Appendix 4

Calculation of the Variances of υ_φ, υ_η, υ_θ

A4.1. Variance of the υ_φ and υ_η variables

From relationship [3.50] in section 3.3.2 of Chapter 3 we know that the azimuth angle φ_i of the waves with random trials is linked to the υ_φ variable by the following expression:

$$\upsilon_\varphi = \sin \varphi_i \qquad \text{[A4.1]}$$

The computation of the pdf established in [3.52] enables us to formulate the sought after variance, i.e.:

$$\sigma^2_{\upsilon_\varphi} = \mathrm{E}\left[\upsilon^2_\varphi\right] = \int_{\mathsf{D}} \upsilon^2_\varphi \, \mathrm{p}_\varphi(\upsilon_\varphi)\, d\upsilon_\varphi \qquad \text{[A4.2]}$$

is the equation in which D is the variation domain of υ_φ, i.e. the interval [*-1 +1*]. Integral [A4.2] can then take the detailed form below:

$$\sigma^2_{\upsilon_\varphi} = \frac{1}{2\pi}\int_{-1}^{+1} \frac{\upsilon^2_\varphi}{\sqrt{1-\upsilon^2_\varphi}}\, d\upsilon_\varphi \qquad \text{[A4.3]}$$

By using expression [A4.1], we carry out the change of variable leading to integral [A4.4]:

$$\sigma_{\upsilon_\varphi}^2 = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 \varphi_i \, d\varphi_i \qquad [A4.4]$$

Analytical solution of this integral allocate the value ½ to the variance:

$$\sigma_{\upsilon_\varphi}^2 = \frac{1}{2} \qquad [A4.5]$$

Regarding υ_η, a similar calculation leads to the determination of the integral [A4.6]:

$$\sigma_{\upsilon_\eta}^2 = -\frac{1}{2\pi}\int_{-1}^{+1} \frac{\upsilon_\eta^2}{\sqrt{1-\upsilon_\eta^2}} d\upsilon_\eta = \frac{1}{2} \qquad [A4.6]$$

A4.2. Variance of the υ_θ variable

We recall that the definition of the υ_θ variable as found in equation [3.50]:

$$\upsilon_\theta = -\sin\theta_i \qquad [A4.7]$$

Knowing that θ_i covers the [0 $+\pi$] interval, amounts to saying that we can divide the domain to conform to the partition suggested in [A4.8]:

$$\theta_i \in [0 + \pi/2] \Rightarrow \upsilon_\theta \in [-1 \quad 0] \quad \text{and} \quad \pi - \theta_i \in [0 + \pi/2] \Rightarrow \upsilon_\theta \in [-1 \quad 0] \qquad [A4.8]$$

The calculation of the moment of the square of υ_θ is consequently similar to the integral of υ_θ on the two joined intervals [-1 0], i.e.:

$$\sigma_{\upsilon_\theta}^2 = \mathrm{E}\left[\upsilon_\theta^2\right] = 2\int_0^{-1} (\upsilon_\theta)^2 \, \mathrm{p}_\theta(\upsilon_\theta) \, d\upsilon_\theta \qquad [A4.9]$$

After insertion of the pdf found in [3.54], the variance can be calculated by the integral below:

$$\sigma_{\upsilon_\theta}^2 = \int_0^{-1} \frac{\upsilon_\theta^3}{\sqrt{1-\upsilon_\theta^2}} d\upsilon_\theta \qquad [A4.10]$$

Solving this integral analytically leads to the following results:

$$\int_0^{-1} \frac{\upsilon_\theta^3}{\sqrt{1-\upsilon_\theta^2}} d\upsilon_\theta = 2\int_0^{-1} \upsilon_\theta \left(1-\upsilon_\theta^2\right)^{\frac{1}{2}} d\upsilon_\theta = \left[-\frac{2}{3}\left(1-\upsilon_\theta^2\right)^{\frac{3}{2}}\right]_0^{-1} = \frac{2}{3} \qquad [A4.11]$$

Coming back to equation [A4.10] finally gives the value of the variance:

$$\sigma_{\upsilon_\theta}^2 = \frac{2}{3} \qquad [A4.12]$$