). On the other hand, we have

$$\frac{\overline{I}\,\overline{I}^*}{2} = I^2_{rms} = \left(i^2(t)\right)_{av} \tag{7.24}$$

Therefore we find

$$\overline{P} = R\,I^2_{rms} + j\omega L I^2_{rms} - j\frac{1}{\omega C}I^2_{rms} \tag{7.25}$$

Further, from (7.15), at the capacitor terminals we have $I_{rms} = \omega C U_{C_{rms}}$.

By taking this into account, the third term on the right hand side of (7.25) can be rewritten as $I^2_{rms}/(\omega C) = \omega C U^2_{C_{rms}}$. Hence, we get

$$\overline{P} = RI^2_{rms} + j2\omega\left(\tfrac{1}{2}LI^2_{rms} - \tfrac{1}{2}CU^2_{C_{rms}}\right)$$

The term $RI^2_{rms} = \left(R\;i^2(t)\right)_{av} = P_J$ is interpreted as the time-averaged power losses due to the Joule effect in the resistor.

The term $\frac{1}{2}LI^2_{rms} = \frac{1}{2}L\left(i^2(t)\right)_{av} = (W_m)_{av}$ is interpreted as the time-averaged magnetic energy stored in the inductor.

Likewise, the term $\frac{1}{2}CU^2_{C_{rms}} = \frac{1}{2}C\left(u^2_C(t)\right)_{av} = (W_e)_{av}$ is interpreted as the time-averaged electric energy stored in the capacitor.

Finally we obtain

$$\overline{P} = P_J + j2\omega\left((W_m)_{av} - (W_e)_{av}\right) \tag{7.26}$$

The general result shown in (7.26) is the complex Poynting theorem, which is of key importance in many electrical engineering applications, including rapid time-varying field phenomena.

Since the complex power, on the left-hand side of (7.26), is given by

$$\overline{P} = \frac{\overline{U}\,\overline{I}^*}{2} = P + jP_Q$$

we see that the active power P can be interpreted as the time-averaged power losses at the resistors ($P = P_J \geq 0$). This conclusion is not new, and confirms the analysis result in (7.21).

Also – and this is new – the reactive power, P_Q, can now be interpreted as a measure of the balance between the time-averaged magnetic and electric energies stored in inductors and capacitors. If magnetic energy predominates, you will have $P_Q > 0$; otherwise, if electric energy predominates, you will have $P_Q < 0$.

If, at a given frequency, the generator voltage and current are in phase, $\varphi = 0$, the reactive power is null, $P_Q = 0$, and consequently, the time-averaged values of the magnetic and electric energies stored must necessarily compensate each other, $(W_m)_{av} = (W_e)_{av}$.

At this point a few remarks are in order:

- The complex Poynting theorem in (7.26) is not to be confused with the former Poynting theorem in (7.20). These theorems are independent of each other.
- The result in (7.26) cannot be deduced from (7.20).
- The Poynting theorem applies to any kind of time regime, harmonic or non-harmonic.
- The complex Poynting theorem applies only to harmonic regimes.

7.2.6 Impedance and Admittance Operators

The concept of impedance is very simple, but very important and useful. Look again at Figure 7.8 where a linear passive circuit is driven by a generator whose sinusoidal voltage and current are given by $u(t) = U\ \cos(\omega t + \alpha_u)$ and $i(t) = I\ \cos(\omega t + \alpha_i)$. The phase shift between u and i is $\varphi = \alpha_u - \alpha_i$.

The impedance of the circuit $\overline{Z}$ (units: Ω, ohm) is the complex operator that transforms the complex amplitude of $i(t)$ into the complex amplitude of $u(t)$

$$\overline{U} = \overline{Z}\,\overline{I}$$

In other words,

$$\overline{Z} = Z\, e^{j\varphi} = \frac{\overline{U}}{\overline{I}} = \frac{U\, e^{j\alpha_u}}{I\, e^{j\alpha_i}} \to Z = |\,\overline{Z}\,| = \frac{U}{I} = \frac{U_{rms}}{I_{rms}};\ \measuredangle\,\overline{Z} = \varphi = \alpha_u - \alpha_i \tag{7.27}$$

The following are important notes that you should never forget:

- The impedance operator is only defined in the context of time-harmonic regimes.
- The impedance is an operator acting over complex amplitudes; it does not operate on time-varying quantities. Writing $u(t) = \overline{Z}\ i(t)$ is complete nonsense!
- The impedance operator cannot be defined for nonlinear circuits.
- For linear passive circuits the impedance angle φ is limited to the range $-\pi/2$ to $\pi/2$.
- In general, the impedance operator is frequency dependent, $\overline{Z} = \overline{Z}(\omega)$.

For exemplification purposes consider the *RLC* circuit in Figure 7.1 and recall the phasor-domain equation in (7.16):

$$\overline{U} = \left(R + j\omega L + \frac{1}{j\omega C}\right)\overline{I}$$

The term in parentheses is the impedance operator for the *RLC* circuit

$$\overline{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\ \exp\left[j\arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)\right]$$

$$Z(\omega) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$\varphi(\omega) = \arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right) \tag{7.28}$$

The admittance operator $\overline{Y}$ (units: S, siemens), which can also be useful, is the inverse of the impedance operator

$$\overline{Y} = \overline{Z}^{-1} = \frac{\overline{I}}{\overline{U}} \tag{7.29}$$

In general, both $\overline{Z}$ and $\overline{Y}$ can be broken down into their constituent real and imaginary parts, that is $\overline{Z} = R + jX$ and $\overline{Y} = G + jS$.

R, G, X and *S* are usually named resistance, conductance, reactance and susceptance, respectively. For passive circuits, where the impedance angle φ is limited to the range $-\pi/2$ to $\pi/2$, you must always find $R \geq 0$ and $G \geq 0$, whereas both X and S can be positive, negative or zero.

7.2.7 Resonance

For time-harmonic regimes, a circuit including resistors, inductors and capacitors is said to be a resonant circuit when its impedance (or admittance) is purely real. Whenever this situation occurs, the magnitude of the driving voltage or current is at a stationary point, at a maximum or at a minimum.

Despite the presence of inductors and capacitors, the generator feeding the circuit interprets the latter as a pure resistor; the phase shift between the voltage and current at the generator terminals is zero, $\varphi = 0$.

Resonance conditions are critically dependent on the working frequency.

Just to give you a simple example, if the impedance angle of the *RLC* circuit in (7.28) is analyzed, and $\varphi = 0$ is enforced, you will at once conclude that the resonance condition for such a particular circuit is $\omega L = 1/(\omega C)$, which implies that $\left|\overline{Z}\right| = Z_{\min} = R$ and $\left|\overline{I}\right| = \left|\overline{I}\right|_{\max} = U/R$. In addition, you may note that the capacitor and inductor voltages of the *RLC* resonant circuit are in phase opposition but have identical amplitudes: $\overline{U}_L = -\overline{U}_C$. This common amplitude, depending on the circuit parameters, may become much higher than the resistor voltage amplitude; their ratio Q is ordinarily termed the quality factor of the circuit at resonance

$$Q = \frac{U_L}{U_R} = \frac{U_C}{U_R} = \frac{\omega_{\text{res}} L}{R} = \frac{1}{\omega_{\text{res}} CR} = \frac{\sqrt{L/C}}{R}$$

The fact that resonant circuits behave as pure resistors, $\varphi = 0$, signifies that the inductor and capacitor effects must cancel each other in some way. According to the complex Poynting theorem in (7.26), the reactive power is zero, and the time-averaged values of the magnetic and electric energies stored compensate for each other, $(W_m)_{av} = (W_e)_{av}$.

The concept of quality factor for an *RLC* resonant circuit was introduced above as a means to quantify circuit overvoltages. However, a more general definition of the quality factor, which applies to any resonant circuit, is usually given in the form

$$Q = \omega_{\text{res}} \frac{(W_m)_{av} + (W_e)_{av}}{(p_J)_{av}} = \omega_{\text{res}} \frac{(W_{em})_{av}}{(p_J)_{av}}$$

where W_{em} denotes the total electromagnetic energy stored.

7.2.8 Application Example (RL ∥ C Circuit)

The *RL* series circuit with a capacitor *C* in parallel finds applications in several areas. It is used in power systems to illustrate the so-called power factor compensation problem; in signal processing it is used as an example of a band reject filter; in instrumentation and measurement it also permits the illustration of the basic functioning principle of a spectrum analyzer.

Here we focus attention on the power factor compensation problem. Later, in Section 7.5, we will deal with the other applications.

Consider the circuit representation in Figure 7.11 where the *RL* branch simulates an electrical installation. The generator voltage is given by $u(t) = U\cos(\omega t)$, with $\omega = 2\pi f$, and $f = 50\,\text{Hz}$. The capacitor of capacitance *C* can be switched on or off.

Data: $U = \sqrt{2}\ 230\,\text{V},\ R = 5\ \Omega,\ L = 59.4\,\text{mH}$.

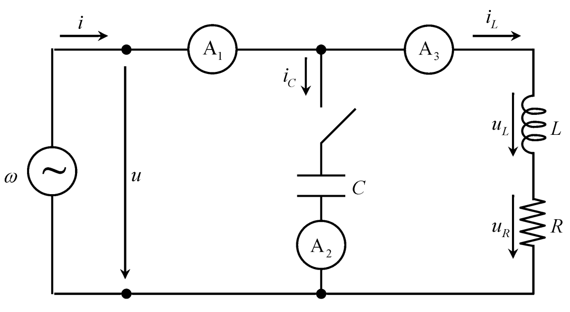

Figure 7.11 Power factor compensation problem (*RL* series circuit with a parallel-connected capacitor)

Questions

Q_1 Consider that the capacitor is switched off.

Write the time-domain and phasor-domain equations of the circuit.
Determine $i(t),\ u_R(t)$ and $u_L(t)$. Determine the active and reactive powers.

Q_2 Consider that the capacitor is switched on.

Write the time-domain and phasor-domain equations of the circuit.

Write an expression for the admittance operator $\overline{Y} = \overline{I}/\overline{U}$ and, from it, determine C such that resonance takes place.
Determine $u_R(t),\ u_L(t),\ i_L(t),\ i_C(t)$ and $i(t)$. Draw the corresponding phasor diagram.
Three ammeters $A_1,\ A_2$ and A_3 are placed in the generator, capacitor and installation branches. What are their rms readings?
Determine the active and reactive powers. Confirm the results obtained by application of the complex Poynting theorem.

Solutions

Q_1

$$i = i_L \rightarrow \overline{I} = \overline{I}_L$$
$$u = u_R + u_L \rightarrow \overline{U} = \overline{U}_R + \overline{U}_L$$
$$u_R = Ri_L \rightarrow \overline{U}_R = R\overline{I}_L$$
$$u_L = L\frac{di_L}{dt} \rightarrow \overline{U}_L = j\omega L\overline{I}_L$$
$$u = Ri_L + L\frac{di_L}{dt} \rightarrow \overline{U} = (R + j\omega L)\,\overline{I}_L$$

Now, evaluate the impedance $\overline{Z}_{RL}$ of the electrical installation:

$$\overline{Z}_{RL} = R + j\omega L = Z_{RL}\, e^{j\varphi} = 5.0 + j18.7 = 19.32\, e^{j75^o}\ \Omega$$

The current in the installation provided by the generator is found next:

$$\overline{I} = \overline{I}_L = \frac{\overline{U}}{\overline{Z}_{RL}} = \sqrt{2}\,11.9\, e^{-j75^o}\ \mathrm{A} \rightarrow i(t) = \sqrt{2}\,11.9\ \cos(\omega t - 5\pi/12)\ \mathrm{A}$$

The voltages across R and L are

$$\overline{U}_R = \sqrt{2}\,59.5\, e^{-j75^o}\ \mathrm{V} \rightarrow u_R(t) = \sqrt{2}\,59.5\ \cos(\omega t - 5\pi/12)\ \mathrm{V}$$
$$\overline{U}_L = \sqrt{2}\,222.1\, e^{j15^o}\ \mathrm{V} \rightarrow u_L(t) = \sqrt{2}\,222.1\ \cos(\omega t + \pi/12)\ \mathrm{V}$$

Taking into account that the phase shift φ between u and i is 75°, the active power is evaluated as

$$P = \underbrace{U_{rms}I_{rms}}_{P_S}\cos\varphi = 230 \times 11.9 \times 0.259 = 0.71\,\mathrm{kW}$$

(Note: the power ratio $P/P_S = \cos\varphi$ is called the 'power factor'.)
The reactive power is $P_Q = U_{rms}I_{rms}\sin\varphi = 230 \times 11.9 \times 0.966 = 2.65\,\mathrm{kVAr}$.

Q_2 Now the capacitor C is switched on and, therefore, $i \neq i_L$.

All the time-domain and phasor-domain equations obtained in Q_1 remain valid here, with the exception of the first one, $i = i_L \rightarrow \overline{I} = \overline{I}_L$, which clearly does not apply and must be replaced by

$$i = i_L + i_C \rightarrow \overline{I} = \overline{I}_L + \overline{I}_C \quad \text{and} \quad i_C = C\frac{du}{dt} \rightarrow \overline{I}_C = j\omega C\overline{U}$$

At this stage you must realize that the functioning of the RL branch is not minimally affected by the presence of the parallel-connected capacitor. The latter will simply affect the current that the generator must provide.

Because $\overline{I} = \overline{I}_L + \overline{I}_C$ you are probably guessing that the generator current is going to increase compared to the situation analyzed in Q_1. Well, your intuition is failing you! To your surprise the current is indeed going to decrease (that is exactly the goal of inserting the capacitor in this circuit...). The point is – we repeat – summing vectors and summing scalars are different things.

Let us evaluate the admittance of the global circuit:

$$\begin{aligned}\overline{Y} = \frac{\overline{I}}{\overline{U}} = \frac{\overline{I}_C + \overline{I}_L}{\overline{U}} &= j\omega C + \frac{1}{R + j\omega L} \\ &= j\omega C + \frac{1}{R + j\omega L} \times \frac{R - j\omega L}{R - j\omega L} = \frac{R}{Z_{RL}^2} + j\omega\left(C - \frac{L}{Z_{RL}^2}\right)\end{aligned}$$

where $Z_{RL}^2 = R^2 + (\omega L)^2$.

Resonance occurs when the angle of $\overline{Y}$ goes through zero or, which is the same, when the imaginary part of the admittance is zero, $\Im\left(\overline{Y}\right) = 0$. Consequently, the resonance condition for this circuit is

$$C = \frac{L}{Z_{RL}^2} = \frac{L}{R^2 + (\omega L)^2} \tag{7.30}$$

Numerically, you obtain $C = 159.1\,\mu\text{F}$.

With this value for the capacitance you find $\overline{I}_C = j\omega C\overline{U} = \sqrt{2}\ 11.5\ e^{j90°}$ A, from which you get $i_C(t) = \sqrt{2}\ 11.5\ \cos(\omega t + \pi/2)$ A.

The quantities $u_R(t)$, $u_L(t)$ and $i_L(t)$, remain unchanged.

Finally – and this is the most important point – the new current in the generator is

$$\overline{I} = \overline{I}_C + \overline{I}_L = \overline{Y}\,\overline{U} = \frac{R\overline{U}}{Z_{RL}^2} = \sqrt{2}\,3.08\text{ A} \ \rightarrow\ i(t) = \sqrt{2}\,3.08\ \cos(\omega t)\text{ A}$$

which is almost four times smaller in magnitude than the one calculated in Q_1 when the capacitor was disconnected.

Figure 7.12 shows the phasor diagram for this problem, illustrating the existing relations among the voltages and currents in the resonant circuit.

The ammeters A_1, A_2 and A_3, placed in the generator, capacitor and installation branches, read 3.08 A, 11.5 A and 11.9 A respectively.

Evaluation of the complex power $\overline{P} = \overline{U}\,\overline{I}^*/2 = P + jP_Q$ gives $P = 0.71$ kW and $P_Q = 0$.

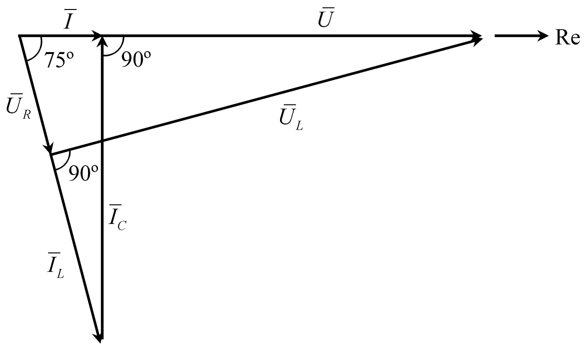

Figure 7.12 Illustrative phasor diagram for the $RL \parallel C$ circuit. When C is chosen appropriately (perfect power factor compensation), not only are the generator voltage and current in phase, but also the current magnitude is minimal

The active power P remains invariant because the Joule losses in the RL branch remain the same, $P_J = RI^2_{L_{rms}} = 0.71\,\text{kW}$.

The reactive power dropped to zero because the power factor has been compensated ($\varphi = 0 \rightarrow \cos\varphi = 1,\ \sin\varphi = 0$). In other words, the time-averaged electric energy in the capacitor is equal to the time-averaged magnetic energy in the inductor:

$$(W_m)_{av} = \tfrac{1}{2}LI^2_{L\,rms} = 4.2\text{ J},\ (W_e)_{av} = \tfrac{1}{2}CU^2_{rms} = 4.2\,\text{J}$$

7.3 Transformer Analysis

So far we have been dealing with circuit examples where magnetic coupling is absent. Now it is time to turn our attention to transformer circuits where magnetic coupling is a key issue. Due to space limitations, here we will only examine two-winding transformers with a linear core, such as the one shown in Figure 7.13 (remember that linearity is a prerequisite for the usage of inductance coefficients).

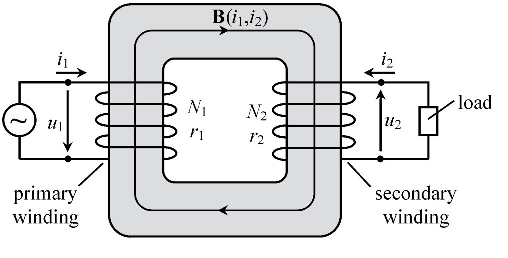

Figure 7.13 Single-core transformer representation, showing the convention for reference signs of the voltages and currents in the primary and secondary windings

Two windings are present: the primary winding is the one connected to the generator; the secondary winding is the one connected to the load (the latter is assumed to be a linear passive device). The transformer windings, with N_1 and N_2 turns, are characterized respectively by their internal resistances r_1 and r_2.

Before we proceed to the analysis of the transformer equations, a point worth emphasizing is that the writing of the equations is critically dependent on the arbitrary conventional reference directions assigned to the currents and voltages in play. Here, we adhere to the following convention. Primary and secondary currents are oriented so as to create concordant magnetic fields in the core ($L_M > 0$). Primary and secondary voltages are oriented so as to produce positive power flows from the generator to the transformer and from the transformer to the load.

Time-domain equations for the transformer circuit are obtained using the induction law from Chapter 5. Using a circulation path $\mathbf{s}_1$ oriented along i_1, going through the primary winding conductor and closing at the generator terminals, we obtain

$$\underbrace{\oint_{\mathbf{s}_1} \mathbf{E} \cdot d\mathbf{s}_1}_{r_1 i_1 - u_1} = -\frac{d\psi_1}{dt} \rightarrow u_1(t) = r_1\ i_1(t) + \frac{d\psi_1(t)}{dt} \tag{7.31}$$

where the primary linkage flux depends on both currents $\psi_1 = \psi_1(i_1, i_2)$.

Using a circulation path $\mathbf{s}_2$ oriented along i_2, going through the secondary winding conductor and closing at the load terminals, we obtain

$$\underbrace{\oint_{\mathbf{s}_1} \mathbf{E} \cdot d\mathbf{s}_2}_{r_2 i_2 + u_2} = -\frac{d\psi_2}{dt} \rightarrow -u_2(t) = r_2\ i_2(t) + \frac{d\psi_2(t)}{dt} \tag{7.32}$$

where the secondary linkage flux depends on both currents $\psi_2 = \psi_2(i_1, i_2)$.

The load equation, which at this stage cannot be written explicitly (because the load has not yet been specified), can be put in the form

$$u_2 = u_2(i_2) \tag{7.33}$$

For time-harmonic regimes, the above time-domain equations transform to the phasor domain as

$$\overline{U}_1 = r_1 \overline{I}_1 + j\omega\overline{\psi}_1; \quad -\overline{U}_2 = r_2 \overline{I}_2 + j\omega\overline{\psi}_2; \quad \overline{U}_2 = \overline{Z}_2 \overline{I}_2 \tag{7.34}$$

where $\overline{Z}_2$ is the impedance operator characterizing the load placed at secondary winding terminals.

Assuming, as before, that the transformer core displays a linear behavior, then the magnetic linkage fluxes can be written as linear combinations of the currents in play (Chapter 4):

$$\begin{cases} \psi_1(t) = L_{11} i_1(t) + L_M i_2(t) \\ \psi_2(t) = L_M i_1(t) + L_{22} i_2(t) \end{cases} \rightarrow \begin{cases} \overline{\psi}_1 = L_{11} \overline{I}_1 + L_M \overline{I}_2 \\ \overline{\psi}_2 = L_M \overline{I}_1 + L_{22} \overline{I}_2 \end{cases} \tag{7.35}$$

By substituting (7.35) into (7.34) we get the phasor-domain equations governing the transformer:

$$\overline{U}_1 = (r_1 + j\omega L_{11})\overline{I}_1 + j\omega L_M \overline{I}_2 \tag{7.36a}$$

$$-\overline{U}_2 = (r_2 + j\omega L_{22})\overline{I}_2 + j\omega L_M \overline{I}_1 \tag{7.36b}$$

$$\overline{U}_2 = \overline{Z}_2 \overline{I}_2 \tag{7.36c}$$

Note that, according to the conventions adopted, all the inductance coefficients L_{11}, L_{22} and L_M are positive quantities.

7.3.1 The Ideal Transformer

The ideal transformer does not exist. It is a fictitious device whose properties listed below cannot be fulfilled in nature. Nonetheless, some practical engineers may make use of it as a zeroth-order model to get coarse estimations of the transformer behavior.

The properties of a single-core ideal transformer are:

- Primary and secondary windings are made of perfect conductors with $\sigma \to \infty$.
- The transformer core is made of a perfect magnetic material with $\mu \to \infty$.

The first simplifying condition implies $r_1 = r_2 = 0$. The second simplifying condition ensures that the core is a perfect tube for the flux of **B** lines, that is dispersion is absent,

$$\psi_1 = N_1 \phi, \;\; \psi_2 = N_2 \phi$$

and magnetic coupling is perfect ($k = 1$). Moreover, since $\mu \to \infty$, the magnetic field **H** in the transformer core is zero, $\mathbf{H} = \mathbf{B}/\mu = 0$.

By using the above properties in (7.31) and (7.32) we obtain

$$\begin{cases} u_1 = N_1 \; d\phi/dt \\ -u_2 = N_2 \; d\phi/dt \end{cases} \rightarrow \quad \frac{u_1(t)}{u_2(t)} = -\frac{N_1}{N_2} \tag{7.37}$$

On the other hand, the application of Ampère's law to a closed circulation path inside the transformer core (Chapter 4) yields

$$0 = \oint_{\mathbf{S}} \underbrace{\mathbf{H}}_{0} \cdot d\mathbf{s} = \int_{S_s} \mathbf{J} \cdot \mathbf{n}_s \; dS = N_1 \; i_1 + N_2 \; i_2 \;\; \rightarrow \;\; \frac{i_1(t)}{i_2(t)} = -\frac{N_2}{N_1} \tag{7.38}$$

In short, apart from a minus sign, the ratio of the transformer voltages is equal to the corresponding winding turns ratio, whereas the ratio of the transformer currents is the reverse of the winding turns ratio.

The time-domain equations in (7.37) and (7.38) have a corresponding phasor-domain counterpart that reads as

$$\frac{\overline{U}_1}{\overline{U}_2} = -\frac{N_1}{N_2}, \;\; \frac{\overline{I}_1}{\overline{I}_2} = -\frac{N_2}{N_1} \tag{7.39}$$

By using (7.39) you can readily conclude that the input impedance $\overline{Z}_1$ of the ideal transformer (measured at the primary winding terminals) is equal to the load impedance $\overline{Z}_2$ multiplied by the factor $(N_1/N_2)^2$:

$$\overline{Z}_1 = \frac{\overline{U}_1}{\overline{I}_1} = \left(\frac{N_1}{N_2}\right)^2 \frac{\overline{U}_2}{\overline{I}_2} = \left(\frac{N_1}{N_2}\right)^2 \overline{Z}_2 \tag{7.40}$$

7.3.2 Transformer Impedance, Pseudo Lenz's Law

The zeroth-order result in (7.40) is not a very reliable one as far as actual transformers are concerned, since their properties hardly match those of ideal transformers. Therefore, we need to find $\overline{Z}_1$ based on (7.36).

To start with, we determine the relationship between $\overline{I}_2$ and $\overline{I}_1$. Eliminating $\overline{U}_2$ using (7.36b) and (7.36c) yields

$$\overline{I}_2 = -\frac{j\omega L_M}{r_2 + j\omega L_{22} + \overline{Z}_2}\overline{I}_1 \tag{7.41}$$

Substituting (7.41) into (7.36a) you get

$$\overline{Z}_1 = (r_1 + j\omega L_{11}) + \left(\frac{(\omega L_M)^2}{r_2 + j\omega L_{22} + \overline{Z}_2}\right) \tag{7.42}$$

The first term on the right-hand side of this equation is to be interpreted as the primary winding self-impedance, that is the one that is observed when the secondary winding is left open ($\overline{I}_2 = 0,\ \overline{Z}_2 \to \infty$). The second term on the right-hand side, resulting from magnetic coupling, is to be interpreted as the influence of the secondary winding (load included) on the input impedance of the transformer.

It is appropriate at this stage to go back to the pseudo Lenz's law to which we referred earlier in Problem 5.13.2.

From (7.42) you can see that when the transformer load is disconnected, the inductance observed at the generator terminals coincides with $L_{11} = (1/\omega)\Im\left(\overline{Z}_1\right)_{\overline{Z}_2=\infty}$.

According to common sense (based on Lenz's law), when the load is plugged in, the transformer core is expected to demagnetize (due to the alleged counteraction of the secondary current) and, consequently, the equivalent inductance $L'_{11} = (1/\omega)\Im\left(\overline{Z}_1\right)_{\overline{Z}_2\neq\infty}$ measured at the generator terminals is expected to be smaller than L_{11} itself.

We are going to show that this is not always true.

Let us write the load impedance as $\overline{Z}_2 = R_2 + jX_2$, where $X_2 < 0$ for capacitive loads. From (7.42) we have

$$\overline{Z}_1 = (r_1 + j\omega L_{11}) + \left(\frac{(\omega L_M)^2}{(r_2 + R_2) + j(\omega L_{22} + X_2)}\right)$$

After simple algebraic manipulation we get

$$\overline{Z}_1 = (r_1 + j\omega L_{11}) + \underbrace{\left(\frac{(\omega L_M)^2}{(r_2 + R_2)^2 + (\omega L_{22} + X_2)^2}\right)}_{K \geq 0} \times [(r_2 + R_2) - j(\omega L_{22} + X_2)]$$

Then we conclude that $\Re\left(\overline{Z}_1\right) = r_1 + K(r_2 + R_2) \geq r_1$; that is, the effect of the secondary winding is always to increase the equivalent input resistance of the transformer.

As for the equivalent input inductance of the transformer, we find

$$L'_{11} = \frac{1}{\omega}\Im(\overline{Z}_1) = L_{11} - K\left(L_{22} + \frac{X_2}{\omega}\right) \tag{7.43}$$

If the transformer load $\overline{Z}_2$ includes a capacitor with reactance $X_2 = -1/(\omega C_2)$, such that $C_2 < 1/(\omega^2 L_{22})$, then the term $K(L_{22} + X_2/\omega)$ in (7.43) becomes negative, and consequently, we see that the equivalent inductance at the primary terminals increases in magnitude, $L'_{11} > L_{11}$, which is in clear contradiction with the ordinary postulate of Lenz's law.

7.3.3 Equivalent Circuits

Some computer programs devoted to circuit analysis do not handle easily problems where magnetic coupling among inductors is present, as in the case of real transformers. Fortunately, you can circumvent such a problem by making use of equivalent circuits where magnetic coupling issues are absent.

This subsection addresses the topic of transformer equivalent circuits.

Consider the 'T' circuit shown in Figure 7.14, consisting of three unknown uncoupled impedances, $\overline{Z}_\alpha$, $\overline{Z}_\beta$ and $\overline{Z}_0$, which is cascaded with an ideal transformer characterized by a transformation ratio $\upsilon = n_1/n_2$.

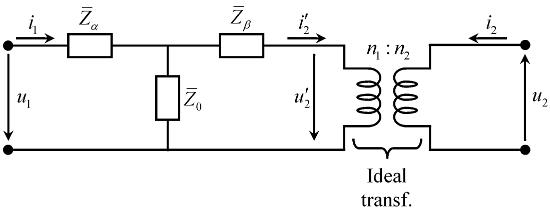

Figure 7.14 Equivalent circuit representation of the transformer

An important note to bear in mind is that the turns ratio n_1/n_2 of the ideal transformer is an arbitrary parameter of your choice that does not have to coincide with the turns ratio N_1/N_2 of the real transformer that you want to simulate.

Taking into account the results in Section 7.3.1 concerning the ideal transformer, you have here

$$\overline{U}'_2 = -\upsilon\,\overline{U}_2 \quad \text{and} \quad \overline{I}'_2 = -\frac{1}{\upsilon}\,\overline{I}_2 \tag{7.44}$$

The phasor-domain equations for the 'T' circuit are

$$\overline{U}_1 = \overline{Z}_\alpha \overline{I}_1 + \overline{Z}_0(\overline{I}_1 - \overline{I}'_2)$$
$$\overline{U}'_2 = \overline{Z}_0(\overline{I}_1 - \overline{I}'_2) - \overline{Z}_\beta \overline{I}'_2$$

By substituting (7.44) into the above results, and rearranging terms, you get

$$\overline{U}_1 = (\overline{Z}_\alpha + \overline{Z}_0)\overline{I}_1 + \frac{\overline{Z}_0}{v}\overline{I}_2 \tag{7.45a}$$

$$-\overline{U}_2 = \frac{\overline{Z}_\beta + \overline{Z}_0}{v^2}\overline{I}_2 + \frac{\overline{Z}_0}{v}\overline{I}_1 \tag{7.45b}$$

Next, all you have to do is to establish a term-by-term comparison between the preceding circuit equations and those describing the real transformer in (7.36a) and (7.36b). Then you will find

$$\overline{Z}_0 = j\omega \underbrace{vL_M}_{l_0}, \quad \overline{Z}_\alpha = r_1 + j\omega \underbrace{(L_{11} - vL_M)}_{l_1}, \quad \overline{Z}_\beta = v^2 r_2 + j\omega v^2 \underbrace{(L_{22} - L_M/v)}_{l_2} \tag{7.46}$$

The element described by $\overline{Z}_0$ is a pure inductor whose inductance is $l_0 = vL_M$. The element described by $\overline{Z}_\alpha$ is the series connection of a resistor and an inductor, whose resistance and inductance are, respectively, r_1 and l_1. Similarly, the element described by $\overline{Z}_\beta$ is the series connection of a resistor and an inductor, whose resistance and inductance are, respectively, $r_\beta = v^2 r_2$ and $l_\beta = v^2 l_2$. These results are summarized in Figure 7.15.

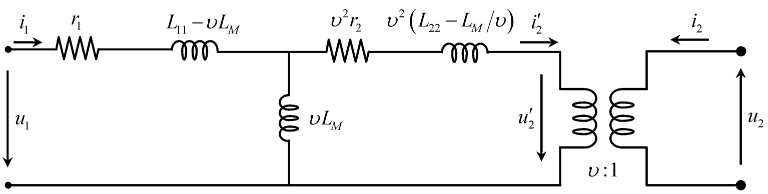

Figure 7.15 Detailed representation of the transformer equivalent circuit

At this point a few remarks are in order.

- Although the transformer's equivalent scheme in Figure 7.15 has been deduced using phasor-domain equations, it is also valid for time-domain analysis, provided that v is chosen to be real. This is so because its internal components are described by frequency-independent parameters under the assumption of slow time-varying phenomena.
- Given the fact that v is an arbitrary parameter, you really do not have one equivalent circuit, but an infinite number of equivalent circuits at your choice. If, for instance, you make $v = L_{11}/L_M$ then the impedance $\overline{Z}_\alpha$ will turn into a pure resistor $\overline{Z}_\alpha = r_1$; likewise, if you decide to make $v = L_M/L_{22}$ then the impedance $\overline{Z}_\beta$ will turn into a pure resistor $\overline{Z}_\beta = v^2 r_2$.
- The real transformer in Figure 7.13 and the circuit in Figure 7.15 are formally equivalent from the viewpoint of their accessible voltages $(u_1,\ u_2)$ and currents $(i_1,\ i_2)$. However, from an internal perspective there is no sort of correspondence between the real transformer and the equivalent circuit – it suffices to say that the internal circuit components are quite arbitrary since they depend on your own particular choice of v.

Suppose now that your goal is to materialize an equivalent circuit so that you can run some laboratory tests on it as a way of predicting the primary and secondary voltages and currents of the real transformer. This presents you with the problem of the physical realizability of the circuit; in other words, you are going to need to buy the actual components of the circuit (resistors with positive resistances, inductors with positive inductances). This objective naturally puts some constraints on your choices regarding υ, namely:

$$l_0 > 0 \rightarrow \upsilon \in \Re,\ \ \upsilon > 0$$

$$\begin{cases} l_1 = L_{11} - \upsilon L_M \geq 0 \\ l_2 = L_{22} - L_M/\upsilon \geq 0 \end{cases} \rightarrow \quad \frac{L_M}{L_{22}} \leq \upsilon \leq \frac{L_{11}}{L_M} \tag{7.47}$$

As a parenthetical remark, you should note that the interval defined for υ is a closed interval. In fact, the condition defined in (7.47) is compatible with the inequality $L_M^2 \leq L_{11}L_{22}$, which is always true and is nothing more than a restatement that the magnetic coupling factor between any two inductors cannot exceed unity (4.42).

If you have an inquisitive mind, you must now have an additional question drumming in your head. Where are you going to buy the ideal transformer in Figure 7.15?

Nowhere is the answer. But we may add that you do not need to. In fact, all you have to do is to replace the ideal transformer and its load by the corresponding input impedance measured at the terminals where u_2' and i_2' are defined, that is, from (7.40)

$$\overline{Z}_2' = \upsilon^2 \overline{Z}_2 \tag{7.48}$$

Finally you get the physically realizable circuit shown in Figure 7.16(a).

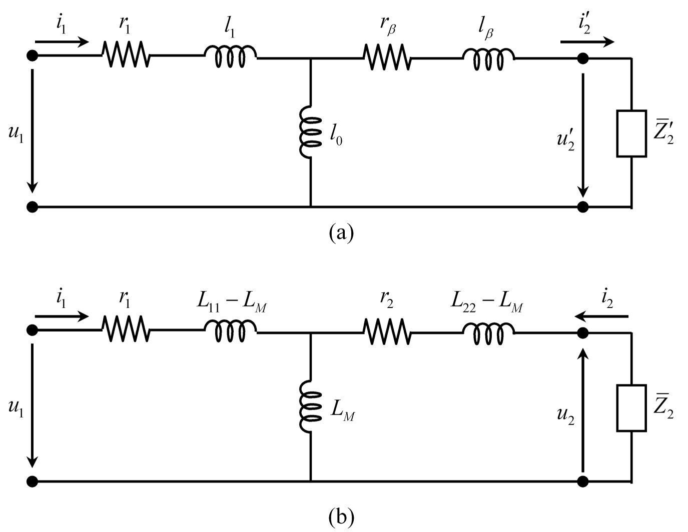

Figure 7.16 Transformer equivalent circuit with the ideal transformer removed. (a) The influence of the load impedance is taken into account through $\overline{Z}_2' = \upsilon^2\overline{Z}_2$. (b) Simplification arising from the choice $\upsilon = 1$.

Of course you can now argue that you have lost direct access to the secondary winding quantities u_2 and i_2. You are right. But the thing is that this detail is unimportant, because from (7.44) you can always retrieve such information from u_2' and i_2':

$$u_2 = -u_2'/v, \quad i_2 = -v\, i_2'$$

However, if your transformer happens to be such that $L_{11} \geq L_M$ and $L_{22} \geq L_M$, then, according to (7.47), you will be allowed to choose $v = 1$, a choice that gives you direct access to the secondary winding quantities themselves (Figure 7.16(b)).

7.3.4 Application Example (Capacitively Loaded Transformer)

Consider a given transformer whose winding resistances are negligibly small ($r_1 = r_2 = 0$). In order to evaluate the transformer induction coefficients, two laboratory experiments were conducted – see Figure 7.17.

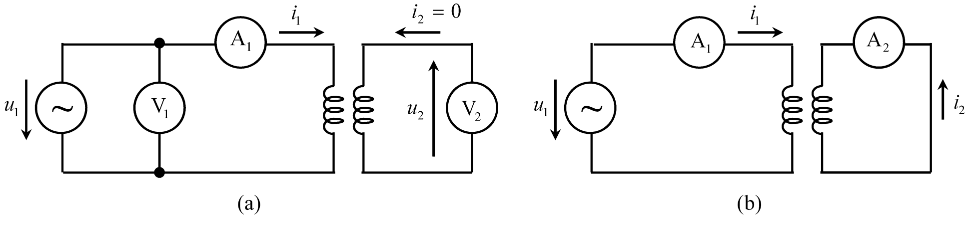

Figure 7.17 Experimental determination of the transformer inductances. (a) Secondary open. (b) Secondary short-circuited

Firstly, with the transformer left open ($i_2 = 0$), rms values for i_1, u_1 and u_2 were measured. The results obtained are, respectively, $A_1 \equiv 1\,\text{A}$, $V_1 \equiv 157.1\,\text{V}$ and $V_2 \equiv 62.8\,\text{V}$. Secondly, with the transformer short-circuited ($u_2 = 0$), rms values for i_1 and i_2 were measured. The results obtained are, respectively, $A_1 \equiv 1\,\text{A}$ and $A_2 \equiv 1.6\,\text{A}$.

At normal functioning the transformer's secondary winding is loaded with a capacitor of capacitance $C_2 = 63.66\,\mu\text{F}$. The voltage across the load is given by $u_2(t) = 200\cos(\omega t)$ V.

Assume $f = 50\,\text{Hz}$ in all your calculations.

Questions

Q_1 Find the inductance coefficients L_{11}, L_M and L_{22} of the transformer windings.

Q_2 Draw an equivalent circuit for the transformer using the choice $v = 2$ (check if this choice is a permissible one from the viewpoint of the physical realizability of the circuit).

Q_3 Determine the phasor-domain voltages and currents of the secondary and primary windings of the transformer (employ the equivalent circuit).

Q_4 Determine the input impedance of the loaded transformer and compare the equivalent self-inductance at the primary terminals to the measured one when the transformer was left open ($i_2 = 0$).

Solutions

Q_1 When the transformer is left open you obtain, from (7.36), $\overline{U}_1 = j\omega L_{11}\overline{I}_1$, and $\overline{U}_2 = -j\omega L_M \overline{I}_1$. Therefore you find:

$$L_{11} = \frac{U_{1_{rms}}}{\omega I_{1_{rms}}} = 0.5\,\text{H} \quad \text{and} \quad L_M = \frac{U_{2_{rms}}}{\omega I_{1_{rms}}} = 0.2\,\text{H}$$

When the transformer is short-circuited ($\overline{Z}_2 = 0$) you obtain, from (7.41),

$$\overline{I}_2 = -\frac{L_M}{L_{22}}\overline{I}_1$$

Therefore you find $L_{22} = L_M\ I_{1_{rms}}/I_{2_{rms}} = 0.125\,\text{H}$.

Q_2 $\dfrac{L_M}{L_{22}} \le v \le \dfrac{L_{11}}{L_M} \quad \rightarrow \quad 1.6 \le v \le 2.5$

So you see that $v = 2$ is a permissible choice.

Parameter evaluation, from (7.46):

$$\begin{cases} l_\alpha = l_1 = L_{11} - vL_M = 0.1\ \text{H} \\ l_\beta = v^2 l_2 = v^2\left(L_{22} - L_M/v\right) = 0.1\ \text{H} \\ l_0 = vL_M = 0.4\ \text{H} \end{cases}$$

As for the input impedance of the ideal transformer, you have

$$\overline{Z}_2' = \frac{1}{j\omega C_2'} = v^2\frac{1}{j\omega C_2} \quad \rightarrow \quad C_2' = \frac{C_2}{v^2} = 15.92\ \mu\text{F}$$

Figure 7.18 shows the particular equivalent circuit for this application example.

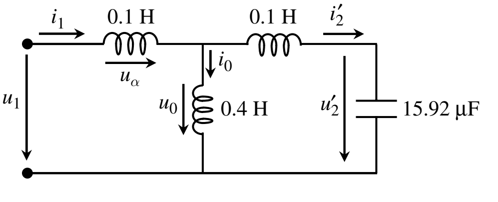

Figure 7.18 One possible equivalent circuit for the transformer examined in Section 7.3.4

Q_3

$$\overline{U}_2 = 200\ \text{V} \rightarrow \overline{U}_2' = -v\overline{U}_2 = 400\ e^{j\pi}\ \text{V}$$

$$\overline{I}_2 = j\omega C_2 \overline{U}_2 = 4\ e^{j\pi/2}\ \text{A} \rightarrow \overline{I}_2' = -\overline{I}_2/v = 2\ e^{-j\pi/2}\ \text{A}$$

$$\overline{U}_0 = j\omega l_\beta \overline{I}_2' + \overline{U}_2' = 337.2\ e^{j\pi}\ \text{V}$$

$$\overline{I}_0 = \frac{\overline{U}_0}{j\omega l_0} = 2.68\ e^{j\pi/2}\ \text{A}$$

$$\overline{I}_1 = \overline{I}_0 + \overline{I}_2' = 0.68\ e^{j\pi/2}\ \text{A}$$

$$\overline{U}_\alpha = j\omega l_\alpha \overline{I}_1 = 21.4\ e^{j\pi}\ \text{V}$$

$$\overline{U}_1 = \overline{U}_\alpha + \overline{U}_0 = 358.6\ e^{j\pi}\ \text{V}$$

You may note that the primary and secondary currents are in phase, which means that they do give rise to concordant magnetizing effects.

Q_4 $\overline{Z}_1 = \overline{U}_1/\overline{I}_1 = j\omega L_{11}' = j527.4\ \Omega \rightarrow L_{11}' = 1.68\,\text{H}$ (remember that $L_{11} = 0.5\,\text{H}$).
When the transformer is capacitively loaded, the generator at the primary winding observes an increased inductance (contrary to expectations based on Lenz's law.)

7.4 Transient Regimes

Sections 7.2 and 7.3 have been dedicated to steady-state harmonic regimes. Now, the time has come to shift our attention to the analysis of transient phenomena, that is the phenomena subsequent to switching operations which, as time elapses, should tend to stabilize in a steady-state solution.

7.4.1 Free-Regime and Steady-State Solutions

As depicted in Figure 7.19, a generator is switched on, at $t = 0$, in a linear passive circuit containing a number of resistors, inductors and capacitors.

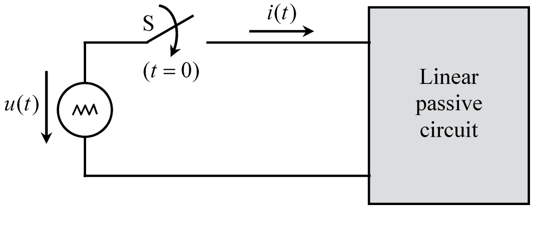

Figure 7.19 A voltage generator switched on to a linear passive circuit

By application of the time-domain fundamental laws governing both magnetic and electric induction phenomena, you will obtain a description of the generator current $i(t)$ in the typical form of a linear differential equation with constant coefficients of order n:

$$\text{For } t > 0: \; a_n \frac{d^n i}{dt^n} + \cdots + a_k \frac{d^k i}{dt^k} + \cdots + a_1 \frac{di}{dt} + a_0 i = f\,(u(t)) \tag{7.49}$$

where the a_k coefficients (with $k = 0$ to n) are real, and the function $f(u)$ on the right-hand side of the equation depends on the generator voltage. The order n of the equation – depending on the complexity of the circuit – has an upper limit determined by the total number of inductors and capacitors pertaining to the circuit.

The solution to (7.49) is obtained by breaking it down into two sub-solutions, the steady-state solution $i_S(t)$ and the free-regime solution $i_F(t)$.

The steady-state solution is a particular solution of the complete equation in (7.49). The steady-state solution is the one which the current $i(t)$ converges to as time goes on

$$\lim_{t\to\infty} i(t) \to i_S(t)$$

For example, when the driving voltage $u(t)$ is time harmonic, the solution for $i_S(t)$ is also time harmonic, and you can determine it by using the phasor-domain technique developed in Sections 7.2 and 7.3.

The free-regime solution is the general solution of the homogeneous equation corresponding to the one in (7.49); that is, when you make $f(u) = 0$,

$$a_n \frac{d^n i}{dt^n} + \cdots + a_k \frac{d^k i}{dt^k} + \cdots + a_1 \frac{di}{dt} + a_0 i = 0 \tag{7.50}$$

The free-regime solution derives its name from the fact that the solution of (7.50) is free from the influence of the generator.

Since the generator's influence has been removed, the solution of (7.50) must tend to zero as time elapses:

$$\lim_{t\to\infty} i_F(t) \to 0$$

Equations of the type shown in (7.50) are known to have solutions in the form of linear combinations of exponential time-decaying functions. As a matter of fact, if in (7.50) you substitute Ie^{st} for $i(t)$ you will get an algebraic polynomial equation (the so-called characteristic equation)

$$a_n s^n + \cdots + a_k s^k + \cdots + a_1 s = 0 \tag{7.51}$$

whose roots $s_1, \ldots, s_k, \ldots, s_n$ will enable you to write the free-regime solution as a sum of n independent exponentials (we are assuming that multiple roots are absent)

$$i_F(t) = \sum_{k=1}^{n} I_k e^{s_k t} \tag{7.52}$$

The roots of the characteristic equation, $s_1, \ldots, s_k, \ldots, s_n$, can be real or complex but, in either case, their real parts cannot be positive, otherwise the amplitude of the free-regime solution will increase with time, which is a physically impossible situation with generators removed.

As a parenthetical note, and for completion purposes, it should be added that in some (rare) circumstances multiple roots can occur in (7.51). When that is the case, the solution for $i_F(t)$ is more complicated. For instance, if a root s_k with multiplicity m $(m \leq n)$ is found to exist then its contribution to (7.52) will take the form

$$\left(I_{k,1} + I_{k,2}t + \cdots + I_{k,m}t^{m-1}\right) e^{s_k t} \tag{7.53}$$

7.4.2 Initial Conditions

In (7.52) a total of n unknown amplitudes I_k need to be determined. For that purpose you have to take into account a set of n initial conditions that should be enforced on the so-called state variables of the problem.

Independently of the switching operations (closing or opening) that take place in a given circuit, certain quantities (state variables) can never change suddenly, namely capacitor voltages u_C and inductor currents i_L.

Time discontinuities in $u_C(t)$ would mean that the electric energy stored in the capacitor would change instantaneously; likewise, time discontinuities in $i_L(t)$ would mean that the magnetic energy stored in the inductor would change instantaneously. These energy jumps would require infinite amounts of power at those components

$$p_C = \frac{d}{dt} W_e(t) \to \infty; \quad p_L = \frac{d}{dt} W_m(t) \to \infty$$

but since this is a physical impossibility, you cannot avoid the obvious conclusion that u_C and i_L ought to remain unchanged immediately before and after the switching operation at $t = 0$,

$$u_C(0^+) = u_C(0^-); \quad i_L(0^+) = i_L(0^-) \tag{7.54}$$

The consideration of n initial conditions, as in (7.54), allows you finally to solve the original problem stated in (7.49).

7.4.3 Analysis of the Capacitor Charging Process

In order to illustrate the above theoretical considerations, we are now going to examine the very simple transient phenomena resulting from the switching on of a DC generator in an RC circuit as described in Figure 7.20, where the capacitor is initially discharged, $u_C(0^-) = 0$.

For $t > 0$, application of the induction law to the clockwise-oriented closed path $\mathbf{s}$ yields

$$\oint_{\mathbf{s}} \mathbf{E} \cdot d\mathbf{s} = -\frac{d\psi_{S_s}}{dt} \to -U + Ri(t) + u_C(t) \approx 0 \tag{7.55}$$

where the time derivative of the linkage magnetic flux across S_s has been neglected.

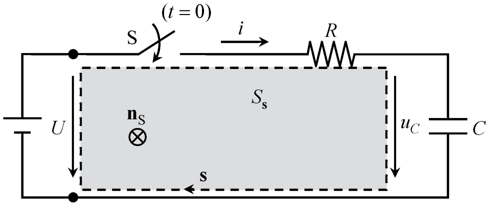

Figure 7.20 A DC voltage generator switched on in an RC series circuit. Ordinarily, the magnetic flux linkage across the shaded surface S_s is negligibly small

Since the state variable of the circuit under analysis is the capacitor voltage, we should use (6.7) to replace $i(t)$ by its expression in terms of $u_C(t)$, that is $i = C\ du_C/dt$. On doing this we find a first-order differential equation in the variable $u_C(t)$

$$RC\frac{du_C(t)}{dt} + u_C(t) = U \tag{7.56}$$

A particular solution of the preceding equation is $u_C(t) = U$, where U is a constant (the DC voltage of the generator). Therefore, the steady-state solution for the problem is

$$(u_C)_S = U \tag{7.57}$$

The free-regime solution $(u_C)_F$ is obtained upon examination of the homogeneous equation

$$RC\frac{du_C(t)}{dt} + u_C(t) = 0$$

Since the corresponding characteristic equation is $RCs + 1 = 0$, with a root

$$s = -\frac{1}{\tau} = -\frac{1}{RC} \tag{7.58}$$

where $\tau = RC$ is the so-called time constant of the circuit, we find that the free-regime solution takes the form of a time-decaying exponential function

$$(u_C(t))_F = U'\ e^{st} = U'\ e^{-t/\tau} \tag{7.59}$$

By summing the steady-state solution in (7.57) with the free-regime solution in (7.59) we get

$$u_C(t) = U + U'\ e^{-t/\tau}$$

The unknown U' is determined by enforcing the initial condition $u_C(0^+) = u_C(0^-) = 0$, that is $0 = U + U'\ e^0$, from which $U' = -U$ is obtained. Hence

$$\text{For } t > 0:\ u_C(t) = U(1 - e^{-t/\tau}) \tag{7.60}$$

If you wish to know the transient current in the circuit you merely have to find the time derivative of the above result,

$$i(t) = C\frac{du_C}{dt} = CU\left(\frac{e^{-t/\tau}}{\tau}\right) = \frac{U}{R}\ e^{-t/\tau} \tag{7.61}$$

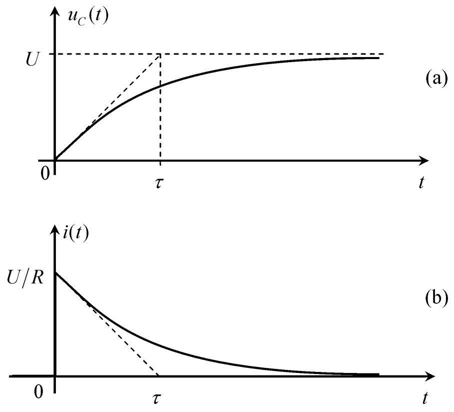

Figure 7.21 Transient response of the *RC* series circuit. The characteristic duration of transient phenomena is determined by the time constant $\tau = RC$. (a) Plot of the capacitor voltage against time. (b) Plot of the generator current against time

Graphical plots illustrating the results in (7.60) and (7.61) are shown in Figure 7.21.

An interesting point connected with the preceding results is the energy balance. As you already know, the final energy stored in the capacitor is $W_e = \frac{1}{2}CU^2$. Let us now evaluate the energy W expended by the generator as well as the energy W_J dissipated in the resistor during the charging process:

$$W = \int_0^\infty p(t)dt = U\int_0^\infty i(t)dt = CU^2; \quad W_J = R\int_0^\infty i^2(t)dt = \tfrac{1}{2}CU^2 \tag{7.62}$$

The amazing conclusion is that, irrespective of the resistance value, the DC generator always has to expend an amount of energy that is twice the value of the final energy stored in the capacitor, meaning that the yield factor of the charging process is exactly 50 %.

Suppose now that the resistor is removed from the circuit. You face a paradox, don't you? Where has the missing energy gone?

Some people will answer that it has been radiated away. But if you insist and ask for the calculation of the radiated power, you will most probably have no reply.

Well, things are simpler than they look, at first glance.

Observe again the results depicted in Figure 7.21. As you make R go to zero, the derivative *di/dt* increases greatly, the same thing happening to the derivative $d\psi_{S_s}/dt$. The real problem with the limit case $R = 0$ is that (7.56), the equation that we have used to model the problem, is no longer valid. You have to go back to (7.55) and drop the approximation $d\psi_{S_s}/dt \approx 0$.

Although the inductance L of the closed loop describing the circuit is only small, L must be taken into account when $i(t)$ undergoes rapid changes. In conclusion, for the analysis of the capacitor charging process, with $R = 0$, the governing equation to be employed is

$$\oint_{\mathbf{s}} \mathbf{E}\cdot d\mathbf{s} = -\frac{d\psi_{S_s}}{dt} \to -U + u_C(t) = -L\frac{di(t)}{dt}$$

or, which is the same (using $i = C\ du_C/dt$),

$$LC\frac{d^2u_C(t)}{dt^2} + u_C(t) = U \tag{7.63}$$

As before, the steady-state solution is given by (7.57), $(u_C)_S = U$. The crucial difference resides with the free-regime solution, whose characteristic equation now reads $LCs^2 + 1 = 0$, and whose two roots are purely imaginary numbers

$$s_{1,2} = \pm j\omega_0, \quad \text{with } \omega_0 = 1/\sqrt{LC} \tag{7.64}$$

Consequently, the free-regime solution is now described by

$$(u_C(t))_F = U_1 e^{j\omega_0 t} + U_2 e^{-j\omega_0 t} \tag{7.65}$$

At this stage you should note that the left-hand side of (7.65) is a real-valued function, and, of course, the same thing should happen to the right-hand side of (7.65). For this to be possible the unknown constants U_1 and U_2 must be complex conjugates of each other

$$U_1 = \overline{U}_0 = U_0 e^{j\theta};\ \ U_2 = \overline{U}_0^* = U_0 e^{-j\theta}$$

Substituting the preceding results into (7.65) we obtain

$$(u_C(t))_F = U_0\left(e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}\right) = 2U_0 \cos(\omega_0 t + \theta)$$

Finally, the transient solution for u_C, that is the sum of the steady-state and free-regime solutions, is written as

$$u_C(t) = U + 2U_0 \cos(\omega_0 t + \theta) \tag{7.66}$$

The corresponding current $i = C\ du_C/dt$ is evaluated as

$$i(t) = -2CU_0\omega_0 \sin(\omega_0 t + \theta) \tag{7.67}$$

The constants U_0 and θ are to be determined using two initial conditions. One has already been stated above, $u_C(0) = 0$. The second, new, one is $i(0) = 0$, because the current in a loop containing an inductance L cannot have time discontinuities.

The latter condition, together with (7.67), leads to $\theta = 0$. The first initial condition, combined with (7.66), leads to $U_0 = -U/2$.

In conclusion, in the limit case $R = 0$, the capacitor charging process turns out to be an undamped periodic oscillation, whose periodicity is critically dependent on the self-inductance of the loop described by the circuit; the capacitor voltage and current being given by

$$u_C(t) = U\,(1 - \cos(\omega_0 t));\ \ i(t) = CU\omega_0 \sin(\omega_0 t) \tag{7.68}$$

See the corresponding graphical plots in Figure 7.22.

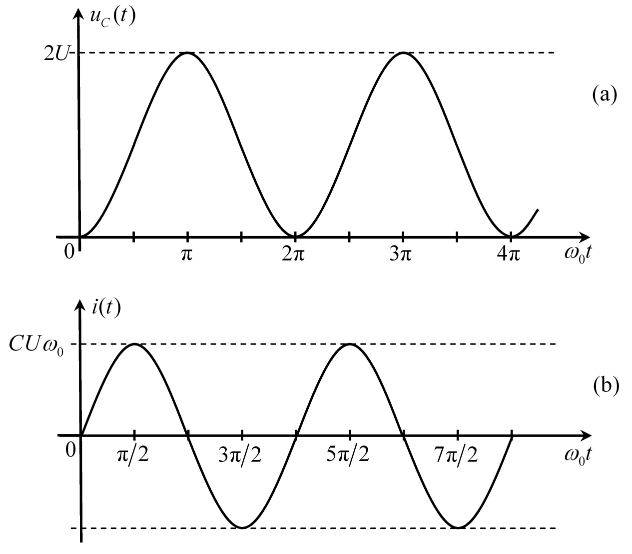

Figure 7.22 Transient response of the *RC* series circuit, with $R = 0$, but taking into account the small inductance L of the circuit's closed loop. The oscillation characterizing the transient phenomena is determined by the angular frequency $\omega_0 = 1/\sqrt{LC}$. (a) Capacitor voltage. (b) Generator current

At any given instant t' the energy W brought into play by the generator is the sum of the magnetic energy stored in the magnetic field of the loop plus the electric energy stored in the capacitor:

$$\begin{cases} W(t') = \int\limits_0^{t'} p(t)\,dt = CU^2\,(1-\cos(\omega_0 t')) \\ W_m(t') = \dfrac{CU^2}{2}\sin^2(\omega_0 t') \\ W_e(t') = \dfrac{CU^2}{2}\,(1-\cos(\omega_0 t'))^2 \end{cases} \rightarrow W(t') = W_m(t') + W_e(t')$$

where, from (7.64), $\omega_0 = 1/\sqrt{LC}$.

7.4.4 Connecting an Inductive Load to an AC Generator

Next, let us examine the transient regime occurring when an AC generator is switched in an *RL* circuit – see Figure 7.23. As you will see, studying this case brings out novel aspects that we have not yet treated, providing you with further insights into transient phenomena.

As you should know already, application of the induction law to the circuit in Figure 7.23 yields the following governing equation:

$$\text{For } t > 0: \; L\frac{di(t)}{dt} + Ri(t) = u(t), \quad \text{with } \; u(t) = U\cos(\omega t + \alpha_u) \qquad (7.69)$$

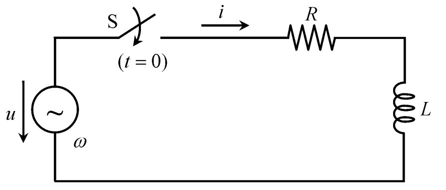

Figure 7.23 Connecting an *RL* series circuit to an AC generator

The steady-state solution is determined by resorting to the phasor-domain technique described earlier in Section 7.2. The impedance of the circuit is

$$\overline{Z} = R + j\omega L = Z\, e^{j\varphi};\;\; Z = \sqrt{R^2 + (\omega L)^2};\;\; \varphi = \arctan(\omega L/R)$$

The phasor associated with the sinusoidal current is obtained from

$$\overline{I} = \frac{\overline{U}}{\overline{Z}} = I\, e^{j\alpha_i} \rightarrow \begin{cases} I = U/Z \\ \alpha_i = \alpha_u - \varphi \end{cases} \tag{7.70}$$

From (7.70), the steady-state solution for the circuit current is written in the time domain as

$$(i(t))_S = I\cos(\omega t + \alpha_i) \tag{7.71}$$

The analysis of the free-regime starts with the homogeneous equation associated with (7.69)

$$L\frac{di(t)}{dt} + Ri(t) = 0$$

The corresponding characteristic equation is $Ls + R = 0$, and its negative root is

$$s = -\frac{1}{\tau} = -\frac{R}{L} \tag{7.72}$$

where the time constant of the circuit is $\tau = L/R$. The free-regime solution is a time-decaying exponential function

$$(i(t))_F = I'\, e^{st} = I'\, e^{-t/\tau} \tag{7.73}$$

The transient current is obtained by summing the sub-solutions in (7.71) and (7.73):

$$\text{For } t > 0:\;\; i(t) = I\cos(\omega t + \alpha_i) + I'\, e^{-t/\tau} \tag{7.74}$$

The unknown amplitude I' of the free-regime solution is found by consideration of the initial condition pertaining to this problem, which is $i(0^+) = i(0^-) = 0$. Consequently, from (7.74), we have: $0 = I\cos(\alpha_i) + I'$. Hence

$$I' = -I\cos(\alpha_i) = -I\cos(\alpha_u - \varphi) \tag{7.75}$$

Combining (7.74) and (7.75), we get the final solution:

$$\text{For } t > 0:\;\; i(t) = I\left[\cos(\omega t + \alpha_i) - \cos(\alpha_i)\, e^{-t/\tau}\right]$$

Noting, from (7.75), that the initial amplitude I' of the free-regime depends on the initial phase α_u of the generator voltage, several interesting points arise and deserve to be pointed out.

If $\alpha_u = \varphi \pm \pi/2$ then the free regime will be absent. You switch on the circuit and a purely sinusoidal current establishes itself immediately, $i(t) = I\cos(\omega t \pm \pi/2)$. Clearly, this is the most desirable situation.

On the contrary, if $\alpha_u = \varphi$ or $\alpha_u = \varphi + \pi$ then the free regime will have a maximum initial amplitude. For $\alpha_u = \varphi$ you get

$$i(t) = I\left(\cos(\omega t) - e^{-t/\tau}\right) \tag{7.76a}$$

For $\alpha_u = \varphi + \pi$ you get

$$i(t) = I\left(e^{-t/\tau} - \cos(\omega t)\right) \tag{7.76b}$$

The cases described by (7.76) are the least desirable situations, because after the closing of the switch, the circuit current displays a distorted asymmetrical shape. Furthermore, if the time constant τ happens to be much longer than the sinusoid period T, then several repetitive overcurrent peaks will occur for quite a long time. To better understand what we are talking about, look for example at (7.76b) and assume that the exponential decay is extremely slow – that is, consider the approximation $\exp(-t/\tau) \approx 1$. At instants of time such that $t = t_m = mT/2$, with m an odd number, you will get $i(t_m) \approx 2I$, which signifies that the transient current may reach an intensity that is twice the predicted one for the steady-state regime (this can not only blow protective fuses, but even endanger the equipment itself).

7.4.5 Disconnecting an Inductive Load

Have you ever tried to unplug a running inductive appliance (like a washing machine or an electric fan) from its wall socket? What did you notice? Most probably you saw an arc discharge (sparks) occurring. Do you know why that happened? If not, you will learn about it now.

Take the same circuit we analyzed earlier (see Figure 7.23), and consider that enough time has elapsed after the switching-on operation, so that the final steady state has been reached

$$i(t) = (i(t))_S = I\cos(\omega t + \alpha_i) \tag{7.77}$$

At $t = t_0$ you decide to open the switch, disconnecting the load. If you are very lucky, that is if by chance $\omega t_0 + \alpha_i = \pm\pi/2$, then you will be interrupting a null current, $i(t_0^+) = i(t_0^-) = 0$. The initial condition for inductor currents is not violated and nothing unusual happens.

However, most likely the opening of the switch will occur when $i(t_0^-) \neq 0$. In this case, although the switch is open, the circuit current refuses to go to zero immediately (inductor currents can never have time discontinuities).

By analyzing the circuit in Figure 7.24, you find for the voltage $u_S(t)$ at the switch terminals

$$\text{For } t > t_0: \; u_S = u - Ri - L\frac{di}{dt} \tag{7.78}$$

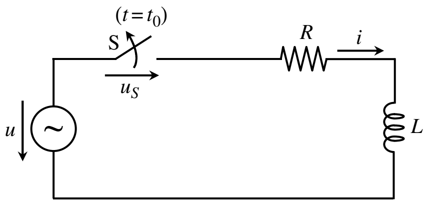

Figure 7.24 Disconnecting an inductive load

Subsequent to the switching operation, and since you are trying to interrupt $i(t)$ abruptly, the derivative *di/dt* increases dramatically. This has several consequences related to overvoltage problems. The first is that the inductor itself can be damaged (the winding insulation may fail). Secondly, a sudden overvoltage appears in u_S leading to a very intense electric field $\mathbf{E}(t)$ between the switch contacts, which in turn gives rise to breakdown phenomena in the air, originating an arc discharge channel through which conduction currents are allowed to flow until $i(t)$ reaches zero and the phenomenon ceases naturally.

A technical solution to avoid arc discharges consists of inserting the switch inside a vacuum enclosure (by definition a vacuum cannot be ionized, and therefore dielectric breakdown cannot occur). The question, for the most curious among you, is how does the electric current manage to flow in a vacuum?

Do you have any idea? Maybe you can find the answer in Chapter 6. . .

If it helps, you may imagine that a small capacitance exists between the terminals of the open switch. In fact, the current flow is ensured in the form of a displacement current, $\varepsilon_0\ \partial\mathbf{E}/\partial t$, across the open switch.

We have learnt that inductor currents cannot be interrupted without harmful consequences. Having said that, let us analyze a simple protection scheme that can be used to prevent the problems mentioned above.

As shown in Figure 7.25, a protective resistor R_0 is placed in parallel with the RL circuit. When the switch is open, the current $i(t)$ suffers no interruption, since it can still flow in the closed loop formed by R_0, R and L.

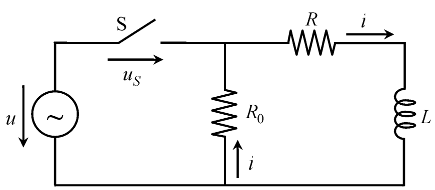

Figure 7.25 Disconnecting an inductive load protected by a parallel-connected resistor R_0

The governing equations of the new circuit for $t > t_0$ are

$$L\frac{di(t)}{dt} + (R + R_0)\, i(t) = 0 \tag{7.79}$$

$$u_S(t) = u(t) + R_0 i(t) \tag{7.80}$$

Since (7.79) is itself a homogeneous differential equation, the steady-state solution for $i(t)$ is zero. In other words, the transient regime is completely described by the free-regime solution, that is

$$\text{For } t > t_0: \ i(t) = I_0\, e^{-(t-t_0)/\tau_0}, \ \text{with } \tau_0 = L/(R+R_0) \tag{7.81}$$

The unknown constant I_0 is determined from (7.81) and (7.77), by enforcing the initial condition at $t = t_0: \ I_0 = i(t_0^+) = i(t_0^-) = I\cos(\omega t_0 + \alpha_i)$. Finally we find

$$\begin{aligned} i(t) &= I\cos(\omega t_0 + \alpha_i)\, e^{-(t-t_0)/\tau_0} \\ u_S(t) &= U\cos(\omega t + \alpha_u) + R_0 I\cos(\omega t_0 + \alpha_i)\, e^{-(t-t_0)/\tau_0} \end{aligned} \tag{7.82}$$

As $t \to \infty$, the load current goes exponentially to zero, whereas the switch voltage tends to follow the generator's sinusoidal voltage.

If the protective resistor R_0 in Figure 7.25 is removed, $R_0 \to \infty$, you will see from (7.82) that u_S increases to infinity, leading to breakdown phenomena at the switch terminals, as we discussed at the beginning of this subsection.

7.4.6 Application Example (Switching Off a Transformer Protected by a Capacitor)

A transformer, characterized by its intrinsic parameters $r_1, \ r_2, \ L_{11}, \ L_{22}$ and L_M, has its secondary winding open ($i_2 = 0$). A protective capacitor of capacitance C is connected in parallel with the primary winding – see Figure 7.26.

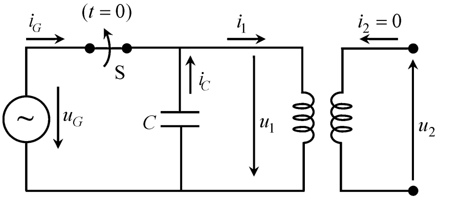

Figure 7.26 Switching off a transformer protected by a parallel-connected capacitor

The generator's applied voltage is $u_G(t) = U_G\cos(\omega_G t + \pi/2)$.

Data: $r_1 = r_2 = 10\ \Omega, \ L_{11} = L_{22} = 17.32\,\text{mH}, \ L_M = 10\,\text{mH}, \ C = 46.41\,\mu\text{F}, \ U_G = 10\,\text{V}, \ \omega_G = 1\,\text{krad/s}$.

Questions

Q_1 Assume that the switch S is closed for a long time. Determine the phasors associated with the voltages and currents marked in Figure 7.26.

Q_2 Analyze the transient regime resulting from opening the switch at $t = 0$; in particular, determine the time evolution of $i_1(t), \ u_1(t)$ and $u_2(t)$.

Solutions

Q$_1$ $\overline{U}_G = \overline{U}_1 = 10\, e^{j90°}$ V.

$$\overline{I}_1 = \frac{\overline{U}_1}{r_1 + j\omega_G L_{11}} = 500\, e^{j30°}\ \text{mA};\ \overline{I}_C = -j\omega_G C \overline{U}_1 = 464.1\,\text{mA};$$

$$\overline{I}_G = \overline{I}_1 - \overline{I}_C = 252\, e^{j97°}\ \text{mA};\ \overline{U}_2 = -j\omega_G L_M \overline{I}_1 = 5\, e^{-j60°}\ \text{V}.$$

Q$_2$ By opening the switch you make $i_G = 0$.

Because of the initial conditions that we will need to employ later, it is recommended that you first find the initial values of the state variables of the problem, that is the voltage u_1 across the capacitor and the current i_1 in the primary winding:

$$u_1(0) = U_{10} = \Re\left(\overline{U}_1\right) = 0; \quad i_1(0) = I_{10} = \Re\left(\overline{I}_1\right) = 500\cos(30^o) = 433\,\text{mA} \tag{7.83}$$

The capacitor current i_C (which is not a state variable) coincides with i_1, for $t > 0$.

By application of the induction law to the primary winding you find

$$u_1(t) = r_1 i_1(t) + L_{11}\frac{di_1(t)}{dt} \tag{7.84}$$

Further, for $t > 0$, at the capacitor terminals you have

$$i_1(t) = -C\frac{du_1(t)}{dt} \tag{7.85}$$

By combining the two preceding equations you immediately obtain a second-order homogeneous differential equation in the state variable $u_1(t)$:

$$\frac{d^2}{dt^2}u_1(t) + \frac{r_1}{L_{11}}\frac{d}{dt}u_1(t) + \frac{1}{L_{11}C}u_1(t) = 0$$

This result can be rewritten in the canonical form

$$\frac{d^2}{dt^2}u_1(t) + 2\beta\frac{d}{dt}u_1(t) + \omega_0^2 u_1(t) = 0 \tag{7.86}$$

where

$$\beta = \frac{r_1}{2L_{11}}, \quad \omega_0 = \frac{1}{\sqrt{L_{11}C}} \tag{7.87}$$

The constant β is called the damping factor (units: Np/s), and ω_0 is called the undamped angular frequency (units: rad/s). The reason for this terminology will become clear very soon.

The characteristic equation corresponding to (7.86), as well as its roots s_1 and s_2, is

$$s^2 + 2\beta s + \omega_0^2 = 0,\ \text{with roots}:\ s_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{7.88}$$

Three distinct situations can now happen:

1. Two negative real roots are found when $\beta > \omega_0$. The solution, which is said to be overdamped, takes the form: $u_1(t) = U' \ e^{-t/\tau_1} + U'' \ e^{-t/\tau_2}$, where $\tau_1 = -1/s_1$ and $\tau_2 = -1/s_2$.
2. One double real root is found when $\beta = \omega_0$. The solution, which is said to be critically damped, takes the form $u_1(t) = (U' + tU'')\, e^{-\beta t}$.
3. Two complex conjugate roots are found when $\beta < \omega_0$. The solution, which is said to be underdamped, takes the oscillatory form

$$u_1(t) = U \ e^{-\beta t} \cos(\omega t + \theta) \tag{7.89}$$

In our problem we have, from (7.87), $\beta = 288.7\,\mathrm{Np/s}$ and $\omega_0 = 1115.4\,\mathrm{rad/s}$. Two complex conjugate characteristic roots with negative real part are encountered:

$$s = -\beta + j\omega = \omega_0 \ e^{j\delta} \text{ and } s^* = -\beta - j\omega = \ \omega_0 \ e^{-j\delta}$$

where $\omega = \sqrt{\omega_0^2 - \beta^2}$, the so-called damped angular frequency, is $\omega = 1077.4\,\mathrm{rad/s}$ and $\delta = 105° = 7\pi/12$.

The transient regime for $u_1(t)$, which is a purely free regime, can then be written as $u_1(t) = U' \ e^{st} + U'' \ e^{s^* t}$, where U' and U'' must necessarily be complex conjugate numbers (otherwise $u_1(t)$ would not be a real-valued function).

By making $U' = (U'')^* = \frac{1}{2}\overline{U}$, with $\overline{U} = Ue^{j\theta}$, we obtain

$$u_1(t) = \left(\frac{\overline{U}}{2} \ e^{st}\right) + \left(\frac{\overline{U}}{2} \ e^{st}\right)^* = \Re\left(\overline{U} \ e^{st}\right)$$

Substituting $(-\beta + j\omega)$ for s, and substituting $Ue^{j\theta}$ for $\overline{U}$, we find

$$\Re\left(\overline{U} \ e^{st}\right) = U \ e^{-\beta t} \cos(\omega t + \theta)$$

which confirms the expression in (7.89). In conclusion, we can write the transient oscillatory solution for $u_1(t)$ as

$$u_1(t) = \begin{cases} \Re\left(\overline{U} \ e^{st}\right) \\ U \ e^{-\beta t} \cos(\omega t + \theta) \end{cases} \tag{7.90}$$

The result on the top, with complex quantities, is more compact and more useful for calculations (since the variable t appears a single time). The result on the bottom, involving real functions, is, however, more appealing from a physical point of view, as it permits a straightforward interpretation of the phenomenon – an oscillation of angular frequency ω whose amplitude decreases with time according to the damping factor β.

Going back to the problem under analysis, the initial condition $u_1(0) = 0$ applied to (7.90) leads to $\theta = \pi/2$. For the determination of the unknown U we have to resort to the second initial condition in (7.83), $i_1(0) = I_{10} = 433\,\text{mA}$, which leads us to an examination of $i_1(t)$ in (7.85):

$$i_1(t) = -C\frac{du_1(t)}{dt} = -C\,\Re\left(\frac{d}{dt}Ue^{j\pi/2}e^{st}\right) = -CU\,\Re\left(jse^{st}\right) \tag{7.91a}$$

Taking into account that $s = -\beta + j\omega = \omega_0\, e^{j\delta}$, we may rewrite (7.91a) as

$$i_1(t) = \omega_0 CU\,\Re\left(e^{-\beta t}\, e^{j(\omega t+\delta-\pi/2)}\right) = \omega_0 CU\, e^{-\beta t}\cos(\omega t+\delta-\pi/2) \tag{7.91b}$$

For $t = 0$, we get

$$I_{10} = -CU\,\Re\,(js) = \omega_0 CU \sin\delta$$

from which the constant U is evaluated,

$$U = \sqrt{\frac{L_{11}}{C}}\frac{I_{10}}{\sin\delta} = 8.66\ \text{V}$$

The voltage across the secondary winding is finally obtained from $u_2 = -L_M\, di_1/dt$. By using (7.91a) we obtain, for $t > 0$,

$$u_2(t) = L_M CU\,\Re\left(js\frac{d}{dt}e^{st}\right) = L_M CU\,\Re\left(js^2e^{st}\right) = \omega_0^2 L_M CU\,\Re\left(e^{-\beta t}e^{j(\omega t+2\delta+\pi/2)}\right)$$

$$u_2(t) = \frac{L_M}{L_{11}}U\, e^{-\beta t}\cos\left(\omega t + 2\delta + \tfrac{\pi}{2}\right)$$

Let us summarize the results obtained for the transient regime subsequent to the opening of the switch:

$$\text{For } t>0:\ \begin{cases} u_1(t) = U_1\, e^{-\beta t}\cos(\omega t+\pi/2), & \text{with } U_1 = 8.66\ \text{V} \\ i_1(t) = I_1\, e^{-\beta t}\cos(\omega t+\pi/12), & \text{with } I_1 = 448.3\ \text{mA} \\ u_2(t) = U_2\, e^{-\beta t}\cos(\omega t-\pi/3), & \text{with } U_2 = 5.00\ \text{V} \end{cases}$$

where $\beta = 288.7\,\text{Np/s}$ and $\omega = 1077.4\,\text{rad/s}$.

The nature of the oscillatory regime is explained by the flowing back and forth of the energy between the capacitor and the transformer. At $t = 0$ the system's energy resides entirely in the transformer core (magnetic energy). After a few instants have elapsed, at $\omega t = 5\pi/12$, the energy stored in the transformer vanishes; most of the initial energy is now found to be stored in the capacitor (electric energy). During the transfer process some energy is transformed into heat due to Joule losses in the resistor (damping). The interchange of energy between the two reactive components of the circuit repeats itself periodically, lasting until no more electromagnetic energy exists. This idea is schematically represented in Figure 7.27.

A final remark: although the voltage u_2 at the transformer's secondary winding is not a state variable, you can check that $u_2(0^+) = u_2(0^-) = 2.5\,\text{V}$. Can you figure out why this must indeed happen in our circuit (Figure 7.26)?

(Hint: Analyze the continuity of the function di_1/dt.)

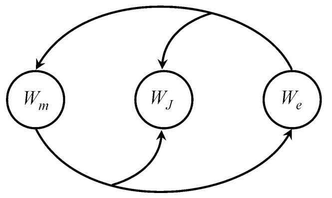

Figure 7.27 The oscillatory nature of the transient regime can be interpreted as the result of the periodic interchange of electromagnetic energy between the transformer and the capacitor, the resistor being responsible for the damping effect (Joule losses)

7.5 Proposed Homework Problems

Problem 7.5.1

Figure 7.28(a) shows the basic idea of a radio receiver tuning circuit, which essentially consists of two coupled inductors and a variable capacitor. The primary inductor current $i_a(t)$ is fed from a receiving antenna. The mutual inductance between inductors is L_M. The secondary inductor is characterized by its self-inductance L and by its internal resistance R.

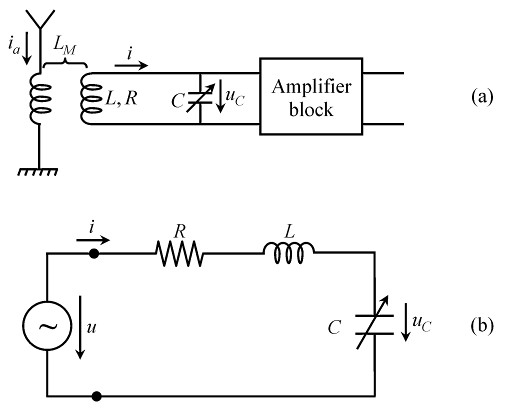

Figure 7.28 Radio receiver tuning circuit. (a) Schematic diagram. (b) Equivalent *RLC* series circuit, where the applied voltage is given by $u(t) = -L_M \; di_a/dt$

The antenna current includes contributions from distinct broadcasting stations, $i_a = \sum_k \; i_k$, but we are interested in tuning only to the station operating at $f_1 = 1\,\text{MHz}$ (medium-frequency band). The 1 MHz component of $i_a(t)$ is given by $i_1(t) = I_1 \cos(\omega_1 t)$, with $I_1 = 2.25\,\mu\text{A}$.

Assume that the amplification block is ideal ($\overline{Z}_{\text{amp}} \to \infty$).

Data: $L_M = 40\,\mu\text{H}, \;\; L = 84.4\,\mu\text{H}, \;\; R = 10\;\Omega$.

Q_1 Show that the *RLC* circuit in Figure 7.28(b) is equivalent to the tuning circuit in Figure 7.28(a). In particular, determine the equivalent generator voltage $u(t)$.

Q_2 For $f = f_1$, determine the tuning capacitance of the circuit which maximizes the amplitude of the voltage u_C.

Q_3 For $f = f_1$, determine the resonance capacitance of the circuit which maximizes the amplitude of the current i.

Q_4 Show that the situations in Q_2 and Q_3 converge to each other when $R \ll \omega L$.

Q_5 For the situation in Q_3, considering $i_a(t) = i_1(t) = I_1 \cos(\omega_1 t)$, determine the phasors, $\overline{U}$, $\overline{I}$ and $\overline{U}_C$.

Answers

Q_1

$$u(t) = -L_M \frac{di_a(t)}{dt}$$

Q_2

$$C_{tuning} = \frac{L}{R^2 + (\omega_1 L)^2} = 300.01\,\text{pF}$$

Q_3

$$C_{res} = \frac{1}{\omega_1^2 L} = 300.12\ \text{pF}$$

Q_4

$$C_{res} = \lim_{R \to 0} C_{tuning}$$

Q_5

$$\overline{U} = 0.566\, e^{-j\pi/2}\ \text{mV};\ \ \overline{I} = 56.6\, e^{-j\pi/2}\ \mu\text{A};\ \ \overline{U}_C = 30\, e^{j\pi}\ \text{mV}.$$

Problem 7.5.2

As shown in Figure 7.29, an $L \parallel C$ circuit is inserted between an AC voltage generator and a resistive load R. Data: $L = 10\,\text{mH},\ C = 100\,\text{pF}$.

Q_1 Determine analytically the transfer function $\overline{T}(\omega) = \overline{U}_2/\overline{U}_1$.

Q_2 Find the frequency f_0 for which $i = 0$, and therefore $T(\omega) = 0$.

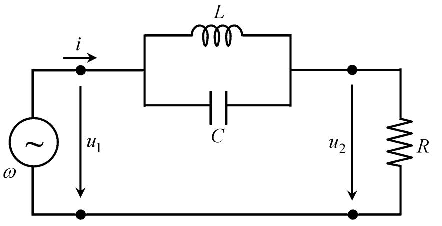

Figure 7.29 Notch filter made of an $L \parallel C$ circuit

Q_3 Draw a sketch of $T(\omega)$, for $\omega \in [0; \infty[$, showing that the circuit under analysis behaves as a notch filter (a notch filter passes all frequencies except those in a stopband centered at ω_0).

Answers

Q_1

$$\overline{T}(\omega) = \frac{1}{1 + j\dfrac{\omega L/R}{1 - \omega^2 LC}}$$

Q_2

$$\omega_0 = 1/\sqrt{LC} = 1\,\mathrm{Mrad/s};\;\; f_0 = \omega_0/(2\pi) = 159.15\,\mathrm{kHz}. \tag{7.92}$$

Q_3

$$T(0) = T(\infty) = 1;\;\; T(\omega_0) = 0. \quad \text{See sketch in Figure 7.30.} \tag{7.93}$$

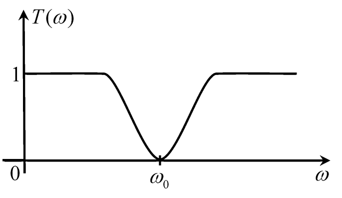

Figure 7.30 Magnitude plot of the filter transfer function, showing the center frequency $\omega_0 = 1/\sqrt{LC}$

Problem 7.5.3

Consider the circuit shown in Figure 7.31, which illustrates the basic principle of a spectrum analyzer. The input voltage signal is defined as

$$u(t) = \sum_{k=1}^{5} u_k(t) = \sum_{k=1}^{5} U_k \cos(k\omega t + \alpha_k)$$

where $\omega = 10\,\text{krad/s}$.

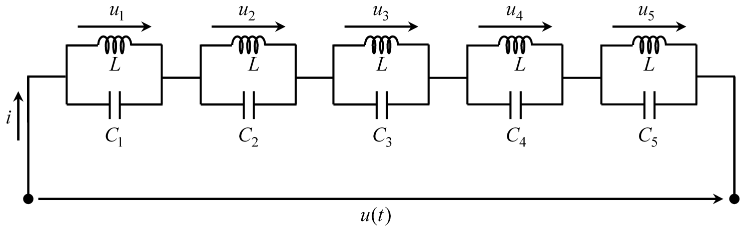

Figure 7.31 Basic principle of a spectrum analyzer

Q_1 Taking into account the results from Problem 7.5.2, show that the voltage components $u_1,\ u_2,\ u_3,\ u_4$ and u_5 appear separately across each $L \parallel C$ block, provided that each block respectively resonates at $\omega_1 = \omega,\ \omega_2 = 2\omega,\ \omega_3 = 3\omega,\ \omega_4 = 4\omega$ and $\omega_5 = 5\omega$.

Q_2 Assuming that the uncoupled inductors are equal, with $L = 10\,\text{mH}$, determine the required values for $C_1,\ C_2,\ C_3,\ C_4$ and C_5.

Answers

Q_1 By using the superposition principle (valid for linear circuits) each frequency component can be analyzed separately. For ω_k the resonant $L \parallel C_k$ block behaves as an open circuit ($i = 0$), but the other blocks, having a finite impedance, will show zero voltage across them. Consequently, the voltage across the resonant $L \parallel C_k$ block is $u_k(\omega_k)$.

Q_2 $C_k = 1/(\omega_k^2 L);\ C_1 = 1000\,\text{nF};\ C_2 = 250\,\text{nF};\ C_3 = 111.1\,\text{nF};\ C_4 = 62.5\,\text{nF};\ C_5 = 40\,\text{nF}.$

Problem 7.5.4

Consider the circuit depicted in Figure 7.32, where the ideal ammeters $A_R,\ A_C,\ A_L$ and A_0 read rms values. The applied voltage is $u(t) = \sqrt{2}\ U_{rms} \cos(\omega t)$.

Data: $U_{rms} = 5\,\text{V},\ \omega = 10\,\text{krad/s},\ R = 50\ \Omega$ and $L = 5\,\text{mH}$.

Q_1 Write the time-domain and phasor-domain equations that govern the steady-state harmonic regime of the circuit.

Q_2 Determine the capacitor's capacitance $C = C_{\text{res}}$ that brings the circuit to a resonant situation.

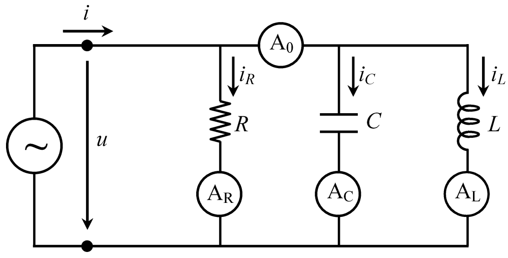

Figure 7.32 An *RLC* parallel circuit with ideal ammeters placed for measuring rms current intensities

Q_3 Assume $C = C_{\text{res}}$.

Determine numerically the phasor representation of $u,\ i_R,\ i_C,\ i_L$ and i.
Indicate the readings of all the ammeters included in the circuit.
Determine the active and reactive powers at the generator terminals. Check the results using the complex Poynting theorem.

Answers

Q_1 $i_R = u/R \rightarrow \overline{I}_R = \overline{U}/R;\ i_C = C\ du/dt \rightarrow \overline{I}_C = j\omega C\overline{U};\ u = L\ di_L/dt \rightarrow \overline{I}_L = \overline{U}/(j\omega L)$.

$$i = i_R + i_C + i_L \rightarrow \overline{I} = \overline{I}_R + \overline{I}_C + \overline{I}_L = \overline{U}\left(\frac{1}{R} + j\omega C + \frac{1}{j\omega L}\right)$$

Q_2 $C = C_{\text{res}} = 2\,\mu\text{F}$.

Q_3 $\overline{U} = \sqrt{2}\,5\,\text{V};\ \overline{I}_R = \sqrt{2}\,0.1\,\text{A};\ \overline{I}_L = \sqrt{2}\,0.1\,e^{-j\pi/2}\,\text{A};\ \overline{I}_C = \sqrt{2}\,0.1\,e^{+j\pi/2}\,\text{A};\ \overline{I} = \sqrt{2}\,0.1\,\text{A}$

$\text{A}_\text{R} = \text{A}_\text{C} = \text{A}_\text{L} = 100\,\text{mA};\ \text{A}_0 \equiv 0$ (note that $i_C(t) + i_L(t) = 0$).

$P = 5\,\text{W};\ P_Q = 0;\ P_J = RI^2_{R_{rms}} = 5\,\text{W};\ (W_m)_{av} = (W_e)_{av} = 25\,\mu\text{J}$.

Problem 7.5.5

A residential four-wire installation is fed by a three-phase voltage generator (see Problem 5.13.6). As shown in Figure 7.33, both the generator and the load are star connected. The generator phase-to-neutral voltages are defined as $u_k(t) = \sqrt{2}\ U_{rms}\ \cos(\omega t - \alpha_k)$, where $U_{rms} = 230\,\text{V}$, and $\alpha_k = (k-1)2\pi/3$, for $k = 1, 2, 3$. The frequency is $f = 50\,\text{Hz}$.

Q_1 Determine the phase-to-phase generator voltages.

Q_2 Assume an unbalanced load whose impedances are $\overline{Z}_1 = 20\ \Omega,\ \overline{Z}_2 = 30\ e^{j\pi/3}\ \Omega$ and $\overline{Z}_3 = 40\ e^{j\pi/6}\ \Omega$. Determine the phasors characterizing the installation currents, including the neutral, that is $\overline{I}_1, \overline{I}_2, \overline{I}_3$ and $\overline{I}_N$.

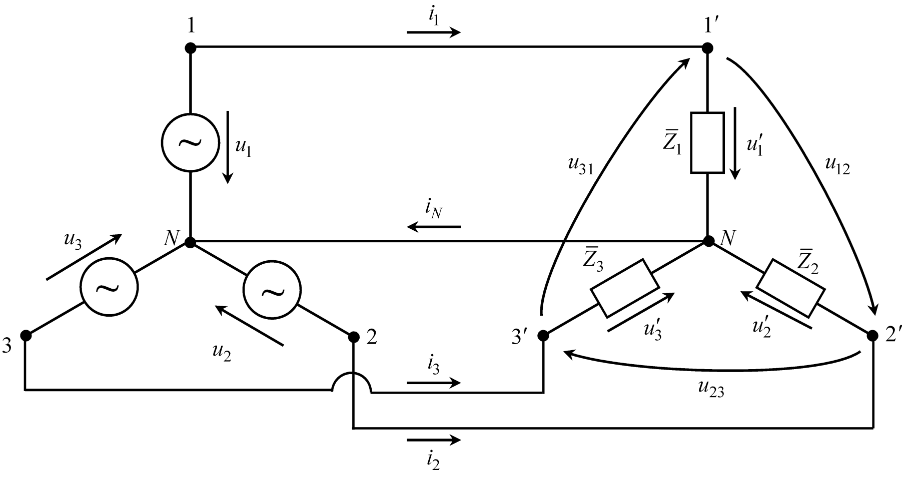

Figure 7.33 Four-wire, three-phase installation. The generator and the unbalanced load are star connected

Q_3 Keep considering the unbalanced load described in Q_2 but suppose that, by accident, the neutral wire is interrupted, $\overline{I}_N = 0$. Find the new voltages $\overline{U}_1'$, $\overline{U}_2'$ and $\overline{U}_3'$ applied to the load impedances. Show that the load voltage u_2' exceeds its nominal value (230 V) by about 30 %.

Q_4 Assume that the protective circuit breaker associated with load $\overline{Z}_2$ opens, a situation that from the generator's point of view is equivalent to making $\overline{Z}_2 = \infty$. Recompute the new voltage phasors $\overline{U}_1'$, $\overline{U}_2'$ and $\overline{U}_3'$. Check that a subsequent new overvoltage appears across $\overline{Z}_3$.

Answers

Q_1 $\overline{U}_{12} = \overline{U}_1 - \overline{U}_2 = \sqrt{2}\left(\sqrt{3}\ 230\right)\ e^{j\pi/6}\ \text{V} \ \rightarrow\ u_{12}(t) = \sqrt{2}\ 398.4\ \cos(\omega t + \pi/6)\ \text{V}.$

$\overline{U}_{23} = \overline{U}_2 - \overline{U}_3 = \sqrt{2}\left(\sqrt{3}\ 230\right)\ e^{-j\pi/2}\ \text{V} \ \rightarrow\ u_{23}(t) = \sqrt{2}\ 398.4\ \cos(\omega t - \pi/2)\ \text{V}.$

$\overline{U}_{31} = \overline{U}_3 - \overline{U}_1 = \sqrt{2}\left(\sqrt{3}\ 230\right)\ e^{j5\pi/6}\ \text{V} \ \rightarrow\ u_{31}(t) = \sqrt{2}\ 398.4\ \cos(\omega t + 5\pi/6)\ \text{V}.$

Q_2 With the neutral wire present we have $\overline{U}_1' = \overline{U}_1,\ \overline{U}_2' = \overline{U}_2,\ \overline{U}_3' = \overline{U}_3$.

$$\overline{I}_k = \frac{\overline{U}_k'}{\overline{Z}_k} = \begin{cases} \overline{I}_1 = \sqrt{2}\ 11.50\ \text{A} \\ \overline{I}_2 = \sqrt{2}\ 7.67\, e^{j180^o}\,\text{A}\,, \\ \overline{I}_3 = \sqrt{2}\ 5.75\, e^{j90^o}\,\text{A} \end{cases} \qquad \overline{I}_N = \overline{I}_1 + \overline{I}_2 + \overline{I}_3 = \sqrt{2}\ 6.91\, e^{j56.3^o}\,\text{A}$$

Q_3

$$\begin{bmatrix} \overline{Z}_1 & -\overline{Z}_2 & 0 \\ 0 & \overline{Z}_2 & -\overline{Z}_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \end{bmatrix} = \begin{bmatrix} \overline{U}_{12} \\ \overline{U}_{23} \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \end{bmatrix} = \begin{bmatrix} \overline{Z}_1 & -\overline{Z}_2 & 0 \\ 0 & \overline{Z}_2 & -\overline{Z}_3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \overline{U}_{12} \\ \overline{U}_{23} \\ 0 \end{bmatrix}$$

$$\overline{U}_1' = \overline{Z}_1 \overline{I}_1 = \sqrt{2}\ 230.3\, e^{-j17.7^o}\ \text{V};\ \overline{U}_2' = \overline{Z}_2 \overline{I}_2 = \sqrt{2}\ 297.1\, e^{-j115.0^o}\ \text{V};$$
$$\overline{U}_3' = \overline{Z}_3 \overline{I}_3 = \sqrt{2}\ 180.1\, e^{j134.2^o}\ \text{V}$$

Q_4

$$\overline{U}_1' = \sqrt{2}\ 136.9\, e^{-j50.1^\circ}\ \text{V};\ \overline{U}_2' = \sqrt{2}\ 398.4\, e^{-j130.2^\circ}\ \text{V};\ \overline{U}_3' = \sqrt{2}\ 273.9\ e^{j159.9^\circ}\ \text{V}.$$

Problem 7.5.6

A three-phase generator feeds a perfectly balanced load $\overline{Z}_1 = \overline{Z}_2 = \overline{Z}_3 = \overline{Z} = Z\ e^{j\varphi}$. The load is star connected – see Figure 7.34(a).

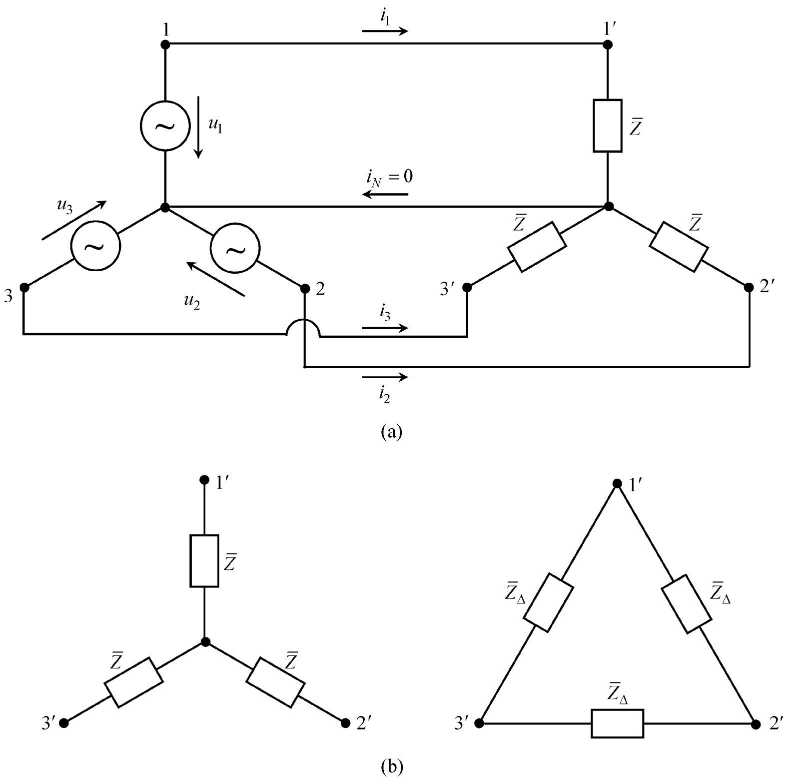

Figure 7.34 (a) Four-wire, three-phase installation with a star-connected balanced load (the neutral current is zero). (b) Equivalence between a star-connected balanced load and a delta-connected balanced load, $\overline{Z}_\Delta = 3\overline{Z}$

Q_1 Check that the neutral wire is not needed, by showing that $\overline{I}_N = 0$.

Q_2 Show that the instantaneous three-phase power $p(t) = p_1(t) + p_2(t) + p_3(t)$ is time invariant.

Q_3 The star load connection can be replaced by an equivalent delta connection as shown in Figure 7.34(b). Show that the relationship between $\overline{Z}_\Delta$ and $\overline{Z}$ is given by $\overline{Z}_\Delta = 3\overline{Z}$.

Answers

Q_1

$$\overline{I}_N = \frac{\overline{U}_1 + \overline{U}_2 + \overline{U}_3}{\overline{Z}} = 0$$

Q_2

$$p(t) = \sum_{k=1}^{3} u_k(t) i_k(t) = 2\frac{U_{rms}^2}{Z} \sum_{k=1}^{3} \cos(\omega t - \alpha_k)\cos(\omega t - \alpha_k - \varphi)$$

$$p(t) = \frac{U_{rms}^2}{Z}\left(\sum_{k=1}^{3}\cos\varphi + \underbrace{\sum_{k=1}^{3}\cos(2\omega t - 2\alpha_k - \varphi)}_{0}\right) = 3\frac{U_{rms}^2}{Z}\cos\varphi$$

Q_3 At node $1'$:

$$\begin{cases} \text{Star connection} : \overline{I}_1 = \dfrac{\overline{U}_1}{\overline{Z}} \\ \text{Delta connection} : \overline{I}_1 = \dfrac{\overline{U}_{12} - \overline{U}_{31}}{\overline{Z}_\Delta} \end{cases} \rightarrow \quad \overline{Z}_\Delta = \overline{Z} \times \frac{\overline{U}_{12} - \overline{U}_{31}}{\overline{U}_1} = 3\overline{Z}$$

At the remaining nodes the same result is obtained, $\overline{Z}_\Delta = 3\overline{Z}$.

Problem 7.5.7

The transformer shown in Figure 7.35 is subjected to a steady-state harmonic regime with $f = 50\,\text{Hz}$.

In order to determine its constitutive parameters two experiments were conducted. Firstly, in Figure 7.36(a), the secondary winding was left open ($i_2 = 0$). Secondly, in Figure 7.36(b), the primary and secondary windings were connected in series ($i_1 = i_2$).

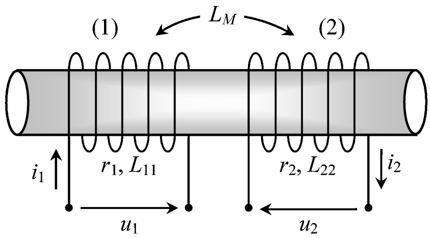

Figure 7.35 Two-windings transformer

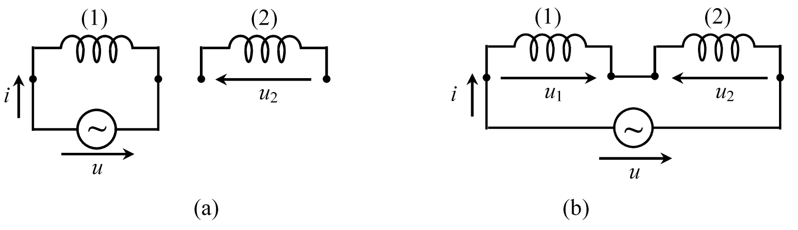

Figure 7.36 Two experiments for the characterization of the transformer parameters. (a) Secondary open. (b) Primary and secondary windings connected in series

In both experiments, rms readings of the generator voltage, generator current and secondary voltage were recorded; the generator active power P was also measured in both situations:

first experiment: $U_{rms} = 230\,\mathrm{V},\ I_{rms} = 6.976\,\mathrm{A},\ U_{2rms} = 109.6\,\mathrm{V},\ P = 486.7\,\mathrm{W}$;

second experiment: $U_{rms} = 230\,\mathrm{V},\ I_{rms} = 2.838\,\mathrm{A},\ P = 161.1\,\mathrm{W}$.

Q_1 Write the phasor-domain equations concerning the first experiment. Determine the parameters $r_1,\ L_{11}$ and L_M.

Q_2 Write the phasor-domain equations concerning the second experiment. Determine the parameters r_2 and L_{22}.

Q_3 For the second experiment, obtain $U_{2_{rms}}$.

Answers

Q_1 $$\overline{U} = \overline{U}_1;\ \overline{I} = \overline{I}_1;\ \overline{I}_2 = 0\ \rightarrow\ \overline{U} = (r_1 + j\omega L_{11})\overline{I};\ -\overline{U}_2 = j\omega L_M \overline{I}.$$

$$P = r_1 I_{rms}^2 \rightarrow r_1 = \frac{P}{I_{rms}^2} = 10\ \Omega;\ \frac{U_{rms}}{I_{rms}} = \sqrt{r_1^2 + (\omega L_{11})^2} \rightarrow L_{11} = 100\,\mathrm{mH}$$

$$U_{2_{rms}} = \omega L_M I_{rms}\ \rightarrow\ L_M = U_{2_{rms}}/(\omega I_{rms}) = 50\,\mathrm{mH}.$$

Q$_2$ $$\overline{U} = \overline{U}_1 - \overline{U}_2;\ \overline{I} = \overline{I}_1 = \overline{I}_2 \ \rightarrow\ \overline{U} = [(r_1 + r_2) + j\omega(L_{11} + 2L_M + L_{22})]\,\overline{I}.$$

$$P = (r_1 + r_2) I_{rms}^2 \ \rightarrow\ r_2 = \frac{P}{I_{rms}^2} - r_1 = 10\ \Omega$$

$$\frac{U_{rms}}{I_{rms}} = \sqrt{(r_1 + r_2)^2 + \omega^2 (L_{11} + 2L_M + L_{22})^2} \ \rightarrow\ L_{22} = 50\,\text{mH}$$

Q$_3$ $$-\overline{U}_2 = (r_2 + j\omega L_{22})\overline{I} + j\omega L_M \overline{I} \ \rightarrow\ U_{2_{rms}} = |r_2 + j\omega(L_M + L_{22})|\, I_{rms} = 93.6\,\text{V}.$$

Problem 7.5.8

The transformer shown in Figure 7.37(a) is driven by a 50 Hz voltage u_1. The load connected to the secondary winding is a capacitor of capacitance $C_2 = 55.36\,\mu$F, across which a voltage u_2, given by $u_2(t) = \sqrt{2}\ 115\ \cos(\omega t)$ V, is found to exist.

Figure 7.37(b) shows an equivalent circuit of the real transformer, where the following parameters are defined $R = 5\ \Omega,\ L = 732\,$mH and $C_2' = 13.84\,\mu$F.

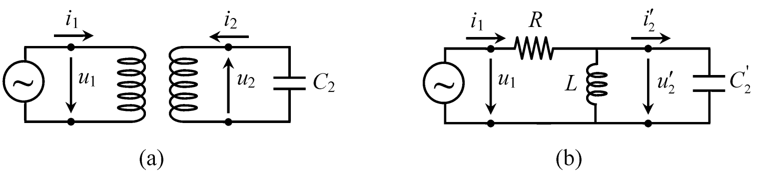

Figure 7.37 A capacitively loaded transformer. (a) Schematic diagram. (b) Equivalent circuit

Q$_1$ Obtain the value of the turns ratio υ of the ideal transformer associated with the equivalent circuit. Next, determine the constitutive parameters of the real transformer, $r_1,\ r_2,\ L_{11},\ L_M$ and L_{22}.

Q$_2$ Determine the primary and secondary voltages and currents in the phasor domain. In addition, determine the auxiliary quantities $\overline{U}_2'$ and $\overline{I}_2'$.

Q$_3$ Compute the active and reactive powers brought into play by the generator. Verify the results obtained using the complex Poynting theorem.

Answers

Q$_1$ $C_2/C_2' = \upsilon^2 \rightarrow \upsilon = 2.$

$$r_1 = 5\ \Omega,\ r_2 = 0;\ L_{11} = 732\,\text{mH};\ L_M = 366\,\text{mH};\ L_{22} = 183\,\text{mH}.$$

Q$_2$ $\overline{U}_2 = \sqrt{2}\ 115\,\text{V};\ \overline{U}_2' = \sqrt{2}\ 230\,e^{j\pi}\ \text{V};\ \overline{I}_2 = \sqrt{2}\ 2\,e^{j\pi/2}\ \text{A};\ \overline{I}_2' = \sqrt{2}\ e^{-j\pi/2}\ \text{A}.$

$$\overline{I}_1 = 0;\ \overline{U}_1 = \sqrt{2}\ 230\ e^{j\pi}\ \text{V}.$$

Q_3 $P = P_Q = 0$.

$$P = P_J = 0;\ P_Q = 2\omega[(W_m)_{av} - (W_e)_{av}];\ (W_m)_{av} = (W_e)_{av} = 366\,\text{mJ}.$$

Problem 7.5.9

The circuit in Figure 7.38 refers to the transient phenomena of discharging a capacitor over an RL circuit. The switch S is closed at $t = 0$ when $u(0) = U_0 = 86.6\,\text{V}$.

Data: $R = 10\ \Omega,\ L = 1\,\text{mH},\ C = 10\,\mu\text{F}$.

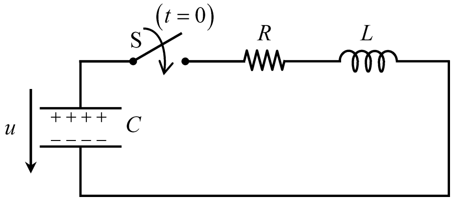

Figure 7.38 A charged capacitor is discharged over an *RL* circuit

Q_1 Determine the differential equation that governs $u(t)$ for $t > 0$.

Q_2 Making use of the initial conditions pertaining to this problem, find $u(0^+)$ and

$$\left(\frac{du}{dt}\right)_{t=0^+}$$

Q_3 Show that the transient regime solution for $u(t)$ is a damped periodic oscillation, that is $u(t) = Ue^{-\beta t}\ \cos(\omega t + \theta)$. Obtain $\beta,\ \omega,\ \theta$ and U.

Answers

Q_1

$$\frac{d^2u}{dt^2} + 2\beta\frac{du}{dt} + \omega_0^2 u = 0;\ \beta = \frac{R}{2L};\ \omega_0 = \frac{1}{\sqrt{LC}}$$

Q_2

$$u(0^+) = U_0 = 86.6\,\text{V};\ i(0^-) = i(0^+) = 0 \rightarrow \left(\frac{du}{dt}\right)_{t=0^+} = 0$$

Q_3 $\beta = 5000\,\text{Np/s};\ \omega_0 = 10\,\text{krad/s}.\ \beta < \omega_0 \rightarrow$ damped periodic oscillation.
$\omega = 8.66\,\text{krad/s};\ \theta = -\pi/6;\ U = 100\,\text{V}.$

Problem 7.5.10

Consider the circuit represented in Figure 7.39, whose parameters R, L and C are known. The DC generator voltage is $U_G = 12\,\text{V}$. The circuit, which is operating for a long time, is interrupted at $t = 0$.

Data: $R = 120\ \Omega,\ L = 1\,\text{H},\ C = 10\,\text{nF}$.

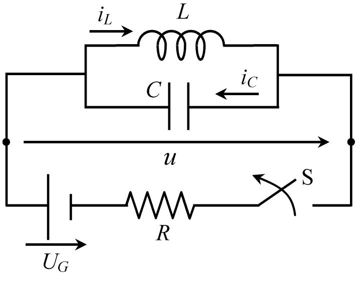

Figure 7.39 The opening of the switch S gives rise to a high-voltage harmonic regime

Q_1 Determine the differential equation that governs $u(t)$ for $t > 0$.

Q_2 Making use of the initial conditions pertaining to this problem, find $u(0^+)$ and

$$\left(\frac{du}{dt}\right)_{t=0^+}$$

Q_3 Show that the transient regime solution for $u(t)$ is a purely periodic oscillation, that is $u(t) = U\ \cos(\omega_0 t + \theta)$. Obtain $\omega_0,\ \theta$ and U.

Q_4 Check the result obtained for U using energy balance considerations.

Answers

Q_1

$$\frac{d^2u}{dt^2} + \frac{1}{LC}u = 0; \quad (\beta = 0)$$

Q_2

$$u(0^+) = u(0^-) = 0;\ i_C(0^+) = i_L(0^+) = i_L(0^-) = 0.1\text{A} \rightarrow \left(\frac{du}{dt}\right)_{t=0^+} = -10^7\,\text{V/s}$$

Q_3

$$u(t) = U\ \cos(\omega_0 t + \theta);\ \omega_0 = 1/\sqrt{LC} = 10\,\text{krad/s};\ \theta = \pi/2;\ U = 1\,\text{kV}.$$

Q_4

$$(W_e)_{\max} = \tfrac{1}{2} C U^2 = (W_m)_{\max} = 5\,\text{mJ} \rightarrow U = \sqrt{\frac{2\,(W_m)_{\max}}{C}} = 1\,\text{kV}$$

Problem 7.5.11

Take the transformer in Figure 7.40(a) whose constitutive parameters are known:

$$r_1 = r_2 = 25\ \Omega,\ L_{11} = L_{22} = 300\,\text{mH and } L_M = 50\,\text{mH}$$

Now consider the transformer connections shown in Figure 7.40(b), where $U = 100\,\text{V}$ and $C = 50\,\mu\text{F}$. The capacitor is initially discharged. The switch S closes at $t = 0$.

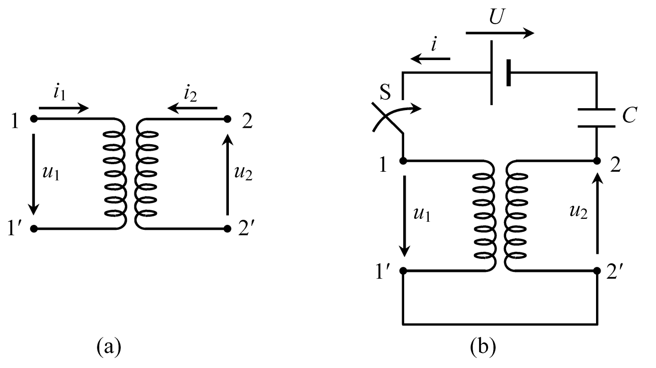

Figure 7.40 (a) Transformer characterized by $r_1,\ r_2,\ L_{11},\ L_{22}$ and L_M. (b) Switching on a DC voltage generator over a circuit containing a capacitor and a transformer whose windings are connected in series

Q_1 Bearing in mind the time-domain equations of the transformer, and taking into account that $i_1 = -i_2 = i$, determine the differential equation that governs $i(t)$ for $t > 0$.

Q_2 Define the initial conditions of the problem.

Q_3 Show that the transient regime solution for $i(t)$ is a damped periodic oscillation, that is $i(t) = I\ e^{-\beta t}\ \cos(\omega t - \theta)$. Obtain $\beta,\ \omega,\ \theta$ and I.

Answers

Q_1

$$U = Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt \rightarrow L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$

where $R = r_1 + r_2 = 50\ \Omega$ and $L = L_{11} + L_{22} - 2L_M = 0.5\,\text{H}$.

Q_2

$$i(0) = 0;\;\; u_C(0) = 0 \rightarrow \left(\frac{di}{dt}\right)_{t=0^+} = \frac{U}{L} = 200\,\mathrm{A/s}$$

Q_3

$$\beta = \frac{R}{2L} = 50\ \mathrm{Np/s};\;\; \omega_0 = \frac{1}{\sqrt{LC}} = 200\,\mathrm{rad/s}$$

$\beta < \omega_0 \rightarrow$ damped periodic oscillation.
$\omega = 193.7\,\mathrm{rad/s};\;\; \theta = \pi/2;\;\; I = 1.033\,\mathrm{A}.$