Appendix 2

Formulas of the Quality Factor of a Rectangular Cavity

A2.1. Quality factor of the $TM_{m\,n\,p}$ mode

We will reproduce the formula established by Liu of the $TM_{m\,n\,p}$ mode [LIU 83]:

$$Q_{m\,n\,p} = \frac{Z_w\, abd\, k_{xy}^2\, k_{mnp}}{4R_s\left[b(a+d)\,k_x^2 + a(b+d)\,k_y^2\right]} \quad \text{[A2.1]}$$

The notation conventions are recalled below:

$$k_x = m\frac{\pi}{a} \quad k_y = n\frac{\pi}{b} \quad k_z = p\frac{\pi}{d} \quad k_{x\,y} = \sqrt{k_x^2 + k_y^2} \quad \text{[A2.2]}$$

$$k_{mnp} = \sqrt{k_x^2 + k_y^2 + k_z^2} \quad \text{[A2.3]}$$

We find in [A2.1] the impedance of the Z_w plane wave and the surface resistance of the walls of the R_s cavity:

$$Z_w = \sqrt{\frac{\mu_0}{\varepsilon_0}} \quad \text{and} \quad R_s = \frac{1}{\sigma\,\delta} \quad \text{with} \quad \delta = \sqrt{\frac{2}{\omega\,\mu_0\mu_r\sigma}} \quad \text{[A2.4]}$$

a, b and d are the dimensions of the cavity.

A2.2. Calculation of the average $\tilde{Q}$ quality factor

We will not recall the whole development of the Liu calculation, which is reproduced in detail in [LIU 83]. This appendix will limit itself to the framework of the calculation.

The computation is established for an oversized cavity, in the bandwidth occupied by a mode interfering with other modes, whose population will only increase with the frequency of the RF emission. According to the description proposed in section 2.3.2, we consider that the $\Delta f_{m\,n\,p}$ bandwidth belongs with the mode tuned on the $f_{m\,n\,p}$ frequency. Knowing that ΔN modes enter this band, the average $< Q >$ quality factor results from the estimate produced below:

$$\frac{1}{\tilde{Q}} = \frac{1}{\Delta N} \sum_{i} \frac{1}{Q_i} \qquad \text{[A2.5]}$$

The formulation of the integral found in this formula is justified by the composition of the losses analyzed in section 4.2.3 of Chapter 4. A writing convention is added to these physical considerations, where the index i includes the m, n, p triplet, as well as the nature of the considered modes, depending on whether there are TM or TE kinds.

In highly oversized operating, the modal density is very high. Consequently, we can merge the integer $m\ n\ p$ indices with continuous variables. Under this assumption, we can replace [A2.5] with the calculation of the integral expressed in [A2.6]:

$$\frac{1}{\tilde{Q}} = \frac{1}{\Delta N} \iiint_{\mathsf{D}} \frac{1}{Q_i}\, dm\ dn\ dp \qquad \text{[A2.6]}$$

In order to restrict the integration domain D, we go to the domain of the wave numbers, which are briefly described in section 2.3.8. The continuous variables replacing the $m\ n\ p$ indices will be the projections of the k_{mnp} wave number, which can currently be merged with the k_0 wave number. The latter depends on the ω_0 excitation angular frequency of the chamber, i.e.:

$$k_{mnp} \cong k_0 = \frac{\omega_0}{c} \qquad \text{[A2.7]}$$

By using spherical coordinates, the projections of the wave number take the respective expressions:

$$\frac{\pi}{a}m = k_0 \sin\theta\cos\varphi, \quad \frac{\pi}{b}n = k_0 \sin\theta\sin\varphi, \quad \frac{\pi}{d}p = k_0 \cos\theta \qquad \text{[A2.8]}$$

The variation domains of the k wave number and of the θ and φ variables is defined as follows:

$$k \in \left[k_0 - \Delta k/2 \quad k_0 + \Delta k/2\right], \quad \theta \in \left[0 \quad \pi\right], \quad \varphi \in \left[0 \quad 2\pi\right] \qquad \text{[A2.9]}$$

The D domain is thus considered as the gap contained between two concentric spheres distant from Δk. Let us specify that the symmetry conditions imposed on the field distributions enable us to reduce the integration domain to one eighth of a sphere.

Under these conditions, calculation of integral [A2.6] takes the development:

$$\frac{1}{\tilde{Q}_i} = \int_{k_0-\Delta k/2}^{k_0+\Delta k/2} \int_0^{\pi} \int_0^{2\pi} \frac{1}{Q_i} k^2 \sin\theta \; dk\, d\theta\, d\varphi \qquad \text{[A2.10]}$$

Under the spherical coordinates and use of formula [A2.1] attached to the $\mathrm{TM}_{m\,n\,p}$ mode, we obtain a detailed form of the function ($1 / Q_i$) appearing in the previous integral i.e.

$$\frac{1}{Q_{\mathrm{TM}_{mnp}}} = \frac{4R_s}{Z_w}\frac{1}{kV}\left[ab + d(b\cos^2\varphi + a\sin^2\varphi)\right] \qquad \text{[A2.11]}$$

The V parameter then represents the volume of the chamber. After solving integral [A2.11], we reach the expression of the average quality factor for the considered mode:

$$\frac{1}{\tilde{Q}_{\mathrm{TM}_{mnp}}} = \frac{2R_s\, k_0}{Z_w\, \pi^2}\left[ab + \frac{1}{2}d\,(a+b)\right]\frac{\Delta k}{\Delta N} \qquad \text{[A2.12]}$$

According to the use of the Weyl formula approximation [2.62] found in section 2.3.5, we can deduce the ratio $\Delta k / \Delta N$ appearing in the previous expression, i.e.:

$$\frac{\Delta k}{\Delta N} \cong \frac{\pi^2}{V}\frac{1}{k_0^2} \qquad \text{[A2.13]}$$

After carrying out this calculation on the five classes of modes of the rectangular cavity, Liu reaches a compact formula:

$$\frac{1}{\tilde{Q}} = \sum_{i=1}^{5} \frac{1}{\tilde{Q}_i} \quad \rightarrow \quad \tilde{Q} = \frac{V}{S\,\delta}\,\frac{3}{2}\,\frac{1}{1+\dfrac{3\pi}{8k_0}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{d}\right)} \tag{A2.14}$$

This formula successively includes the volume of the V chamber, its a, b, d, dimensions, the surface S of the metal walls, and the skin depth δ of the high frequency currents.

If we admit that the second term of the denominator remains much lower than one, we obtain the simplified expression of the average quality factor:

$$\frac{3\pi}{8k_0}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) << 1 \quad \rightarrow \quad \tilde{Q} \cong \frac{3}{2}\,\frac{V}{S\,\delta} \tag{A2.15}$$

A2.3. Bibliography

[LIU 83] LIU B.H., CHANG D.C., MA M.T., Eigen modes and the composite quality factor of a reverberation chamber, NBS Technical Notes 1066, August 1983.