

Chapter 1

Scalar Hyperbolic Conservation Laws in One Dimension of Space

1.1. Definitions

1.1.1. *Hyperbolic scalar conservation laws*

A one-dimensional hyperbolic scalar conservation law is a Partial Differential Equation (PDE) that can be written in the form:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S \quad [1.1]$$

where t and x are respectively the time- and space-coordinates, U is the so-called conserved variable, F is the flux and S is the source term. Equation [1.1] is said to be the conservation, or divergent, form of the conservation law. The following definitions are used:

- the flux F is the amount of U that passes at the abscissa x per unit time due to the fact that U (also called the transported variable) is being displaced;
- the source term S is the amount of U that appears per unit time and per unit volume, irrespective of the amount transported via the flux F . If U represents the concentration in a given chemical substance, the source term may express degradation phenomena, or radioactive decay. S is positive when the conserved variable appears in the domain, negative if U disappears from the domain;

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– the conservation law is said to be scalar because it deals with only one dependent variable. When several equations in form [1.1] are satisfied simultaneously, the term “system of conservation laws” is used. Systems of conservation laws are dealt with in Chapter 2.

Only hyperbolic conservation laws are dealt with in what follows. The conservation law is said to be hyperbolic if the flux F is a function of U (and none of its derivatives) and, possibly, of x and t . Such a dependence is expressed in the form:

$$\left. \begin{aligned} F &= F(U, x, t) \\ S &= S(U, x, t) \end{aligned} \right\} \quad [1.2]$$

The function $F(U, x, t)$ is called the “flux function”.

NOTE.– The expression $F(U, x, t)$ in equation [1.2] indicates that F depends on U at the abscissa x at the time t and does not depend on such quantities as derivatives of U with respect to time or space. For instance, the following expression:

$$F = aU \quad [1.3]$$

is a permissible expression [1.2] for F , while the following, diffusion flux:

$$F = -D \frac{\partial U}{\partial x} \quad [1.4]$$

where D is the diffusion coefficient, does not yield a hyperbolic conservation law because the flux F is a function of the first-order derivative of U with respect to space.

In the case of a zero source term, equation [1.1] becomes

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad [1.5]$$

In such a case (see section 1.1.2), U is neither created nor destroyed over the domain. The total amount of U over the domain varies only due to the difference between the incoming and outgoing fluxes at the boundaries of the domain.

Depending on the expression of the flux function, the conservation law is said to be convex, concave or non-convex (Figure 1.1):

– the law is convex when the second-order derivative $\partial^2 F / \partial U^2$ of the flux function with respect to U is positive for all U ;

- the law is concave when the second-order derivative $\partial^2 F / \partial U^2$ of the flux function with respect to U is negative for all U ;
- the law is said to be non-convex when the sign of the second-order derivative $\partial^2 F / \partial U^2$ of the flux function with respect to U changes with U .

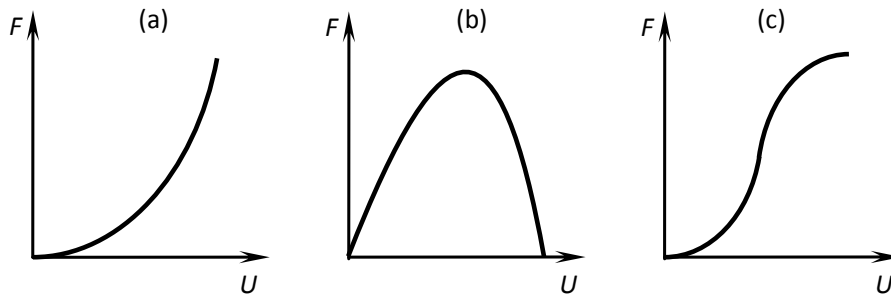


Figure 1.1. Typical examples of flux functions: convex (a), concave (b), non-convex (c)

1.1.2. Derivation from general conservation principles

The conservation form [1.1] is derived from a balance over a control volume of unit section defined between x_0 and $x_0 + \delta x$ (Figure 1.2). The balance is carried out over the control volume between two times t_0 and $t_0 + \delta t$. The variation in the total amount of U contained in the control volume is then related to the derivatives $\partial U / \partial t$ and $\partial F / \partial x$ in the limit of vanishing δt and δx .

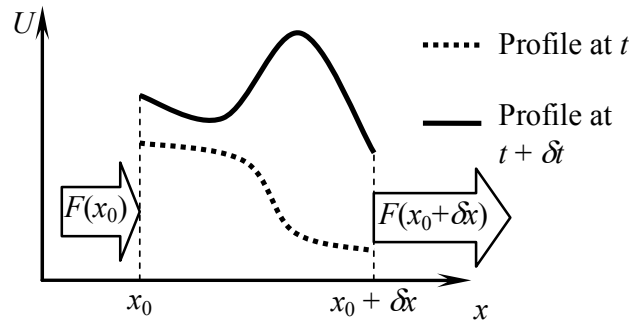


Figure 1.2. Definition sketch for the balance over a control volume

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The total amount $M(t_0)$ of U contained in the control volume at $t = t_0$ is defined as:

$$M(t_0) = \int_{x_0}^{x_0 + \delta x} U(x, t_0) dx \quad [1.6]$$

At $t = t_0 + \delta t$, the total amount of U contained in the control volume is:

$$M(t_0 + \delta t) = \int_{x_0}^{x_0 + \delta x} U(x, t_0 + \delta t) dx \quad [1.7]$$

The variation δS in the amount of U induced by the source term S over the domain between t_0 and $t_0 + \delta t$ is given by:

$$\delta S = \int_{t_0}^{t_0 + \delta t} \int_{x_0}^{x_0 + \delta x} S(U, x, t) dx dt \quad [1.8]$$

The amount $\delta F(x_0)$ of U brought by the flux F across the left-hand side boundary of the control volume between t_0 and $t_0 + \delta t$ is given by:

$$\delta F(x_0) = \int_{t_0}^{t_0 + \delta t} F(x_0, t) dt \quad [1.9]$$

A quantity $\delta F(x_0 + \delta x)$ leaves the domain across the right-hand side boundary:

$$\delta F(x_0 + \delta x) = \int_{t_0}^{t_0 + \delta t} F(x_0 + \delta x, t) dt \quad [1.10]$$

Stating the conservation of U over the control volume $[x_0, x_0 + \delta x]$ between t_0 and $t_0 + \delta t$, the following equality is obtained:

$$M(t_0 + \delta t) = M(t_0) + \delta F(x_0) - \delta F(x_0 + \delta x) + \delta S \quad [1.11]$$

Substituting equations [1.6] – [1.10] into equation [1.11] leads to:

$$\begin{aligned} \int_{x_0}^{x_0 + \delta x} [U(x, t_0 + \delta t) - U(x, t_0)] dx &= \int_{t_0}^{t_0 + \delta t} [F(x_0, t) - F(x_0 + \delta x, t)] dt \\ &+ \int_{t_0}^{t_0 + \delta t} \int_{x_0}^{x_0 + \delta x} S(x, t) dx dt \end{aligned} \quad [1.12]$$

A first-order Taylor series expansion around (x_0, t_0) gives:

$$\left. \begin{aligned} U(x_0, t_0 + \delta t) - U(x_0, t_0) &= \delta t \frac{\partial U}{\partial t} + O(\delta t^2) \\ F(x_0, t_0) - F(x_0 + \delta x, t_0) &= -\delta x \frac{\partial F}{\partial x} + O(\delta x^2) \end{aligned} \right\} \quad [1.13]$$

where the quantities $O(\delta t^2)$ and $O(\delta x^2)$ are second- or higher-order polynomials with respect to δt and δx respectively. These polynomials contain the second- and higher-order derivatives of U and F with respect to t and x . When δt and δx tend to zero, the polynomial $O(\delta t^2)$ becomes negligible compared to the quantity $\delta t \partial U / \partial t$ because δt^2 decreases faster than δt . The polynomial $O(\delta x^2)$ becomes negligible compared to $\delta x \partial F / \partial x$ for the same reason. Relationships [1.13] thus become:

$$\left. \begin{aligned} U(x_0, t_0 + \delta t) - U(x_0, t_0) &\underset{\delta t \rightarrow 0}{\approx} \delta t \frac{\partial U}{\partial t} \\ F(x_0, t_0) - F(x_0 + \delta x, t_0) &\underset{\delta x \rightarrow 0}{\approx} -\delta x \frac{\partial F}{\partial x} \end{aligned} \right\} \quad [1.14]$$

A similar reasoning leads to the following equivalence:

$$\int_{t_0}^{t_0 + \delta t} \int_{x_0}^{x_0 + \delta x} S(x, t) \, dx \, dt \underset{\substack{\delta t \rightarrow 0 \\ \delta x \rightarrow 0}}{\approx} \delta t \, \delta x \, S \quad [1.15]$$

Substituting equations [1.14] and [1.15] into equation [1.12] leads to

$$\delta t \frac{\partial U}{\partial t} \delta x = -\delta x \frac{\partial F}{\partial x} \delta t + \delta t \delta x \, S \quad [1.16]$$

Dividing equation [1.16] by $\delta t \delta x$ yields the conservation form [1.1], recalled here:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S$$

The following remarks can be made:

– the Partial Differential Equation (PDE) [1.1] is a particular case of the more general, integral equation [1.12]. Equation [1.1] is obtained from equation [1.12] using the assumption that δt and δx tend to zero. Equation [1.12] is the so-called weak form of equation [1.1] (see Chapter 3 for more details);

– the conservation form [1.1] is based on the implicit assumption that F is differentiable with respect to x and U is differentiable with respect to t . Consequently, [1.1] is meaningful only when U is continuous in space and time. In contrast, equation [1.12] is meaningful even when U is discontinuous in space and/or time. This has consequences on the calculation of discontinuous solutions, as shown in Chapter 3.

1.1.3. Non-conservation form

Equation [1.1] can be rewritten in the so-called non-conservation form that involves only derivatives of U . The non-conservation form of equation [1.1] is:

$$\frac{\partial U}{\partial t} + \lambda \frac{\partial U}{\partial x} = S' \quad [1.17]$$

where λ is called the wave speed, and S' is a source term that may be identical (but not necessarily) to the source term S in equation [1.1]. Equation [1.17] is obtained from equation [1.1] by rewriting the derivative $\partial F / \partial x$ as:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial U} \frac{\partial U}{\partial x} + F' \quad [1.18]$$

where the term $F' = (\partial F / \partial x)_{U=\text{Const}}$ contains all the derivatives of F other than the derivative with respect to U . The expression of F being known, $\partial F / \partial U$ and F' are easily determined. Substituting equation [1.18] into equation [1.1] yields:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial U} \frac{\partial U}{\partial x} + F' = S \quad [1.19]$$

that is:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial U} \frac{\partial U}{\partial x} = S - F' \quad [1.20]$$

Comparing equation [1.20] to equation [1.17] leads to the following definitions for λ and S' :

$$\left. \begin{aligned} \lambda &= \frac{\partial F}{\partial U} \\ S' &= S - \left(\frac{\partial F}{\partial x} \right)_{U=\text{const}} \end{aligned} \right\} \quad [1.21]$$

The expressions of F and S being known, the knowledge of U at any point in time and space allows λ and S' to be calculated directly. From definition [1.21], in the case where the variations in F are due to variations in U only, $F' = 0$ and S' is identical to S .

Example: assume that the flux function F is defined as in equation [1.3], recalled here:

$$F = aU$$

where a is a function of x and t . Equation [1.18] then becomes:

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x} [a(x, t)U] = a \frac{\partial U}{\partial x} + U \frac{\partial a}{\partial x} \quad [1.22]$$

and λ and F' are given by:

$$\left. \begin{aligned} \lambda &= a \\ F' &= U \frac{\partial a}{\partial x} \end{aligned} \right\} \quad [1.23]$$

If a does not depend on x , $F' = 0$ because $\partial a / \partial x = 0$.

1.1.4. Characteristic form – Riemann invariants

Writing a conservation law in non-conservation form leads to the notions of characteristic form and the Riemann invariant. Such notions are essential to the understanding of hyperbolic conservation laws. A very convenient way of determining the behavior of the solutions of hyperbolic conservation laws consists of identifying invariant quantities (that is, quantities that do not change) along certain trajectories, also called the “characteristic curves” (or more simply the “characteristics”). The solution is calculated by “following” the invariants along the

characteristics, which allows the value of U to be determined at any point. To do so, the non-conservation form [1.17] is used:

$$\frac{\partial U}{\partial t} + \lambda \frac{\partial U}{\partial x} = S'$$

The purpose is to derive the expression of the variation δU in U observed by an observer travelling at a given speed v . A small time interval δt is considered, over which the traveler moves by a distance $\delta x = v \delta t$. The variation δU “seen” by the observer is given by:

$$\delta U = \frac{\partial U}{\partial t} \delta t + \frac{\partial U}{\partial x} \delta x = \left(\frac{\partial U}{\partial t} + v \frac{\partial U}{\partial x} \right) \delta t \quad [1.24]$$

Note that from the observer’s point of view, U is a function of time only, because the observer’s location $x(t)$ is defined by $dx/dt = v$. When δt tends to zero, the ratio $\delta U / \delta t$ tends to the so-called total derivative dU/dt . Therefore equation [1.24] becomes:

$$\frac{\delta U}{\delta t} \underset{\delta t \rightarrow 0}{\approx} \frac{dU}{dt} = \frac{\partial U}{\partial t} + v \frac{\partial U}{\partial x} \quad \text{for } \frac{dx}{dt} = v \quad [1.25]$$

In the particular case of an observer moving at a speed λ , equation [1.25] becomes:

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \lambda \frac{\partial U}{\partial x} \quad \text{for } \frac{dx}{dt} = \lambda \quad [1.26]$$

Comparing equations [1.26] and [1.17] leads to:

$$\frac{dU}{dt} = S' \quad \text{for } \frac{dx}{dt} = \lambda \quad [1.27]$$

Equation [1.27] is the so-called characteristic form of equation [1.1]. The trajectory, the equation of which is $dx/dt = \lambda$, is called a characteristic. λ is called the wave speed.

S' being a function of U , x and t , its value may be calculated at any point (x, t) if the value of U is known. The first-order Ordinary Differential Equation (ODE) [1.27] is applicable along the characteristic.

In the (important) particular case where the source term S' is zero, equation [1.17] becomes:

$$\frac{\partial U}{\partial t} + \lambda \frac{\partial U}{\partial x} = 0 \quad [1.28]$$

and equation [1.27] becomes:

$$\frac{dU}{dt} = 0 \quad \text{for} \quad \frac{dx}{dt} = \lambda \quad [1.29]$$

Equation [1.29] can also be written as:

$$U = \text{Const} \quad \text{for} \quad \frac{dx}{dt} = \lambda \quad [1.30]$$

Consequently, the quantity U is invariant to an observer moving at the speed λ . U is called a Riemann invariant.

The physical meaning of the wave speed is the following. The wave speed is the speed at which the variations in U (and not U itself) propagate. A perturbation appearing in the profile of U at a given time propagates at the speed λ . The wave speed can be viewed as the speed at which “information”, or “signals” created by variations in U , propagate in space.

1.2. Determination of the solution

1.2.1. Representation in the phase space

The phase space is a very useful tool in the determination of the behavior of the solutions of hyperbolic conservation laws. The term “phase space” indicates the (x, t) plane formed by the space coordinate x and the time coordinate t (Figure 1.3).

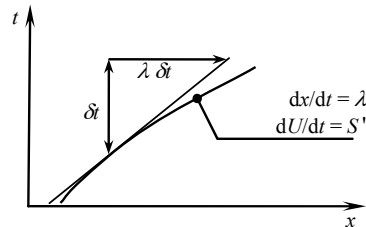


Figure 1.3. Representation of characteristic curves in the phase space

The trajectory $dx/dt = \lambda$ is represented by a curve in the phase space. The distance δx covered by the characteristic over a time interval δt is given by $\delta x = \lambda \delta t$, therefore the slope of the line is $\delta t / \delta x = 1/\lambda$. Note that the sign of λ may change with time depending on the variations in U and the expressions of λ and S' . When λ becomes zero the tangent to the characteristic curve is vertical in the phase space (Figure 1.4a). In contrast, an extremum with respect to time is not physically permissible (Figure 1.4b) because “travelling backwards in time” is not possible.

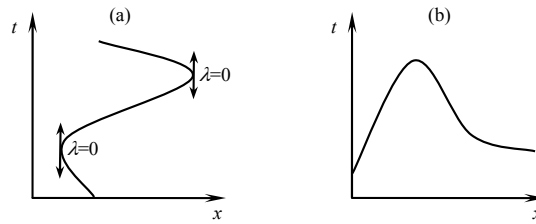


Figure 1.4. Physically permissible (a) and non-permissible (b) characteristics

The representation in the phase space may be used to determine the behavior of the solutions of conservation law [1.1] using the so-called “method of characteristics”. The following simple case is considered:

- the source term S in equation [1.1] is zero;
- the flux depends only on U , therefore $F' = 0$ in equations [1.18] – [1.20].

The characteristic form [1.27] then reduces to equation [1.30], recalled here:

$$U = \text{Const for } \frac{dx}{dt} = \lambda$$

F being a function of U only, λ is also a function of U only. Consequently, if U is constant along a characteristic line, λ is also constant and the characteristic is a straight line in the phase space (Figure 1.5). Assume that the profile $U(x, t_0)$ is known for all x at the time t_0 . The purpose is to determine the profile $U(x, t_1)$ for all x at the time $t_1 > t_0$. Consider the point A, the abscissa of which is denoted by x_A , at which the value of U at (x_A, t_0) is denoted by U_A . Since the wave speed λ depends on U only, the characteristic passing at A is a straight line. Its (constant) wave speed is $\lambda_A = \partial F / \partial U (U_A)$. At time t_1 , the characteristic has moved to point A', the abscissa $x_{A'}$ of which is given by:

$$x_{A'} = x_A + (t_1 - t_0)\lambda_A \quad [1.31]$$

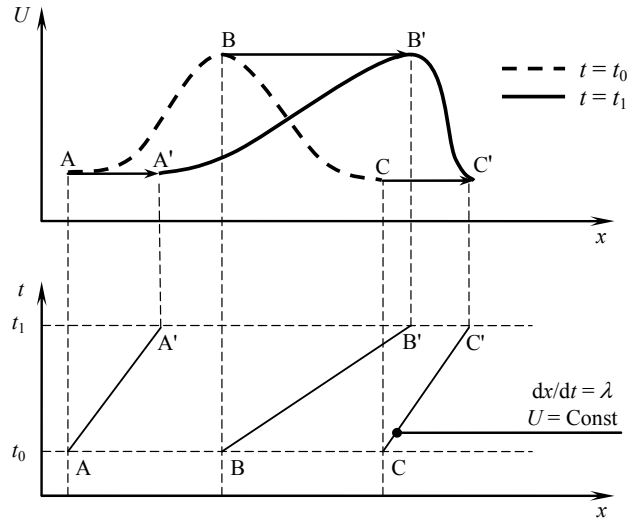


Figure 1.5. Representation of the characteristics in the phase space (bottom) and behavior of the physical profile (top) in the particular case $F' = S = 0$

From the property of invariance of U along the characteristic, U remains unchanged between A and A':

$$U_{A'} = U_A \quad [1.32]$$

Extending the reasoning above to any value of x , the following relationship is obtained:

$$U(x + \lambda \Delta t, t + \Delta t) = U(x, t) \quad [1.33]$$

where Δt represents the quantity $(t_1 - t_0)$ and λ is estimated at (x, t) .

Figure 1.5 shows how the method of characteristics can be used to determine the evolution of a given profile [ABC]. The figure is drawn assuming that λ is an increasing function of U . Therefore, point B moves faster than points A and C because U_B is larger than U_A and U_C . Consequently, the region [AB] tends to spread in time, while the region [BC] becomes narrower. After a certain amount of time point B catches up with point C and the solution becomes discontinuous at point $B' = C'$. The derivatives $\partial U / \partial t$ and $\partial U / \partial x$ are no longer defined and a specific treatment must be applied to determine the solution at later times. Such a treatment is detailed in Chapter 3.

In the general case, S and F' are non-zero. Then relationship [1.33] cannot be used because:

- U is not invariant along a characteristic line;
- the characteristics are therefore curved lines, the slope of which depends on the local value of x and U .

Therefore, no simple relationship can be derived between the initial profile at $t = t_0$ and the final profile at $t = t_1$. In most cases, the solution must be computed approximately using numerical methods. Such methods are dealt with in Chapters 6 and 7.

1.2.2. Initial conditions, boundary conditions

In practical applications, the solution of equation [1.1] is sought over a domain of finite length. A key issue is the amount of information needed for the calculation of U at a point $M(x, t)$ in the domain. This question is best answered using the phase space (Figure 1.6). The solution domain is assumed to extend from $x = 0$ to $x = L$.

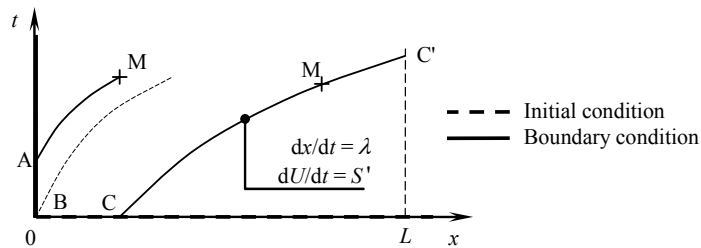


Figure 1.6. Initial and boundary conditions in the phase space

For the sake of clarity, the wave speed λ is assumed to be positive over the entire domain (the case where the sign of the wave speed changes is examined at the end of the section). Two possibilities arise:

- If point M is located on the right-hand side of the characteristic that passes at point B ($x = 0, t = 0$), there exists a point C on the line ($t = 0$) such that the characteristic passing at C passes at M . Point C is called the foot of the characteristic at $t = 0$. If the value of U is known at point C , U can be computed along the characteristic line by solving the characteristic form [1.27] using any analytical or numerical method. Therefore, the value of U can be computed at any point M located on the right-hand side of the characteristic that passes at B ($0, 0$), provided

that $U(x, 0)$ is known for all x between 0 and L . The function that describes the profile $U(x, 0)$ is called the initial condition. It is expressed as follows:

$$U_0(x) = U(x, t = 0), \quad x \in [0, L] \quad [1.34]$$

– If point M is located on the left-hand side of the characteristic passing at B, the value of U at M cannot be calculated from the initial condition and the knowledge of the value of U at all points A along the line ($x = 0$) is necessary. The function that describes the profile $U(0, t)$ is called a boundary condition. In the case of a positive λ , the characteristics enter the domain on the left-hand side and the left boundary condition must be used. It is expressed as follows:

$$U_b(t) = U(x = 0, t), \quad t > 0 \quad [1.35]$$

Note that a boundary condition can be prescribed only if the characteristics enter the domain. In the situation illustrated by Figure 1.6, prescribing a boundary condition at the point C' would be meaningless because the value of U at C' is entirely determined by the initial condition at C via the characteristic form [1.27] and cannot be prescribed independently of it. Depending on the variations of λ with U , x and t , the number of boundary conditions needed to determine U uniquely over the domain $[0, L]$ may be 0, 1 or 2 (see Figure 1.7).

In configuration (a) the characteristics leave the domain at both boundaries ($x = 0$) and ($x = L$). The value of U at both boundaries is determined entirely by the initial condition $U(x, 0)$. In configuration (b) the left-hand boundary condition is needed because the characteristics enter the domain at $x = 0$ and the value of U at this location cannot be determined from the values inside the domain. In contrast, the knowledge of U at the right-hand boundary is not required because U is determined uniquely from the value of U inside the domain. In configuration (c) the characteristics enter the domain at both $x = 0$ and $x = L$. Consequently, two boundary conditions are needed, one at each end of the domain, because $U(0, t)$ and $U(L, t)$ cannot be determined from inside the domain and must therefore be specified independently in the form of boundary conditions.

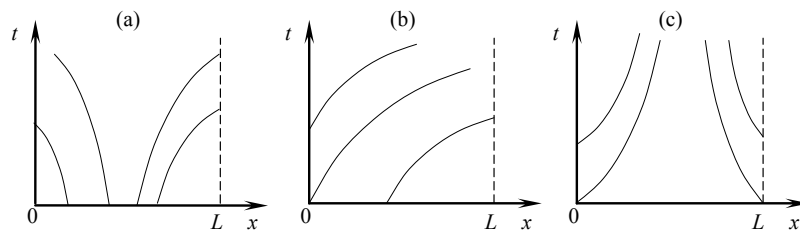


Figure 1.7. Number of boundary conditions needed depending on the variations of the wave speed: none (a), one (b), two (c)

1.3. A linear law: the advection equation

1.3.1. *Physical context – conservation form*

The linear advection equation is the simplest possible hyperbolic conservation law. It is found in many domains of fluid mechanics because it expresses a widespread phenomenon, the transport of a given quantity in a moving fluid. The transported variable may be the temperature of the fluid, the concentration in a given chemical, etc. The expression “advection” is often understood as the advection of a passive scalar, that is, a quantity that does not influence the behavior of the flow by which it is transported. In a number of cases however, the transported quantity influences the velocity field, a phenomenon known as coupling. This is the case of the inviscid Burgers equation dealt with in section 1.4.

In this section, a passive scalar is considered. The example of a chemical substance dissolved in water with a concentration variable in space and time is used. The water is assumed to flow in a channel, the transverse dimensions of which are assumed to be negligible compared to the longitudinal dimension. The channel may be an open channel (a river, a canal) or a closed channel (a conduit) with a cross-sectional area variable in space and time. The assumption of negligible transverse dimensions for the channel allows the assumption of a one-dimensional, longitudinal flow and transport process to be used. The channel is represented as a one-dimensional object. The space coordinate is the curvilinear abscissa (Figure 1.8).

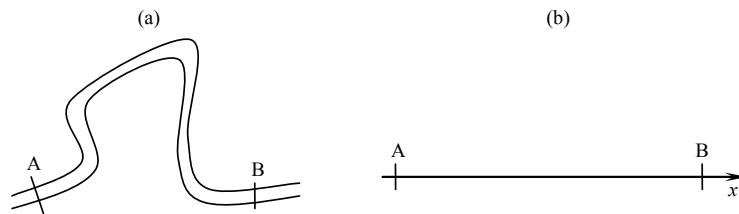


Figure 1.8. *One-dimensional representation of a channel, the transverse dimension of which can be considered negligible. Reality (a) and model (b)*

The governing PDE for the one-dimensional transport of a dissolved substance is derived by carrying out a balance as in section 1.1.2. The total quantity $M(t)$ of substance (the “mass” as introduced in section 1.1.2) over an elementary channel slice of length δx (Figure 1.9) is given by

$$M = C \delta V \quad [1.36]$$

where C is the concentration of the dissolved substance and δV is the volume of the elementary channel slice, given by:

$$\delta V = A \delta x \quad [1.37]$$

where A is the cross-sectional area of the channel (Figure 1.9).

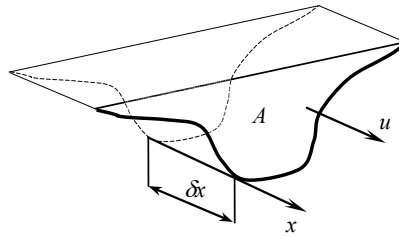


Figure 1.9. Perspective view of an elementary channel section

The amount $\delta F(x_0)$ of dissolved chemical that passes at x_0 during an elementary time interval δt is given by:

$$\delta F(x_0) = AuC \delta t \quad [1.38]$$

where u is the flow velocity. Using the same reasoning as in equations [1.11] to [1.16] with a zero source term, the PDE that describes the conservation of mass (also called the continuity equation) is obtained:

$$\frac{\partial}{\partial t} (AC) + \frac{\partial}{\partial x} (AuC) = 0 \quad [1.39]$$

Equation [1.39] can be written in the form [1.1] by defining the conserved variable U , the flux F and the source term S as:

$$\left. \begin{aligned} U &= AC \\ F &= QC \\ S &= 0 \end{aligned} \right\} \quad [1.40]$$

where $Q = Au$ is the so-called liquid discharge.

1.3.2. Characteristic form

Several approaches may be used to rewrite equation [1.39] in characteristic form. A first approach consists of defining the conserved quantity as AC and rewriting equations [1.39] to [1.40] as:

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(uU) = 0 \quad [1.41]$$

noting that $\partial / \partial x(uU) = u \partial U / \partial x + U \partial u / \partial x$, equation [1.41] becomes:

$$\frac{\partial U}{\partial t} + u \frac{\partial U}{\partial x} = -U \frac{\partial u}{\partial x} \quad [1.42]$$

As shown in section 1.1.4 (see equations [1.24] to [1.27]), equation [1.42] is equivalent to:

$$\frac{dU}{dt} = -U \frac{\partial u}{\partial x} \quad \text{for } \frac{dx}{dt} = u \quad [1.43]$$

Equation [1.43] is of limited interest because U does not appear as an invariant quantity along a characteristic line.

In a second approach, equation [1.39] is rewritten with respect to the concentration C by developing the derivatives:

$$A \frac{\partial C}{\partial t} + C \frac{\partial A}{\partial t} + Q \frac{\partial C}{\partial x} + C \frac{\partial Q}{\partial x} = 0 \quad [1.44]$$

Equation [1.44] is rewritten as:

$$A \frac{\partial C}{\partial t} + Q \frac{\partial C}{\partial x} = - \left(\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} \right) C \quad [1.45]$$

Noting that the continuity equation for the flow can be written as:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad [1.46]$$

substituting equation [1.46] into equation [1.45] yields the following equation:

$$A \frac{\partial C}{\partial t} + Q \frac{\partial C}{\partial x} = 0 \quad [1.47]$$

Dividing by A and noting that $Q/A = u$ leads to:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = 0 \quad [1.48]$$

From the developments carried out in section 1.1.4 (see equations [1.24] to [1.27]), equation [1.48] is known to be equivalent to the following characteristic form:

$$\frac{dC}{dt} = 0 \quad \text{for} \quad \frac{dx}{dt} = u \quad [1.49]$$

Equation [1.49] is equivalent to

$$C = \text{Const} \quad \text{for} \quad \frac{dx}{dt} = u \quad [1.50]$$

Equation [1.50] is an interesting alternative to equation [1.43] because it allows a Riemann invariant to be derived. The Riemann invariant is the concentration of the dissolved substance. Note that the conserved quantity (the mass AC per unit length of channel) is not identical to the invariant quantity (the concentration).

1.3.3. Example: movement of a contaminant in a river

The difference between a conserved quantity and an invariant quantity is best illustrated by the following example. Consider a river in which a contaminant is transported without degradation or external inflow. The cross-sectional area of the river is variable, with a sudden narrowing at point N. The cross-sectional areas upstream and downstream of the narrowing are denoted by A_1 and A_2 respectively. The discharge Q is assumed to be constant in both space and time. The numerical parameters are summarized in Table 1.1.

Symbol	Meaning	Value
A_1	Cross-sectional area of the channel upstream of the narrowing	100 m ²
A_2	Cross-sectional area of the channel downstream of the narrowing	25 m ²
C_1	Concentration upstream of the narrowing	1 g/l
C_{us}	Upstream boundary condition ($x = 0$)	0 g/l
L_1	Length of the contaminant cloud upstream of the narrowing	100 m
Q	Liquid discharge	10 m ³ /s

Table 1.1. *Parameters of the problem*

The initial contaminant concentration is assumed to be zero everywhere, except over a segment [AB] where it is uniformly equal to C_1 . The segment [AB] is assumed to be located upstream of N at $t = 0$ (Figure 1.10). The upstream boundary condition is a zero concentration, so that no additional contaminant enters the domain from the upstream boundary.

As shown by the data in Table 1.1 the flow velocity is 0.1 m/s upstream of the narrowing and 0.25 m/s downstream of it. Denoting by $t_{A'}$ and $t_{B'}$ the times at which the characteristics passing at A and B reach the narrowing N, the following relationship holds:

$$\frac{x_B - x_A}{t_{A'} - t_{B'}} = u_1 = \frac{Q}{A_1} \quad [1.51]$$

Equation [1.51] can be rewritten as:

$$t_{A'} - t_{B'} = \frac{(x_B - x_A)A_1}{Q} = \frac{A_1 L_1}{Q} \quad [1.52]$$

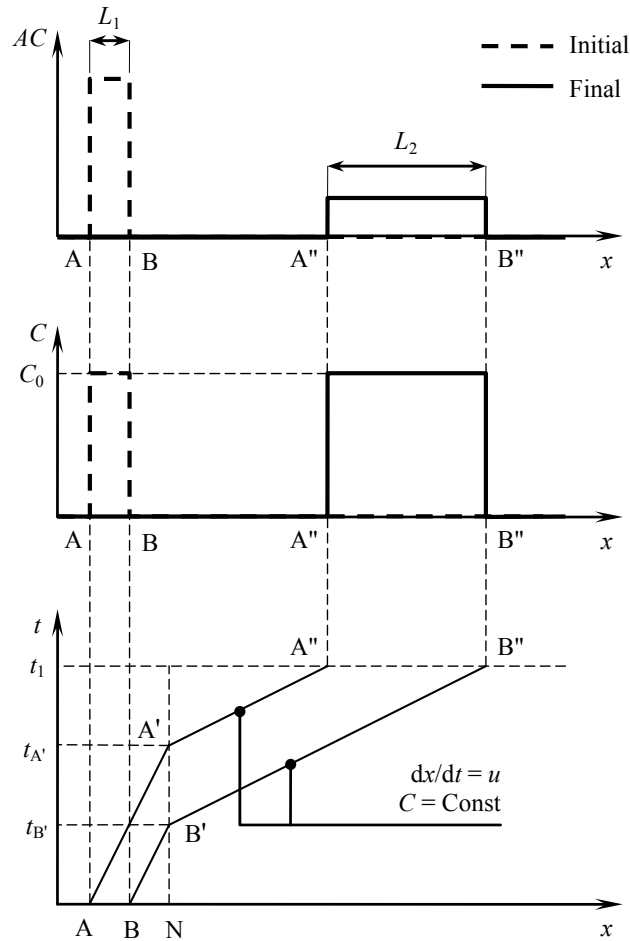


Figure 1.10. Behavior of the solution in the physical space (top, middle) and the phase space (bottom)

Consider now a time t_1 after the contaminant cloud has left point R . The front and rear of the cloud are denoted by A'' and B'' respectively. The abscissas $x_{A''}$ and $x_{B''}$ are related to the times $t_{A'}$ and $t_{B'}$ as follows:

$$\frac{x_{B''} - x_{A''}}{t_{A'} - t_{B'}} = u_2 = \frac{Q}{A_2} \quad [1.53]$$

Denoting with L_2 the width of the cloud downstream of the narrowing, the following relationship holds:

$$L_2 = \frac{Q}{A_2} (t_{A'} - t_{B'}) = \frac{A_1}{A_2} L_1 \quad [1.54]$$

From the parameters in Table 1.1 the cloud is 4 times larger downstream of the narrowing than upstream of it. Note that:

– the concentration C is invariant along a characteristic line. Therefore the concentration profile remains piecewise constant. The integral of the concentration with respect to x at $t = 0$ is given by:

$$\int_0^{+\infty} C(x,0) dx = L_1 C_0 \quad [1.55]$$

The integral at $t = t_1$ is given by:

$$\int_0^{+\infty} C(x,t_1) dx = L_2 C_0 = \frac{A_1}{A_2} L_1 C_0 \quad [1.56]$$

where equation [1.54] is used to relate L_2 and L_1 . Although the source term is zero and no contaminant enters the domain across the boundary, the integral of C with respect to x is not constant. The concentration is an invariant quantity, it is not a conserved quantity;

– the mass of contaminant per unit length is equal to AC . At $t = 0$ the integral of AC with respect to x is:

$$\int_0^{+\infty} A(x)C(x,0) dx = A_1 L_1 C_0 \quad [1.57]$$

At $t = t_1$ the integral is equal to:

$$\int_0^{+\infty} A(x)C(x,t_1) dx = A_2 L_2 C_0 = A_2 \frac{A_1}{A_2} L_1 C_0 = A_1 L_1 C_0 \quad [1.58]$$

where equation [1.54] is used again to provide a relationship between L_1 and L_2 . In contrast with the integral of C , the integral of AC (that is, the total mass of solute contained in the river reach) is conserved. However, AC is equal to $A_1 C_0$ upstream of N and is equal to $A_2 C_0$ downstream of N. Consequently, AC is a conserved quantity, not an invariant quantity.

1.3.4. Summary

The conservation form of the linear advection equation for a contaminant in a channel is equation [1.39]. It can be written in the form [1.1] by defining the conserved variable, the flux and the source terms as in equation [1.40].

The non-conservation form of equation [1.39] is equation [1.48] and its characteristic form is equation [1.49].

The conserved variable is the mass of contaminant AC per unit length of channel. The invariant quantity is the contaminant concentration C .

1.4. A convex law: the inviscid Burgers equation

1.4.1. Physical context – conservation form

The Burgers equation was first introduced [BUR 48] as a simple model to account for nonlinear advection and diffusion. A transformation was proposed independently by Hopf [HOP 50] and Cole [COL 51], whereby the Burgers equation is transformed into a linear PDE. In what follows, only the inviscid form of the equation is considered. If the initial and boundary conditions are not too complex, an analytical solution can be derived for the inviscid Burgers equation. This equation is often used to assess the performance of numerical methods for hyperbolic PDEs.

The inviscid Burgers equation can be viewed as a restriction of the Euler equations of gas dynamics introduced in Chapter 2. It is derived from the momentum equation under the assumption of negligible external forces, pressure gradients and momentum diffusion. If this is the case, the variations in the fluid velocity are due only to the initial distribution of momentum in the fluid. In contrast with the advection equation dealt with in section 1.3, the inviscid Burgers equation describes the advection of an active scalar, in that the value of u influences its own propagation speed.

Consider a slice of fluid, the (infinitesimal) length of which is denoted by δx and the cross-sectional area of which is equal to unity (Figure 1.11). The mass δm and the momentum δq of the slice of fluid are given by:

$$\left. \begin{aligned} \delta m &= \rho \delta x \\ \delta q &= \rho u \delta x = u \delta m \end{aligned} \right\} \quad [1.59]$$

where u and ρ are the fluid velocity and density respectively.

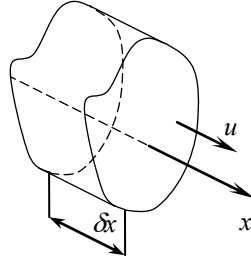


Figure 1.11. Perspective view of an elementary volume of width δx and unit cross-sectional area

Both the density and the momentum are transported at the speed u of the fluid. The mass that crosses the unit section over an elementary time interval δt is given by:

$$\delta F_m = \rho u \delta t = q \delta t \quad [1.60]$$

The momentum that crosses the unit section over the time interval δt is given by:

$$\delta F_q = qu \delta t = \rho u^2 \delta t \quad [1.61]$$

Applying the reasoning [1.11–16] to the conservation of the density ρ leads to the so-called continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \quad [1.62]$$

while the principle of conservation of the momentum $q = \rho u$ leads to the so-called momentum equation:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) = 0 \quad [1.63]$$

1.4.2. Characteristic form

The characteristic form of the inviscid Burgers equation is obtained by developing equation [1.63] into:

$$\rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} + u \frac{\partial}{\partial x}(\rho u) + \rho u \frac{\partial u}{\partial x} = 0 \quad [1.64]$$

Substituting equation [1.62] into equation [1.64] yields the following equation:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = 0 \quad [1.65]$$

Dividing by the density leads to an equation in u in non-conservation form:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad [1.66]$$

As shown in section 1.1.4 (see equations [1.24] to [1.27]), equation [1.66] is equivalent to:

$$\frac{du}{dt} = 0 \quad \text{for} \quad \frac{dx}{dt} = u \quad [1.67]$$

that is:

$$u = \text{Const} \quad \text{for} \quad \frac{dx}{dt} = u \quad [1.68]$$

The conserved quantities are the density ρ and the momentum $q = \rho u$, the invariant quantity is u . Note however that u may be seen as a conserved quantity in equation [1.66], which can be rewritten in conservation form by noting that $u \, du = d(u^2/2)$:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = 0 \quad [1.69]$$

Equation [1.69] is the so-called conservation form of the inviscid Burgers equation. It can be written in the form [1.1] by defining U and F as follows:

$$\left. \begin{aligned} U &= u \\ F &= \frac{U^2}{2} \end{aligned} \right\} \quad [1.70]$$

Equation [1.69] is a convex conservation law because the flux function $F(U)$ is a quadratic function with a positive second-order derivative. The quantity U is conserved and invariant. Moreover, the wave speed is equal to $dF/dU = U = u$. Since u is an invariant along the characteristic curves, these are straight lines in the phase space.

1.4.3. Example: propagation of a perturbation in a fluid

The current example deals with the variations in the fluid density and velocity arising from an initial perturbation in the velocity profile. In what follows, the time t is assumed to be small enough for the density and the velocity profiles to remain continuous, that is, the derivatives of ρ and u with respect to time and space are assumed to take finite values. If this is not the case, the analysis hereafter becomes invalid and a specific treatment must be applied to the discontinuities (also called shocks). Such a treatment is not detailed in this chapter. It is covered in detail in Chapter 3.

Consider the following initial condition. At $t = 0$, the density is uniformly equal to ρ_0 . The velocity profile is uniformly equal to u_0 , except over a segment [AC], where it is triangular (Figure 1.12). The initial velocity profile is symmetrical with a width $2L$ and the maximum value (also called peak velocity) is denoted by u_1 . The parameters of the problem are summarized in Table 1.2. The velocity being positive everywhere, the characteristics move from left to right. Therefore the upstream boundary is the left-hand boundary, $x = 0$.

Symbol	Meaning	Value
u_0	Initial flow velocity outside the perturbed region [AC]	1 m/s
u_1	Initial peak velocity	2 m/s
u_{us}	Flow velocity at the upstream boundary ($x = 0$)	1 m/s
L	Half-length of the initial velocity perturbation	1 m
ρ_0	Initial density	1 kg/m ³
ρ_{us}	Fluid density at the upstream boundary ($x = 0$)	1 kg/m ³

Table 1.2. Problem parameters

The characteristic form [1.67] allows the behavior of the flow velocity to be determined. Points A and C move at the speed u_0 , while point B moves at the speed u_1 . Therefore point B moves faster than A and C. The profile becomes smoother between A and B and steeper between B and C.

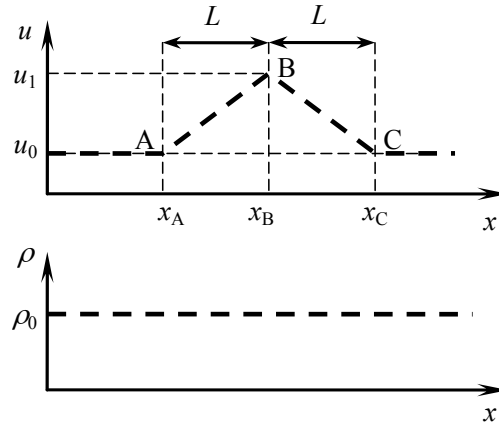


Figure 1.12. Initial profiles for the velocity (top) and the density (bottom)

The profile can easily be shown to remain triangular. Indeed, differentiating equation [1.66] with respect to x leads to the following equation:

$$\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} \right) = 0 \quad [1.71]$$

Swapping the derivatives in the first term of equation [1.71] and developing the second term yields:

$$\frac{\partial^2 u}{\partial t \partial x} + \left(\frac{\partial u}{\partial x} \right)^2 + u \frac{\partial^2 u}{\partial x^2} = 0 \quad [1.72]$$

which gives the following PDE in $\partial u / \partial x$:

$$\frac{\partial u}{\partial t} \left(\frac{\partial u}{\partial x} \right) + u \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = - \left(\frac{\partial u}{\partial x} \right)^2 \quad [1.73]$$

The reasoning developed in section 1.1.4 allows equation [1.73] to be rewritten in the form:

$$\frac{d}{dt} \left(\frac{\partial u}{\partial x} \right) = - \left(\frac{\partial u}{\partial x} \right)^2 \quad \text{for } \frac{dx}{dt} = u \quad [1.74]$$

The solution of equation [1.74] is:

$$\frac{\partial u}{\partial x}(t) = \frac{u'_0}{1 + (t - t_0) u'_0} \quad \text{for } \frac{dx}{dt} = u \quad [1.75]$$

where u'_0 is the value of $\partial u / \partial x$ at $t = t_0$ along the characteristic line (Figure 1.13).

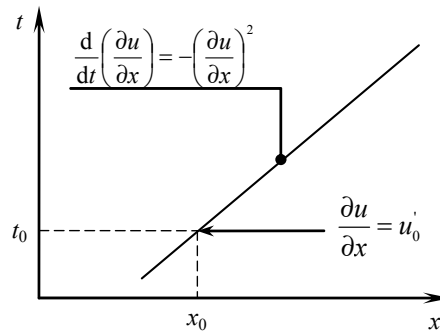


Figure 1.13. Determining the variations in $\partial u / \partial x$ along a characteristic from the initial velocity profile

The function $u'_0(x)$ is given by:

$$u'_0(x) = u(x, t_0) = \begin{cases} 0 & \text{for } x < x_A \\ \frac{u_1 - u_0}{L} & \text{for } x_A \leq x < x_B \\ \frac{u_0 - u_1}{L} & \text{for } x_B \leq x < x_C \\ 0 & \text{for } x \geq x_C \end{cases} \quad [1.76]$$

This function is piecewise constant and remains such at later times. Applying equation [1.75] along a characteristic line and using equation [1.76] for the initial condition leads to:

$$\frac{\partial u}{\partial x}(x, t) = \begin{cases} 0 & \text{for } x < x_{A'} \\ \frac{u_1 - u_0}{L + (u_1 - u_0)t} & \text{for } x_{A'} \leq x < x_{B'} \\ \frac{u_0 - u_1}{L + (u_0 - u_1)t} & \text{for } x_{B'} \leq x < x_{C'} \\ 0 & \text{for } x \geq x_{C'} \end{cases} \quad [1.77]$$

where the abscissas $x_{A'}$, $x_{B'}$ and $x_{C'}$ are the abscissas at the time t of the characteristics passing at A, B and C at the time $t_0 = 0$ (Figure 1.14):

$$\left. \begin{aligned} x_{A'} &= x_A + u_0 t \\ x_{B'} &= x_B + u_1 t \\ x_{C'} &= x_C + u_0 t \end{aligned} \right\} \quad [1.78]$$

Equation [1.77] is that of a piecewise constant derivative. Consequently, the profile of u is piecewise linear. Note that, since $u_1 > u_0$, the quantity $L + (u_1 - u_0)t$ in the second equation [1.77] increases with time and the magnitude of $\partial u / \partial x$ decreases with time between A' and B'. Conversely, $L + (u_0 - u_1)t$ in the third equation [1.77] decreases with time and the magnitude of $\partial u / \partial x$ increases with time between B' and C'. In other words, the profile [A'B'] becomes flatter, while the profile [B'C'] becomes steeper. From the third equation [1.77] the derivative becomes infinite for a time $t = t_{\max}$ given by:

$$L + (u_0 - u_1) t_{\max} = 0 \quad [1.79]$$

that is:

$$t_{\max} = \frac{L}{u_1 - u_0} \quad [1.80]$$

At $t = t_{\max}$ the segment [B'C'] becomes vertical. The velocity profile is no longer continuous, and both the conservation form [1.1] and the non-conservation form [1.17] become invalid. A special treatment is needed to allow the behavior of the solution to be analyzed at later times. Such a treatment is described in Chapter 3.

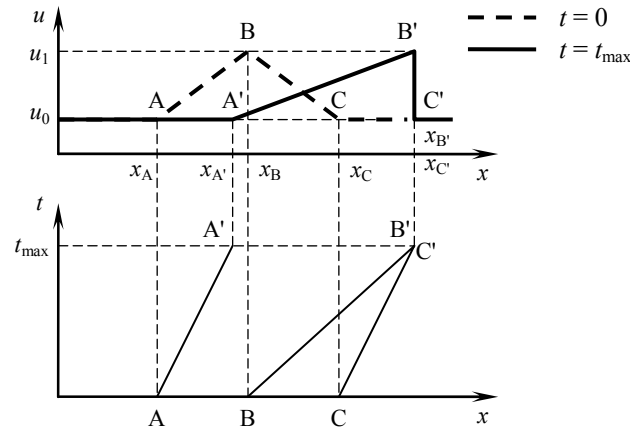


Figure 1.14. Velocity profile in the physical space (top) and in the phase space (bottom) for $t = 0$ and $t = t_{max}$.

1.4.4. Summary

The conservation form of the inviscid Burgers equation is given by equation [1.69]. It can be written as in equation [1.1] by defining the conserved variable and the flux as in equations [1.70].

The non-conservation and characteristic forms of the equation are given by equations [1.66] and [1.67] respectively.

Both the conserved variable and the Riemann invariant are equal to the fluid velocity u .

1.5. Another convex law: the kinematic wave for free-surface hydraulics

1.5.1. Physical context – conservation form

The kinematic wave is a simplified form of the open channel flow equations (Figure 1.15). It is a scalar equation, which makes it easy to solve. A summary of the conditions under which the kinematic wave provides a valid approximation to open channel flow can be found in [LIG 55]. The underlying assumptions of the kinematic wave equations are the following:

- Assumption (A1): the transverse dimensions of the channel are negligible compared to its longitudinal dimensions and the flow can be considered as one-dimensional.

– Assumption (A2): inertia is negligible and the channel slope is steep enough for the energy slope to be equivalent to the channel bed slope.

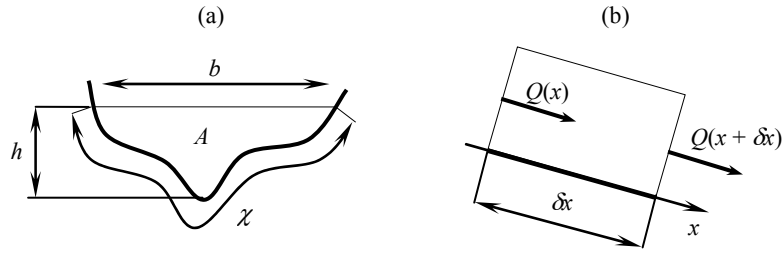


Figure 1.15. Flow in an open channel. Cross-sectional view (a), side view (b)

Under such conditions, the energy loss originating from friction against the walls is balanced with the energy gained from the slope. The friction term is assumed to obey Strickler's law:

$$Q = K_{\text{Str}} A R_H^{2/3} S_f^{1/2} \quad [1.81]$$

where A is the cross-sectional area of the channel, K_{Str} is Strickler's friction coefficient, R_H is the hydraulic radius and S_f is the slope of the energy line. The hydraulic radius is defined as the ratio of the cross-sectional area A to the wetted perimeter χ :

$$R_H = A / \chi \quad [1.82]$$

Since the slope of the energy line is assumed to be equal to the bottom slope S_0 , equation [1.81] becomes:

$$Q = K_{\text{Str}} \frac{A^{5/3}}{\chi^{2/3}} S_0^{1/2} \quad [1.83]$$

A mass balance over a control volume defined between abscissas x_0 and $x_0 + \delta x$ (Figure 1.15b) leads to the continuity equation (see section 1.2):

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad [1.84]$$

where Q is given by equation [1.83]. Note that Q is a known function of A at all points, because S_0 is known at all points and the law that relates the wetted perimeter χ to A is a known function of the geometry of the channel.

1.5.2. Non-conservation and characteristic forms

Equation [1.84] can be written in non-conservation form by developing the derivative of Q so as to involve derivatives of A , K_{Str} and S_0 :

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial A} \frac{\partial A}{\partial x} + \frac{\partial Q}{\partial K_{\text{Str}}} \frac{\partial K_{\text{Str}}}{\partial x} + \frac{\partial Q}{\partial S_0} \frac{\partial S_0}{\partial x} = 0 \quad [1.85]$$

Note that the purpose is to rewrite equation [1.85] in the form [1.17], recalled here:

$$\frac{\partial U}{\partial t} + \lambda \frac{\partial U}{\partial x} = S'$$

To do so, it is sufficient to define U as the cross-sectional area A and λ and S' as follows:

$$\left. \begin{aligned} \lambda &= \frac{\partial Q}{\partial A} \\ S' &= - \frac{\partial Q}{\partial K_{\text{Str}}} \frac{\partial K_{\text{Str}}}{\partial x} - \frac{\partial Q}{\partial S_0} \frac{\partial S_0}{\partial x} \end{aligned} \right\} \quad [1.86]$$

Equation [1.85] becomes:

$$\frac{\partial A}{\partial t} + \lambda \frac{\partial A}{\partial x} = S' \quad [1.87]$$

The variations in the cross-sectional area A propagate at the speed λ , as illustrated by the characteristic form of the equation:

$$\frac{dA}{dt} = S' \quad \text{for} \quad \frac{dx}{dt} = \lambda \quad [1.88]$$

Note that the cross-sectional area is a conserved variable (see equation [1.84]) but it is not necessarily an invariant quantity (note the source term in equation [1.88]). If S' is non-zero, A is not invariant along a characteristic line.

Also note that, Q being a function of A via relationship [1.81], its variations also propagate at the speed λ . This can be shown in a more rigorous way by rewriting equation [1.87] as:

$$\frac{\partial A}{\partial Q} \frac{\partial Q}{\partial t} + \lambda \frac{\partial A}{\partial Q} \frac{\partial Q}{\partial x} = S' \quad [1.89]$$

Dividing by $\partial A / \partial Q$ yields:

$$\frac{\partial Q}{\partial t} + \lambda \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial A} S' \quad [1.90]$$

Substituting the first equation [1.86] into equation [1.90] leads to:

$$\frac{\partial Q}{\partial t} + \lambda \frac{\partial Q}{\partial x} = \lambda S' \quad [1.91]$$

which leads to the characteristic form for Q :

$$\frac{dQ}{dt} = \lambda S' \quad \text{for} \quad \frac{dx}{dt} = \lambda \quad [1.92]$$

1.5.3. Expression of the wave speed

The wave speed is equal to $\lambda = \partial Q / \partial A$. This derivative is difficult to estimate in the general case because the relationship between the wetted perimeter and the cross-sectional area can rarely be determined analytically. The difficulty can be removed by involving the water depth h (that is, the difference between the elevation of the free surface and the lowest point of the river bed):

$$\lambda = \frac{\partial Q}{\partial h} \frac{\partial h}{\partial A} = \frac{\partial Q}{\partial h} \left(\frac{\partial A}{\partial h} \right)^{-1} \quad [1.93]$$

The analytical expression of $\partial A / \partial h$ is derived very easily by noting that an infinitesimal variation dh in the water depth results in an infinitesimal variation dA in the cross-sectional area (Figure 1.16):

$$dA = b \, dh \quad [1.94]$$

hence the relationship:

$$\frac{\partial A}{\partial h} = b \quad [1.95]$$

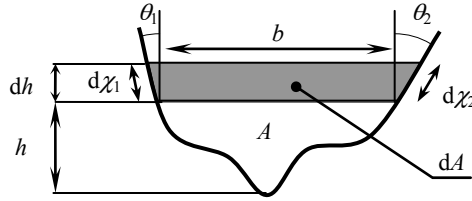


Figure 1.16. Definition sketch for the derivatives of A and R_H with respect to h

The derivative $\partial Q / \partial h$ is obtained from equation [1.83]:

$$\begin{aligned} \frac{\partial Q}{\partial h} &= K_{\text{Str}} S_0^{1/2} \frac{\partial}{\partial h} \left(\frac{A^{5/3}}{\chi^{2/3}} \right) \\ &= K_{\text{Str}} S_0^{1/2} \left[\frac{5}{3} \left(\frac{A}{\chi} \right)^{2/3} \frac{\partial A}{\partial h} - \frac{2}{3} \left(\frac{A}{\chi} \right)^{5/3} \frac{\partial \chi}{\partial h} \right] \end{aligned} \quad [1.96]$$

The expression of $\partial \chi / \partial h$ is determined by considering the variation $d\chi$ caused by an infinitesimal variation dh in the water depth:

$$d\chi = \left(\frac{1}{\cos \theta_1} + \frac{1}{\cos \theta_2} \right) dh \quad [1.97]$$

which leads to:

$$\frac{\partial \chi}{\partial h} = \frac{1}{\cos \theta_1} + \frac{1}{\cos \theta_2} \quad [1.98]$$

Substituting equations [1.95] and [1.98] into equations [1.96] and [1.93] yields the final expression of λ :

$$\lambda = K_{\text{Str}} S_0^{1/2} \left[\frac{5}{3} R_H^{2/3} - \frac{2R_H^{5/3}}{3b} \left(\frac{1}{\cos \theta_1} + \frac{1}{\cos \theta_2} \right) \right] \quad [1.99]$$

Note that the flow velocity u is given by:

$$u = \frac{Q}{A} = K_{\text{Str}} S_0^{1/2} R_H^{2/3} \quad [1.100]$$

This allows equation [1.99] to be rewritten as:

$$\lambda = \frac{5}{3}u - \frac{2R_H}{3b} \left(\frac{1}{\cos \theta_1} + \frac{1}{\cos \theta_2} \right) u \quad [1.101]$$

The following remarks can be made:

– The wave speed is different from the flow velocity. In the general case, λ is larger than u . In the particular case of a very wide rectangular channel, the ratio of the wave speed to the flow velocity is exactly 5/3 (see section 1.5.4).

– The steeper the slope, the faster the wave for a given water depth. A large Strickler coefficient, which corresponds to a small friction term, also induces a large wave speed. Friction therefore contributes to a reduced wave speed.

– For a given value of the hydraulic radius, non-vertical embankments contribute to reduce the wave speed. Indeed, the milder the embankments, the larger the tangents of the angles θ_1 and θ_2 and, from equation [1.99], the smaller the value of λ . Note that this is true not only when the channel widens from the bottom to the top (i.e. for positive θ_1 and θ_2), but also when the channel narrows down (i.e. for negative θ_1 and θ_2). This is because, for a given increase in h , non-vertical embankments induce a larger increase in the wetted perimeter (hence, in the friction) than vertical embankments would. Friction that is exerted at the walls increases faster when the embankments are not vertical, which contributes to reducing the wave speed.

– However, a section that widens from the bottom to the top yields a larger wave speed than a section that narrows down from the bottom to the top because the hydraulic radius increases faster with h .

– In the general case, the profile of a perturbation that propagates in the channel is subject to deformation in time. This is because λ is a function of R_H and b and therefore of A . Consequently, if A is variable, λ is also variable, which induces a deformation in the profile $A(x)$. This also means that the profile shape of Q is also altered as it travels in the channel. The assumption of a constant shape for the perturbation is valid only if the amplitude of the perturbation is small enough for the wave speed to be considered constant.

– A perturbation with a small amplitude may also be subjected to deformation as it travels in the channel. Indeed, if the Strickler coefficient or the bed slope are

variable in space, the source term S' in equation [1.17] is non-zero (see equation [1.86]) and A is not an invariant along a characteristic line.

To summarize, the assumption of zero deformation for a perturbation is valid only if the amplitude of the perturbation is small enough and if the parameters that govern friction (i.e. the Strickler coefficient and the channel slope) are constant in space.

1.5.4. Particular case: flow in a rectangular channel

Consider a rectangular channel, the (constant) width b of which is very large compared to the water depth h . The Strickler coefficient and the slope are assumed to be constant. In this case the area and the hydraulic radius are given by:

$$\left. \begin{aligned} A &= bh \\ R_H &= \frac{bh}{b+2h} \end{aligned} \right\} \quad [1.102]$$

If h is very small compared to b , the hydraulic radius is equivalent to the water depth h . This assumption is known as the “wide channel approximation”. The expression for the discharge is then simplified into:

$$Q = K_{\text{Str}} b S_0^{1/2} h^{5/3} = \frac{K_{\text{Str}} S_0^{1/2}}{b^{2/3}} A^{5/3} \quad [1.103]$$

Substituting equation [1.103] into equation [1.84] leads to the conservation form:

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} \left(\frac{K_{\text{Str}} S_0^{1/2}}{b^{2/3}} A^{5/3} \right) = 0 \quad [1.104]$$

The non-conservation form of equation [1.104] is:

$$\frac{\partial A}{\partial t} + \frac{5}{3} \frac{K_{\text{Str}} S_0^{1/2} A^{2/3}}{b^{2/3}} \frac{\partial A}{\partial x} + \frac{A^{5/3}}{b^{2/3}} \frac{\partial}{\partial x} (K_{\text{Str}} S_0^{1/2}) = 0 \quad [1.105]$$

which is equivalent to the non-conservation form [1.87], recalled here:

$$\frac{\partial A}{\partial t} + \lambda \frac{\partial A}{\partial x} = S'$$

provided that λ and S' are defined as:

$$\left. \begin{aligned} \lambda &= \frac{5}{3} \frac{K_{\text{Str}} S_0^{1/2} A^{2/3}}{b^{2/3}} \\ S' &= -\frac{A^{5/3}}{b^{2/3}} \frac{\partial}{\partial x} (K_{\text{Str}} S_0^{1/2}) = 0 \end{aligned} \right\} \quad [1.106]$$

From equation [1.103], we have:

$$\lambda = \frac{5}{3} \frac{Q}{A} = \frac{5}{3} u \quad [1.107]$$

1.5.5. Summary

The conservation form of the kinematic wave equation is given by equation [1.84]. Its non-conservation form is equation [1.87]. Its characteristic form is equation [1.88].

The cross-sectional area A and the discharge Q propagate at the same wave speed λ given by equations [1.99] and [1.101]. The wave speed increases with the channel slope, the Strickler coefficient and the slope of the embankments. It is an increasing function of the water depth h .

The cross-sectional area is the conserved variable in the conservation equation. It is not an invariant along the characteristic lines when the channel slope or the Strickler coefficient are variable in space.

1.6. A non-convex conservation law: the Buckley-Leverett equation

1.6.1. Physical context – conservation form

The Buckley-Leverett equation [BUC 42] is one of the simplest existing models for the propagation of non-aqueous phase liquids (typically, hydrocarbons) and water in porous media. The displacement of a single fluid in a porous medium can be described by Darcy's law, initially derived for water:

$$V_d = -K \frac{\partial H}{\partial x} \quad [1.108]$$

where K is the hydraulic conductivity, V_d is the so-called Darcy velocity and H is the hydraulic head, defined as:

$$H = \frac{p}{\rho g} + z + \frac{v^2}{2g} \quad [1.109]$$

where g is the gravitational acceleration, p is the pressure and ρ is the density. Darcy's law reflects the assumption that the water flows slowly enough in the soil pores for the head loss to be proportional to the velocity.

In other words, the flow regime is assumed to be laminar, in contrast with the turbulent regime, where the head loss is assumed to be proportional to the square of the velocity. When the flow velocity is very small, the term $v^2/(2g)$ can be neglected in equation [1.109] and the expression of the hydraulic head becomes:

$$H \approx \frac{p}{\rho g} + z \quad [1.110]$$

Note that the hydraulic conductivity K is a function of both the geometric properties of the porous medium and the physical characteristics of the fluid. A smaller pore size induces a higher friction against the grains, which means that a larger pressure gradient is needed for the fluid to move at a given velocity.

Darcy's law is applicable to soils saturated with water, but also to porous media saturated with other liquids, with the difference that the conductivity K must be changed to reflect the difference between the density and viscosity of the fluid and that of water.

When the porous medium contains both water and a hydrocarbon, Darcy's law is not applicable. There are two reasons for this:

- the water and the hydrocarbon are said to be immiscible. They are present as two separate fluids in the soil. No dissolution or mixing occurs between these two liquids;
- the surface tension of a hydrocarbon is different from that of water. The interface between the water and the hydrocarbon is curved and the pressure is different on both sides of the interface. The pressure being larger in the hydrocarbon, the hydrocarbon appears to be “trapped” in the aqueous phase (Figure 1.17). The hydrocarbon is said to be a non-wetting fluid.

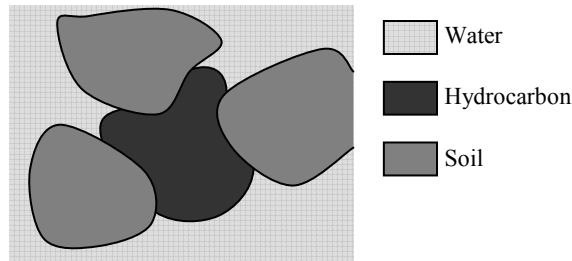


Figure 1.17. *Hydrocarbon-water interface in a porous medium*

Since the pressure is different in the water and in the hydrocarbon, the motion of these two fluids should not be expected to obey the same equation. In principle, an equation such as [1.108] should be needed for each of the liquids, with an additional relationship between the pressure jump across the water-hydrocarbon interface and the saturation in each liquid. Such models exist and are used on a daily basis in petroleum reservoir simulation. The most sophisticated models available describe three-phase behaviors: the water, the hydrocarbon and a gas phase with a volatilized fraction of hydrocarbon. Such models are very time-consuming and require powerful computers. In 1942, Buckley and Leverett had nevertheless proposed a single equation that accounts for the essential properties of water and hydrocarbon motion in porous media.

The Buckley-Leverett equation is a scalar law. Its flux function takes the form:

$$F = \frac{s^2}{s^2 + (1-s)^2 b_{BL}} V_d \tag{1.111}$$

where F is the water flux, b_{BL} is a shape parameter (see Figure 1.8) and s is the saturation, defined as the ratio of the volume filled with water to the porosity (that is, the total volume that can be occupied by a fluid).

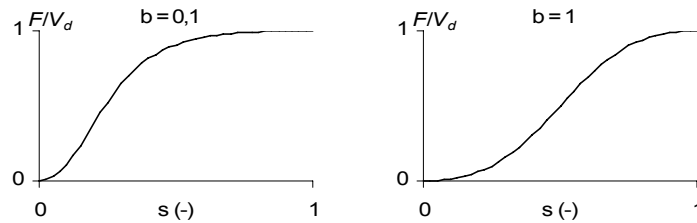


Figure 1.18. *Flux function for the Buckley-Leverett equation*

By definition, the saturation s lies within the interval $[0, 1]$. The volume occupied by the hydrocarbon is $1 - s$. Figure 1.18 illustrates the variations of the flux function F with the saturation for two values of the coefficient b_{BL} . When $b_{BL} = 1$ the flux function is symmetrical with respect to the point $s = 1/2$. An important feature of F is that its derivative with respect to s is equal to zero for $s = 0$ and $s = 1$. This results in specific aspects of hydrocarbon mobility, as detailed in section 1.6.2.

The Buckley-Leverett equation is derived from a balance over a slice of soil of unit cross-sectional area, the width of which is denoted by δx (Figure 1.11). The mass M of the water contained in this volume is given by:

$$M = \rho s \delta x \quad [1.112]$$

where ρ is the water density (assumed constant hereafter). The mass of water that passes at the abscissa x_0 per unit time is given by:

$$\delta F(x_0) = \rho F(x_0) \delta x \quad [1.113]$$

By reasoning as in section 1.1.2 (see equations [1.11] to [1.16]), the following equation is obtained:

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x}(\rho F) = 0 \quad [1.114]$$

Dividing by ρ yields the conservation form of the Buckley-Leverett equation:

$$\frac{\partial s}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad [1.115]$$

where F is defined as in equation [1.111].

1.6.2. Characteristic form

The characteristic form of the equation is obtained by first rewriting equation [1.115] in non-conservation form

$$\frac{\partial s}{\partial t} + \lambda \frac{\partial s}{\partial x} = 0 \quad [1.116]$$

where $\lambda = \partial F / \partial s$. The expression of λ is obtained by differentiating equation [1.111] with respect to s :

$$\lambda = \frac{\partial F}{\partial s} = 2 \frac{(1-s)s}{[s^2 + (1-s)^2 b_{BL}]^2} b_{BL} V_d \quad [1.117]$$

Figure 1.19 illustrates the variations of λ for two different values of the shape parameter b_{BL} . Note that for $b_{BL} = 1$, the curve $\lambda(s)$ is symmetrical with respect to the axis $s = 1/2$.

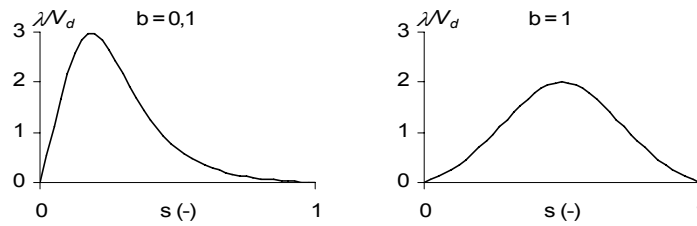


Figure 1.19. Wave speed as a function of the water saturation for the Buckley-Leverett equation

A similar reasoning to that of section 1.1.4 (see equations [1.24] to [1.27]) allows equation [1.116] to be rewritten in characteristic form as:

$$\frac{ds}{dt} = 0 \quad \text{for} \quad \frac{dx}{dt} = \lambda \quad [1.118]$$

an alternative writing for which is:

$$s = \text{Const} \quad \text{for} \quad \frac{dx}{dt} = \lambda \quad [1.119]$$

The water saturation s is a conserved quantity. It is also a Riemann invariant along the characteristics. From a physical point of view, λ is the speed at which the interface between the water and the hydrocarbon moves. The following points may be noted:

- the wave speed λ is equal to 0 for $s = 0$. This reflects a well-known property of hydrocarbons as non-wetting fluids, the mobility of which is very low at small water saturations;
- λ is maximum for and intermediate value s_{\max} of s between 0 and 1. The hydrocarbon achieves its maximum mobility for intermediate values of s ;

– λ is equal to 0 for $s = 1$. As a well-known consequence of hydrocarbon mobility properties, decontaminating totally an aquifer initially contaminated by hydrocarbons is almost impossible. A classical decontamination method consists of injecting pure water at one or several points in the aquifer, while the immiscible mixture is pumped from the aquifer at other points. Such a method is not totally efficient in that the mobility of the hydrocarbon decreases with its saturation. Removing the remaining 2% of hydrocarbons from an aquifer may be more time-consuming than the removal of the first 98% in some cases.

1.6.3. Example: decontamination of an aquifer

Consider an aquifer of width L partly contaminated by a hydrocarbon. The initial water saturation in the aquifer is assumed to be uniformly equal to s_0 . For the sake of clarity, s_0 is assumed to be larger than the saturation s_{\max} for which λ is maximum. Pure water is injected at the left-hand boundary of the aquifer ($x = 0$) from $t = 0$ onwards. The Darcy velocity is denoted by V_d . The water saturation thus rises instantaneously to $s = 1$ at the left-hand boundary. The saturation profile is made continuous by assuming that s decreases linearly from $s = 1$ to $s = 0$ between $x = 0$ and $x = \varepsilon$ (Figure 1.20).

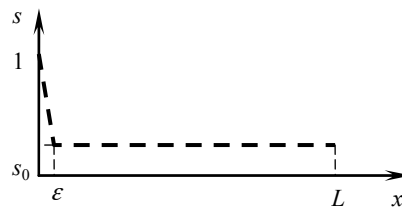


Figure 1.20. Using a continuous profile between $x = 0$ and $x = \varepsilon$ to represent a discontinuous saturation profile

The width ε of the saturation ramp can be made arbitrarily small, a discontinuous profile being obtained for $\varepsilon = 0$. The solution at $t > 0$ is obtained using a diagram in the phase space (Figure 1.21). Consider a point A initially located at $x = x_A$, such that $s_0 < s_A < 1$. The saturation being invariant along a characteristic line, λ is also invariant and the characteristics are straight lines in the phase space.

$$x_{A'} = x_A + \lambda(s_A)t = \lambda(s_A)t \quad [1.120]$$

Since s is an invariant along the characteristic line, the saturation at A' is identical to the saturation at A. The coordinates $(x_{A'}, s_{A'})$ of A' in the phase space are given by:

$$\left. \begin{aligned} x_{A'} &= \lambda(s_A) t \\ s_{A'} &= s_A \end{aligned} \right\} \quad [1.121]$$

As a consequence of equation [1.121], the point $s = 1$ does not move because $\lambda = 0$ for $s = 1$. The saturation profile decreases from $s = 1$ at $x = 0$ to $s = s_0$ at $x = \lambda(s_0) t$. The saturation remains uniformly equal to s_0 for all $x > x_0$ (Figure 1.21).

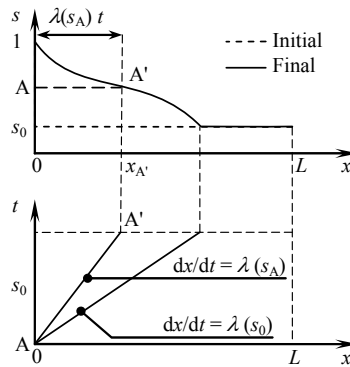


Figure 1.21. Representation of the solution in the physical space (top) and the phase space (bottom)

The complete equation of the profile at $t > 0$ is:

$$s = \begin{cases} \lambda^{-1}(x/t) & \text{for } x \leq \lambda(s_0) t \\ s_0 & \text{for } x \geq \lambda(s_0) t \end{cases} \quad [1.122]$$

where λ^{-1} denotes the reciprocal function of $\lambda(s)$. $\lambda^{-1}(x/t)$ represents the value of s for which $\lambda = x/t$.

Note that the solution depends on x/t only. The solution is said to be self-similar. The self-similarity property of the solution is an important feature of the Riemann problem, an initial-value problem used in a number of numerical methods for wave propagation (see Chapter 4 for more details).

1.6.4. Summary

The conservation form of the Buckley-Leverett equation is given by equation [1.115]. It can be written in the general form [1.1] by defining U as the water saturation and the flux F as in equation [1.111].

The non-conservation form of the equation is given as in equation [1.116–117]. Its characteristic form is given by equation [1.118].

The water saturation is the conserved quantity. It is also a Riemann invariant for the Buckley-Leverett equation.

1.7. Advection with adsorption/desorption

1.7.1. Physical context – conservation form

The linear advection equation is treated in detail in section 1.2. It expresses the advective transport of a non-reactive, conservative substance (also called a “tracer”). The situation is now considered where the transported substance is subjected to adsorption/desorption on a substratum. This is typically the case with pesticides in groundwater.

Adsorption/desorption is a physical-chemical phenomenon that reflects the ability of the transported substance to interact with a substratum and, possibly, to remain fixed on it. The substance is transported at a concentration C_T in a liquid or gas phase that is in direct contact with the substratum (Figure 1.22). Part of the substance reacts with the substratum and is immobilized. The quantity of immobile substance may be expressed as a mass concentration (that is, the mass of adsorbed substance per unit mass of substratum) denoted by C_A . As in any chemical reaction, the adsorption phenomenon does not take place instantaneously. It is conditioned by a reaction kinetics. In many cases however, the adsorption kinetics is rapid enough to justify the assumption that the concentrations in the fluid and the adsorbed substance are at equilibrium.

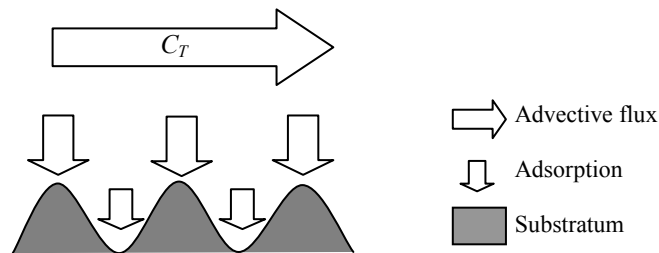


Figure 1.22. Definition sketch of the adsorption phenomenon

The concentration C_A in the immobile phase is generally not proportional to the transported concentration C_T because the number of adsorption sites in the substratum is not infinite. Moreover, all of the available adsorption sites are not

equally accessible to the dissolved substance. For small values of C_T , the most accessible adsorption sites are occupied first and C_A increases quickly with C_T . As C_T increases, the adsorption sites become less and less accessible and the function that relates C_A to C_T tends to an asymptote. Three main models are available for adsorption/desorption (Figure 1.23):

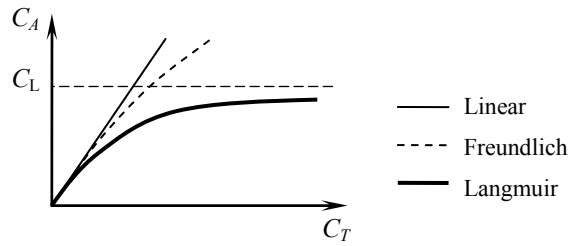


Figure 1.23. Linear, Freundlich and Langmuir adsorption isotherms

– in the linear law C_A is assumed to be directly proportional to C_T :

$$C_A = k_{\text{lin}} C_T \quad [1.123]$$

where k_{lin} is a constant;

– the Freundlich adsorption model uses the implicit assumption that the number of adsorption sites is infinite. However, the adsorption sites become less and less accessible as the concentration in the immobile phase (and therefore in the mobile phase) increases:

$$C_A = k_F C_T^b \quad [1.124]$$

where b is an exponent smaller than unity and k_F is a constant;

– the Langmuir adsorption model is based on the assumption of a limited number of equally accessible adsorption sites. The number of sites being limited, there is a physical limit to the adsorbed concentration C_A . The maximum concentration is reached asymptotically when the transported concentration C_T tends to infinity. The resulting law takes the form:

$$C_A = C_L \frac{k_L C_T}{1 + k_L C_T} \quad [1.125]$$

where C_L is the maximum permissible concentration in the immobile phase and k_L is a constant.

Consider a substance dissolved in water. The water flows in a soil with which the dissolved substance is likely to react (Figure 1.24). It is assumed that the soil is fully saturated with water, with no gas phase into which the substance may volatilize. The substance is dissolved at a concentration C_T and immobilized on the substratum at a mass concentration C_A . The conservation form of the transport equation is derived as detailed in section 1.1.2 via a balance on a control volume of unit section, the length δx of which is made arbitrarily small to lead to the differential form of the equation. The Darcy velocity (i.e. the liquid discharge per unit cross-section in the soil) is assumed to be constant in what follows.

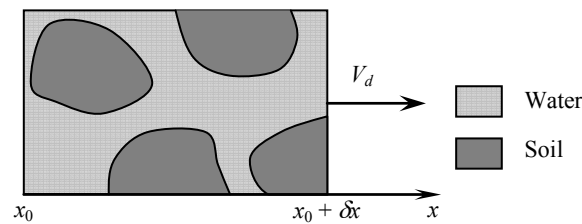


Figure 1.24. Transport and adsorption of a reactive substance in a soil

The mass M of substance contained in the control volume is the sum of two terms:

- the mass M_T of the product in the mobile phase (that is, the water) is given by

$$M_T = \theta C_T \delta x \quad [1.126]$$

where θ is the water content;

- the mass M_A of substance contained in the immobile phase is:

$$M_A = \rho_A C_A \delta x \quad [1.127]$$

where ρ_A is the soil density. The total mass of substance contained in the control volume is:

$$M = M_T + M_A = (C_T + \rho_A C_A) \delta x \quad [1.128]$$

The advection flux is given by the product of the liquid discharge and the concentration in the mobile phase, that is:

$$F = V_d C_T \quad [1.129]$$

Using the same reasoning as in section 1.1.2, the following equation is obtained for mass conservation:

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial x} (V_d C_T) = 0 \quad [1.130]$$

Since C_A is a (known) function of C_T , the quantity $C_T + \rho_A C_A$ is related to C_T by a one-to-one relationship. The so-called retardation factor R_F is introduced:

$$R_F = \frac{M}{\theta C_T} = \frac{\theta C_T + \rho_A C_A}{\theta C_T} = 1 + \frac{\rho_A C_A}{\theta C_T} \quad [1.131]$$

Equation [1.130] becomes:

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial x} \left(\frac{V_d}{\theta R_F} M \right) = 0 \quad [1.132]$$

1.7.2. Characteristic form

Differentiating the flux F with respect to the conserved variable M leads to the characteristic form of equation [1.132]:

$$\frac{\partial M}{\partial t} + \frac{V_d}{\theta R_F} \frac{\partial M}{\partial x} - \frac{V_d M}{\theta R_F^2} \frac{\partial R_F}{\partial x} = 0 \quad [1.133]$$

The derivative of the retardation factor R_F is expressed as a function of $\partial M / \partial x$:

$$\frac{\partial R_F}{\partial x} = \frac{dR_F}{dM} \frac{\partial M}{\partial x} \quad [1.134]$$

The quantity dF/dM is determined solely from M because R_F is a function of C_T (see equation [1.131]) and C_T is related to M by a one-to-one relationship like equations [1.123] to [1.125]. Substituting equation [1.134] into equation [1.133] leads to the following equation in non-conservation form:

$$\frac{\partial M}{\partial t} + \left(\frac{1}{R_F} - \frac{M}{R_F^2} \frac{dR_F}{dM} \right) \frac{V_d}{\theta} \frac{\partial M}{\partial x} = 0 \quad [1.135]$$

that can be rewritten in the form [1.17] by defining the wave speed λ as follows:

$$\lambda = \left(\frac{1}{R_F} - \frac{M}{R_F^2} \frac{dR_F}{dM} \right) \frac{V_d}{\theta} \quad [1.136]$$

Note that V_d/θ is the so-called pore velocity, that is, the speed at which the water molecules travel in the soil.

Also note that the total mass M and the concentrations C_T and C_A move at the same speed. Indeed, equation [1.135] can also be rewritten in the form [1.17] as:

$$\frac{\partial M}{\partial t} + \lambda \frac{\partial M}{\partial x} = 0 \quad [1.137]$$

Introducing the derivative of M with respect to C_T leads to:

$$\frac{\partial M}{\partial C_T} \frac{\partial C_T}{\partial t} + \lambda \frac{\partial M}{\partial C_T} \frac{\partial C_T}{\partial x} = 0 \quad [1.138]$$

which can be simplified into:

$$\frac{\partial C_T}{\partial t} + \lambda \frac{\partial C_T}{\partial x} = 0 \quad [1.139]$$

Equation [1.139] is equivalent to the following characteristic form:

$$\frac{dM}{dt} = \frac{dC_T}{dt} = 0 \quad \text{for} \quad \frac{dx}{dt} = \lambda \quad [1.140]$$

When chemicals such as pesticides are dealt with, the solute concentration remains small enough for the retardation factor to be reasonably considered to be almost constant. The wave speed λ can therefore be approximated as follows:

$$\lambda \approx \frac{V_d}{\theta R_F} \quad [1.141]$$

It should be remembered however that equation [1.141] is only an approximation and that the complete expression of R_F is given by equation [1.136]. It is clearly visible from equations [1.136] and [1.141] that the speed at which the concentration propagates is not the pore velocity but the pore velocity divided by a factor that can be approximated with R_F . This is why R_F is called the retardation factor.

1.7.3. Summary

The conservation form of the advection equation with adsorption-desorption is given by equation [1.132]. Its non-conservation form is equation [1.137], its characteristic form is equation [1.140].

The conserved variable is the total mass of product (in the mobile and in the immobile phase) per unit volume of soil. This quantity is also an invariant along the characteristic curves.

The concentration in the liquid phase being related solely to the total mass of product per unit volume of soil, it is also an invariant that moves at the same speed.

The wave speed of the concentration wave is smaller than that of the liquid in which it is transported. The wave speed is equal to the velocity of the liquid phase divided by the retardation factor.

1.8. Summary of Chapter 1

1.8.1. *What you should remember*

Scalar hyperbolic conservation laws may be written in conservation, non-conservation or characteristic form. Such equations express the movement of a given quantity at a finite speed, called the wave speed.

The conservation form is derived from a balance over a control volume, the size of which is made arbitrarily small, over a time interval that is decreased to zero so as to allow the time and space derivatives of the conserved quantity and the flux function to be introduced into the equation (see section 1.1.2).

The expression of the wave speed is obtained by differentiating the flux function with respect to the conserved variable (see section 1.1.4). The trajectory, the speed of which is the wave speed, is called a characteristic. A quantity that remains constant along a characteristic is called a Riemann invariant (see section 1.1.4).

The solution can be determined solely over a subdomain $[0, L]$ of space provided that the initial condition (that is, the value of the variable at all points of the domain) and the boundary conditions are known. Zero, one or two boundary conditions may be needed depending on the flow configuration (see section 1.2.2).

The conserved variable is not necessarily an invariant along the characteristic lines (see the examples of the linear advection in section 1.3, the inviscid Burgers equation in section 1.4, the kinematic wave in section 1.5). However, the conserved

variable may also be a Riemann invariant under certain conditions (see sections 1.5, 1.6 and 1.7).

In the general case the wave speed is different from the fluid velocity (see sections 1.5 to 1.7), although exceptions exist (see sections 1.3 and 1.4).

1.8.2. Application exercises

1.8.2.1. Exercise 1.1: the inviscid Burgers equation

Solve the Burgers equation for the following initial conditions:

- the flow velocity is uniformly equal to u_1 for $x < x_1$. It is uniformly equal to u_2 for $x > x_2 > x_1$. It increases linearly from $u_1 < u_2$ to u_2 between x_1 and x_2 ;
- the density is uniformly equal to ρ_0 over the domain.

1) Derive the analytical formula for the profiles $u(x)$ and $\rho(x)$ at $t > 0$.

2) Sketch the behavior of the solution in the physical space and in the phase space.

Indications and searching tips for the solution of this exercise can be found at the following URL: <http://vincentguinot.free.fr/waves/exercises.htm>.

1.8.2.2. Exercise 1.2: the kinematic wave equation

Derive the analytical formula for the wave speed λ in a channel, the cross-section of which is a trapezium of bottom width b_0 , the embankments of which make an angle θ with the vertical (Figure 1.25). Provide a graphical representation of λ as a function of the water depth h and carry out the numerical application for the set of parameters provided in Table 1.3. Note that the case $b_0 = 0$ corresponds to a triangular channel, in which case only positive values of θ are meaningful.

Symbol	Meaning	Value
b_0	Channel bottom width	0 m, 10 m, 40 m
K_{Str}	Strickler coefficient	40
S_0	Channel bed slope	1%
θ	Angle between the embankments and the vertical	$-30^\circ, 0^\circ, 30^\circ, 60^\circ$

Table 1.3. Parameters for Exercise 1.2

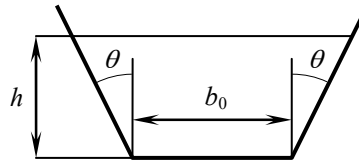


Figure 1.25. Definition sketch for the trapezoidal channel

Indications and searching tips for the solution of this exercise can be found at the following URL: <http://vincentguinot.free.fr/waves/exercises.htm>.

1.8.2.3. Exercise 1.3: the kinematic wave equation

Consider a rectangular channel where the width b and the Strickler coefficient K_{Str} are constant. The channel is divided into three zones, the boundaries of which are located at x_1 and x_2 , with $x_1 < x_2$ (Figure 1.26):

- For $x < x_1$ the slope is uniformly equal to $S_{0,1}$.
- For $x > x_2$ the slope is uniformly equal to $S_{0,2}$.
- For x between x_1 and x_2 the slope varies continuously from $S_{0,1}$ to $S_{0,2}$. No analytical expression is available for the variations of the slope as a function of the longitudinal coordinate.

At $t = 0$, the discharge Q is uniform over the domain, which is equivalent to assuming steady state.

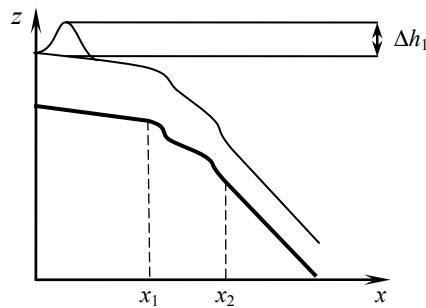


Figure 1.26. Propagation of a wave in a channel with variable slope

A discharge perturbation ΔQ_1 appears at the upstream end of the channel and propagates downstream (in the direction of positive x). The discharge perturbation is

assumed to be very small compared to the initial discharge Q , with the consequence that the wave speed can be assumed to be independent from time in the various parts of the channel.

1) Express the height Δh_1 of the perturbation in the upstream part of the channel ($x < x_1$) as a function of b , K_{Str} , Q , $S_{0,1}$ and ΔQ_1 .

2) Determine the height Δh_2 of the perturbation when it reaches $x = x_2$ and the corresponding discharge perturbation ΔQ_2 .

Carry out the numerical application for the parameter values given in Table 1.4.

Symbol	Meaning	Value
b	Channel width	40 m
K_{Str}	Strickler coefficient	40
Q	Initial discharge	$10 \text{ m}^3/\text{s}$
$S_{0,1}$	Channel slope upstream of x_1	0.1%
$S_{0,2}$	Channel slope downstream of x_2	1%
ΔQ_1	Discharge perturbation	$1 \text{ m}^3/\text{s}$

Table 1.4. Parameters for Exercise 1.3

Indications and searching tips for the solution of this exercise can be found at the following URL: <http://vincentguinot.free.fr/waves/exercises.htm>.

1.8.2.4. Exercise 1.4: the Buckley-Leverett equation

Consider an aquifer of length L contaminated by a hydrocarbon at an initial water saturation of 50% (Figure 1.27). The remediation technique consists of injecting pure water at the left-hand boundary of the aquifer ($x = 0$). The Darcy velocity V is constant.

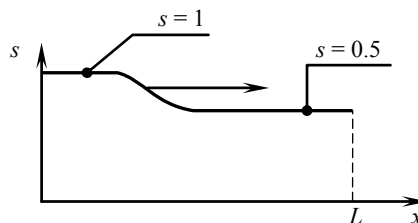


Figure 1.27. Injecting pure water to decontaminate an aquifer

1) Determine the shape and the expression of the saturation profile as a function of space (hint: try to express x as a function of s as opposed to the usual formulation $s(x)$).

2) Determine the time after which the average hydrocarbon saturation is 5%, 1% and 0.5%. Carry out the numerical application for the parameter values in Table 1.5.

Symbol	Meaning	Value
b_{BL}	Shape parameter for the Buckley-Leverett flux function	1
L	Length of the aquifer	200 m
s_0	Initial water saturation in the aquifer	0.5
s_L	Water saturation at the left-hand boundary ($x = 0$)	1
V	Darcy velocity	1 m/day

Table 1.5. Parameters for Exercise 1.4

Indications and searching tips for the solution of this exercise can be found at the following URL: <http://vincentguinot.free.fr/waves/exercises.htm>.

1.8.2.5. Exercise 1.5: linear advection with adsorption-desorption

Consider an aquifer of length L initially contaminated by a solute with an initial concentration C_0 . The solute is adsorbed on the soil grains following Langmuir's adsorption law, the parameters of which can be found in Table 1.6.

Symbol	Meaning	Value
C_0	Initial concentration in the liquid phase	1.5 kg/m ³
C_L	Maximum adsorbed mass concentration	10 ⁻⁴ g/g
k_L	Constant in Langmuir's law	5 l/g
L	Length of the aquifer	200 m
V	Darcy velocity	1 m/day
θ	Water content (porosity) in the aquifer	0.25
ρ_A	Soil bulk density	1,500 kg/m

Table 1.6. Parameters for Exercise 1.6

The aquifer is decontaminated by injecting pure water at the left-hand boundary of the aquifer starting at $t = 0$ (Figure 1.28). The Darcy velocity is denoted by V . The approximation [1.141] is assumed to be valid.

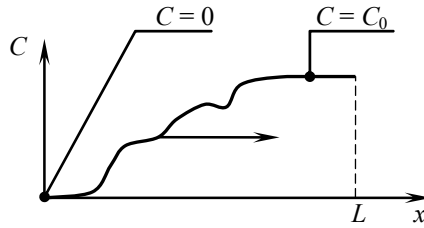


Figure 1.28. *Decontaminating an aquifer by injecting pure water*

1) Determine the expression and the numerical value of the time T_d at which the contaminant concentration starts to decrease at the right-hand boundary of the aquifer ($x = L$).

2) Determine the expression for the profile for $t > 0$ (searching tip: try to express x as a function of C_T as opposed to the more classical approach $C_T(x)$).

Indications and searching tips for the solution of this exercise can be found at the following URL: <http://vincentguinot.free.fr/waves/exercises.htm>.