Chapter 2

Hyperbolic Systems of Conservation Laws in One Dimension of Space

2.1. Definitions

2.1.1. *Hyperbolic systems of conservation laws*

A system of conservation laws is a system of partial differential equations (PDEs) that can be written in conservation form as

$$\left.\begin{aligned} &\frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial x} = S_1 \\ &\qquad\vdots \\ &\frac{\partial U_p}{\partial t} + \frac{\partial F_p}{\partial x} = S_p \\ &\qquad\vdots \\ &\frac{\partial U_m}{\partial t} + \frac{\partial F_m}{\partial x} = S_m \end{aligned}\right\} \qquad [2.1]$$

where the variables $U_1, U_2, \ldots, U_m$ are the m conserved variables, the quantities $F_1, F_2, \ldots, F_m$ are the corresponding fluxes and $S_1, S_2, \ldots, S_m$ are the corresponding source terms. System [2.1] of m equations for m unknowns is often referred to as an "$m \times m$ system". In most publications dealing with systems of conservation laws, the following vector notation is used:

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$$\frac{\partial \mathrm{U}}{\partial t}+\frac{\partial \mathrm{F}}{\partial x}=\mathrm{S} \tag{2.2}$$

where U, F and S are vectors of size *m* defined as:

$$\mathrm{U}=\begin{bmatrix} U_1 \\ \vdots \\ U_p \\ \vdots \\ U_m \end{bmatrix}, \qquad \mathrm{F}=\begin{bmatrix} F_1 \\ \vdots \\ F_p \\ \vdots \\ F_m \end{bmatrix}, \qquad \mathrm{S}=\begin{bmatrix} S_1 \\ \vdots \\ S_p \\ \vdots \\ S_m \end{bmatrix} \tag{2.3}$$

The components of the vectors F and S are functions of the components of U and, possibly, of *t* and *x*. The following notation may also be used:

$$\left.\begin{aligned} \mathrm{U} &= \left[U_1,\ldots,U_p,\ldots,U_m\right]^T \\ \mathrm{F} &= \left[F_1,\ldots,F_p,\ldots,F_m\right]^T \\ \mathrm{S} &= \left[S_1,\ldots,S_p,\ldots,S_m\right]^T \end{aligned}\right\} \tag{2.4}$$

where *T* denotes the transposition operator. The notations [2.1] and [2.2] and [2.3] are equivalent in that the derivative of a vector is the vector formed by the derivatives of its individual components. Any reader who is not familiar with the notation of linear algebra should refer to Appendix A, which summarizes the basic notation and principles of linear algebra.

Equation [2.2] is called the conservation form of the system by analogy with the scalar conservation form [1.1] introduced in Chapter 1. The conservation form [2.2] can be rewritten in conservation form as:

$$\frac{\partial \mathrm{U}}{\partial t}+\mathrm{A}\frac{\partial \mathrm{U}}{\partial x}=\mathrm{S'} \tag{2.5}$$

where A is an $m \times m$ matrix and S' is a source term, not necessarily equal to *S*. The non-conservation form [2.5] is equivalent to the conservation form [2.2] provided that A and S' are defined as follows:

$$\left.\begin{aligned} &\mathrm{dF} = \mathrm{A}\ \mathrm{dU} \\ &\mathrm{S'} = \mathrm{S}-\left(\frac{\partial \mathrm{F}}{\partial x}\right)_{\mathrm{U=Const}} \end{aligned}\right\} \tag{2.6}$$

By definition, A is the Jacobian matrix of F with respect to U. A is calculated by differentiating the components of F with respect to the components of U:

$$\mathrm{A} = \begin{bmatrix} \partial F_1/\partial U_1 & \cdots & \partial F_1/\partial U_p & \cdots & \partial F_1/\partial U_m \\ \vdots & \ddots & \vdots & & \vdots \\ \partial F_p/\partial U_1 & \cdots & \partial F_p/\partial U_p & \cdots & \partial F_p/\partial U_m \\ \vdots & & \vdots & \ddots & \vdots \\ \partial F_m/\partial U_m & \cdots & \partial F_m/\partial U_p & \cdots & \partial F_1/\partial U_m \end{bmatrix} \qquad [2.7]$$

It can be checked easily that definition [2.7] verifies the first equation [2.6]. The vector S' contains the components of the vector S and the derivatives of F that do not depend on U. S' and S are not identical in the general case.

2.1.2. *Hyperbolic systems of conservation laws – examples*

A hyperbolic system of conservation laws is a conservation law system with the following properties:

– the components of F and S are functions of the components of U and possibly of x and t, but F does not contain any derivative of U with respect to x or t;

– the Jacobian matrix A has m real, distinct eigenvalues.

Example 1: consider a river reach, the cross-section of which is rectangular, where the cross-sectional area A and the liquid discharge Q verify the kinematic wave equation [1.84] seen in section 1.5. A contaminant with a concentration C is subjected to pure advection, as described by equation [1.39] (see section 1.3). This 2×2 system can be rewritten in the vector conservation form [2.2] by defining U, F and S as follows:

$$\mathrm{U} = \begin{bmatrix} A \\ AC \end{bmatrix}, \qquad \mathrm{F} = \begin{bmatrix} Q \\ QC \end{bmatrix} = \begin{bmatrix} K_{\mathrm{Str}} b^{-2/3} S_0^{1/2} A^{5/3} \\ K_{\mathrm{Str}} b^{-2/3} S_0^{1/2} A^{5/3} C \end{bmatrix}, \qquad \mathrm{S} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad [2.8]$$

Equation [2.8] can be rewritten in the form [2.3] by defining the components of U and F as:

$$\left.\begin{aligned} U_1 &= A \\ U_2 &= AC \\ F_1 &= K_{\mathrm{Str}} b^{-2/3} S_0^{-1/2} U_1^{5/3} \\ F_2 &= K_{\mathrm{Str}} b^{-2/3} S_0^{-1/2} U_1^{2/3} U_2 \end{aligned}\right\} \qquad [2.9]$$

The Jacobian matrix A of F with respect to U is given by:

$$\mathrm{A} = \begin{bmatrix} \dfrac{5K_{\mathrm{Str}}S_0^{-1/2}}{3b^{2/3}}U_1^{2/3} & 0 \\ \dfrac{2K_{\mathrm{Str}}S_0^{-1/2}}{3b^{2/3}}U_1^{-1/3}U_2 & \dfrac{K_{\mathrm{Str}}S_0^{-1/2}}{b^{2/3}}U_1^{2/3} \end{bmatrix} \qquad [2.10]$$

It is easy to check that A has the following eigenvalues:

$$\left.\begin{aligned} \lambda^{(1)} &= \frac{K_{\mathrm{Str}}S_0^{-1/2}}{b^{2/3}}U_1^{2/3} = \frac{K_{\mathrm{Str}}S_0^{-1/2}}{b^{2/3}}A^{2/3} \\ \lambda^{(2)} &= \frac{5K_{\mathrm{Str}}S_0^{-1/2}}{3b^{2/3}}U_1^{2/3} = \frac{5}{3}\frac{K_{\mathrm{Str}}S_0^{-1/2}}{b^{2/3}}A^{2/3} \end{aligned}\right\} \qquad [2.11]$$

Note that the first eigenvalue is the flow velocity u (see equation [1.103]) and the second eigenvalue is the wave speed λ of the kinematic wave (see equation [1.106]). The eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$ are real and distinct, therefore the system is a 2×2 system of hyperbolic conservation laws.

Example 2: consider the system formed by the continuity equation [1.62] and the momentum equation [1.63]. Remember that these equations form the basis for the derivation of the inviscid Burgers equation in section 1.4.1. System [1.62–63] can be rewritten in the form [2.2] by defining the components of U and F as:

$$\left.\begin{aligned} U_1 &= \rho \\ U_2 &= \rho u \\ F_1 &= \rho u = U_1 \\ F_2 &= \rho u^2 = U_2^2 / U_1 \end{aligned}\right\} \qquad [2.12]$$

The Jacobian matrix of F with respect to U is given by:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ -\left(\dfrac{U_2}{U_1}\right)^2 & 2\dfrac{U_2}{U_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -u^2 & 2u \end{bmatrix} \qquad [2.13]$$

This matrix has the following double eigenvalue:

$$\lambda^{(1)} = \lambda^{(2)} = u \qquad [2.14]$$

System [1.62–63] is a system of conservation laws but it is not hyperbolic because its two eigenvalues are not distinct. Such a system is said to be “linearly degenerate”.

2.1.3. *Characteristic form – Riemann invariants*

The purpose is to find trajectories (called the characteristics) along which some quantities (the Riemann invariants) are constant. The present section outlines a general method for the derivation of the invariants from the Jacobian matrix A.

The starting point is the non-conservation form [2.5], recalled here:

$$\frac{\partial \mathrm{U}}{\partial t} + \mathrm{A}\frac{\partial \mathrm{U}}{\partial x} = \mathrm{S}'$$

Since the system is hyperbolic, the matrix A has m real, distinct eigenvalues with the corresponding m eigenvectors. The pth eigenvector (i.e. the eigenvector associated with the pth eigenvalue $\lambda^{(p)}$ of A) is denoted by $\mathrm{K}^{(p)}$ hereafter. The eigenvectors $\mathrm{K}^{(1)}$, $\mathrm{K}^{(2)}$, …, $\mathrm{K}^{(m)}$ can be seen as the columns of the $m \times m$ matrix of eigenvectors of A. This matrix is denoted by K.

$$\mathrm{K} = \left[\mathrm{K}^{(1)}, \mathrm{K}^{(2)}, \ldots, \mathrm{K}^{(m)}\right] = \begin{bmatrix} K_1^{(1)} & \cdots & K_1^{(m)} \\ \vdots & & \vdots \\ K_m^{(1)} & \cdots & K_m^{(m)} \end{bmatrix} \qquad [2.15]$$

where $K_i^{(p)}$ is the ith component of the pth eigenvector of A. An interesting property of K is that it allows A to be transformed into a diagonal matrix using the following matrix product

$$\mathrm{K}^{-1}\mathrm{AK} = \Lambda \qquad [2.16]$$

where K^{-1} is the inverse of K and Λ is the diagonal matrix, the elements of which are the eigenvalues of A:

$$\Lambda = \begin{bmatrix} \lambda^{(1)} & & & & \\ & \ddots & & 0 & \\ & & \lambda^{(p)} & & \\ & 0 & & \ddots & \\ & & & & \lambda^{(m)} \end{bmatrix} \qquad [2.17]$$

The Riemann invariants are introduced by left-multiplying the non-conservation form [2.5] by K^{-1}:

$$K^{-1}\frac{\partial U}{\partial t}+K^{-1}A\frac{\partial U}{\partial x}=K^{-1}S' \qquad [2.18]$$

Noting that $K\,K^{-1}$ is equal to the identity matrix, equation [2.18] can be rewritten as:

$$K^{-1}\frac{\partial U}{\partial t}+K^{-1}AK\,K^{-1}\frac{\partial U}{\partial x}=K^{-1}S' \qquad [2.19]$$

Substituting equation [2.16] into equation [2.19] leads to:

$$K^{-1}\frac{\partial U}{\partial t}+\Lambda\,K^{-1}\frac{\partial U}{\partial x}=K^{-1}S' \qquad [2.20]$$

The vectors W and S" are defined as:

$$\left.\begin{aligned} d\,W &= K^{-1}\,d\,U \\ S'' &= K^{-1}S \end{aligned}\right\} \qquad [2.21]$$

The definitions [2.21] allow equation [2.20] to be rewritten as:

$$\frac{\partial W}{\partial t}+\Lambda\frac{\partial W}{\partial x}=S'' \qquad [2.22]$$

an alternative writing for which is:

$$\frac{\partial}{\partial t}\begin{bmatrix} W_1 \\ \vdots \\ W_p \\ \vdots \\ W_m \end{bmatrix}+\begin{bmatrix} \lambda^{(1)} & & & & \\ & \ddots & & 0 & \\ & & \lambda^{(p)} & & \\ & 0 & & \ddots & \\ & & & & \lambda^{(m)} \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} W_1 \\ \vdots \\ W_p \\ \vdots \\ W_m \end{bmatrix}=\begin{bmatrix} S''_1 \\ \vdots \\ S''_p \\ \vdots \\ S''_m \end{bmatrix} \qquad [2.23]$$

where W_p is the pth component of the vector W. Equation [2.23] is equivalent to the following system of independent equations in non-conservation form:

$$\frac{\partial W_p}{\partial t}+\lambda^{(p)}\frac{\partial W_p}{\partial x}=S''_p \quad \forall p=1,2,\ldots,m \qquad [2.24]$$

As shown in section 1.1.4, equation [2.24] is equivalent to the following ordinary differential equation

$$\frac{\mathrm{d}W_p}{\mathrm{d}t} = S''_p \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda^{(p)} \qquad \forall p = 1,2,\ldots,m \qquad [2.25]$$

The m equations [2.25] form the characteristic form of equation [2.2]. When S" = 0, system [2.25] simplifies into:

$$W_p = \text{Const} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda^{(p)} \qquad \forall p = 1,2,\ldots,m \qquad [2.26]$$

The quantity W_p is called the pth Riemann invariant. Note that if U is known at a given point (x, t) in the phase space, the m Riemann invariants can be calculated by integrating the first equation [2.21]. Conversely, if the m Riemann invariants are known, U can be determined by integrating the reciprocal relationship:

$$\mathrm{dU} = \mathrm{K\ dW} \qquad [2.27]$$

Equation [2.27] can also be rewritten as:

$$\mathrm{dU}_p = \sum_{k=1}^{m} K_{p,k}\ \mathrm{dW}_k = \sum_{k=1}^{m} K_p^{(k)}\ \mathrm{dW}_k \qquad [2.28]$$

where $K_{p,k} = K_k^{(p)}$ denotes the element on the pth row and the kth column of K, that is, the pth component of the kth eigenvector of A. The set of equivalent relationships [2.21] and [2.27–28] indicate that the variation dW is nothing but the expression of the variation dU in the base of eigenvectors of A.

2.2. Determination of the solution

2.2.1. *Domain of influence, domain of dependence*

The characteristic relationships [2.25] naturally lead to the notions of domain of influence and domain of dependence. Consider a point A (x_0, t_0) in the phase space at which U is known. Along each of the m characteristic lines that pass through A, a relationship [2.25] is verified (Figure 2.1). U being known at A, each of the invariants W_p, defined as in equation [2.21], are also known at A. If the ordinary differential equation [2.25] can be solved for a given p, the value of W_p can be determined at any time $t_1 > t_0$ along the pth characteristic.

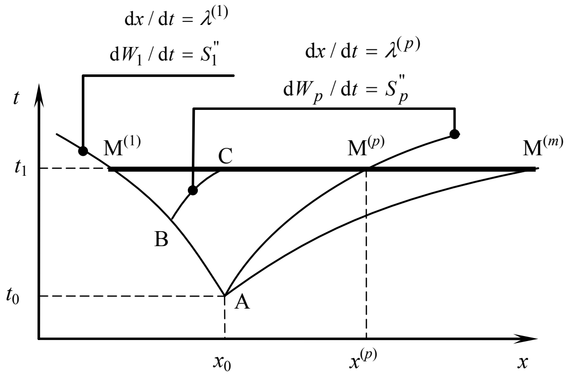

Figure 2.1. *Domain of influence (bold line). Representation in the phase space*

In what follows, $M^{(p)}$ denotes the intersection between the pth characteristic and the line $t = t_1$ in the phase space (Figure 2.1). The wave speeds $\lambda^{(p)}$ being ranked in ascending order, the points $M^{(1)}$ and $M^{(m)}$ are respectively the leftmost and rightmost points on the line $t = t_1$. The following reasoning is made:

– The values of all the invariants W_p at the point A are determined directly from the value of U at A because the variations in U and W obey equation [2.21]. Therefore the value of W_1 stems directly from that of U.

– The value of W_p at the point $M^{(1)}$ is a direct function of its value at A via equation [2.25].

– The value of W_p at $M^{(1)}$ influences that of U via equation [2.27].

Consequently, the value of U at the point A influences the value of U at the point $M^{(1)}$ via the first Riemann invariant. Reproducing this reasoning for any characteristic $dx/dt = \lambda^{(p)}$ leads to the conclusion that the value of U at A influences the value of U at all the points $M^{(p)}$.

Consider now a point B located on the first characteristic passing at A. The point B is located at a time between t_0 and t_1. m characteristics can be drawn from B. The pth characteristic issued from B intersects the line $t = t_1$ at the point C. Reproducing the reasoning above along (AB) and (BC) successively leads to the conclusion that the value of U at A influences the value of U at C. This reasoning can be generalized to all the points B located on all the characteristics issued from A and to all the points C located on all the characteristics issued from all the possible locations of B. Since the possible locations of C span the segment $[M^{(1)}M^{(m)}]$, the point A influences the value of U over this segment. The segment $[M^{(1)}M^{(m)}]$ is called the domain of influence of point A.

In most analytical and numerical solution methods, the important issue is to determine the region of space over which U should be known at a given time t_0 in order to be computable at a later time t_1. The set of points that influence point A is called the domain of dependence (Figure 2.2).

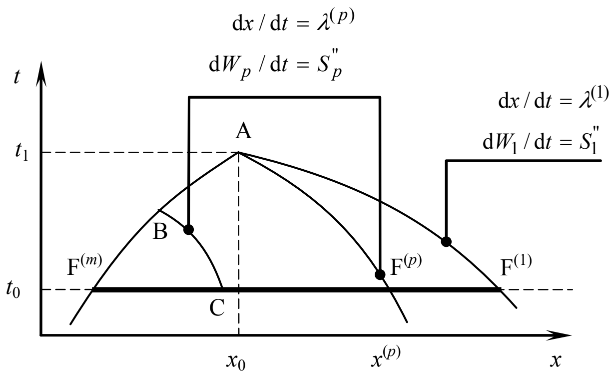

Figure 2.2. *Domain of dependence (bold line). Representation in the phase space*

The domain of dependence is determined in the same way as the domain of influence, except that the characteristics are followed backward in time. Let $F^{(p)}$ be the point at the intersection between the *p*th characteristic that passes through A and the line $t = t_0$. The point $F^{(p)}$ is called the foot of the *p*th characteristic. Using the same reasoning as in the previous two sections leads to the conclusion that the value of U at point A is influenced by the value of U at all the points between $F^{(1)}$ and $F^{(m)}$. The segment $[F^{(m)}F^{(1)}]$ is called the domain of dependence of A.

2.2.2. *Existence and uniqueness of solutions – initial and boundary conditions*

This section deals with the existence and uniqueness of the solutions of hyperbolic systems of conservation laws. Assume that U is to be determined at the point $A(x_0, t_1)$ in Figure 2.2. To do so, the *m* components $U_1, U_2, \ldots, U_m$ must be determined uniquely. Therefore, *m* independent equations in the form [2.25] must be written for these *m* components.

Most practical applications deal with a finite domain of length *L*, called the computational domain. For the sake of clarity, the point $x = 0$ is located at the left-hand boundary of the domain, the right-hand boundary of which is located at $x = L$ (Figure 2.3). Two configurations may occur:

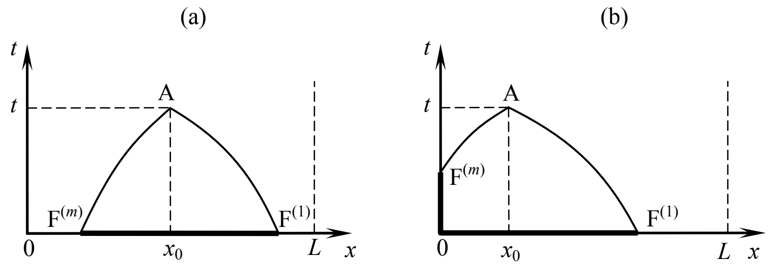

Figure 2.3. *Determination of the solutions when the domain of dependence is included entirely in the computational domain (a) and when the domain of dependence includes the domain boundary (b)*

– The computational domain $[0, L]$ contains entirely the domain of dependence of A (Figure 2.3a). In this case, the knowledge of U over the domain of dependence at $t = 0$ allows the value of U to be determined uniquely at A by writing the m characteristic equations [2.25] between the feet $F^{(p)}$ and A. The m equations [2.25] allow the m Riemann invariants to be determined at A. The knowledge of the m Riemann invariants leads to that of U via relationships [2.27].

– There exists points (in particular near the boundaries of the computational domain), the domain of dependence of which is not entirely included in the segment $[0, L]$ (see Figure 2.3b). In such a case, knowing the value of U over the segment $[0, L]$ is insufficient because there is at least one (and possibly more than one) characteristic entering the domain across a boundary. In Figure 2.3, the mth invariant can be determined at A only if its value is known at the left-hand boundary. The value of the invariant is known at $F^{(m)}$ provided that U is known at $F^{(m)}$. U is known at $F^{(m)}$ only if the m Riemann invariants are known. Let n_e denote the number of Riemann invariants that enter the domain at the boundary. In order to determine U uniquely at the boundary, n_e conditions must be supplied on the components of U, the remaining $m - n_e$ conditions being supplied by the invariants that travel to $F^{(m)}$ from within the domain.

To summarize, the following conditions must be prescribed for the existence and uniqueness of the solution to be guaranteed over the domain:

– the initial condition U $(x, 0)$ (that is, all the components U_p of U) must be known over the entire domain $[0, L]$ at $t = 0$;

– boundary conditions must be prescribed at the boundaries of the computational domain. The number of boundary conditions needed at a given boundary is equal to the number of characteristics that enter the domain.

2.3. A particular case: compressible flows

2.3.1. *Definition*

Compressible flows represent a vast majority of the flows that can be described using hyperbolic systems of conservation laws. The term “compressible flow” refers to flows with the following characteristics:

– the governing equations contain at least the continuity and the momentum equations (there may be additional equations in the system);

– the momentum equation includes a term that accounts for the pressure forces within the fluid;

– the pressure in the fluid is a function of at least the fluid density (or the mass of fluid per unit length of volume if a one-dimensional configuration is dealt with; the mass per unit surface if a two-dimensional configuration is dealt with). The law that relates the pressure to the density (and possibly to other variables) is called an equation of state;

– the system of conservation laws is hyperbolic.

2.3.2. *Conservation form*

The simplest possible system for compressible flows is a 2×2 system, that is, a system with two equations (the continuity equation and the momentum equation) in two independent variables. The equations for such a system are written by defining a control volume between the abscissas x_0 and $x_0 + \delta x$ and carrying out a mass and momentum balance between the times t_0 and $t_0 + \delta t$ (Figure 2.4).

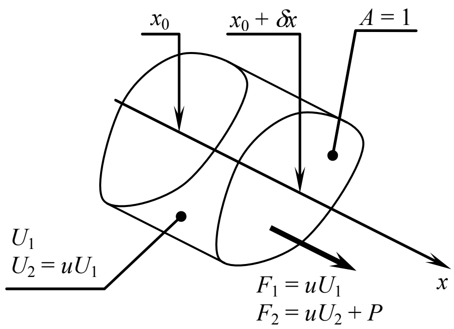

Figure 2.4. *Mass and momentum balance over a control volume*

The continuity equation is derived as in Chapter 1 by noting that the mass per unit length of domain (denoted by U_1 hereafter) is transported at the flow velocity u, yielding a momentum flux F_1:

$$F_1 = uU_1 \tag{2.29}$$

As shown in Chapter 1, the continuity equation is written in conservation form as:

$$\frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial x} = 0 \tag{2.30}$$

The momentum U_2 per unit length in the domain is defined as:

$$U_2 = uU_1 \tag{2.31}$$

Consequently the total amount of momentum in the control volume at time t is given by:

$$\delta U_2(t) = \int_{x_0}^{x_0+\delta x} U_2(x,t)\mathrm{d}x \tag{2.32}$$

Between the times t_0 and $t_0 + \delta x$ an amount $\delta F_M(x_0)$ of momentum enters the control volume at the left-hand boundary:

$$\delta F_{\mathrm{QdM}}(x_0) = \int_{t_0}^{t_0+\delta t} uU_2(x_0,t)\mathrm{d}t \tag{2.33}$$

During the same time interval, an amount $\delta F_M(x_0 + \delta x)$ of momentum leaves the control volume at the right-hand boundary:

$$\delta F_{\mathrm{QdM}}(x_0 + \delta x) = \int_{t_0}^{t_0+\delta t} uU_2(x_0 + \delta x, t)\mathrm{d}t \tag{2.34}$$

Denoting by $P(x_0)$ the pressure force exerted on the left-hand boundary ($x = x_0$), the pressure force exerted on the right-hand boundary ($x = x_0 + \delta x$) is $-P(x_0 + \delta x)$. The fundamental principle of dynamics can be written as:

$$\begin{aligned}\delta U_2(t_0 + \delta t) - \delta U_2(t_0) = {} & \delta F_{\mathrm{QdM}}(x_0) - \delta F_{\mathrm{QdM}}(x_0 + \delta x) \\ & + \int_{t_0}^{t_0+\delta t} [P(x_0,t) - P(x_0 + \delta x, t)]\mathrm{d}t \\ & + \int_{t_0}^{t_0+\delta t} \int_{x_0}^{x_0+\delta x} S_2(x,t)\mathrm{d}x\mathrm{d}t\end{aligned} \tag{2.35}$$

where S_2 is a source term due to external forces such as volume or wall forces. When δt and δx tend to zero the following equivalences hold:

$$\left.\begin{aligned}
\delta U_2(t_0+\delta t)-\delta U_2(t_0) &\underset{\substack{\delta x\to 0\\ \delta t\to 0}}{\approx} \delta x\,\delta t\,\frac{\partial U_2}{\partial t}\\
\delta F_{\mathrm{QdM}}(x_0)-\delta F_{\mathrm{QdM}}(x_0+\delta x) &\underset{\substack{\delta x\to 0\\ \delta t\to 0}}{\approx} -\delta x\,\delta t\,\frac{\partial}{\partial x}(uU_2)\\
\int_{t_0}^{t_0+\delta t}[P(x_0,t)-P(x_0+\delta x,t)]\mathrm{d}t &\underset{\substack{\delta x\to 0\\ \delta t\to 0}}{\approx} -\delta x\,\delta t\,\frac{\partial P}{\partial x}\\
\int_{t_0}^{t_0+\delta t}\int_{x_0}^{x_0+\delta x} S_2(x,t)\mathrm{d}x\,\mathrm{d}t &\underset{\substack{\delta x\to 0\\ \delta t\to 0}}{\approx} \delta x\,\delta t\,S_2
\end{aligned}\right\}\qquad [2.36]$$

Substituting equations [2.36] into equation [2.35] and simplifying by δt and δx yields:

$$\frac{\partial U_2}{\partial t}+\frac{\partial F_{\mathrm{QdM}}}{\partial x}+\frac{\partial P}{\partial x}=S_2 \qquad [2.37]$$

The flux F_2 (also known as the "specific force" [CHO 73]) is defined as:

$$F_2=F_{\mathrm{QdM}}+P=uU_2+P \qquad [2.38]$$

Equation [2.38] allows equation [2.37] to be written in conservation form as:

$$\frac{\partial U_2}{\partial t}+\frac{\partial F_2}{\partial x}=S_2 \qquad [2.39]$$

Equations [2.30] and [2.39] can be written in the vector form [2.1], recalled here:

$$\frac{\partial \mathrm{U}}{\partial t}+\frac{\partial \mathrm{F}}{\partial x}=\mathrm{S}$$

by defining U, F and S as:

$$\mathrm{U}=\begin{bmatrix}U_1\\ U_2\end{bmatrix},\qquad \mathrm{F}=\begin{bmatrix}uU_1\\ uU_2+P\end{bmatrix},\quad \mathrm{S}=\begin{bmatrix}0\\ S_2\end{bmatrix} \qquad [2.40]$$

with the additional relationship [2.29], that can be rewritten as:

$$\frac{U_2}{U_1} = u \qquad [2.41]$$

2.3.3. *Characteristic form*

The characteristic form is derived as follows. In a first step the expression of the Jacobian matrix of F with respect to U is determined. To do this, the components of F must be rewritten as functions of the components of U. Substituting equation [2.41] into definition [2.40] leads to the following expression:

$$\mathrm{F} = \begin{bmatrix} U_2 \\ U_2^2 / U_1 + P \end{bmatrix} \qquad [2.42]$$

Under the assumption of a compressible flow, the pressure force P is a function of U_1. The equation of state $P = P(U)$ is assumed to be known. The conservation form [2.2] is equivalent to the non-conservation form [2.5], recalled here:

$$\frac{\partial \mathrm{U}}{\partial t} + \mathrm{A}\frac{\partial \mathrm{U}}{\partial x} = \mathrm{S}'$$

provided that A is the Jacobian matrix of F with respect to U. The expression of A stems directly from equation [2.42]:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ \dfrac{\partial P}{\partial U_1} - \left(\dfrac{U_2}{U_1}\right)^2 & 2\dfrac{U_2}{U_1} \end{bmatrix} \qquad [2.43]$$

The quantity c is defined as:

$$c = \left(\frac{\partial P}{\partial U_1}\right)^{1/2} \qquad [2.44]$$

Substituting equations [2.41] and [2.44] into equation [2.43] leads to:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix} \qquad [2.45]$$

The eigenvalues of A are:

$$\left.\begin{aligned} \lambda^{(1)} &= u - c \\ \lambda^{(2)} &= u + c \end{aligned}\right\} \qquad [2.46]$$

with corresponding eigenvectors $K^{(1)}$ and $K^{(2)}$ given by:

$$K^{(1)} = \begin{bmatrix} 1 \\ u - c \end{bmatrix}, \quad K^{(1)} = \begin{bmatrix} 1 \\ u + c \end{bmatrix} \qquad [2.47]$$

Therefore the matrix K and its inverse are:

$$K = \begin{bmatrix} 1 & 1 \\ u - c & u + c \end{bmatrix}, \qquad K^{-1} = \frac{1}{2c}\begin{bmatrix} c + u & -1 \\ c - u & 1 \end{bmatrix} \qquad [2.48]$$

The Riemann invariants are given by the first relationship [2.21]:

$$\left.\begin{aligned} \mathrm{d}W_1 &= \left(\frac{1}{2} + \frac{u}{2c}\right)\mathrm{d}U_1 - \frac{1}{2}\mathrm{d}U_2 \\ \mathrm{d}W_2 &= \left(\frac{1}{2} - \frac{u}{2c}\right)\mathrm{d}U_1 + \frac{1}{2}\mathrm{d}U_2 \end{aligned}\right\} \qquad [2.49]$$

2.3.4. *Physical interpretation*

Consider a fluid initially at rest ($u = 0$). Due to some reason (for example, a transient originating from an external forcing), a perturbation ΔU_1 appears in the fluid at the abscissa x_0. This perturbation triggers a perturbation in the Riemann invariants W_1 and W_2. The perturbation in W_1 propagates at a speed $\lambda^{(1)}$, while the perturbation in W_2 propagates at a speed $\lambda^{(2)}$. If ΔU_1 is small, u remains negligible compared to c, and $\lambda^{(1)}$ and $\lambda^{(2)}$ are approximately equal to $-c$ and $+c$ respectively. Consequently, the initial perturbation in U_1 triggers two perturbations that propagate at the same speed in opposite directions (Figure 2.5). Since the pressure is a function of P, c is also called the speed of the pressure waves. In the case of a compressible gas or liquid, c is referred to as the speed of sound.

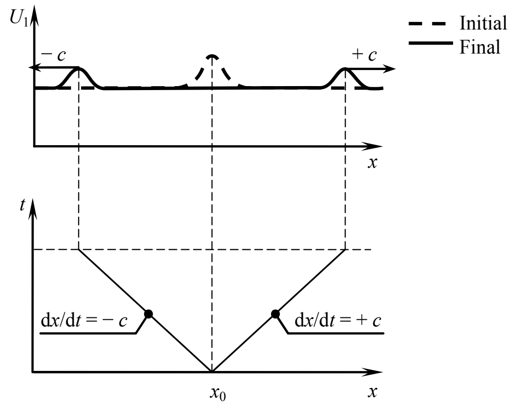

Figure 2.5. *Compressible flow. Wave propagation in a fluid initially at rest*

2.4. A linear 2×2 system: the water hammer equations

2.4.1. *Physical context – assumptions*

The water hammer equations are the simplest existing equations for compressible systems. Indeed, these equations are linear and the momentum equation is simplified by neglecting inertia. Although the first version of the complete equations is usually attributed to Joukowski [JOU 98], they are best known as the Allievi equations [ALL 03]. The derivation of these equations has been widely addressed in the literature [JAE 33, JAE 77, WYL 77].

Water hammer is a wave propagation phenomenon that occurs in systems of pressurized conduits (e.g. a drinking water supply network, pipes in hydropower plants) under rapid changes in the flow conditions. Such changes may result from the sudden opening (or closing) of a valve, pipe failure, etc., that leads to rapid variations in the discharge or pressure (or both) at a given point in the network. When the flow conditions are modified almost instantaneously, dynamical equilibrium is no longer ensured. The fluid is subjected to strong accelerations within a few hundredths or tenths of a second. Such accelerations result in local mass accumulation or deficit, depending on the case. Owing to the small compressibility of the fluid and the rigidity of the pipe material, the locally accumulating water is subjected to a local compression, while deficit regions are characterized by a pressure drop. The pressure fluctuations propagate in the pipe to yield pressure waves. The pressure variations may easily reach several atmospheres or tens of atmospheres, which may damage the pipe network severely. The simulation of water hammer episodes resulting from pipe, pump or valve failure is an important task in the design of protection devices for pressurized pipe networks.

The following example illustrates the water hammer phenomenon in a schematic way. Consider steady state water flow in a pipe at a uniform pressure p_0 and a constant discharge Q_0. The water is pumped into the pipe at the left-hand end and leaves the pipe at the right-hand end (Figure 2.6a). At $t = 0$ the right-hand side of the pipe is closed instantaneously. The water is still being pumped at a discharge Q_0 at the left-hand end of the pipe and the discharge is still equal to Q_0 everywhere in the pipe, including immediately upstream of the downstream valve. Continuity imposes that the water that reaches the valve accumulates in the pipe next to the valve, thus creating a local pressure rise and the inflation of the conduit (Figure 2.6b). The pressure in the inflated zone is higher than the initial pressure p_0, while continuity with the downstream section of the pipe imposes that the discharge be zero. Since the water keeps flowing to the upstream end of the inflated zone, it accumulates at its boundary. The boundary of the inflated zone where the water is immobile moves to the left to accommodate the incoming water (Figure 2.6b). The inflated zone eventually reaches the left-hand end of the pipe (Figure 2.6c). The pressure wave propagates at the speed $-c$, the typical order of magnitude of which is a kilometer per second.

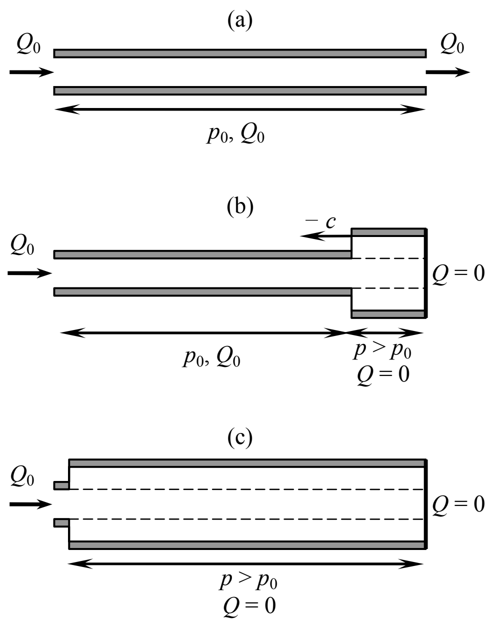

Figure 2.6. *Sudden closure of a valve at the downstream end of a pipe and resulting water hammer phenomenon. Initial state (a), next to the closure of the valve (b), after the wave reaches the upstream end of the pipe (c). The dashed line indicates the initial size of the pipe*

The governing assumptions of the water hammer phenomenon are the following:

– Assumption (A1). The pressure variations in a pipe may reach several millions of Pa, while the typical variations in the flow speed may be 1 m/s. An order of magnitude analysis indicates that the inertial term is negligible compared to the pressure term in the momentum equation [2.37].

– Assumption (A2). The fluid is (slightly) compressible and the conduit is (slightly) elastic. This yields a linear dependence between the pressure and the mass of fluid per unit length. The corresponding relationship, expressed as in equation [2.44], leads to the conclusion that the speed of sound c is a constant. Note that a typical order of magnitude for c is 1 km/s.

– Assumption (A3). Owing to high rigidity, the conduit is only slightly deformable, even under sever transients. The variations in the pressure force P are mainly due to the variations in the pressure, not to the variations in the cross-sectional area of the pipe.

– Assumption (A4). Owing to the weak compressibility of the fluid, the relative variations in the fluid density are very small. Consequently, the variations in the mass discharge are mainly due to the variations in the flow speed.

2.4.2. *Conservation form*

2.4.2.1. *Notation*

The water hammer phenomenon can be described using the continuity and momentum equations in one dimension of space. The assumption of a one-dimensional flow is valid because of the high contrast between the length (typically, tens to hundreds of meters, sometimes kilometers) and the diameter of the pipes (typically a few centimeters to a meter). The equations are derived in conservation form for pipes of arbitrary shape, with a variable cross-section and a non-horizontal axis. Figure 2.7 illustrates the notation and the coordinate system used.

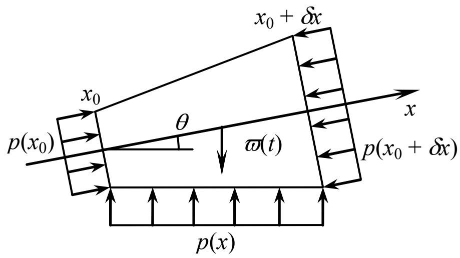

Figure 2.7. *Derivation of the water hammer equations. Definition sketch for the geometry*

The x-axis is that of the pipe. The angle between the x-axis and the horizontal is denoted by θ, with the convention that θ is positive when the elevation increases with x. The cross-section of the pipe is denoted by A. A mass and momentum balance is carried out between the times t_0 and $t_0 + \delta t$ over a slice of pipe delineated by $x = x_0$ and $x = x_0 + \delta x$.

2.4.2.2. *Continuity equation*

The mass balance can be written as follows:

$$\delta U_1(t_0 + \delta t) - \delta U_1(t_0) = \delta F_1(x_0) - \delta F_1(x_0 + \delta x) \qquad [2.50]$$

where $\delta U_1(t)$ is the mass of fluid contained in the control volume at the time t and $\delta F_1(x)$ is the mass of fluid that passes at the abscissa x over the time interval δt. Equation [2.50] expresses that the variation in the mass of the control volume is due to the difference between the mass that enters and the mass that leaves the control volume. By definition, $\delta U_1(t)$ and $\delta F_1(x)$ are given by:

$$\left.\begin{aligned} \delta U_1(t) &= \int_{x_0}^{x_0+\delta x} (\rho A)(x,t)\,\mathrm{d}x \\ \delta F_1(x) &= \int_{t_0}^{t_0+\delta t} (\rho u A)(x,t)\,\mathrm{d}x \end{aligned}\right\} \qquad [2.51]$$

where u is the fluid velocity and ρ is the fluid density. Substituting equation [2.51] into equation [2.50], introducing the derivatives with respect to time and space leads, in the limit of vanishing δt and δx, to the so-called continuity equation (see equations [1.12–16] for details of the proof):

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho u A) = 0 \qquad [2.52]$$

Defining the first component U_1 of the conserved variable as $U_1 = \rho A$, equation [2.52] can be rewritten as

$$\left.\begin{aligned} \frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial x} &= 0 \\ U_1 &= \rho A \\ F_1 &= u U_1 \end{aligned}\right\} \qquad [2.53]$$

2.4.2.3. *Momentum equation*

The momentum balance is given by the fundamental principle of dynamics:

$$\begin{aligned}\delta U_2(t_0+\delta t)-\delta U_2(t_0) = {} & \delta F_2(x_0)-\delta F_2(x_0+\delta x)\\ & +\delta P(x_0)-\delta P(x_0+\delta x)+\delta F_V+\delta F_P\end{aligned} \quad [2.54]$$

where $\delta U_2(t)$ is the total momentum of the fluid over the control volume at the time t and $\delta F_2(x)$ is the momentum of the volume of fluid that passes at x over the time interval δt. $\delta P(x)$ is the integral of the pressure force exerted on the pipe cross-section at x over the time interval δt. δF_V and δF_W are respectively the integrals of the volume forces and the forces exerted by the walls on the control volume over the time interval δt. By definition, δU_2 and δF_2 are given by:

$$\left.\begin{aligned}\delta U_2(t) &= \int_{x_0}^{x_0+\delta x}(\rho uA)(x,t)\mathrm{d}x\\ \delta F_2(x) &= \int_{t_0}^{t_0+\delta t}(\rho u^2A)(x,t)\mathrm{d}x\end{aligned}\right\} \quad [2.55]$$

The pressure force $P(x)$ being defined as the product of the pipe cross-section and the average pressure over the cross-section (Figure 2.7), δP is given by:

$$\delta P(t) = \int_{t_0}^{t_0+\delta t}(Ap)(x,t)\mathrm{d}t \quad [2.56]$$

The only volume force is the weight of the control volume:

$$\varpi(t) = \int_{x_0}^{x_0+\delta x} g(\rho A)(x,t)\mathrm{d}x \quad [2.57]$$

The weight of the control volume is a vector collinear with the vertical. Since the momentum balance [2.54] is written in the x-direction, δF_V is the projection of the integral of $\varpi(t)$ between t_0 and $t_0+\delta t$ onto the x–axis:

$$\delta F_V(t) = \int_{t_0}^{t_0+\delta t} -\varpi(t)\sin\theta\,\mathrm{d}t = -g\sin\theta\int_{t_0}^{t_0+\delta t}\int_{x_0}^{x_0+\delta x}(\rho A)(x,t)\mathrm{d}x\,\mathrm{d}t \quad [2.58]$$

The integral of the forces exerted by the walls can be broken into two terms:

$$\delta F_W(t) = \int_{t_0}^{t_0+\delta t} \int_{x_0}^{x_0+\delta x} (R_p + R_f)(x,t)\,\mathrm{d}x \qquad [2.59]$$

where R_f and R_p are respectively the forces due to friction against the pipe wall and the projection of the reaction of the wall onto the x-axis. Under the assumption of turbulent flow, R_f is classically assumed to be proportional to the square of the flow velocity. The friction force is exerted in the opposite direction to that of the flow, which leads to an expression in the form:

$$R_f = -k|u|u \qquad [2.60]$$

The term R_p is equal to the projection of the reaction of the wall to the pressure force onto the x-axis. The reaction of the wall stems from the pressure exerted by the fluid in the direction of the normal unit vector to the wall. According to the theorem of action and reaction, the reaction exerted by the wall on the fluid is the opposite of the force exerted by the fluid on the wall (Figure 2.8).

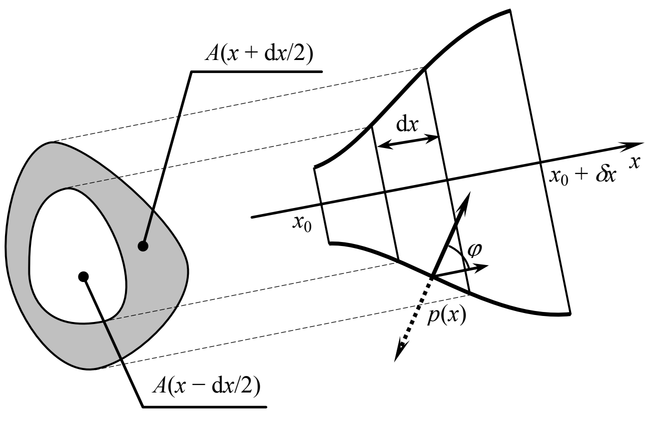

Figure 2.8. *Pressure force exerted by the fluid on the wall (dashed arrow) and reaction of the wall on the fluid (solid arrow)*

The projection of R_p onto the x-axis is determined by multiplying its norm by the cosine of the angle φ between the wall and the x-axis (Figure 2.8). The expression of the integral of the projected vector over the wall of the entire control volume may become rather complex when the shape of the pipe is complex. However, the developments are kept simple by noting that the only quantity of interest is the projection of the reaction onto the x-axis. The x-component of the pressure force exerted on a slice of pipe of infinitesimal length dx centered around the abscissa x is the product of the pressure $p(x)$ over an infinitesimal area dA in the plane normal to the x-axis. dA is the variation of the cross-sectional area between x and x + dx:

$$\mathrm{d}A = A(x + \mathrm{d}x/2) - A(x - \mathrm{d}x/2) = \frac{\partial A}{\partial x}\mathrm{d}x \quad [2.61]$$

Integrating the pressure force over the control volume leads to:

$$\int_{x_0}^{x_0+\delta x} R_p(x,t)\mathrm{d}x = \int_{x_0}^{x_0+\delta x} \frac{\partial A}{\partial x} p(x,t)\mathrm{d}x \quad [2.62]$$

Substituting equations [2.60] and [2.62] into equation [2.59] leads to:

$$\delta F_P(t) = \int_{t_0}^{t_0+\delta t}\int_{x_0}^{x_0+\delta x}\left(-k|u|u + \frac{\partial A}{\partial x}p\right)(x,t)\mathrm{d}x\,\mathrm{d}t \quad [2.63]$$

Substituting equations [2.55], [2.56], [2.58] and [2.53] into [2.54] and reasoning as in equations [1.12–16], the momentum equation is written as:

$$\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}(\rho u^2 A + P) = p\frac{\partial A}{\partial x} - g\rho A\sin\theta - k|u|u \quad [2.64]$$

where the pressure force P is equal to the product Ap. Recalling definition [2.53], the expression of the second component of the conserved variable becomes:

$$\left.\begin{aligned} U_2 &= \rho Q = F_1 \\ \rho u^2 A &= \rho\frac{Q^2}{A} = \frac{U_2^2}{U_1} \end{aligned}\right\} \quad [2.65]$$

2.4.2.4. *Simplification – vector form*

The expression of the second component F_2 of the flux vector can be simplified by recalling Assumption (A1) stated in section 2.4.1. Neglecting the momentum flux $\rho Q^2/A$ with respect to the pressure terms in the momentum equation leads to:

$$\frac{\partial}{\partial x}\left(\rho\frac{Q^2}{A}\right) << \frac{\partial P}{\partial x} \quad [2.66]$$

and equation [2.64] is simplified into:

$$\frac{\partial}{\partial t}(\rho Au) + \frac{\partial P}{\partial x} = p\frac{\partial A}{\partial x} - g\rho A\sin\theta - k|u|u \qquad [2.67]$$

Equations [2.52] and [2.67] can be written in the vector form [2.2], recalled here:

$$\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} = \mathrm{S}$$

provided that U, F and S are defined as:

$$\mathrm{U} = \begin{bmatrix} \rho A \\ \rho Q \end{bmatrix}, \quad \mathrm{F} = \begin{bmatrix} \rho Q \\ Ap \end{bmatrix}, \quad \mathrm{S} = \begin{bmatrix} 0 \\ p\ \partial A / \partial x - \rho g A \sin\theta - k|u|u \end{bmatrix} \qquad [2.68]$$

2.4.3. *Characteristic form – Riemann invariants*

System [2.2] can be rewritten in the non-conservation form [2.5], recalled here:

$$\frac{\partial \mathrm{U}}{\partial t} + \mathrm{A}\frac{\partial \mathrm{U}}{\partial x} = \mathrm{S}'$$

where A is the Jacobian matrix of F with respect to U and S' = S. The definitions [2.68] lead to the following expression for A:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ c^2 & 0 \end{bmatrix} \qquad [2.69]$$

where c is defined as in equation [2.44], recalled here:

$$c = \left(\frac{\partial P}{\partial U_1}\right)^{1/2} = \left(\frac{\partial (Ap)}{\partial (\rho A)}\right)^{1/2}$$

From Assumption (A2) stated in section 2.4.1, the variations in the pressure force Ap are proportional to those in the mass per unit length ρA. Consequently, c does not depend on the flow variables, it is a constant that depends only on the local properties of the pipe, such as its diameter, thickness and material.

The matrix A has the following eigenvalues and eigenvectors:

$$\lambda^{(1)} = -c, \qquad \lambda^{(2)} = c$$
$$\mathrm{K}^{(1)} = \begin{bmatrix} 1 \\ -c \end{bmatrix}, \qquad \mathrm{K}^{(2)} = \begin{bmatrix} 1 \\ c \end{bmatrix} \qquad [2.70]$$

The eigenvalues being real and distinct, system [2.2, 2.68] is hyperbolic. The variations in the pressure force being related to those of the mass per unit length, the system is compressible.

The characteristic form is obtained by multiplying equation [2.5] by the matrix K^{-1} and noting that A is transformed into the diagonal matrix Λ in the base formed by its eigenvectors (see equations [2.18–22] in section 2.1.3). This leads to equation [2.22], recalled here:

$$\frac{\partial \mathrm{W}}{\partial t} + \Lambda \frac{\partial \mathrm{W}}{\partial x} = \mathrm{S}'$$

where the vector dW is defined as in the first equation [2.21], recalled here:

$$\mathrm{d\,W} = \mathrm{K}^{-1}\,\mathrm{d\,U}$$

Substituting equations [2.68] and [2.70] into equation [2.21] and multiplying by $2c$, the following expression is obtained for dW and S":

$$\left.\begin{aligned} \mathrm{dW} &= 2c\begin{bmatrix} \frac{1}{2} & -\frac{1}{2c} \\ \frac{1}{2} & \frac{1}{2c} \end{bmatrix}\begin{bmatrix} \mathrm{d}U_1 \\ \mathrm{d}U_2 \end{bmatrix} = \begin{bmatrix} c\ \mathrm{d}(\rho A) - \mathrm{d}(\rho Q) \\ c\ \mathrm{d}(\rho A) + \mathrm{d}(\rho Q) \end{bmatrix} = \begin{bmatrix} \mathrm{d}W_1 \\ \mathrm{d}W_2 \end{bmatrix} \\ \mathrm{S}'' &= 2c\mathrm{K}^{-1}\mathrm{S} = \frac{1}{2c}\begin{bmatrix} k|u|u + \rho g A \sin\theta - \frac{\partial A}{\partial x}p \\ -k|u|u - \rho g A \sin\theta + \frac{\partial A}{\partial x}p \end{bmatrix} \end{aligned}\right\} \qquad [2.71]$$

Equation [2.22] can be rewritten as:

$$\frac{\partial W_p}{\partial t} + \lambda^{(p)} \frac{\partial W_p}{\partial x} = S''_p, \quad p = 1{,}2 \qquad [2.72]$$

The mass per unit length and the pressure force are not directly measurable in practice. Engineers usually deal with the pressure and the liquid discharge that can be easily measured using manometers and flow meters. The Riemann invariants are rewritten so as to involve p and Q. Equations [2.44] allow the following relationships to be written:

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(\rho A) = \frac{1}{c^2}\frac{\partial}{\partial t}(Ap)\\&\frac{\partial}{\partial x}(\rho A) = \frac{1}{c^2}\frac{\partial}{\partial x}(Ap) = \frac{A}{c^2}\frac{\partial p}{\partial x} + \frac{p}{c^2}\frac{\partial A}{\partial x}\end{aligned}\right\} \quad [2.73]$$

From Assumption (A3), the variations in the pressure force (Ap) are mainly due to the variations in p. Equation [2.73] simplifies into:

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(\rho A) \approx \frac{A}{c^2}\frac{\partial p}{\partial t}\\&\frac{\partial}{\partial x}(\rho A) = \frac{A}{c^2}\frac{\partial p}{\partial x} + \frac{p}{c^2}\frac{\partial A}{\partial x}\end{aligned}\right\} \quad [2.74]$$

Assumption (A4) allows the following expression to be derived for the derivatives of the mass discharge:

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(\rho Q) \approx \rho\frac{\partial Q}{\partial t}\\&\frac{\partial}{\partial x}(\rho Q) \approx \rho\frac{\partial Q}{\partial x}\end{aligned}\right\} \quad [2.75]$$

Substituting equations [2.74] and [2.75] into equations [2.71] and [2.71], introducing equation [2.70] yields the following expressions:

$$\left.\begin{aligned}&\frac{A}{c}\frac{\partial p}{\partial t} - \rho\frac{\partial Q}{\partial t} - \left(\frac{A}{c}\frac{\partial p}{\partial x} + \frac{p}{c}\frac{\partial A}{\partial x} - \rho\frac{\partial Q}{\partial x}\right)c = k|u|u + \rho gA\sin\theta - p\frac{\partial A}{\partial x}\\&\frac{A}{c}\frac{\partial p}{\partial t} + \rho\frac{\partial Q}{\partial t} + \left(\frac{A}{c}\frac{\partial p}{\partial x} + \frac{p}{c}\frac{\partial A}{\partial x} + \rho\frac{\partial Q}{\partial x}\right)c = -k|u|u - \rho gA\sin\theta + p\frac{\partial A}{\partial x}\end{aligned}\right\} \quad [2.76]$$

Note that the terms $p\,\partial A/\partial x$ on both sides of the equal sign are canceled out. Simplifying by A/c leads to the following equations:

$$\left.\begin{aligned}&\frac{\partial p}{\partial t} - \frac{\rho c}{A}\frac{\partial Q}{\partial t} - \left(\frac{\partial p}{\partial x} - \frac{\rho c}{A}\frac{\partial Q}{\partial x}\right)c = \frac{c}{A}k|u|u + \rho gc\sin\theta\\&\frac{\partial p}{\partial t} + \frac{\rho c}{A}\frac{\partial Q}{\partial t} + \left(\frac{\partial p}{\partial x} + \frac{\rho c}{A}\frac{\partial Q}{\partial x}\right)c = -\frac{c}{A}k|u|u - \rho gc\sin\theta\end{aligned}\right\} \quad [2.77]$$

that can be rewritten as:

$$\left.\begin{aligned}\frac{\partial p}{\partial t}-c\frac{\partial p}{\partial x}-\frac{\rho c}{A}\left(\frac{\partial Q}{\partial t}-c\frac{\partial Q}{\partial x}\right)=\frac{c}{A}k|u|u+\rho gc\sin\theta\\ \frac{\partial p}{\partial t}+c\frac{\partial p}{\partial x}+\frac{\rho c}{A}\left(\frac{\partial Q}{\partial t}+c\frac{\partial Q}{\partial x}\right)=-\frac{c}{A}k|u|u-\rho gc\sin\theta\end{aligned}\right\}\qquad [2.78]$$

The following characteristic equations are obtained:

$$\left.\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t}-\frac{\rho c}{A}\frac{dQ}{\mathrm{d}t}=\left(k|u|u+\rho gA\sin\theta\right)\frac{c}{A}\qquad&\text{for }\frac{\mathrm{d}x}{\mathrm{d}t}=-c\\ \frac{\mathrm{d}p}{\mathrm{d}t}+\frac{\rho c}{A}\frac{dQ}{\mathrm{d}t}=\left(-k|u|u-\rho gA\sin\theta\right)\frac{c}{A}\qquad&\text{for }\frac{\mathrm{d}x}{\mathrm{d}t}=c\end{aligned}\right\}\qquad [2.79]$$

In the particular case where the cross-sectional area and the wave speed are constant in space, $\partial A/\partial x=0$ and c can be taken out of the derivative. This remark also holds for A and ρ because of Assumption (A4) that allows the following approximation to be made:

$$\mathrm{d}\left(\frac{\rho c}{A}Q\right)=\frac{\rho c}{A}\left(\frac{\mathrm{d}\rho}{\rho}+\frac{\mathrm{d}c}{c}-\frac{\mathrm{d}A}{A}\right)Q+\frac{\rho c}{A}\mathrm{d}Q\approx\frac{\rho c}{A}\mathrm{d}Q\qquad [2\ .80]$$

Introducing the above simplifications, equation [2.79] can be rewritten as:

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(p-\frac{\rho c}{A}Q\right)=\frac{kc}{A}|u|u+\rho gc\sin\theta\qquad&\text{for }\frac{\mathrm{d}x}{\mathrm{d}t}=-c\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(p-\frac{\rho c}{A}Q\right)=-\frac{kc}{A}|u|u-\rho gc\sin\theta\qquad&\text{for }\frac{\mathrm{d}x}{\mathrm{d}t}=c\end{aligned}\right\}\qquad [2.81]$$

Noting that $u = Q/A$, equation [2.81] becomes:

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(p-\rho cu)=\frac{kc}{A}|u|u+\rho gc\sin\theta\qquad&\text{for }\frac{\mathrm{d}x}{\mathrm{d}t}=-c\\ \frac{\mathrm{d}}{\mathrm{d}t}(p+\rho cu)=-\frac{kc}{A}|u|u-\rho gc\sin\theta\qquad&\text{for }\frac{\mathrm{d}x}{\mathrm{d}t}=c\end{aligned}\right\}\qquad [2.82]$$

Consequently, the vector of Riemann invariants and the source term S" are defined as:

$$\mathrm{W} = \begin{bmatrix} p - \rho c u \\ p + \rho c u \end{bmatrix}, \quad \mathrm{S}'' = \frac{kc}{A} |u| u \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad [2.83]$$

For the sake of clarity (or to make the derivation of analytical solutions easier), the slope of the pipe and the friction term are sometimes assumed to be zero. In this case the source term S" becomes zero. Note that equation [2.83] leads to a very simple relationship between W and the state vector (p, u):

$$\left.\begin{aligned} p &= \frac{1}{2}(W_1 + W_2) \\ u &= \frac{1}{2\rho c}(W_2 - W_1) \end{aligned}\right\} \qquad [2.84]$$

NOTE. – The assumption of a constant area and speed of sound is essential in the derivation of the Riemann invariants [2.83]. When A and c are not constant in space (that is, when the properties of the pipe are not homogeneous), an analytical expression must be provided for their variations for equation [2.79] to be integrated.

2.4.4. *Calculation of the solution*

2.4.4.1. *Treatment of internal points*

For the sake of simplicity, the pipe is assumed to be horizontal and friction is assumed to be negligible. The properties of the material and the cross-sectional area are assumed to be constant in space. The pipe extends from $x = 0$ to $x = L$. The present section focuses on the calculation of the flow variables p and u at any internal point M of the domain, to the exclusion of the boundaries (Figure 2.9). The issue of boundaries is discussed in the next section. Denoting by A and B the feet of the characteristics $\mathrm{d}x/\mathrm{d}t = -c$ and $\mathrm{d}x/\mathrm{d}t = +c$ passing at M respectively, the knowledge of the invariant W_1 at B and that of the invariant W_2 at A allows the solution to be determined uniquely at M. Indeed, using equation [2.82] along the characteristics (AM) and (BM) under the assumption of zero friction and slope ($k = 0$, $\theta = 0$) leads to:

$$\left.\begin{aligned} W_1(\mathrm{M}) &= W_1(\mathrm{B}) \\ W_2(\mathrm{M}) &= W_2(\mathrm{A}) \end{aligned}\right\} \qquad [2.85]$$

If p and u are known at A and B, equation [2.83] allows W_1(B) and W_2(A) to be determined. W_1(M) and W_2(M) are derived using equation [2.85]. The pressure p_M and the velocity u_M at the point M are determined using relationships [2.84].

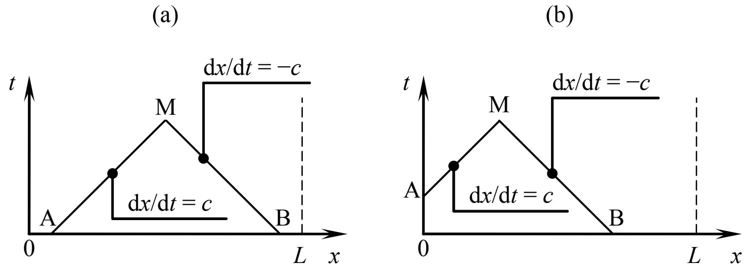

Figure 2.9. *Determination of the solution at internal points when the domain of dependence is included entirely in the computational domain (a) and the domain of dependence contains a boundary (b)*

$$\left.\begin{aligned} p_M &= \frac{1}{2}[W_1(B) + W_2(A)] = \frac{p_A + p_B}{2} + \rho c \frac{u_A - u_B}{2} \\ u_M &= \frac{1}{2\rho c}[W_2(A) - W_1(B)] = \frac{p_A - p_B}{2\rho c} + \frac{u_A + u_B}{2} \end{aligned}\right\} \quad [2.86]$$

Note that if u_A is larger than u_B, the water accumulates between A and B, which causes an increase in the pressure at the intermediate point M, hence the term (u_A – u_B) in the first equation [2.86]. Conversely, it p_A is larger than p_B, the points located between A and B are subjected to a positive acceleration and u_M increases, hence the term ($p_A - p_B$) in the second equation [2.86].

The coordinates of points A and B do not need to be identical for relationships [2.86] to be applicable. Two cases may be encountered depending on the value of t_M:

– if the domain of dependence of the point M is included in the segment [0, L] (Figure 2.9a), the knowledge of the initial condition is sufficient for the determination of p_M and u_M;

– if the domain of dependence of M contains a boundary (Figure 2.9b), only one invariant (here, the invariant W_1) reaches the point M from inside the computational domain. The second invariant (here W_2) travels along the characteristic (AM) issued from the left-hand boundary and requires that a boundary condition be prescribed at $x = 0$ for the solution to be determined uniquely. Boundary conditions are dealt with in section 2.4.4.2.

2.4.4.2. *Treatment of boundary conditions*

As shown in the previous section, only one Riemann invariant is known at the boundary of the domain because there is only one characteristic coming from the domain. For the solution to be unique at the boundary, additional information should be supplied in the form of a boundary condition.

Consider first the left-hand boundary (Figure 2.10a). The pressure p_A and the flow velocity u_A are determined uniquely at A provided that the invariants W_1 and W_2 are known at A. Then, equation [2.84] can be applied. W_1(A) can be determined from the initial condition at the internal point A', that is the foot of the characteristic dx/dt = – c that passes at A. The following equality holds:

$$W_1(\mathrm{A}) = W_1(\mathrm{A}') \qquad [2.87]$$

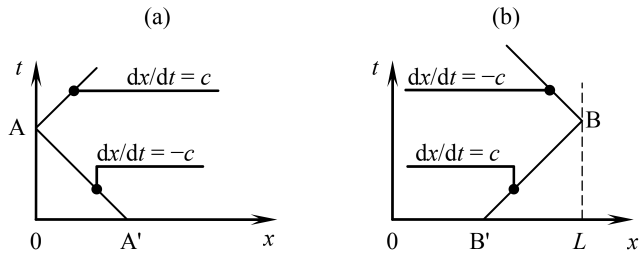

Figure 2.10. *Determination of the solution at the boundaries: left-hand boundary (a); right-hand boundary (b)*

However, W_2(A) cannot be computed from the initial condition because the characteristic dx/dt = + c does not reach point A from inside the domain. The missing information must be supplied in the form of an additional boundary condition. The general expression for such a condition is:

$$f(p_A, u_A, t) = 0 \qquad [2.88]$$

where the function f is known *a priori*. Equations [2.87–88] form a 2×2 system, the solution of which is unique.

The main three types of boundary condition met in engineering practice are the following:

– prescribed pressure p_b at the left-hand boundary. System [2.87–88] can be rewritten as:

$$\left.\begin{aligned} p_{\mathrm{A}} - \rho c u_{\mathrm{A}} &= p_{\mathrm{A'}} - \rho c u_{\mathrm{A'}} \\ p_{\mathrm{A}} &= p_b(t) \end{aligned}\right\} \qquad [2.89]$$

The pressure p_{A} is known and the flow velocity u_{A} is obtained from the first equation [2.89]:

$$u_{\mathrm{A}} = \frac{p_b(t) - p_{\mathrm{A'}}}{\rho c} + u_{\mathrm{A'}} \qquad [2.90]$$

– prescribed velocity u_b at the left-hand boundary (this condition is equivalent to prescribing a discharge because the cross-sectional area A of the pipe is known). System [2.87–88] can be rewritten as:

$$\left.\begin{aligned} p_{\mathrm{A}} - \rho c u_{\mathrm{A}} &= p_{\mathrm{A'}} - \rho c u_{\mathrm{A'}} \\ u_{\mathrm{A}} &= u_b(t) \end{aligned}\right\} \qquad [2.91]$$

which leads to the following expression for p_{A}:

$$p_{\mathrm{A}} = p_{\mathrm{A'}} + [u_b(t) - u_{\mathrm{A'}}]\rho c \qquad [2.92]$$

– known relationship between the flow velocity u_b (or the discharge Q_b) and the pressure p_b. Such a relationship may express the head loss across singularities such as valves, bends, sudden changes in the cross-sectional area, as well as head gains across pumps, etc. In most cases, the function f is nonlinear and system [2.87–88] must be solved using iterative methods.

Reasoning by symmetry, the following formulations are obtained for a right-hand boundary condition:

– prescribed pressure p_b at the right-hand boundary. u_{B} is given by:

$$u_{\mathrm{B}} = \frac{p_{\mathrm{B'}} - p_b(t)}{\rho c} + u_{\mathrm{B'}} \qquad [2.93]$$

– prescribed velocity u_b at the right-hand boundary. The pressure p_{B} is given by:

$$p_{\mathrm{B}} = p_{\mathrm{B'}} + [u_{\mathrm{B'}} - u_b(t)]\rho c \qquad [2.94]$$

2.4.4.3. *Bergeron's graphical method*

Bergeron's graphical method uses a representation of the characteristic relationships in the (p, u) plane instead of the (x, t) phase space. The principle of the method is outlined for horizontal pipes with a constant cross-sectional area where the wave speed is constant and friction is negligible. Under such assumptions, equations [2.85] can be rewritten as:

$$\left.\begin{aligned} p_M - \rho c u_M &= p_B - \rho c u_B \\ p_M + \rho c u_M &= p_A + \rho c u_A \end{aligned}\right\} \qquad [2.95]$$

These are the equations of straight lines in the (p, u) coordinate system (Figure 2.11). Indeed, system [2.95] can be rewritten as:

$$\left.\begin{aligned} u_M &= \frac{p_M - p_B}{\rho c} + u_B \\ u_M &= \frac{p_A - p_M}{\rho c} + u_A \end{aligned}\right\} \qquad [2.96]$$

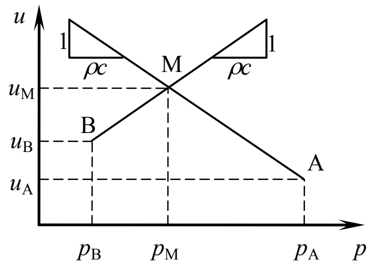

Figure 2.11. *Principle of Bergeron's graphical method.*

The slopes of the straight lines are $-1/(\rho c)$ and $1/(\rho c)$. The solution (p_M, u_M), that satisfies equations [2.96], is therefore located at the intersection between the straight lines in the plane (p, u). The solution is determined graphically by locating the points A and B in the (p, u) plane, drawing the straight lines with slopes $-1/(\rho c)$ and $1/(\rho c)$ that pass at A and B respectively and finding their intersection M (Figure 2.11). Note that the method can be applied to more complex physical situations, in particular when friction is not negligible or when a boundary condition is supplied in the form of a relationship between the pressure and the flow velocity. Application examples can be found in [CAR 72] and [LEN 96].

2.4.5. *Summary*

The water hammer equations are based on the assumptions of a weakly compressible fluid and a weakly deformable pipe, where inertia is negligible and the relationship between the pressure and the mass per unit length is linear (see Assumptions (A1–4) in section 2.4.1).

The water hammer equations are written in conservation form [2.2] by defining the variable vector, the flux and the source term as in equation [2.68].

The characteristic form of the water hammer equations is given by equation [2.79]. When the physical properties of the pipe are homogeneous (constant wave speed and cross-sectional area) equation [2.79] can be simplified into equation [2.82]. The differential dW can then be integrated and W and S are given by equation [2.83].

The solution is computed at internal points using equation [2.86]. The existence and uniqueness of the solution requires that a boundary condition should be prescribed at each end of the computational domain. Boundary conditions in the form of prescribed pressure or velocity are given by equations [2.89–94].

2.5. A nonlinear 2×2 system: the Saint Venant equations

2.5.1. *Physical context – assumptions*

The open channel flow equations, also known as the Saint Venant equations, describe the behavior of flows in open channels such as rivers, canals or pipes where the water flows freely (in other words, the flow is not pressurized). Such equations are one-dimensional because the transverse dimensions of open channels are often very small compared to their longitudinal dimension. The Saint Venant equations are based on the following set of assumptions:

– Assumption (A1). The water is assumed to be incompressible within the usual range of pressure. Its density ρ is constant.

– Assumption (A2). The transverse and vertical acceleration of the water particles can be neglected compared to the longitudinal component of the acceleration. This is equivalent to assuming that the streamlines are only weakly curved. Consequently the pressure field is hydrostatic in a given channel cross-section.

– Assumption (A3). The flow regime is turbulent. The head loss, mainly due to friction against the channel walls, is proportional to the square of the velocity.

– Assumption (A4). The slope of the channel is small enough for the longitudinal coordinate to coincide with the horizontal axis.

If we compare the Saint Venant equations to the water hammer equations studied in section 2.4, several differences arise:

– the water in an open channel is not assumed to be compressible because the range of the pressure variations in an open channel is much smaller than in pressurized pipes;

– the inertial terms cannot be neglected in the momentum equation. They may even become locally predominant and trigger discontinuities in the flow, such as hydraulic jumps, that appear when a rapid flow enters a zone where the flow is slower.

A detailed analysis of the Saint Venant equations, as well as their numerical solution by standard, commercially available numerical models, can be found in [CUN 80].

2.5.2. ***Conservation form***

2.5.2.1. *Notation*

The following notation is used (Figure 2.12). $A(x)$ is the cross-sectional area of the channel at the abscissa x, $b(x)$ is the width of the free surface, $h(x)$ is the water depth, that is, the vertical distance between the free surface and the lowest point of the section (called the bottom of the channel), $W(x, z)$ is the width of the channel at the abscissa x and the elevation z, $z_b(x)$ is the elevation of the bottom, $\zeta(x)$ is the elevation of the free surface and χ is the wetted perimeter.

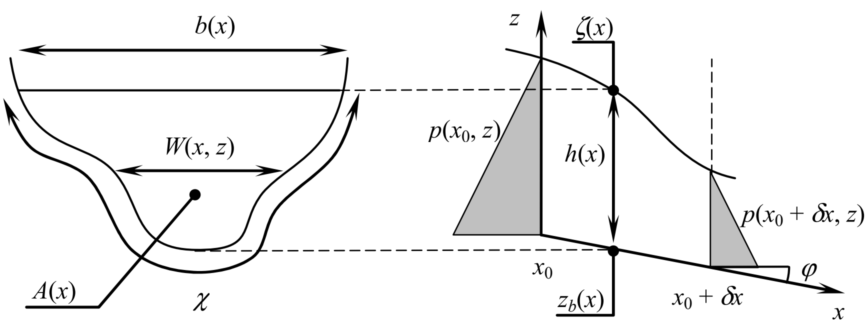

Figure 2.12. *Definition sketch for the Saint Venant equations. The pressure forces exerted on the control volume are given by the areas of the gray-shaded triangles*

Note that by definition, the following relationship holds between A and b:

$$\mathrm{d}A = b\ \mathrm{d}z \qquad [2.97]$$

This relationship is used in section 2.5.3.3 to derive the expression of the wave speed.

2.5.2.2. *Continuity equation*

A mass balance is carried out between the times t_0 and $t_0 + \delta t$ over an elementary control volume extending from x_0 to $x_0 + \delta x$:

$$\delta U_1(t_0 + \delta t) - \delta U_1(t_0) = \delta F_1(x_0) - \delta F_1(x_0 + \delta x) \qquad [2.98]$$

where $\delta U_1(t)$ is the mass of the fluid contained in the control volume at the time t and $\delta F_1\,(x)$ is the mass of fluid that passes at the abscissa x during the time interval δt. Equation [2.98] states that the variation of the amount of water contained in the control volume is equal to the difference between the amount that flows in and the amount that flows out. By definition, δU_1 and δF_1 are given by:

$$\left.\begin{aligned} \delta U_1(t) &= \int_{x_0}^{x_0+\delta x} (\rho A)(x,t)\,\mathrm{d}x \\ \delta F_1(x) &= \int_{t_0}^{t_0+\delta t} (\rho u A)(x,t)\,\mathrm{d}x \end{aligned}\right\} \qquad [2.99]$$

where A is the cross-sectional area, u is the fluid velocity and ρ is the density of the fluid. Substituting equation [2.99] into equation [2.98] in the limit of small δt and δx leads to the following equation (see equations [1.12–16] for a detailed proof):

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho u A) = 0 \qquad [2.100]$$

Using Assumption (A2) of an incompressible fluid allows equation [2.100] to be divided by the (constant) density ρ. Noting that $Q = Au$, equation [2.100] is simplified into:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad [2.101]$$

Defining the first component of the vector variable as A, equation [2.101] can be written in conservation form as:

$$\left.\begin{aligned} \frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial x} &= 0 \\ U_1 &= A \\ F_1 &= Q \end{aligned}\right\} \qquad [2.102]$$

2.5.2.3. *Momentum equation*

The momentum balance is obtained by applying the fundamental principle of dynamics to a control volume of length δx over a time interval δt:

$$\begin{aligned} \delta U_2(t_0+\delta t) - \delta U_2(t_0) = \delta F_2(x_0) - \delta F_2(x_0+\delta x) \\ + \delta P(x_0) - \delta P(x_0+\delta x) + \delta F_W \end{aligned} \qquad [2.103]$$

where $\delta P(x)$ is the integral between t and $t + \delta t$ of the pressure force exerted on the cross-section at the abscissa x, $\delta U_2(t)$ is the momentum of the fluid contained in the slice of channel of length δx, and δF_W is the x-component of the reaction exerted by the channel walls on the control volume. By definition, δU_2 and δF_2 are given by equation [2.55], recalled here:

$$\left.\begin{aligned} \delta U_2(t) &= \int_{x_0}^{x_0+\delta x} (\rho u A)(x,t)\,\mathrm{d}x \\ \delta F_2(x) &= \int_{t_0}^{t_0+\delta t} (\rho u^2 A)(x,t)\,\mathrm{d}x \end{aligned}\right\}$$

The pressure force $P(x)$ is equal to the integral of the pressure over the cross-section. From Assumption (A2), the pressure field is hydrostatic, that is, it is proportional to the distance between the point under consideration and the free surface. The pressure $p(x, z)$ at the abscissa x and at the elevation z is therefore:

$$p(x,z) = (\zeta - z)\rho g \qquad [2.104]$$

where both z and ζ depend on x (this is omitted in the notation for the sake of clarity). The pressure $p(x, z)$ is exerted on an elementary slice of height δz and width $W(x, z)$. The pressure force $P(x)$ is therefore given by:

$$P(x) = \int_{z_b(x)}^{\zeta(x)} p(x,z)W(x,z)\,\mathrm{d}z = \int_{z_b(x)}^{\zeta(x)} [\zeta(x) - z]\rho g W(x,z)\,\mathrm{d}z \qquad [2.105]$$

The force δF_W exerted by the walls of the channel on the control volume is expressed as the sum of three forces:

$$\delta F_P(t) = \int_{x_0}^{x_0+\delta x} (R_p + R_f + R_b)(x,t)\mathrm{d}x \qquad [2.106]$$

where R_b, R_f and R_w are respectively the reaction of the bottom on the control volume in the vertical plane, the friction force and the reaction of the walls on the control volume in the horizontal plane. The friction force R_f is usually expressed in terms of the "energy slope", or "energy grade line" S_f. S_f is positive when energy is lost in the direction of positive x. The following equivalence holds between R_f and S_f:

$$R_f = -\rho g h S_f \qquad [2.107]$$

Note that R_f is exerted in a direction that is parallel to the channel bottom and not along the horizontal. In addition, owing to the non-zero bottom slope, R_f is exerted over a length that is slightly larger than δx. However, Assumption (A4) allows R_f to be approximated as its projection on the x-axis, while the length over which the force is exerted is approximated with δx. Several formulae are available for S_f. All of them use Assumption (A3) of a turbulent flow regime, hence the assumption that the slope of the energy line is proportional to the square of the flow velocity u. The most frequently used laws are:

$$\begin{aligned} S_f &= \frac{u^2}{C^2 R_H} && \text{(Chezy)} \\ S_f &= \frac{u^2}{K_{\mathrm{Str}}^2 R_H^{4/3}} && \text{(Strickler)} \\ S_f &= n_M^2 \frac{u^2}{R_H^{4/3}} && \text{(Manning)} \end{aligned} \qquad [2.108]$$

where C, K_{Str} and n_{M} are respectively the Chezy, Strickler and Manning friction coefficients. R_H is the hydraulic radius, defined as the ratio of the cross-sectional area to the wetted perimeter:

$$R_H = \frac{A}{\chi} \qquad [2.109]$$

The Chezy coefficient is often used by coastal engineers, while river engineers usually prefer the Strickler and Manning coefficients. A large value of the Chezy or Strickler coefficients, or a small value of the Manning coefficient, indicate that friction is small. The three coefficients can be related to each other as follows:

$$K_{\mathrm{Str}} = \frac{1}{n_M} = \frac{C}{R_H^{1/6}} \qquad [2.110]$$

As in the water hammer equation, the force R_p is given by the projection of the reaction of the walls of the channel onto the x-axis. In what follows, the reaction of the walls is understood as the horizontal component of the reaction of the walls. The reaction of the walls also has a vertical component that is usually referred to as the reaction R_b of the bottom of the channel. The force R_p is considered first. Consider an elementary volume of channel of height dz, the distance of which to the free surface is denoted by d. The domain extends from one channel wall to the other between the abscissas $x - \mathrm{d}x/2$ and $x + \mathrm{d}x/2$ (Figure 2.13). From Assumption (A2), the (hydrostatic) pressure p is given by equation [2.104]. The x-component of the reaction of the wall onto the elementary volume is therefore:

$$\begin{aligned} \mathrm{d}R_p &= \left[W(x + dx/2) - W(x - dx/2)\right] p(x,z)\ \mathrm{d}z \\ &\approx \left(\frac{\partial W}{\partial x}\right)_{\zeta - z = \mathrm{Const}} (\zeta - z)\rho g\ \mathrm{d}x\ \mathrm{d}z \end{aligned} \qquad [2.111]$$

where the notation $(\partial W / \partial x)_{\zeta - z = \mathrm{Const}}$ indicates that the partial derivative of the width with respect to z is estimated keeping a constant distance below the free surface. The force R_p is obtained by integrating dR_p from z_b to ζ and from x_0 to $x_0 + \delta x$:

$$R_p = \rho g \int_{x_0}^{x_0+\delta x} \int_{z_b(x)}^{\zeta(x)} (\zeta - z) \left(\frac{\partial W}{\partial x}\right)_{\zeta - z = \mathrm{Const}} \mathrm{d}z\ \mathrm{d}x \qquad [2.112]$$

The projection R_b of the reaction of the bottom onto the x-axis is determined by carrying out a balance between the forces exerted onto a slice of channel delineated by $x - \mathrm{d}x/2$ and $x + \mathrm{d}x/2$. The reaction of the bottom onto the control volume is exerted in a vertical plane, in the direction of the normal unit vector to the bottom of the channel (see Figure 2.13, top). The projection of the reaction of the channel onto the z-axis is equal to the opposite of the weight $\rho g A \mathrm{d}x$ of the control volume. Since the bottom of the channel makes an angle φ with the horizontal, the horizontal component of the reaction of the bottom of the channel onto the fluid is:

$$R_b = \rho g A(x)\ \mathrm{tg}\varphi\ \mathrm{d}x \qquad [2.113]$$

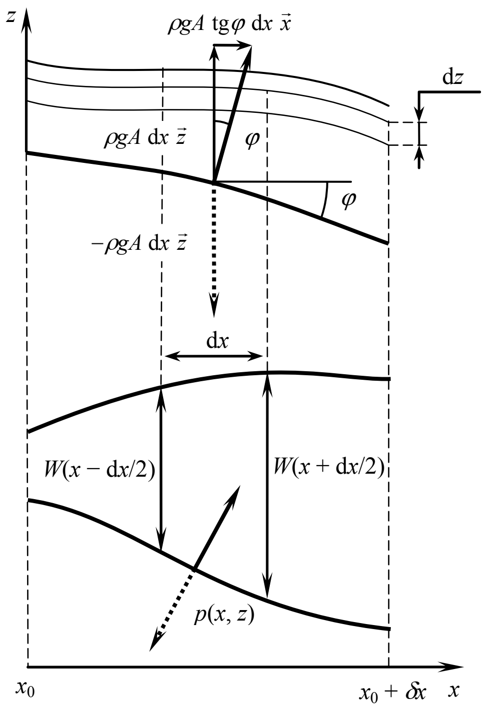

Figure 2.13. *Force exerted by the fluid onto the wall (dashed arrows), reaction of the wall onto the fluid (solid arrows). Vertical projection (top), horizontal projection (bottom)*

Note that the tangent of the angle φ is the slope S_0 of the channel:

$$S_0 = \tan\varphi = -\frac{\partial z_b}{\partial x} \qquad [2.114]$$

Substituting equations [2.55, 2.105, 2.107, 2.112–114] into equation [2.103] in the limit of zero δt and δx, dividing by the constant density ρ leads to:

$$\frac{\partial}{\partial t}(uA) + \frac{\partial}{\partial x}(u^2 A + P/\rho) = (S_0 - S_f)gA + I_p \qquad [2.115]$$

where the integral I_p is given by:

$$I_p = \frac{1}{\rho}\frac{\partial R_p}{\partial x} = g\int_{z_b(x)}^{\zeta(x)} (\zeta - z)\left(\frac{\partial W}{\partial x}\right)_{\zeta - z = \text{Const}} \mathrm{d}z \qquad [2.116]$$

Equation [2.115] can be written in conservation form as:

$$\left.\begin{aligned}&\frac{\partial U_2}{\partial t}+\frac{\partial F_2}{\partial x}=S_2\\&U_2=Q\\&F_2=M=Q^2/A+P/\rho\\&S_2=(S_0-S_f)gA+I_p\end{aligned}\right\}\qquad[2.117]$$

2.5.2.4. *Vector form*

The system formed by equations [2.102] and [2.117] can be written in the vector conservation form [2.2], recalled here:

$$\frac{\partial \mathrm{U}}{\partial t}+\frac{\partial \mathrm{F}}{\partial x}=\mathrm{S}$$

by defining U, F and S as:

$$\mathrm{U}=\begin{bmatrix}A\\Q\end{bmatrix},\ \mathrm{F}=\begin{bmatrix}Q\\M\end{bmatrix}=\begin{bmatrix}Q\\\dfrac{Q^2}{A}+\dfrac{P}{\rho}\end{bmatrix},\ \mathrm{S}=\begin{bmatrix}0\\(S_0-S_f)gA+I_P\end{bmatrix}\qquad[2.118]$$

where $M = Q^2/A + P/\rho$ is the specific force and Q^2/A is the momentum flux.

2.5.3. *Characteristic form – Riemann invariants*

2.5.3.1. *Non-conservation form*

Equation [2.2] is first rewritten in the non-conservation form [2.5], recalled hereafter:

$$\frac{\partial \mathrm{U}}{\partial t}+\mathrm{A}\frac{\partial \mathrm{U}}{\partial x}=\mathrm{S'}$$

where A and S' are given as in equation [2.6], recalled here:

$$\left.\begin{aligned}&\mathrm{dF}=\mathrm{A\ dU}\\&\mathrm{S'}=\mathrm{S}-\left(\frac{\partial \mathrm{F}}{\partial x}\right)_{\mathrm{U=Const}}\end{aligned}\right\}$$

Note from equation [2.118] that A is given by:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix} \qquad [2.119]$$

where c is defined as:

$$c = \left[\frac{\partial (P/\rho)}{\partial A} \right]^{1/2} \qquad [2.120]$$

The expression for c is given in section 2.5.3.3. The source term S' is derived by noting that $(\partial \mathrm{F}/\partial \mathrm{U})_{\mathrm{U=Const}}$ is to be computed for a fixed value of U, that is, for constant A and Q. Consequently:

– the first component of S' is zero because both A and Q are assumed constant;

– the term Q^2/A in the second component M of F is also assumed constant. The only term in M that changes with x for a constant U is the pressure force P. Variations in P arise either from variations in the water depth h for a given channel shape, or from variations in the shape of the channel cross-section for a given h. Differentiating P/ρ with respect to x gives:

$$\frac{\partial}{\partial x}\left(\frac{P}{\rho}\right) = \frac{1}{\rho}\frac{\partial}{\partial x}\int_{z_b}^{\zeta} p(z)W(z)\mathrm{d}z = \frac{1}{\rho}\frac{\partial}{\partial x}\int_{0}^{h} p(z')W(z')\,\mathrm{d}z' \qquad [2.121]$$

where $z' = z - z_b$ is the elevation above the bed level. Using Leibniz' differentiation rule for the integral, noting that $p = 0$ for $z' = h$ and substituting equation [2.104] into equation [2.121] leads to:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{P}{\rho}\right) &= \frac{1}{\rho}\int_0^h \frac{\partial}{\partial x}\left[p(z')\,W(z')\right]\mathrm{d}z' = g\int_0^h \frac{\partial}{\partial x}\left[(h-z')\,W(z')\right]\mathrm{d}z' \\ &= g\int_0^h \frac{\partial h}{\partial x} W(z')\,\mathrm{d}z' + g\int_0^h \frac{\partial W(z')}{\partial x}(h-z')\,\mathrm{d}z' \end{aligned} \qquad [2.122]$$

Note that the second term in equation [2.122] is nothing but the integral I_p as defined in equation [2.116]. Moreover, it is noted that in the case of a prismatic channel, $I_p = 0$. Consequently, the first integral in equation [2.122] is part of the second component of the vector term $(\partial \mathrm{F}/\partial \mathrm{U})_{x=\mathrm{Const}}\,\partial \mathrm{U}/\partial x$ in equation [2.6], while I_p represents the second component of the term $(\partial \mathrm{F}/\partial x)_{\mathrm{U}=\mathrm{Const}}$:

$$\left.\begin{aligned}
&\left[\frac{\partial}{\partial x}\left(\frac{P}{\rho}\right)\right]_{x=\mathrm{Const}} = g\int_0^h \frac{\partial h}{\partial x} W(z')\,\mathrm{d}z' \\
&\left(\frac{\partial M}{\partial x}\right)_{\substack{A=\mathrm{Const}\\ Q=\mathrm{Const}}} = \left[\frac{\partial}{\partial x}\left(\frac{P}{\rho}\right)\right]_{A=\mathrm{Const}} = g\int_0^h \frac{\partial W(z')}{\partial x}(h-z')\,\mathrm{d}z' = I_p
\end{aligned}\right\} \quad [2.123]$$

Consequently, we have:

$$\mathrm{S}' = \mathrm{S} - \left(\frac{\partial \mathrm{F}}{\partial x}\right)_{\mathrm{U}=\mathrm{Const}} = \begin{bmatrix} 0 \\ (S_0 - S_f)gA \end{bmatrix} \quad [2.124]$$

2.5.3.2. *Characteristic form*

The eigenvalues and eigenvectors of A are:

$$\begin{aligned}
&\lambda^{(1)} = u - c, \qquad \lambda^{(2)} = u + c \\
&\mathrm{K}^{(1)} = \begin{bmatrix} 1 \\ u - c \end{bmatrix}, \quad \mathrm{K}^{(2)} = \begin{bmatrix} 1 \\ u + c \end{bmatrix}
\end{aligned} \quad [2.125]$$

Since the Jacobian matrix A has two real and distinct eigenvalues, system [2.2, 2.118] is hyperbolic. The variations of the pressure force are related to those of the mass per unit length; therefore the system describes a compressible behavior.

Recall that the characteristic form is obtained by multiplying the non-conservation form by K^{-1} so as to involve the diagonal matrix Λ formed by the eigenvalues of A:

$$\frac{\partial \mathrm{W}}{\partial t} + \Lambda \frac{\partial \mathrm{W}}{\partial x} = \mathrm{S}''$$

The differential of the vector Riemann invariant dW is defined by the first relationship [2.21]:

$$\mathrm{d}\,\mathrm{W} = \mathrm{K}^{-1}\,\mathrm{d}\,\mathrm{U}$$

Matrices K and K^{-1} are derived from equation [2.125]:

$$K = \begin{bmatrix} 1 & 1 \\ u-c & u+c \end{bmatrix}, \qquad K^{-1} = \frac{1}{2c}\begin{bmatrix} u+c & -1 \\ c-u & 1 \end{bmatrix} \qquad [2.126]$$

Substituting equations [2.119] and [2.126] into equation [2.21] with the definition [2.124] leads to:

$$\left.\begin{aligned} dW &= \frac{1}{2c}\begin{bmatrix} (u+c)dU_1 - dU_2 \\ (c-u)dU_1 + dU_2 \end{bmatrix} = \frac{1}{2c}\begin{bmatrix} (u+c)dA - d(Au) \\ (c-u)dA + d(Au) \end{bmatrix} \\ S'' &= \frac{1}{2c}\begin{bmatrix} -(S_0 - S_f)gA \\ (S_0 - S_f)gA \end{bmatrix} \end{aligned}\right\} \qquad [2.127]$$

The differential dW can be integrated only if the expression for c is known. This is the subject of section 2.5.3.2. The formula for c is then used to derive that of the Riemann invariants in a number of particular cases examined in section 2.5.3.3.

2.5.3.3. *Expression of the speed of the waves in still water*

Definition [2.120] is recalled for c:

$$c = \left[\frac{\partial(P/\rho)}{\partial A}\right]^{1/2}$$

where P is given by equation [2.105]. Dividing equation [2.105] by ρ leads to:

$$\frac{P}{\rho}(x) = \int_{z_b(x)}^{\zeta(z)} (\zeta - z) g W(x,z)\, dz \qquad [2.128]$$

The pressure force P is first shown to be related to the cross-sectional area by a one-to-one relationship. Both the cross-sectional area A and the pressure force P are strictly increasing functions of the elevation ζ of the free surface, provided that b is non-zero. Consequently there is a one-to-one relationship between P and ζ, and there is a one-to-one relationship between A and ζ. Therefore, there exists a one-to-one relationship between A and P and the derivative $\partial P/\partial A$ can be computed as:

$$\frac{\partial(P/\rho)}{\partial A} = \frac{\partial(P/\rho)}{\partial \zeta}\frac{\partial \zeta}{\partial A} = \frac{\partial(P/\rho)}{\partial \zeta}\left(\frac{\partial A}{\partial \zeta}\right)^{-1} \qquad [2.129]$$

The expression of $\partial A / \partial \zeta$ is derived assuming that the elevation of the free surface is subjected to an infinitesimal variation dζ. The corresponding variation dA in the cross-sectional area is:

$$\mathrm{d}A = b\,\mathrm{d}\zeta \qquad [2.130]$$

Consequently:

$$\frac{\partial A}{\partial \zeta} = b \qquad [2.131]$$

The derivative $\partial P / \partial \zeta$ is derived as follows. Consider an infinitesimal variation dζ in the elevation of the free surface, that moves from the point A to the point A' (Figure 2.14).

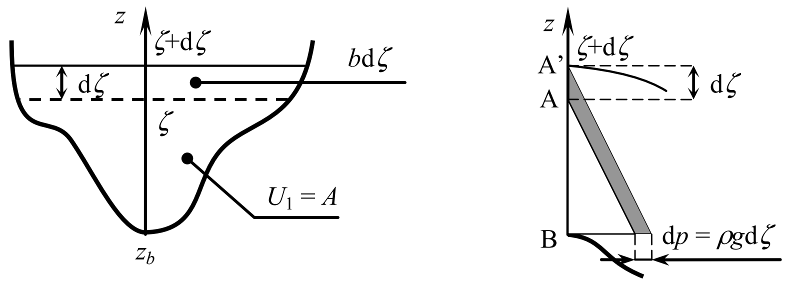

Figure 2.14. *Variation in the pressure force caused by a variation dζ in the water level. The variation dP_1 in the pressure force triggered by the variation dζ is the area of the gray-shaded areas between the triangles that illustrate the pressure field*

Since the pressure is hydrostatic, it increases uniformly by a quantity ρg dz over the entire section. The resulting increase in the pressure field is therefore:

$$\mathrm{d}P_1 = A\mathrm{d}p = A\rho g\mathrm{d}\zeta \qquad [2.132]$$

The additional force brought by the infinitesimal area between the points A and A' in the side view (Figure 2.14, right) is given by:

$$\mathrm{d}P_2 = b\rho g\frac{(\mathrm{d}\zeta)^2}{2} \qquad [2.133]$$

The total variation in the pressure force is given by $dP_1 + dP_2$:

$$dP = \left(A + \frac{b d\zeta}{2} \right) \rho g \ d\zeta \qquad [2.134]$$

In the limit of small $d\zeta$, dP becomes equivalent to $\rho g A \ d\zeta$ and:

$$\frac{\partial(P/\rho)}{\partial \zeta} = \rho g A \qquad [2.135]$$

Substituting equations [2.131] and [2.135] into equation [2.129] leads to:

$$\frac{\partial(P/\rho)}{\partial A} = \frac{gA}{b} \qquad [2.136]$$

Hence the expression of c:

$$c = \left(\frac{gA}{b} \right)^{1/2} \qquad [2.137]$$

In the particular case of a rectangular channel, equation [2.137] simplifies into:

$$c = (gh)^{1/2} \qquad [2.138]$$

This expression is to be connected to a very remarkable behavior. Free-surface waves propagate more rapidly in deep water than in shallow water. This explains in particular the steepening and the breaking of sea waves traveling to the shore. In a more dramatic fashion, equation [2.138] also accounts for the development of tsunamis. The behavior of the solutions of hyperbolic conservation laws in the presence of steep fronts and discontinuities is covered in detail in Chapter 3.

2.5.3.4. *Riemann invariants*

The purpose is to derive a simplified formulation for the Riemann invariants as given by equations [2.127]. Substituting equations [2.135] into equation [2.22] and multiplying by $2c$ leads to the following system:

$$\left.\begin{aligned} c\frac{dA}{dt} - A\frac{du}{dt} &= -(S_0 - S_f)gA \qquad \text{for } \frac{dx}{dt} = u - c \\ c\frac{dA}{dt} + A\frac{du}{dt} &= (S_0 - S_f)gA \qquad \text{for } \frac{dx}{dt} = u + c \end{aligned}\right\} \qquad [2.139]$$

Dividing by A leads to:

$$\left.\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{c}{A}\frac{\mathrm{d}A}{\mathrm{d}t}=(S_0-S_f)g \quad \text{for}\ \frac{\mathrm{d}x}{\mathrm{d}t}=u-c\\ \frac{\mathrm{d}u}{\mathrm{d}t}+\frac{c}{A}\frac{\mathrm{d}A}{\mathrm{d}t}=(S_0-S_f)g \quad \text{for}\ \frac{\mathrm{d}x}{\mathrm{d}t}=u+c\end{aligned}\right\} \qquad [2.140]$$

The above equations can be integrated easily in a number of particular cases. The following three configurations are examined for prismatic channels ($h_{x,A} = 0$):

– *Rectangular channel*. b is constant, A and c are given by:

$$\left.\begin{aligned}c &= (gh)^{1/2}\\ A &= bh = \frac{bc^2}{g}\end{aligned}\right\} \qquad [2.141]$$

The following relationship holds:

$$\mathrm{d}A = b\mathrm{d}h = \frac{2c}{g}\mathrm{d}c \qquad [2.142]$$

Substituting equation [2.142] into equation [2.140] and noting that $c^2 = gh$, the following system is obtained:

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(u-2c)=(S_0-S_f)g \quad \text{for}\ \frac{\mathrm{d}x}{\mathrm{d}t}=u-c\\ \frac{\mathrm{d}}{\mathrm{d}t}(u+2c)=(S_0-S_f)g \quad \text{for}\ \frac{\mathrm{d}x}{\mathrm{d}t}=u+c\end{aligned}\right\} \qquad [2.143]$$

Note that the source term on the right-hand side of the equations is canceled out in horizontal channels with negligible friction or when the flow is uniform ($S_0 = S_f$ by definition). The quantity $u - 2c$ is then constant along the characteristics, the speed of which is $u - c$. The quantity $u + 2c$ is constant along the characteristics of speed $u + c$. The Riemann invariants and the source term in equation [2.22] can be redefined as:

$$\mathrm{W} = \begin{bmatrix} u-2c \\ u+2c \end{bmatrix}, \quad \mathrm{S}'' = \begin{bmatrix} (S_0-S_f)g \\ (S_0-S_f)g \end{bmatrix} \qquad [2.144]$$

– *Triangular channel*. The width b is proportional to the water depth h, A is proportional to the square of b and h:

$$\left.\begin{array}{l} b = 2h \ \mathrm{tg}\theta \\ A = h^2 \ \mathrm{tg}\theta \\ c = (gh/2)^{1/2} \end{array}\right\} \qquad [2.145]$$

where θ is the angle of the embankments with the vertical (the channel is assumed to be symmetric). Equations [2.140] can be rewritten as:

$$\left.\begin{array}{l} \dfrac{\mathrm{d}}{\mathrm{d}t}(u-4c) = (S_0 - S_f)g \quad \text{for} \dfrac{\mathrm{d}x}{\mathrm{d}t} = u - c \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(u+4c) = (S_0 - S_f)g \quad \text{for} \dfrac{\mathrm{d}x}{\mathrm{d}t} = u + c \end{array}\right\} \qquad [2.146]$$

Under uniform conditions, or when the channel is horizontal and friction is negligible, the quantity $u - 4c$ is invariant along the characteristic of speed $u - c$. The quantity $u + 4c$ is invariant along the characteristic of speed $u + c$. The Riemann invariants and the source term in equation [2.22] may be redefined as:

$$\mathrm{W} = \begin{bmatrix} u - 4c \\ u + 4c \end{bmatrix}, \qquad \mathrm{S}'' = \begin{bmatrix} (S_0 - S_f)g \\ (S_0 - S_f)g \end{bmatrix} \qquad [2.147]$$

– Approximation for arbitrary-shaped channels. The following relationships hold:

$$\left.\begin{array}{l} A = \dfrac{bc^2}{g} \\ \mathrm{d}A = \dfrac{c^2}{g}\mathrm{d}b + \dfrac{2b}{g}c\mathrm{d}c = \left(\dfrac{c^2}{g}\dfrac{\mathrm{d}b}{\mathrm{d}c} + \dfrac{2b}{g}c\right)\mathrm{d}c \end{array}\right\} \qquad [2.148]$$

Substituting equation [2.148] into equation [2.140] leads to the following system:

$$\left.\begin{array}{l} \dfrac{\mathrm{d}u}{\mathrm{d}t} - \left(2 + \dfrac{c}{b}\dfrac{\mathrm{d}b}{\mathrm{d}c}\right)\dfrac{\mathrm{d}c}{\mathrm{d}t} = (S_0 - S_f)g \quad \text{for} \dfrac{\mathrm{d}x}{\mathrm{d}t} = u - c \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} + \left(2 + \dfrac{c}{b}\dfrac{\mathrm{d}b}{\mathrm{d}c}\right)\dfrac{\mathrm{d}c}{\mathrm{d}t} = (S_0 - S_f)g \quad \text{for} \dfrac{\mathrm{d}x}{\mathrm{d}t} = u + c \end{array}\right\} \qquad [2.149]$$

If the quantity db/dc can be assumed to be proportional to b/c, then:

$$\frac{c}{b}\frac{\mathrm{d}b}{\mathrm{d}c} = \mathrm{Const} \qquad [2.150]$$

and the term (c/b db/dc) can be passed into the operator d/dt. Equations [2.149] then become:

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[u - \left(2 + \frac{c}{b}\frac{\mathrm{d}b}{\mathrm{d}c}\right)c\right] = (S_0 - S_f)g \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u - c \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left[u + \left(2 + \frac{c}{b}\frac{\mathrm{d}b}{\mathrm{d}c}\right)c\right] = (S_0 - S_f)g \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u + c \end{aligned}\right\} \qquad [2.151]$$

If (c/b db/dc) is not strictly constant, equation [2.151] is only an approximation. This approximation may however prove useful in a number of cases. Note that the rectangular channel (db/dc = 0) and the triangular channel (db/dc = 2) are particular cases of equations [2.151].

The quantity (c/b db/dc) is difficult to estimate in practical applications. It is more conveniently related to the geometry of the channel via the general definition of the propagation speed of the waves in still water:

$$c^2 = \frac{gA}{b} \qquad [2.152]$$

Differentiating equation [2.152] leads to:

$$2\mathrm{c}\ \mathrm{d}c = \left(\frac{\mathrm{d}A}{b} - \frac{A}{b^2}\mathrm{d}b\right)g = \left(\mathrm{d}h - \frac{A}{b^2}\mathrm{d}b\right)g = \left(\frac{\mathrm{d}h}{\mathrm{d}b} - \frac{A}{b^2}\right)g\ \mathrm{d}b \qquad [2.153]$$

Hence the expression of dc/db:

$$\frac{\mathrm{d}c}{\mathrm{d}b} = \left(\frac{\mathrm{d}h}{\mathrm{d}b} - \frac{A}{b^2}\right)\frac{g}{2\mathrm{c}} \qquad [2.154]$$

and:

$$\frac{b}{c}\frac{\mathrm{d}c}{\mathrm{d}b} = \left(\frac{\mathrm{d}h}{\mathrm{d}b} - \frac{A}{b^2}\right)\frac{gb}{2c^2} = \left(\frac{\mathrm{d}h}{\mathrm{d}b} - \frac{A}{b^2}\right)\frac{gb^2}{2gA} = \frac{1}{2}\left(\frac{b^2}{A}\frac{\mathrm{d}h}{\mathrm{d}b} - 1\right) \qquad [2.155]$$

The quantity $c/b\, \mathrm{d}b/\mathrm{d}c$ is obtained directly from equation [2.155]. Note that $\mathrm{d}h/\mathrm{d}b$ can be estimated very easily from the geometry of the channel. The general formula [2.149] becomes:

$$\left.\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}t} - \beta\frac{\mathrm{d}c}{\mathrm{d}t} = (S_0 - S_f)g \qquad && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u - c \\ &\frac{\mathrm{d}u}{\mathrm{d}t} + \beta\frac{\mathrm{d}c}{\mathrm{d}t} = (S_0 - S_f)g \qquad && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u + c \end{aligned}\right\} \qquad [2.156]$$

where β is defined as:

$$\beta = 2 + 2\frac{b^2}{A}\left(\frac{\mathrm{d}h}{\mathrm{d}b} - 1\right)^{-1} \qquad [2.157]$$

If $\mathrm{d}h/\mathrm{d}b$ can be assumed to be proportional to A/b^2, equation [2.151] becomes:

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(u - \beta c) = (S_0 - S_f)g \quad && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u - c \\ &\frac{\mathrm{d}}{\mathrm{d}t}(u + \beta c) = (S_0 - S_f)g \quad && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u + c \end{aligned}\right\} \qquad [2.158]$$

The Riemann invariants and the source term in equation [2.22] can be redefined as:

$$\mathrm{W} = \begin{bmatrix} u - \beta c \\ u + \beta c \end{bmatrix}, \qquad \mathrm{S''} = \begin{bmatrix} (S_0 - S_f)g \\ (S_0 - S_f)g \end{bmatrix} \qquad [2.159]$$

NOTE.– Equation [2.158] leads to equation [2.147]. Equation [2.158] does not lead to equation [2.145] for a rectangular channel because the assumption $\mathrm{d}b \neq 0$ is needed in equation [2.154] to proceed from equation [2.149] to equation [2.158]. This assumption is wrong in the case of a rectangular channel because b is constant.

2.5.4. *Calculation of solutions*

2.5.4.1. *The various possible flow regimes*

As seen in section 2.5.3.1, the propagation speeds of the waves are $u - c$ and $u + c$. In contrast with the water hammer phenomenon, the speeds of the waves are not independent of the flow variables. They can bee seen as the result of the superimposition of two phenomena:

– The propagation of the pressure waves is accounted for the terms $-c$ and $+c$. As in the water hammer phenomenon, a perturbation in the flow gives rise to two pressure waves traveling in opposite directions.

– The pressure waves travel in the fluid (here, water) that moves at the speed u. Therefore, the flow velocity u must be added to the speed of the pressure waves to account for the movement of the water molecules in the channel.

In a Lagrangian coordinate system, that is, a coordinate system that moves at the speed of the flow, the speeds of the waves become $-c$ and $+c$ respectively.

The so-called Froude number is commonly used to characterize the flow regime. This dimensionless number is defined as:

$$\mathrm{Fr} = \frac{|u|}{c} \qquad [2.160]$$

Depending on the value of Fr, the flow regime is said to be subcritical, critical or supercritical:

– For $\mathrm{Fr} < 1$, the flow is said to be subcritical. This corresponds to the condition $|u| < c$. In this case, $u - c$ is negative and $u + c$ is positive. The first wave travels upstream, the second wave travels downstream. As a consequence, the behavior of the flow at a given point in the channel is influenced by the flow conditions downstream of it.

– For $\mathrm{Fr} > 1$, the flow is said to be supercritical. This corresponds to the condition $|u| > c$. If u is positive, both wave speeds are positive. If u is negative, both wave speeds are negative. Both waves propagate downstream and the local behavior of the flow is influenced only by the flow conditions upstream.

– For $\mathrm{Fr} = 1$, the flow is said to be critical. This situation corresponds to the transition between subcritical and supercritical conditions. It is usually restricted to a very small region of the flow. A critical point indicates the limit point in the channel where the flow conditions cease to be influenced by the conditions downstream. Critical conditions are usually encountered at singularities such as sills, weirs or next to bridge piers.

The various possible flow regimes can be illustrated in the phase space as in Figure 2.15. If the flow is subcritical, the slopes of the characteristics have opposite signs (Figure 2.15a). When the flow is critical, one of the characteristics is parallel to the time axis (Figure 2.15b). When the flow is supercritical, the slopes of the two characteristics have the same sign (Figure 2.15c).

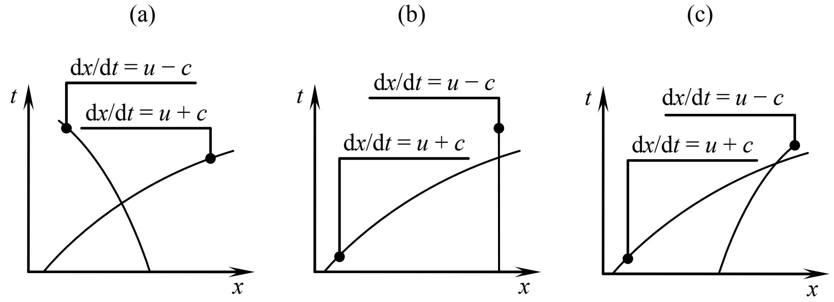

Figure 2.15. *Definition sketch for the various possible flow regimes in the phase space: subcritical (a), critical (b), supercritical (c)*

2.5.4.2. *Treatment of internal points*

Assume first that S" = 0 and that the Riemann invariants can be integrated in the general form [2.159] with the coefficient β defined as in equation [2.157]. The channel reach over which the solution (that is, the couple (A, Q)) is to be determined extends from $x = 0$ to $x = L$ (Figure 2.16). This section focuses on the calculation of the solution at internal points.

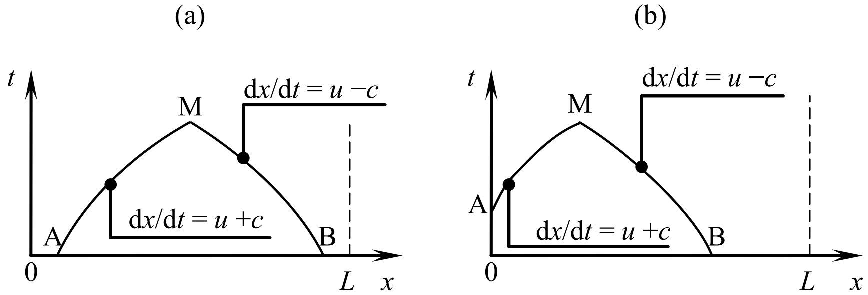

Figure 2.16. *Calculation of the solution at internal points when the domain of dependence is included in the computational domain (a) and when it includes a boundary (b)*

Let A and B denote the feet of the characteristics $\mathrm{d}x/\mathrm{d}t = u + c$ and $\mathrm{d}x/\mathrm{d}t = u - c$ respectively passing at the point M where the solution is sought. A and B may or may not be located at a boundary. A and Q are assumed to be known at both A and B.

As mentioned in section 2.2.2, the solution can be determined uniquely at M provided that the invariant W_1 at B and the invariant W_2 at A are known. Applying relationships [2.158] under the assumption of a zero source term gives:

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(u-\beta c)=0\quad\text{for}\ \frac{\mathrm{d}x}{\mathrm{d}t}=u-c\\ \frac{\mathrm{d}}{\mathrm{d}t}(u+\beta c)=0\quad\text{for}\ \frac{\mathrm{d}x}{\mathrm{d}t}=u+c\end{aligned}\right\}\qquad[2.161]$$

that is:

$$\left.\begin{aligned}u_{\mathrm{M}}-\beta c_{\mathrm{M}}&=u_{\mathrm{B}}-\beta c_{\mathrm{B}}\\ u_{\mathrm{M}}+\beta c_{\mathrm{M}}&=u_{\mathrm{A}}+\beta c_{\mathrm{A}}\end{aligned}\right\}\qquad[2.162]$$

Solving equations [2.162] for u_{M} and c_{M} leads to:

$$\left.\begin{aligned}u_{\mathrm{M}}&=\frac{u_{\mathrm{A}}+u_{\mathrm{B}}}{2}-\frac{c_{\mathrm{A}}-c_{\mathrm{B}}}{2}\beta\\ c_{\mathrm{M}}&=\frac{u_{\mathrm{A}}-u_{\mathrm{B}}}{2\beta}+\frac{c_{\mathrm{A}}+c_{\mathrm{B}}}{2}\end{aligned}\right\}\qquad[2.163]$$

The speed c of the waves in still water is in general related to the cross-sectional area A by a one-to-one relationship. The knowledge of c_{M} allows A_{M} to be determined uniquely.

When the source term S" is non-zero, equations [2.158] are integrated into:

$$\left.\begin{aligned}u_{\mathrm{M}}-\beta c_{\mathrm{M}}&=u_{\mathrm{B}}-\beta c_{\mathrm{B}}+(t_{\mathrm{M}}-t_{\mathrm{B}})S''_{\mathrm{BM}}\\ u_{\mathrm{M}}+\beta c_{\mathrm{M}}&=u_{\mathrm{A}}+\beta c_{\mathrm{A}}+(t_{\mathrm{M}}-t_{\mathrm{A}})S''_{\mathrm{AM}}\end{aligned}\right\}\qquad[2.164]$$

where t_{A}, t_{B} and t_{M} are the time coordinates of the points A, B and M in the phase space respectively and S''_{AM} and S''_{BM} are the average values of the source term S" along the characteristics [AM] and [BM] respectively:

$$\left.\begin{aligned}S''_{\mathrm{AM}}&=\frac{1}{t_{\mathrm{M}}-t_{\mathrm{A}}}\int_{\mathrm{A}}^{\mathrm{M}}\left[(S_0-S_f)g\right]\mathrm{d}t\\ S''_{\mathrm{BM}}&=\frac{1}{t_{\mathrm{M}}-t_{\mathrm{B}}}\int_{\mathrm{B}}^{\mathrm{M}}\left[(S_0-S_f)g\right]\mathrm{d}t\end{aligned}\right\}\qquad[2.165]$$

The solution of system [2.165] is unique:

$$\left.\begin{aligned} u_M &= \frac{u_A + u_B}{2} + \beta\frac{c_A - c_B}{2} + \frac{S''_{AM} + (t_M - t_B)S''_{BM}}{2} \\ c_M &= \frac{u_A - u_B}{2\beta} + \frac{c_A + c_B}{2} + \frac{(t_M - t_A)S''_{AM} - (t_M - t_B)S''_{BM}}{2\beta} \end{aligned}\right\} \quad [2.166]$$

Despite its apparent simplicity, the calculation of the solution as in equation [2.166] is not straightforward. Indeed, it requires that the average values of S" be estimated along the characteristics [AM] and [BM]. From a theoretical point of view, this requires that u and c be known exactly at all points along these characteristics. In arbitrarily-shaped channels, u and c vary along the characteristics. Moreover, the variations in the bed slope S_0 and the friction slope S_f cannot be described analytically. In practical applications, the source term S" must be approximated.

2.5.4.3. *Treatment of boundary conditions*

The treatment of boundary points depends on the flow regime. Three typical situations are examined (Figure 2.17): (1) the flow is subcritical at the boundary, (2) the flow is supercritical, entering the domain, (3) the flow is supercritical, leaving the domain. These cases are detailed hereafter for the treatment of the left-hand boundary of the domain:

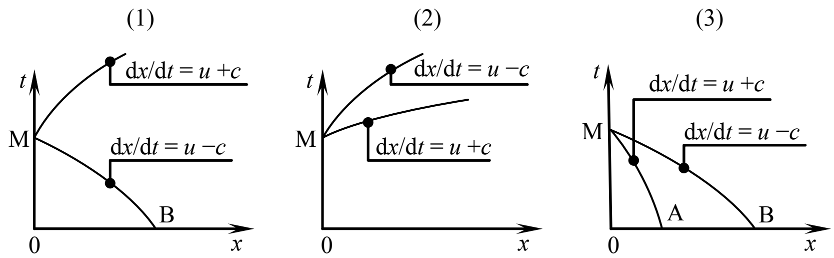

Figure 2.17. *Calculation of the solution at the left-hand boundary: subcritical flow (1), inflowing supercritical flow (2), outflowing supercritical flow (3)*

(1) Subcritical flow. The characteristic $dx/dt = u - c$ leaves the domain, the characteristic $dx/dt = u + c$ enters the domain. The invariant W_1 at any point M on the boundary is known from its value at the internal point B. In contrast, the unknown invariant W_2 cannot be determined from internal points. Additional information must be supplied in the form of a boundary condition. Two equations may be written, the first for the Riemann invariant along the characteristic [BM], the

second for the boundary condition. Except for very simple geometries and boundary conditions, these equations are nonlinear. They can be written in general form as:

$$\left.\begin{aligned} u_M - \beta c_M &= u_B - \beta c_B + (t_M - t_B) S''_{BM} \\ f_b(u_M, c_M, t) &= 0 \end{aligned}\right\} \qquad [2.167]$$

where the (assumed known) function f_b expresses the boundary condition in the form of a time-dependent relationship between u and c. Three main types of boundary conditions are used in practice: prescribed water level (or depth), prescribed discharge, stage-discharge relationship. Their expression for a rectangular channel can be found in section 2.5.4.4. In the general case, system [2.167] must be solved using iterative techniques.

(2) Inflowing supercritical flow. The two characteristics $\mathrm{d}x/\mathrm{d}t = u - c$ and $\mathrm{d}x/\mathrm{d}t = u + c$ enter the domain. The solution at the boundary point M is not influenced by the solution at internal points. Two boundary conditions must be supplied. This is usually done by prescribing the water level and the discharge as functions of time.

(3) Outflowing supercritical flow. The two characteristics $\mathrm{d}x/\mathrm{d}t = u - c$ and $\mathrm{d}x/\mathrm{d}t = u + c$ leave the domain. The solution at the boundary point M is uniquely determined from the internal points. It can be calculated by applying relationships [2.166] between M and the internal points A and B, M being treated exactly in the same way as an internal point (see section 2.5.4.2).

2.5.4.4. *Boundary conditions for a rectangular channel*

The full expression of the main three types of boundary conditions in a rectangular channel where the flow is subcritical is provided hereafter. The channel is assumed to be prismatic, the slope and the friction term are neglected, which leads to a zero source term S". Remember that for a rectangular channel, the speed of the waves in still water is given by $c = (gh)^{1/2}$.

– Prescribed water level h_b at the left-hand boundary. The invariant W_1 is used together with the boundary condition:

$$\left.\begin{aligned} u_M - 2c_M &= u_B - 2c_B \\ c_M &= (gh_b)^{1/2} \end{aligned}\right\} \qquad [2.168]$$

Solving system [2.168] for u_M yields:

$$u_M = u_B - 2c_B + 2(gh_b)^{1/2} \qquad [2.169]$$

– Prescribed discharge Q_b at the left-hand boundary. The invariant W_1 is used again:

$$\left.\begin{aligned} u_M - 2c_M &= u_B - 2c_B \\ u_M A_M &= \frac{b c_M^2 u_M}{g} = Q_b \end{aligned}\right\} \qquad [2.170]$$

Note that the relationship $A = bh = bc^2/g$ is used in the second equation. The nonlinear system [2.170] must be solved iteratively for u_M and c_M.

– Prescribed stage-discharge relationship at the left-hand boundary. The invariant W_1 and the boundary condition are used:

$$\left.\begin{aligned} u_M - 2c_M &= u_B - 2c_B \\ f_b(h_M, u_M A_M, t) &= 0 \end{aligned}\right\} \qquad [2.171]$$

Using the relationship $A = bc^2/g$ leads to:

$$\left.\begin{aligned} u_M - 2c_M &= u_B - 2c_B \\ f_b(c_M, u_M, t) &= 0 \end{aligned}\right\} \qquad [2.172]$$

In the general case system [2.172] is nonlinear and must be solved iteratively.

– Prescribed water depth h_b at the right-hand boundary. The invariant W_2 is used together with the boundary condition:

$$\left.\begin{aligned} u_M + 2c_M &= u_A + 2c_A \\ c_M &= (gh_b)^{1/2} \end{aligned}\right\} \qquad [2.173]$$

Solving system [2.173] for u_M yields:

$$u_M = u_A + 2c_A - 2(gh_b)^{1/2} \qquad [2.174]$$

– Prescribed discharge Q_b at the right-hand boundary. The following system is obtained:

$$\left.\begin{aligned} u_M + 2c_M &= u_A + 2c_A \\ \frac{b c_M^2 u_M}{g} &= Q_b \end{aligned}\right\} \qquad [2.175]$$

The nonlinear system [2.175] must be solved iteratively.

– Prescribed stage-discharge relationship at the right-hand boundary. The invariant W_2 is used together with the boundary condition that is rewritten in the form of a time-dependent relationship between u and c:

$$\left.\begin{aligned} u_{\mathrm{M}} + 2c_{\mathrm{M}} &= u_{\mathrm{A}} + 2c_{\mathrm{A}} \\ f_b(c_{\mathrm{M}}, u_{\mathrm{M}}, t) &= 0 \end{aligned}\right\} \qquad [2.176]$$

In this case again, an iterative solution technique is needed.

2.5.5. *Summary*

The Saint Venant equations are derived on the assumption of incompressible water, of a hydrostatic pressure field, for turbulent flow in nearly horizontal channels.

The structure of the Saint Venant equations is that of a 2×2 compressible system. The wave speeds are $u - c$ and $u + c$. The general formula for the speed c of the waves in still water is given by equation [2.137]. It is given by equation [2.138] for a rectangular channel and by equation [2.145] for a triangular channel.

The general form of the Riemann invariants is defined as in equation [2.140]. Equations [2.149] and equations [2.156–157] provide an alternative writing. These differential relationships may be integrated easily for simple geometries such as rectangular prismatic channels (see equations [2.143–144]) or triangular prismatic channels (see equations [2.146–147]). In prismatic channels with arbitrarily-shaped cross-sections, the (equivalent) formulations [2.151] and [2.158] are applicable, with Riemann invariants defined as in equation [2.159] when $\mathrm{d}h/\mathrm{d}b$ can be assumed to be proportional to A/b^2. If this is not the case, equations [2.151], [2.158] and [2.159] are not exact formulae but approximations of the Riemann invariants.

The flow regime can be characterized using the so-called Froude number Fr, defined as the ratio of the flow velocity u to the speed c of the waves in still water (see equation [2.160]). The flow is said to be subcritical if Fr is smaller than one, supercritical if Fr is larger than one, and critical if Fr is equal to one. The Froude number has the same meaning as the Mach number used for the Euler equations of gas dynamics (see section 2.6).

The solution is unique provided that the initial condition is known at the internal points of the computational domain and that a boundary condition is specified for each characteristic that enters the domain at the boundary of the domain. When the flow is subcritical at a given boundary, only one condition is required. When the flow is supercritical, entering the domain, two boundary conditions are needed. When the flow is supercritical, leaving the domain, no boundary condition is required.

2.6. A nonlinear 3×3 system: the Euler equations

2.6.1. *Physical context – assumptions*

The Euler equations are among the simplest possible systems of governing equations for compressible gas dynamics. They can be seen as a simplified form of the Navier-Stokes equations in the absence of diffusion terms, with the addition of an equation for energy. The Euler equations are derived under the following assumptions:

– Assumption (A1). The gas is compressible. It obeys the equation of state for a perfect gas, whereby the gas internal energy (thus the gas temperature) is a function of the gas pressure and density.

– Assumption (A2). The density, the momentum and the total energy of the gas are assumed to be conserved. The total energy of the gas is formed by the kinetic energy, the internal energy and the energy per unit volume, i.e. the pressure. The total energy should be distinguished from the mechanical energy of the gas, which does not account for thermal effects.

– Assumption (A3). The effects of viscosity, turbulence and heat conduction are neglected. Consequently, the Euler equations do not account for momentum or heat diffusion.

– Assumption (A4). The effects of volume forces such as gravity are negligible.

The Euler equations are used in aerodynamics for far field flow simulations. In near field simulations, a turbulent boundary layer appears in the neighborhood of walls and obstacles that is not accounted for by the Euler equations. Since the continuity and momentum equations in the Euler equations account for the hyperbolic part of the Navier-Stokes equations, their understanding and the understanding of the properties of their solution appears as an important prerequisite in the study of the complete Navier-Stokes equations with energy transport. In fact, the Euler equations allow a number of fundamental phenomena of aerodynamics to be accounted for:

– As shown in sections 2.6.3 and 2.6.4, Assumption (A1) accounts for the fact that the Euler equations describe a compressible flow system. The pressure waves (also called the sound waves) propagate at a finite speed. The Euler equations allow acoustic phenomena to be accounted for.

– The Euler equations allow the occurrence of subsonic, sonic and supersonic flow regimes to be explained.

– The Euler equations allow thermodynamic effects associated with the sudden expansion or compression of gases to be accounted for (deflagrations, etc.).

The study of the Euler equation therefore appears as an indispensable step in the study and practice of aerodynamic modeling.

2.6.2. *Conservation form*

2.6.2.1. *Definitions – notation*

The following notation is used. The gas density, velocity and pressure are denoted by ρ, u and p respectively. The total energy per unit volume, denoted by E, is defined as follows:

$$E = \left(\frac{u^2}{2} + e \right) \rho \qquad [2.177]$$

where e is the internal energy of the gas. From Assumption (A1), e is a function of p and ρ:

$$e = e(p, \rho) \qquad [2.178]$$

The internal energy of a perfect gas is expressed as:

$$e(p, \rho) = \frac{p}{(\gamma - 1)\rho} \qquad [2.179]$$

where γ= 7/5. The entropy s is defined in differential form as:

$$\mathrm{d}s = \frac{\mathrm{d}e + p\mathrm{d}(1/\rho)}{T} \qquad [2.180]$$

where T is the temperature, assumed to satisfy the equation of state for a perfect gas:

$$RT = \frac{p}{\rho} \qquad [2.181]$$

Substituting equations [2.181] and [2.179] into equation [2.180] yields the following expression:

$$\mathrm{d}s = \frac{R}{\gamma - 1}\left(\frac{\mathrm{d}p}{p} - \gamma\frac{\mathrm{d}\rho}{\rho}\right) = \frac{R}{\gamma - 1}\mathrm{d}\left[\ln\left(\frac{p}{\rho^{\gamma}}\right)\right] \qquad [2.182]$$

The conservation form of the one-dimensional continuity, momentum and energy equations is obtained from a balance over a control volume of unit cross-sectional area extending from $x = x_0$ to $x\ \ x_0 + \delta x$ (Figure 2.18).

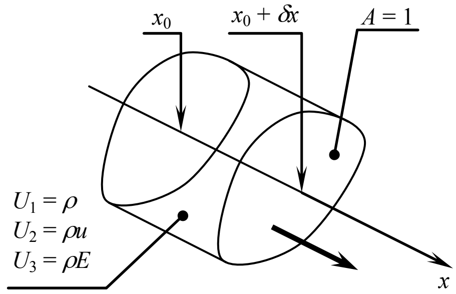

Figure 2.18. *Mass, momentum and energy balance over a control volume*

2.6.2.2. *Continuity equation*

The mass balance can be written as follows:

$$\delta U_1(t_0 + \delta t) - \delta U_1(t_0) = \delta F_1(x_0) - \delta F_1(x_0 + \delta x) \qquad [2.183]$$

where $\delta U_1(t)$ is the mass of fluid contained in the control volume at the time t and $\delta F_1(x)$ is the mass of fluid that passes at the abscissa x during the time interval δt. Equation [2.183] expresses the fact that the variation of the mass contained in the control volume is due to the difference between the inflowing and outflowing mass fluxes across the control sections located at x_0 and $x_0 + \delta x$. δU_1 and δF_1 are defined as:

$$\left.\begin{aligned} \delta U_1(t) &= \int_{x_0}^{x_0+\delta x} (\rho A)(x,t)\mathrm{d}x = A\int_{x_0}^{x_0+\delta x} \rho(x,t)\mathrm{d}x \\ \delta F_1(x) &= \int_{t_0}^{t_0+\delta t} (\rho u A)(x,t)\mathrm{d}x = A\int_{t_0}^{t_0+\delta t} (\rho u)(x,t)\mathrm{d}x \end{aligned}\right\} \qquad [2.184]$$

Substituting equation [2.184] into equation [2.183], dividing by A in the limit of small δt and δx (see equations [1.12–16] for a detailed proof) leads to the following equation for continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \qquad [2.185]$$

Equation [2.185] can be written in the form [2.1] as:

$$\left.\begin{aligned} \frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial x} &= 0 \\ U_1 &= \rho \\ F_1 &= \rho u \end{aligned}\right\} \qquad [2.186]$$

2.6.2.3. *Momentum equation*

A momentum balance over the control volume between t_0 and $t_0 + \delta t$ yields:

$$\begin{aligned} \delta U_2(t_0 + \delta t) - \delta U_2(t_0) = {} & \delta F_2(x_0) - \delta F_2(x_0 + \delta x) \\ & + \delta P(x_0) - \delta P(x_0 + \delta x) \end{aligned} \qquad [2.187]$$

where $\delta U_2(t)$ is the momentum contained in the control volume at the time t and $\delta F_2(x)$ is the momentum attached to the volume of fluid that crosses the abscissa x between t_0 and $t_0 + \delta t$. $\delta P(x)$ represents the integral of the pressure force with respect to time between t_0 and $t_0 + \delta t$. δU_2 and δF_2 are defined as:

$$\left.\begin{aligned} \delta U_2(t) &= \int_{x_0}^{x_0+\delta x} (\rho u A)(x,t)\mathrm{d}x = A \int_{x_0}^{x_0+\delta x} (\rho u)(x,t)\mathrm{d}x \\ \delta F_2(x) &= \int_{t_0}^{t_0+\delta t} (\rho u^2 A)(x,t)\mathrm{d}x = A \int_{t_0}^{t_0+\delta t} (\rho u^2)(x,t)\mathrm{d}x \end{aligned}\right\} \qquad [2.188]$$

The pressure force P is the product of the pressure p and the cross-sectional area A of the control volume:

$$\delta P(t) = \int_{t_0}^{t_0+\delta t} (Ap)(x,t)\mathrm{d}t = A \int_{t_0}^{t_0+\delta t} p(x,t)\mathrm{d}t \qquad [2.189]$$

Substituting equations [2.188–189] into equation [2.187], dividing by A in the limit of small δt and δx (see equations [1.12–16] for details of the proof) leads to the following equation:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = 0 \qquad [2.190]$$

that can be written in the form [2.1] as:

$$\left.\begin{aligned} &\frac{\partial U_2}{\partial t} + \frac{\partial F_2}{\partial x} = 0 \\ &U_2 = \rho u = F_1 \\ &\rho u^2 = \frac{(\rho u)^2}{\rho} = \frac{U_2^2}{U_1} \end{aligned}\right\} \qquad [2.191]$$

2.6.2.4. *Energy equation*

The energy balance over the control volume can be written as:

$$\begin{aligned} \delta U_3(t_0 + \delta t) - \delta U_3(t_0) = {} & \delta F_3(x_0) - \delta F_3(x_0 + \delta x) \\ & + \delta W_p(x_0) - \delta W_p(x_0 + \delta x) \end{aligned} \qquad [2.192]$$

where $\delta U_3(t)$ is the total energy contained in the control volume at the time t, $\delta F_3(x)$ is the total energy attached to the volume of fluid that crosses the abscissa x during the time interval δt and $\delta W_p(x)$ is the work of the pressure force exerted onto the section at the abscissa x during the time interval δt. δU_3 and δF_3 are defined as:

$$\left.\begin{aligned} \delta U_3(t) &= \int_{x_0}^{x_0+\delta x} (AE)(x,t)\,\mathrm{d}x = A\int_{x_0}^{x_0+\delta x} E(x,t)\,\mathrm{d}x \\ \delta F_3(x) &= \int_{t_0}^{t_0+\delta t} (AuE)(x,t)\,\mathrm{d}x = A\int_{t_0}^{t_0+\delta t} (uE)(x,t)\,\mathrm{d}x \end{aligned}\right\} \qquad [2.193]$$

The work $\mathrm{d}W_p(x)$ of the pressure force over an infinitesimal time interval $\mathrm{d}t$ is defined as the product of the pressure force Ap and the displacement $u\,\mathrm{d}t$ of the fluid. Consequently:

$$\delta W_p(t) = A\int_{t_0}^{t_0+\delta t} up(x,t)\,\mathrm{d}t \qquad [2.194]$$

Substituting equations [2.193–194] into equation [2.192], dividing by A in the limit of small δt and δx leads to (see equations [1.12–16] for the details of the reasoning):

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(uE + up) = 0 \qquad [2.195]$$

The energy equation can be rewritten in the conservation form [2.1] as:

$$\left.\begin{aligned} &\frac{\partial U_3}{\partial t} + \frac{\partial F_3}{\partial x} = 0 \\ &U_3 = E \\ &F_3 = (E+p)u = (U_3 + p)\frac{U_2}{U_1} \end{aligned}\right\} \qquad [2.196]$$

2.6.2.5. *Vector conservation form*

Equations [2.186], [2.191] and [2.196] can be rewritten in the vector conservation form [2.2] by defining U and F as:

$$\mathrm{U} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix} = \begin{bmatrix} U_2 \\ U_2^2 / U_1 + p \\ (U_3 + p)U_2 / U_1 \end{bmatrix} \qquad [2.197]$$

2.6.3. *Characteristic form – Riemann invariants*

System [2.2] can be rewritten in the non-conservation form [2.5] by setting the source term to zero:

$$\frac{\partial \mathrm{U}}{\partial t} + \mathrm{A}\frac{\partial \mathrm{U}}{\partial x} = 0 \qquad [2.198]$$

where A is defined as $\partial \mathrm{F} / \partial \mathrm{U}$. Defining the sound speed c as:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\mathrm{d}s=0} \qquad [2.199]$$

leads to the following expression for A:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ c^2 - u^2 & 2u & 0 \\ uc^2 - (E+p)u/\rho & (E+p)/\rho & u \end{bmatrix} \qquad [2.200]$$

A has the following eigenvalues and eigenvectors:

$$\lambda^{(1)} = u - c \qquad \lambda^{(2)} = u \qquad \lambda^{(3)} = u + c$$

$$\mathrm{K}^{(1)} = \begin{bmatrix} 1 \\ u - c \\ (E+p)/\rho - uc \end{bmatrix} \quad \mathrm{K}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathrm{K}^{(3)} = \begin{bmatrix} 1 \\ u + c \\ (E+p)/\rho + uc \end{bmatrix} \qquad [2.201]$$

The vector of the Riemann invariants is defined in differential form as $\mathrm{dW} = \mathrm{K}^{-1}\mathrm{dU}$. K^{-1} is given by:

$$\mathrm{K}^{-1} = \frac{1}{2c}\begin{bmatrix} u + c & -1 & 0 \\ 2[u^2 - (E+p)/\rho]c & -2uc & 2c \\ c - u & 1 & 0 \end{bmatrix} \qquad [2.202]$$

hence the expression of dW:

$$\left.\begin{array}{ll} (u+c)\dfrac{\mathrm{d}\rho}{\mathrm{d}t} - \dfrac{\mathrm{d}}{\mathrm{d}t}(\rho u) = 0 & \text{for } \dfrac{\mathrm{d}x}{\mathrm{d}t} = u - c \\ \left(u^2 - \dfrac{E+p}{\rho}\right)\dfrac{\mathrm{d}\rho}{\mathrm{d}t} - u\dfrac{\mathrm{d}}{\mathrm{d}t}(\rho u) + \dfrac{\mathrm{d}E}{\mathrm{d}t} = 0 & \text{for } \dfrac{\mathrm{d}x}{\mathrm{d}t} = u \\ (c-u)\dfrac{\mathrm{d}\rho}{\mathrm{d}t} + \dfrac{\mathrm{d}}{\mathrm{d}t}(\rho u) = 0 & \text{for } \dfrac{\mathrm{d}x}{\mathrm{d}t} = u + c \end{array}\right\} \qquad [2.203]$$

Noting that $\mathrm{d}(\rho u) = \rho\,\mathrm{d}u + u\,\mathrm{d}\rho$, equations [2.203] become:

$$\left.\begin{array}{ll} c\dfrac{\mathrm{d}\rho}{\mathrm{d}t} - \rho\dfrac{\mathrm{d}u}{\mathrm{d}t} = 0 & \text{for } \dfrac{\mathrm{d}x}{\mathrm{d}t} = u - c \\ \left(-\dfrac{E+p}{\rho}\right)\dfrac{\mathrm{d}\rho}{\mathrm{d}t} - \rho\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{u^2}{2}\right) + \dfrac{\mathrm{d}E}{\mathrm{d}t} = 0 & \text{for } \dfrac{\mathrm{d}x}{\mathrm{d}t} = u \\ c\dfrac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\dfrac{\mathrm{d}u}{\mathrm{d}t} = 0 & \text{for } \dfrac{\mathrm{d}x}{\mathrm{d}t} = u + c \end{array}\right\} \qquad [2.204]$$

The term $E/\rho\,\mathrm{d}\rho$ is removed from the second equation [2.204] by noting that:

$$\mathrm{d}E = \mathrm{d}\left(\rho\frac{E}{\rho}\right) = \frac{E}{\rho}\mathrm{d}\rho + \rho\mathrm{d}\left(\frac{E}{\rho}\right) \qquad [2.205]$$

Substituting equation [2.205] into the second equation [2.204] yields:

$$-\frac{p}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E}{\rho} - \frac{u^2}{2}\right) = 0 \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad [2.206]$$

Using definition [2.177], equation [2.206] is transformed into:

$$p\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\right) + \frac{\mathrm{d}e}{\mathrm{d}t} = 0 \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad [2.207]$$

Using equation [2.180] leads to the following expression:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0 \quad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad [2.208]$$

The density ρ not being measurable directly, the first and third characteristic equations are rewritten so as to involve the pressure p that can be measured directly. This is done by substituting relationship [2.199] into equations [2.204]:

$$\left.\begin{aligned} &\frac{\mathrm{d}p}{\mathrm{d}t} - \rho c\frac{\mathrm{d}u}{\mathrm{d}t} = 0 && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u - c \\ &\frac{\mathrm{d}s}{\mathrm{d}t} = 0 && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u \\ &\frac{\mathrm{d}p}{\mathrm{d}t} + \rho c\frac{\mathrm{d}u}{\mathrm{d}t} = 0 && \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u + c \end{aligned}\right\} \qquad [2.209]$$

Note that definition [2.199] allows for an explicit definition of the speed of sound c. This definition can be used to integrate the Riemann invariants. Indeed, c is defined for a constant entropy, that is, for $\mathrm{d}s = 0$. Using the differential form [2.184], it is easy to check that assuming $\mathrm{d}s = 0$ is equivalent to assuming a constant ratio p^γ/ρ. Consequently, the pressure and the density are related by a law of the type:

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^\gamma \qquad [2.210]$$

where p_0 is a reference pressure for which the density is equal to ρ_0. Substituting equation [2.210] into equation [2.199] yields:

$$c = \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_{\mathrm{d}s=0}^{1/2} = \left(\gamma \frac{p_0}{\rho_0^{\gamma}} \rho^{\gamma-1}\right)^{1/2} = \left(\gamma \frac{p}{\rho}\right)^{1/2} \qquad [2.211]$$

Using [2.210] and [2.211], the following expression is obtained for ρc:

$$\rho c = (\gamma \rho p)^{1/2} = \left[\gamma \rho_0 \left(\frac{p}{p_0}\right)^{1/\gamma} p\right]^{1/2} = \left(\frac{\gamma \rho_0}{p_0^{1/\gamma}}\right)^{1/2} p^{\frac{\gamma+1}{2\gamma}} \qquad [2.212]$$

The first Riemann invariant becomes:

$$\left(\frac{\gamma \rho_0}{p_0^{1/\gamma}}\right)^{1/2} p^{\frac{\gamma+1}{2\gamma}} \mathrm{d}p - \mathrm{d}u = 0 \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u - c \qquad [2.213]$$

Equation [2.213] can be integrated into:

$$\beta_1 p^{\beta_2} - u = \text{Const} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u - c \qquad [2.214]$$

where the coefficients β_1 and β_2 are defined as:

$$\left.\begin{aligned} \beta_1 &= \frac{2\gamma}{\gamma+1}\left(\frac{\gamma \rho_0}{p_0^{1/\gamma}}\right)^{1/2} \\ \beta_2 &= \frac{3\gamma+1}{2\gamma} \end{aligned}\right\} \qquad [2.215]$$

Applying the same reasoning to the third Riemann invariant, the vector W is eventually defined as:

$$\mathrm{W} = \begin{bmatrix} \beta_1 p^{\beta_2} - u \\ s \\ \beta_1 p^{\beta_2} + u \end{bmatrix} \qquad [2.216]$$

The characteristic equations are:

$$\left.\begin{array}{ll}\dfrac{\mathrm{d}}{\mathrm{d}t}(u-\beta_1 p^{\beta_2})=0 & \text{for } \dfrac{\mathrm{d}x}{\mathrm{d}t}=u-c\\ \dfrac{\mathrm{d}s}{\mathrm{d}t}=0 & \text{for } \dfrac{\mathrm{d}x}{\mathrm{d}t}=u-c\\ \dfrac{\mathrm{d}}{\mathrm{d}t}(u+\beta_1 p^{\beta_2})=0 & \text{for } \dfrac{\mathrm{d}x}{\mathrm{d}t}=u+c\end{array}\right\} \qquad [2.217]$$

Note that this expression is valid only because the equation of state [2.179] allows the system to be closed, allowing equation [2.210] to be derived from equation [2.182].

2.6.4. *Calculation of the solution*

2.6.4.1. *The various possible flow regimes*

As seen in section 2.6.3, the three wave speeds are $u-c$, u and $u+c$. They reflect the combination of two physical processes:

– the variables that reflect the local state of the flow (the pressure p, the entropy s, the density ρ) are transported with the local flow velocity u;

– the propagation of the pressure waves is "superimposed" onto the transport phenomenon. The pressure waves, also called the sound waves, arise from the equation of state that provides a relationship between the pressure and the density. From the point of view of an observer moving at the fluid velocity u, the pressure waves propagate in opposite directions at speeds $-c$ and $+c$.

Note that the equation of state between the pressure and the density is essential to the hyperbolic character of the Euler equations. Indeed, the inviscid Burgers equation seen in section 1.4 is derived on the basis of equations [1.62] and [1.63] that do not form a hyperbolic system. In contrast, the continuity and momentum equations [2.186] and [2.191] in the Euler equations form a hyperbolic system (adding the energy equation [2.196] is not necessary for the system to be hyperbolic). Equations [2.186, 2.191] differ from system [1.62–63] only by the pressure term p in equation [2.191]. Therefore the presence of the pressure term is the necessary condition for the system to be hyperbolic.

The flow regime may be characterized by the dimensionless Mach number M, defined as the ratio of the flow velocity to the speed of sound:

$$M = \frac{|u|}{c} \qquad [2.218]$$

The Mach number is the equivalent of the Froude number used for open channel flows (see section 2.5). Three types of flow regimes are distinguished:

– If M is smaller than one, the flow is said to be subsonic. The fluid velocity is smaller than the speed of sound. The characteristic $dx/dt = u - c$ influences the points located upstream, while the characteristics $dx/dt = u$ and $dx/dt = u + c$ travel downstream, thus influencing the points located downstream.

– If M is larger than one, the flow is said to be supersonic. The flow velocity being larger than the speed of sound, the three waves travel in the downstream direction. A perturbation arising in the flow cannot influence the points located upstream of its original location.

– If M is strictly equal to one, the flow is said to be sonic, or transonic.

The subsonic, transonic and supersonic flow regimes are illustrated in Figure 2.19 for a flow directed from left to right.

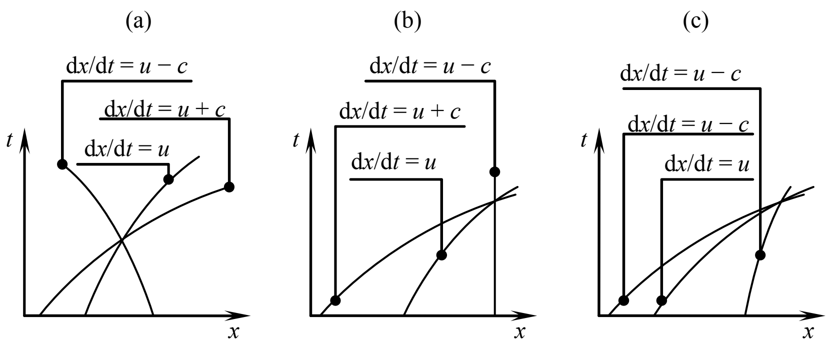

Figure 2.19. *Representation of the various possible flow regimes in the phase space: subsonic (a), transonic (b), supersonic (c). Sketches for a positive flow velocity*

2.6.4.2. *Treatment of internal points*

This section focuses on the solution of the Euler equations at the internal points M of a computational domain that is assumed to extend from $x = 0$ to $x = L$ (Figure 2.20). The feet of the characteristics $u + c$, u and $u + c$ passing at M are denoted by A, B and C respectively. A, B and C may be located on a domain boundary as well as be internal points. The flow variables ρ, u and E are assumed to be known at A, B and C.

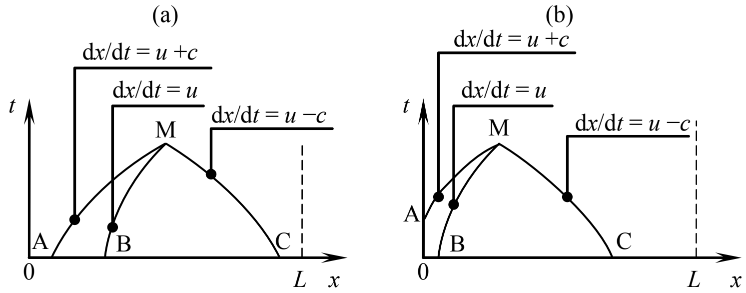

Figure 2.20. *Calculation of the solution at internal points. The domain of dependence may be entirely included in the computational domain (a) or include a boundary (b)*

The Riemann invariants [2.224] allow the following relationships to be written:

$$\left.\begin{aligned} \beta_1 p_M^{\beta_2} - u_M &= \beta_1 p_C^{\beta_2} - u_C \\ s_M &= s_B \\ \beta_1 p_M^{\beta_2} + u_M &= \beta_1 p_A^{\beta_2} + u_A \end{aligned}\right\} \qquad [2.219]$$

Owing to the presence of the nonlinear pressure terms in equations [2.219] and to the dependence of the sound speed c on the pressure, the location of A, B and C cannot be determined analytically in the general case. The calculation of an analytical solution at M is therefore impossible in most practical applications where the initial and boundary conditions are arbitrary functions of space and time. However, analytical or semi-analytical solutions can be found for problems based on simple initial conditions. Such problems include the Riemann problem covered in detail in Chapter 4. The Riemann problem is used in a number of numerical techniques for hyperbolic systems of conservation laws.

Equations [2.219] involve the initial values of p, u and s over the domain of dependence of the solution. The solution can be calculated at internal points only if the initial condition is known over the entire domain $[0, L]$. The knowledge of the initial condition is a necessary condition. It is not a sufficient condition because the knowledge of the solution at the boundaries of the domain is necessary when the domain of dependence of the point M includes the boundaries, as illustrated in Figure 2.20b. The determination of the solution at the boundaries is detailed in the next section.

2.6.4.3. *Treatment of boundary points*

The number of boundary conditions to be supplied depends on the regime and direction of the flow. The following four configurations may occur (Figure 2.21): the flow is subsonic, leaving the domain (Figure 2.21a); the flow is subsonic, entering the domain (Figure 2.21b); the flow is supersonic, entering the domain (Figure 2.21c); or the flow is supersonic, leaving the domain (Figure 2.21d). These configurations are examined for a left-hand boundary, the treatment of a right-hand boundary being deduced by symmetry.

– *Subsonic flow leaving the domain* (Figure 2.1a). The characteristics $\mathrm{d}x/\mathrm{d}t = u - c$ and $\mathrm{d}x/\mathrm{d}t = u$ leave the domain. The flow at the boundary point M is influenced by the Riemann invariants W_2 and W_1 coming respectively from the feet B and C of the characteristics. The missing information on the Riemann invariant W_3 must be supplied in the form of a boundary condition, that is, a possibly time-dependent relationship between the pressure, the flow velocity and the entropy (or the density). This leads to the following system of equations:

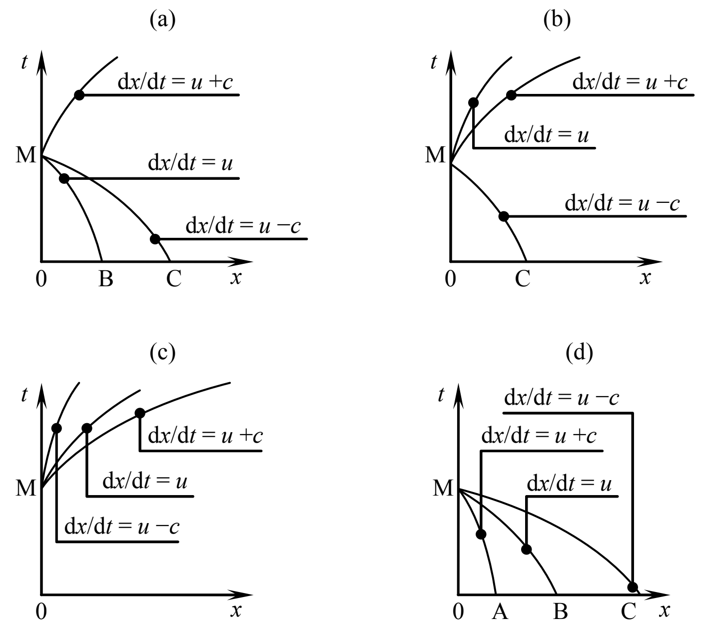

Figure 2.21. *Treatment of boundary points (here on the left-hand boundary). Subsonic flow leaving the domain (a), subsonic flow entering the domain (b), supersonic flow entering the domain (c), supersonic flow leaving the domain (d)*

$$\left.\begin{aligned} & f(p_{\mathrm{M}}, u_{\mathrm{M}}, s_{\mathrm{M}}, t) = 0 \\ & \beta_1 p_{\mathrm{M}}^{\beta_2} - u_{\mathrm{M}} = \beta_1 p_{\mathrm{C}}^{\beta_2} - u_{\mathrm{C}} \\ & s_{\mathrm{M}} = s_{\mathrm{B}} \end{aligned}\right\} \qquad [2.220]$$

– *Subsonic flow entering the domain* (Figure 2.21b). The characteristic $\mathrm{d}x/\mathrm{d}t = u - c$ leaves the domain, the remaining two characteristics enter the domain. The invariant W_3 is known from the initial condition at the foot C of the characteristic. The missing information about the remaining two invariants must be specified in the form of boundary conditions. This results in the following system:

$$\left.\begin{aligned} & f_1(p_{\mathrm{M}}, u_{\mathrm{M}}, s_{\mathrm{M}}, t) = 0 \\ & f_2(p_{\mathrm{M}}, u_{\mathrm{M}}, s_{\mathrm{M}}, t) = 0 \\ & \beta_1 p_{\mathrm{M}}^{\beta_2} - u_{\mathrm{M}} = \beta_1 p_{\mathrm{C}}^{\beta_2} - u_{\mathrm{C}} \end{aligned}\right\} \qquad [2.221]$$

– *Supersonic flow entering the domain* (Figure 2.21c). The three characteristics enter the domain. The behavior of the flow at the boundary is not influenced by the flow conditions inside the domain. Three boundary conditions must be specified:

$$\left.\begin{aligned} p_{\mathrm{M}} &= f_1(t) \\ u_{\mathrm{M}} &= f_2(t) \\ s_{\mathrm{M}} &= f_3(t) \end{aligned}\right\} \qquad [2.222]$$

– *Supersonic flow leaving the domain* (Figure 2.21d). The three characteristics leave the domain. The flow conditions at M are entirely determined by the flow conditions at points A, B and C. System [2.219] is solved exactly as if M was an internal point. No boundary condition is required.

2.6.5. *Summary*

The Euler equations form a 3×3 hyperbolic system of conservation laws. The governing equations are derived from the assumptions of a compressible flow with negligible volume forces, momentum diffusion and heat diffusion.

The Euler equations verify the definitions of a compressible flow system. The speeds of the waves are $u - c$, u and $u + c$. The speed of sound c is given by equations [2.199] and [2.211]. The existence of a finite sound speed reflects the existence of a relationship between the pressure and the density. The hyperbolic character of the Euler equations stems directly from this relationship. If the pressure was independent of the density, the continuity and the momentum equation would be degraded into a simpler formulation of the type [1.62–63] that leads to the inviscid Burgers equation. System [1.62–63] is not hyperbolic.

The Riemann invariants are defined in differential form by equations [2.209]. They can be integrated into equations [2.216]. Their analytical determination is in general impossible because the speed of the characteristics depends on the solution itself, which makes the analytical determination of the feet of the characteristics very difficult, if not impossible.

The flow regime is characterized by the dimensionless Mach number M, defined as in equation [2.218]. The Mach number is the ratio if the flow velocity to the speed of sound. The flow is said to be subsonic when M is smaller than one, sonic or transonic if M is strictly equal to one, and supersonic when M is larger than one.

The solution is determined uniquely over the computational domain provided that (i) the initial condition is known over all the domain and (ii) a boundary condition is supplied for each characteristic that enters the domain. No condition is needed at boundaries where the flow is supersonic, leaving the domain. A boundary with a supersonic inflow requires three conditions. A boundary where the flow is subsonic requires one or two conditions depending on whether the flow leaves or enters the domain.

2.7. Summary of Chapter 2

2.7.1. *What you should remember*

Hyperbolic systems of conservation laws may be expressed in conservation, non-conservation or characteristic form. They reflect the propagation of several waves at different finite speeds.

Coupling several scalar hyperbolic conservation laws does not necessarily yield a hyperbolic system. As shown in section 2.1.2, coupling the scalar hyperbolic equations for continuity and momentum conservation does not lead to a hyperbolic system.

A flow system is said to be compressible if its governing equations form a hyperbolic system of conservation laws with the additional criteria that (i) the system includes at least an equation for the conservation of mass and an equation for the conservation of momentum, (ii) the pressure force is related to the mass per unit volume via an equation of state. The water hammer equations, the Saint Venant equations and the Euler equations describe the behavior of compressible flow systems.

The wave speeds are the eigenvalues of the Jacobian matrix of the flux vector with respect to the conserved variable vector.

The water hammer equations presented in section 2.4 form a 2×2 hyperbolic system of conservation laws. The two waves propagate in opposite directions at the speed of sound. The wave speeds are functions of the local characteristics of the pipe and do not depend on the local flow conditions. The solution can be determined uniquely over a computational domain of finite length provided that the initial condition is known over the entire computational domain and that exactly one boundary condition is specified at each boundary of the domain.

The Saint Venant equations dealt with in section 2.5 form a 2×2 hyperbolic system of conservation laws where the wave speeds are the sum of the local flow velocity and the propagation speed of the waves in still water. The wave speeds are not constant in the general case because they depend on the local flow conditions. Their sign may change depending on whether the flow is subcritical, critical or supercritical. The solution of the Saint Venant equations is determined uniquely over a computational domain of finite length provided that the initial condition is known everywhere over the domain and that one boundary condition is specified for each characteristic that enters the domain at the boundaries.

The Euler equations dealt with in section 2.6 form a 3×3 hyperbolic system of conservation laws, the wave speeds of which are combinations of the local flow velocity and the sound speed. The hyperbolic character of the system stems directly from the presence of the pressure term in the momentum equation. The wave speeds are functions of the local characteristics of the flow. Their sign may change depending on the subsonic, sonic or supersonic nature of the flow. The solution of the Euler equations is determined uniquely over a computational domain of finite length provided that the initial condition is known over the entire domain and that one boundary condition is prescribed for each characteristic that enters the domain.

2.7.2. *Application exercises*

2.7.2.1. *Exercise 2.1: the water hammer equations*

Consider a horizontal pipe of cross-sectional area A, where the pressure waves propagate at the speed c. Friction is assumed to be negligible. The initial flow conditions are steady state conditions, with a pressure and velocity uniformly equal to p_0 and u_0 respectively.

A variation Δp in the pressure appears at the left-hand end of the pipe and propagates to the right at the speed c (Figure 2.22). As a consequence, a variation Δu appears in the flow velocity.

1) Show that Δp and Δu verify the following relationship:

$$\Delta p = \rho c \Delta u \qquad [2.223]$$

2) Assume now that the wave propagates from right to left. Show that the following relationship holds between Δp and Δu:

$$\Delta p = -\rho c \Delta u \qquad [2.224]$$

These equations are called Joukowski's relationships.

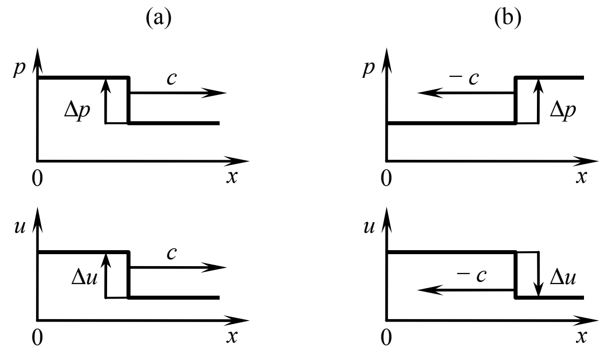

Figure 2.22. *Propagation of a pressure and velocity variation in a pipe. Propagation from left to right (a), from right to left (b)*

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

2.7.2.2. *Exercise 2.2: the water hammer equations*

Consider a horizontal pipe of cross-sectional area A, where the speed of the pressure waves is piecewise constant. The wave speed to the left of the point $x = x_0$ is denoted by c_1, the wave speed to the right of $x = x_0$ is denoted by c_2. The fluid is initially at rest, the pressure is uniformly equal to p_0. The influence of friction is assumed to be negligible.

At $t = 0$ the pressure at the left-hand end of the pipe rises instantaneously to the constant value p_1. The resulting pressure discontinuity propagates to the right at a speed c_1.

1) Derive the expression of the discharge Q_1 on the left-hand side of the pressure discontinuity.

2) The pressure wave reaches the abscissa x_0 where the speed of sound changes to c_2. Considering that the pressure is continuous at $x = x_0$, show that the pressure and the discharge change to new values p_2 and Q_2 when the pressure wave reaches $x = x_0$. Provide the expression of p_2 and Q_2 as functions of p_0, p_1, c_1 and c_2.

3) Show that the pressure surge is amplified if $c_1 < c_2$ (in other words, $|p_2 - p_0| > |p_1 - p_0|$). Conversely, show that the pressure surge is damped if $c_1 > c_2$. Provide a physical interpretation for such a behavior.

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

2.7.2.3. *Exercise 2.3: the water hammer equations*

Consider a horizontal pipe, the cross-sectional area is the following function of the longitudinal coordinate x (Figure 2.23):

– for $x < x_1$ the section is constant, equal to A_1;

– for $x > x_2 > x_1$ the section is constant, equal to A_2;

– for $x_1 < x < x_2$ the section varies continuously from A_1 to A_2.

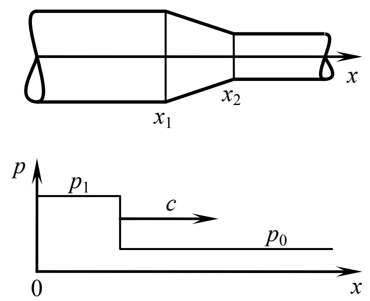

Figure 2.23. *Propagation of a pressure wave in a pipe with variable cross-sectional area*

A_2 may be larger or smaller than A_1. The wave speed c is the same all along the pipe. The effects of friction are assumed to be negligible. The water is initially flowing with a uniform pressure p_0 and a uniform discharge Q_0. At $t = 0$, the pressure at the left-hand end of the pipe changes instantaneously from p_0 to p_1. The resulting pressure discontinuity propagates to the right at the constant wave speed c.

1) Provide the expression of the discharge Q_1 on the left-hand side of the pressure wave before the pressure wave reaches the abscissa x_1.

2) Provide the expression for the pressure p_2 and the discharge Q_2 when the pressure wave reaches the abscissa x_2. What is the effect of a narrowing on the pressure transient? What is the effect of a widening?

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

2.7.2.4. *Exercise 2.4: the Saint Venant equations*

Consider a channel, the width of which decreases from x_1 to x_2 and increases from x_2 to x_3 (Figure 2.24). Assuming steady state, negligible friction and bottom slope, show that:

1) if the flow is subcritical ($u < c$) upstream of the narrowing and supercritical ($u > c$) downstream of it, critical conditions ($u = c$) can be reached only at the narrowest point, $x = x_2$,

2) if the flow is subcritical everywhere in the channel, the water depth reaches a minimum value at $x = x_2$,

3) if the flow is supercritical everywhere in the channel, the water depth reaches a maximum value at $x = x_2$.

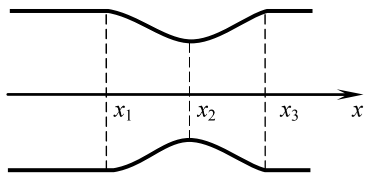

Figure 2.24. *Free surface flow in a channel with a local section narrowing*

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

2.7.2.5. *Exercise 2.5: the Saint Venant equations*

Consider a rectangular channel, the width and slope of which are denoted by b and S_0 respectively. The Strickler coefficient is assumed to be uniform. Steady, uniform flow is assumed, that is, the slope of the energy line is assumed to be identical to the slope of the bottom of the channel.

1) Provide the expression of the wave speed λ of the kinematic wave as a function of the water depth h. The wide channel approximation ($h \ll b$) will be assumed in the calculation of the hydraulic radius for the sake of simplicity.

2) Provide the expressions of the wave speeds $\lambda^{(1)}$ and $\lambda^{(2)}$ in the Saint Venant equations as a function of h, assuming that the assumption of a steady, uniform flow and the wide channel approximation remain valid (the assumption of a uniform flow allows the flow velocity u to be expressed as a function of h).

3) Compare the two expressions and plot the wave speeds as functions of h for the numerical values provided in Table 2.1. Conclude about the validity of the kinematic wave approximation in practical applications.

Symbol	Meaning	Value
b	Channel width	10 m
g	Gravitational acceleration	9.81 m/s^2
K_{Str}	Strickler coefficient	40
S_0	Channel bottom slope	0.1%, 1%, 5%

Table 2.1. *Parameters for Exercise 2.5*

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

2.7.2.6. *Exercise 2.6: the Saint Venant equations*

Consider a rectangular channel, the length and bottom slope of which are denoted by L and S_0 respectively. The elevation of the bottom at the left-hand end of the channel is denoted by z_L. The water is initially at rest. The elevation of the free surface is denoted by ζ_0 (Figure 2.25). The effect of friction is neglected and the perturbations in the free surface elevation are assumed to be small enough for the wave speeds to be considered independent of time. The numerical values of the physical parameters can be found in Table 2.2.

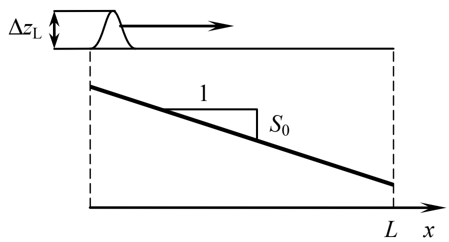

Figure 2.25. *Propagation of a perturbation in a channel with constant bottom slope*

At $t = 0$, a perturbation appears at the left-hand end of the channel. The height of the perturbation is denoted by Δz_L.

1) Provide the expressions of the resulting perturbations Δu_L and ΔQ_L in the velocity and in the discharge.

2) Provide a graphical representation of the characteristic along which the perturbation travels in the phase space. Provide the expression for the time T_R at which the perturbation reaches the right-hand end of the channel.

3) Compute the height Δz_R of the perturbation when it reaches the right-hand end of the channel, as well as the perturbations Δu_R and ΔQ_R in the velocity and in the discharge. What is your conclusion about the validity of the assumption that the wave speed does not depend on time?

Symbol	Meaning	Value
b	Channel width	10 m
g	Gravitational acceleration	9.81 m/s^2
L	Channel length	100 m
S_0	Channel bottom slope	10%
z_G	Elevation of the channel bottom at the left-hand end	0 m
Δz_G	Height of the perturbation at the left-hand end of the channel	0.1 m
ζ_0	Initial elevation of the free surface	1 m

Table 2.2. *Parameters for Exercise 2.6*

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

2.7.2.7. *Exercise 2.7: the Euler equations*

A loudspeaker may be schematized as a plane membrane of cross-sectional area A subjected to a displacement in the direction x normal to the plane (Figure 2.26).

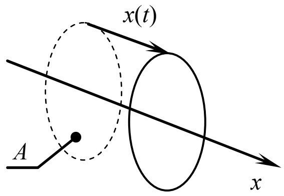

Figure 2.26. *Definition sketch of a loudspeaker membrane*

Both sides of the membrane are in contact with the ambient air (orifices on the rear side of the cabinet to allow for such a contact with the back side of the membrane). Assume that the movement $x(t)$ of the membrane can be described by a periodic, sinusoidal function of time in the form:

$$x(t) = a\cos(2\pi N t) \qquad [2.225]$$

where a is the (constant) amplitude of the movement and N is the (constant) frequency of the sound signal.

1) Assuming a constant speed of sound c, provide the expression for the pressure as a function of time on both sides of the membrane.

2) Determine the average mechanical power needed to move the membrane over a period. Show that the power is proportional to the square of the frequency.

3) Carry out the numerical application for the parameter values in Table 2.3.