Chapter 5

Multidimensional Hyperbolic Systems

5.1. Definitions

5.1.1. *Scalar laws*

A two-dimensional scalar hyperbolic conservation law is a PDE that can be written in conservation form as:

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=S \qquad [5.1]$$

where F and G are respectively the fluxes in the x- and y-direction and S is the source term. As in the one-dimensional case, F and G are functions of U and, possibly, of x and t, but do not contain functions of any of the derivatives of U.

Equation [5.1] can be written in non-conservation form as:

$$\frac{\partial U}{\partial t}+\lambda_x\frac{\partial U}{\partial x}+\lambda_y\frac{\partial U}{\partial y}=S' \qquad [5.2]$$

where λ_x and λ_y are respectively the wave speeds in the x- and y-direction. λ_x, λ_y and $S\,'$ are given by:

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$$\left.\begin{aligned} \lambda_x &= \frac{\partial F}{\partial U} \\ \lambda_y &= \frac{\partial G}{\partial U} \\ S' &= S - \left(\frac{\partial F}{\partial x}\right)_{U=\text{Const}} - \left(\frac{\partial G}{\partial y}\right)_{U=\text{Const}} \end{aligned}\right\} \qquad [5.3]$$

Equation [5.2] is equivalent to the following characteristic form:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = S' \qquad \text{for} \begin{cases} \mathrm{d}x/\mathrm{d}t = \lambda_x \\ \mathrm{d}y/\mathrm{d}t = \lambda_y \end{cases} \qquad [5.4]$$

In the particular case $S' = 0$, equation [5.4] can be integrated into:

$$U = \text{Const} \qquad \text{for} \begin{cases} \mathrm{d}x/\mathrm{d}t = \lambda_x \\ \mathrm{d}y/\mathrm{d}t = \lambda_y \end{cases} \qquad [5.5]$$

Equation [5.1] is extended to three dimensions of space as follows:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = S \qquad [5.6]$$

The non-conservation form of equation [5.6] is:

$$\frac{\partial U}{\partial t} + \lambda_x \frac{\partial U}{\partial x} + \lambda_y \frac{\partial U}{\partial y} + \lambda_z \frac{\partial U}{\partial z} = S' \qquad [5.7]$$

where λ_x, λ_y, λ_z and S' are given by:

$$\left.\begin{aligned} &\lambda_x = \frac{\partial F}{\partial U},\ \lambda_y = \frac{\partial G}{\partial U},\ \lambda_z = \frac{\partial H}{\partial U} \\ &S' = S - \left(\frac{\partial F}{\partial x}\right)_{U=\text{Const}} - \left(\frac{\partial G}{\partial y}\right)_{U=\text{Const}} - \left(\frac{\partial H}{\partial z}\right)_{U=\text{Const}} \end{aligned}\right\} \qquad [5.8]$$

Equation [5.6] can also be written in characteristic form as:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = S' \qquad \text{for} \begin{cases} \mathrm{d}x / \mathrm{d}t = \lambda_x \\ \mathrm{d}y / \mathrm{d}t = \lambda_y \\ \mathrm{d}z / \mathrm{d}t = \lambda_z \end{cases} \tag{5.9}$$

Note that in the particular case $S' = 0$, equation [5.9] is integrated into:

$$U = \text{Const} \qquad \text{for} \begin{cases} \mathrm{d}x / \mathrm{d}t = \lambda_x \\ \mathrm{d}y / \mathrm{d}t = \lambda_y \\ \mathrm{d}z / \mathrm{d}t = \lambda_z \end{cases} \tag{5.10}$$

5.1.2. *Two-dimensional hyperbolic systems*

A two-dimensional, $m \times m$ system of conservation laws is a system of m PDEs in the form [5.1]:

$$\left.\begin{aligned} \frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y} &= S_1 \\ &\vdots \\ \frac{\partial U_m}{\partial t} + \frac{\partial F_m}{\partial x} + \frac{\partial G_m}{\partial y} &= S_m \end{aligned}\right\} \tag{5.11}$$

where U_p, F_p, G_p and S_p ($p = 1, \ldots, m$) are respectively the conserved variable, the flux in the x- and y-direction and the source term for the pth equation. In the general case, F_p, G_p and S_p are functions not only of U_p but also of the other conserved variables $U_1, \ldots, U_m$. System [5.11] can be written in vector conservation form as:

$$\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} + \frac{\partial \mathrm{G}}{\partial y} = \mathrm{S} \tag{5.12}$$

where U, F, G and S are defined as:

$$\mathrm{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix}, \quad \mathrm{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \quad \mathrm{G} = \begin{bmatrix} G_1 \\ \vdots \\ G_m \end{bmatrix}, \quad \mathrm{S} = \begin{bmatrix} S_1 \\ \vdots \\ S_m \end{bmatrix} \tag{5.13}$$

Equation [5.12] can be written in non-conservation form as:

$$\frac{\partial \mathrm{U}}{\partial t} + \mathrm{A}_x \frac{\partial \mathrm{U}}{\partial x} + \mathrm{A}_y \frac{\partial \mathrm{U}}{\partial y} = \mathrm{S}' \qquad [5.14]$$

where A_x, A_y and S' are given by:

$$\left.\begin{aligned} \mathrm{A}_x &= \frac{\partial \mathrm{F}}{\partial \mathrm{U}} \\ \mathrm{A}_y &= \frac{\partial \mathrm{G}}{\partial \mathrm{U}} \\ \mathrm{S}' &= \mathrm{S} - \left(\frac{\partial \mathrm{F}}{\partial x}\right)_{\mathrm{U=Const}} - \left(\frac{\partial \mathrm{G}}{\partial y}\right)_{\mathrm{U=Const}} \end{aligned}\right\} \qquad [5.15]$$

System [5.11] is said to be hyperbolic if any linear combination of the matrices A_x and A_y has m real distinct eigenvalues. This definition is justified and interpreted in section 5.3.1.2.

5.1.3. *Three-dimensional hyperbolic systems*

A three-dimensional, $m \times m$ system of conservation laws is a system of m PDEs in the form [5.6]:

$$\left.\begin{aligned} \frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y} + \frac{\partial H_1}{\partial z} &= S_1 \\ &\vdots \\ \frac{\partial U_m}{\partial t} + \frac{\partial F_m}{\partial x} + \frac{\partial G_m}{\partial y} + \frac{\partial H_m}{\partial z} &= S_m \end{aligned}\right\} \qquad [5.16]$$

where U_p, F_p, G_p, H_p and S_p ($p = 1, \ldots, m$) are the conserved variable, the flux in the x-, y- and z-direction respectively, and the source term for the pth equation. In the general case, F_p, G_p, H_p and S_p are functions not only of U_p but also of the other conserved variables $U_1, \ldots, U_m$. System [5.16] can be written in vector conservation form as:

$$\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} + \frac{\partial \mathrm{G}}{\partial y} + \frac{\partial \mathrm{H}}{\partial z} = \mathrm{S} \qquad [5.17]$$

where the vectors U, F, G, H and S are defined as:

$$\mathrm{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix}, \quad \mathrm{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \quad \mathrm{G} = \begin{bmatrix} G_1 \\ \vdots \\ G_m \end{bmatrix}, \quad \mathrm{H} = \begin{bmatrix} H_1 \\ \vdots \\ H_m \end{bmatrix}, \quad \mathrm{S} = \begin{bmatrix} S_1 \\ \vdots \\ S_m \end{bmatrix} \qquad [5.18]$$

Equation [5.17] can be written in non-conservation form as:

$$\frac{\partial \mathrm{U}}{\partial t} + \mathrm{A}_x \frac{\partial \mathrm{U}}{\partial x} + \mathrm{A}_y \frac{\partial \mathrm{U}}{\partial y} + \mathrm{A}_z \frac{\partial \mathrm{U}}{\partial z} = \mathrm{S'} \qquad [5.19]$$

where A_x, A_y, A_z and S' are defined as:

$$\left.\begin{aligned} &\mathrm{A}_x = \frac{\partial \mathrm{F}}{\partial \mathrm{U}}, \mathrm{A}_y = \frac{\partial \mathrm{G}}{\partial \mathrm{U}}, \mathrm{A}_z = \frac{\partial \mathrm{H}}{\partial \mathrm{U}} \\ &\mathrm{S'} = \mathrm{S} - \left(\frac{\partial \mathrm{F}}{\partial x}\right)_{\mathrm{U=Const}} - \left(\frac{\partial \mathrm{G}}{\partial y}\right)_{\mathrm{U=Const}} - \left(\frac{\partial \mathrm{H}}{\partial z}\right)_{\mathrm{U=Const}} \end{aligned}\right\} \qquad [5.20]$$

The three-dimensional system of conservation laws [5.17] is said to be hyperbolic if any linear combination of the matrices A_x, A_y and A_z has m real, distinct eigenvalues.

5.2. Derivation from conservation principles

This section focuses on the derivation of two-dimensional systems of conservation laws. The generalization to three-dimensional systems is straightforward and will not be detailed hereafter. A system of conservation laws being formed by a set of scalar laws, only the derivation of scalar laws is dealt with hereafter.

Two-dimensional conservation laws are written using a mass balance over a two-dimensional control volume of size $\delta x \times \delta y$ over a time interval δt (Figure 5.1). The balance can be written as:

$$\begin{aligned} \delta U(t_0 + \delta t) - \delta U(t_0) = {} & \delta F(x_0) - \delta F(x_0 + \delta x) \\ & + \delta G(y_0) - \delta G(y_0 + \delta y) + \delta S \end{aligned} \qquad [5.21]$$

where $\delta U(t)$ is the total amount of U contained in the control volume at the time t, $\delta F(x)$ is the total amount of U that crosses the interface of width δy located at the abscissa x over the time interval δt, $\delta G(y)$ is the total amount of U that crosses the interface of width δx located at the ordinate y over the time interval δt, and δS is the total amount of U that appears within the control volume over the time interval δt owing to the source term. By definition:

$$\left.\begin{aligned}
\delta U(t) &= \int_{x_0}^{x_0+\delta x}\int_{y_0}^{y_0+\delta y} U(x,y,t)\,\mathrm{d}y\,\mathrm{d}x \\
\delta F(x) &= \int_{t_0}^{t_0+\delta t}\int_{y_0}^{y_0+\delta y} F(x,y,t)\,\mathrm{d}y\,\mathrm{d}t \\
\delta G(y) &= \int_{t_0}^{t_0+\delta t}\int_{x_0}^{x_0+\delta x} G(x,y,t)\,\mathrm{d}x\,\mathrm{d}t \\
\delta S &= \int_{t_0}^{t_0+\delta t}\int_{x_0}^{x_0+\delta x}\int_{y_0}^{y_0+\delta y} U(x,y,t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\right\} \qquad [5.22]$$

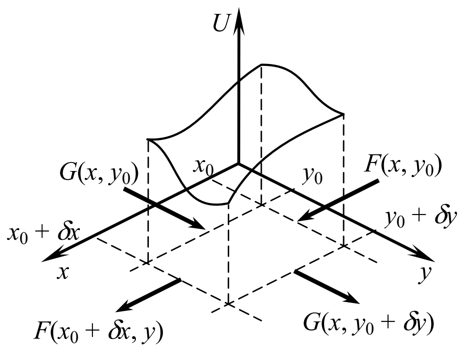

Figure 5.1. *Definition sketch for the derivation of a two-dimensional, scalar conservation law*

When δx, δy and δt tend to zero, the integral of a given function over a surface or a segment tends to the product of the point value of the function and the surface or the segment length. This leads to the following equivalence for δU:

$$\delta U(t) \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\approx} U \delta x \delta y \qquad [5.23]$$

and to the following equivalence for δF and δG:

$$\left.\begin{aligned} \delta F(x) &\underset{\substack{\delta x\to 0\\ \delta t\to 0}}{\approx} F(x,y)\delta y\delta t \\ \delta G(y) &\underset{\substack{\delta x\to 0\\ \delta t\to 0}}{\approx} G(x,y)\delta x\delta t \end{aligned}\right\} \qquad [5.24]$$

The following equivalence holds for δS:

$$\delta S \underset{\substack{\delta x\to 0\\ \delta y\to 0\\ \delta t\to 0}}{\approx} S\delta x\delta y\delta t \qquad [5.25]$$

Substituting equations [5.23–25] into equation [5.21] leads to:

$$\begin{aligned} [U(t_0+\delta t)-U(t_0)]\delta x\delta y &= [F(x_0)-F(x_0+\delta x)]\delta y\delta t \\ &\quad +[G(y_0)-G(y_0+\delta y)]\delta x\delta t + S\delta x\delta y\delta t \end{aligned} \qquad [5.26]$$

The differences in equation [5.26] can be expressed as functions of the derivatives of U, F and G with respect to time and space using the following equivalences:

$$\left.\begin{aligned} U(x,y,t_0+\delta t)-U(x,y,t_0) &\underset{\delta t\to 0}{\approx} \frac{\partial U}{\partial t}\delta t \\ F(x_0+\delta x,y,t)-F(x_0,y,t) &\underset{\delta x\to 0}{\approx} \frac{\partial F}{\partial x}\delta x \\ G(x,y_0+\delta y,t)-G(x,y_0,t) &\underset{\delta y\to 0}{\approx} \frac{\partial G}{\partial y}\delta y \end{aligned}\right\} \qquad [5.27]$$

Substituting equations [5.27] into equation [5.26] and simplifying leads to equation [5.1].

Note that the assumption of a continuous and differentiable solution is not necessary in the derivation of the weak form [5.21], while equation [5.1] is based on such an assumption. The assumption of a differentiable variable is used in equation [5.27] to introduce the derivatives of U, F and G with respect to time and space. As in the one-dimensional case, the "strong form" [5.1] is a particular case of the weak form [5.21]. As in the one-dimensional case, the weak solutions of equation [5.1] may be non-unique. These remarks also hold for three-dimensional scalar laws and for hyperbolic systems of conservation laws.

5.3. Solution properties

5.3.1. *Two-dimensional hyperbolic systems*

5.3.1.1. *The bicharacteristic approach*

This section deals with the properties of the solutions of two-dimensional hyperbolic systems of conservation laws, that is, systems in the form [5.12]. Several approaches may be used to characterize the behavior and properties of the solutions of such systems. The bicharacteristic approach, also called the characteristic surface approach, is one of them.

As in the one-dimensional case, the purpose is to find surfaces in the phase space (x, y, t), over which certain quantities are invariant (Figure 5.2). The knowledge of the solution U at a given point (x_0, y_0) in the characteristic surface should allow the value of U to be computed at any other point that belongs to the surface. Conversely, a given characteristic surface cannot provide any information about the value of U at a point that does not belong to it. This property is used in the derivation of the characteristic form.

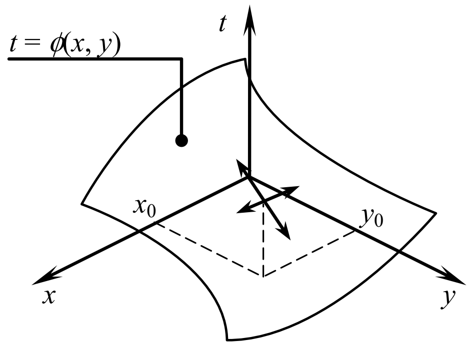

Figure 5.2. *Definition sketch for a characteristic surface in the phase space for a two-dimensional hyperbolic system*

Assume that U is known at a point (x_0, y_0, t_0) in the phase space. The value of U at this point is denoted by U_0. U verifies the conservation form [5.12] and its non-conservation form [5.14]. The equation of the characteristic surface that passes at (x_0, y_0, t_0) is sought in the form:

$$t = \phi(x, y) \qquad [5.28]$$

Assume that U is known over all the surface. The value of U at the point $(x, y, \phi(x, y))$ is denoted by $U_s(x, y)$. The differential dU_s is defined as:

$$dU_s = \frac{\partial U}{\partial t}d\phi + \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \qquad [5.29]$$

The differential $d\phi$ is given by:

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy \qquad [5.30]$$

Substituting equation [5.29] into equation [5.30] leads to:

$$dU_s = \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy\right)\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \qquad [5.31]$$

Hence the derivatives of U_s with respect to x and y:

$$\left.\begin{aligned} \frac{\partial U_s}{\partial x} &= \frac{\partial \phi}{\partial x}\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} \\ \frac{\partial U_s}{\partial y} &= \frac{\partial \phi}{\partial y}\frac{\partial U}{\partial t} + \frac{\partial U}{\partial y} \end{aligned}\right\} \qquad [5.32]$$

The system [5.32] can be rewritten as:

$$\left.\begin{aligned} \frac{\partial U}{\partial x} &= \frac{\partial U_s}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial U}{\partial t} \\ \frac{\partial U}{\partial y} &= \frac{\partial \phi}{\partial y}\frac{\partial U}{\partial t} + \frac{\partial U_s}{\partial y} \end{aligned}\right\} \qquad [5.33]$$

Substituting equation [5.33] into the non-conservation form [5.14] leads to:

$$\left(I - \frac{\partial \phi}{\partial x}A_x - \frac{\partial \phi}{\partial y}A_y\right)\frac{\partial U}{\partial t} = S' - A_x\frac{\partial U_s}{\partial x} - A_y\frac{\partial U_s}{\partial y} \qquad [5.34]$$

where I is the $m \times m$ identity matrix. U may be calculated at any point that does not belong to the surface $t = \phi(x, y)$ using a first-order expansion:

$$U(x, y, t) = U_s(x, y) + [\phi(x, y) - t]\frac{\partial U}{\partial t} \qquad [5.35]$$

U cannot be calculated at a point that does not belong to the characteristic surface if $\partial U / \partial t$ cannot be computed. From equation [5.34], such a condition is equivalent to stating that the matrix $\mathrm{I} - \partial\phi / \partial x \, \mathrm{A}_x - \partial\phi / \partial y \, \mathrm{A}_y$ has no inverse, that is, if its determinant is equal to zero:

$$\left| \mathrm{I} - \frac{\partial \phi}{\partial x} \mathrm{A}_x - \frac{\partial \phi}{\partial y} \mathrm{A}_y \right| = 0 \qquad [5.36]$$

Equation [5.36] provides a necessary condition on the x- and y-slopes of the surfaces $t = \phi(x, y)$. As shown in section 5.4, such surfaces may be approximated locally with cones or straight lines in the phase space.

The equation of the characteristic surfaces allows the generalization of the Riemann invariants to be derived along the surfaces. Equation [5.36] means that at least one of the rows in the matrix $\mathrm{I} - \partial\phi / \partial x \, \mathrm{A}_x - \partial\phi / \partial y \, \mathrm{A}_y$ is a linear combination of the others. In other words, there exists a row vector v such that:

$$\left.\begin{aligned} \mathrm{v}\left(\mathrm{I} - \frac{\partial \phi}{\partial x} \mathrm{A}_x - \frac{\partial \phi}{\partial y} \mathrm{A}_y \right) \frac{\partial \mathrm{U}}{\partial t} - \mathrm{v}\left(\mathrm{S}' - \mathrm{A}_x \frac{\partial \mathrm{U}_s}{\partial x} - \mathrm{A}_y \frac{\partial \mathrm{U}_s}{\partial y} \right) = 0 \\ \mathrm{v}\left(\mathrm{I} - \frac{\partial \phi}{\partial x} \mathrm{A}_x - \frac{\partial \phi}{\partial y} \mathrm{A}_y \right) = 0 \end{aligned}\right\} \qquad [5.37]$$

Substituting equation [5.32] into the first equation [5.37] leads to:

$$\mathrm{v}\left(\frac{\partial \mathrm{U}}{\partial t} - \mathrm{A}_x \frac{\partial \mathrm{U}}{\partial x} - \mathrm{A}_y \frac{\partial \mathrm{U}}{\partial y} - \mathrm{S}' \right) = 0 \qquad [5.38]$$

which leads to a differential relationship between the various components of U.

5.3.1.2. *The secant plane approach*

The classical equations of fluid mechanics are invariant by rotation. The secant plane approach consists of finding the characteristic form of the restriction of the equations to a plane parallel to the time axis for any value of the angle θ between the plane and the x-axis (Figure 5.3). For a given value of θ, the intersection between the plane and the characteristic surface gives one or several characteristic curves. As in the one-dimensional case, Riemann invariants can be derived along these curves.

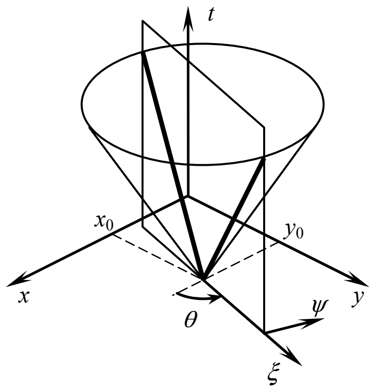

Figure 5.3. *Definition sketch for the secant plane approach. Secant plane and characteristic surface (thin lines), characteristic curves in the secant planes (bold lines)*

A local coordinate system (ξ, ψ) is attached to the secant plane (Figure 5.3). The following relationships hold between the local and global coordinate systems:

$$\left.\begin{aligned} \mathrm{d}\xi &= \cos\theta\ \mathrm{d}x + \sin\theta\ \mathrm{d}y \\ \mathrm{d}\psi &= -\sin\theta\ \mathrm{d}x + \cos\theta\ \mathrm{d}y \end{aligned}\right\} \qquad [5.39]$$

The partial derivatives with respect to x and y are related to those with respect to ξ and ψ via:

$$\left.\begin{aligned} \frac{\partial \mathrm{U}}{\partial x} &= \frac{\partial \mathrm{U}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \mathrm{U}}{\partial \psi}\frac{\partial \psi}{\partial x} = \cos\theta \frac{\partial \mathrm{U}}{\partial \xi} - \sin\theta \frac{\partial \mathrm{U}}{\partial \psi} \\ \frac{\partial \mathrm{U}}{\partial y} &= \frac{\partial \mathrm{U}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \mathrm{U}}{\partial \psi}\frac{\partial \psi}{\partial y} = \sin\theta \frac{\partial \mathrm{U}}{\partial \xi} + \cos\theta \frac{\partial \mathrm{U}}{\partial \psi} \end{aligned}\right\} \qquad [5.40]$$

Substituting equation [5.40] into equation [5.14] leads to the following equation:

$$\frac{\partial \mathrm{U}}{\partial t} + (\cos\theta\ \mathrm{A}_x + \sin\theta\ \mathrm{A}_y)\frac{\partial \mathrm{U}}{\partial \xi} + (-\sin\theta\ \mathrm{A}_x + \cos\theta\ \mathrm{A}_y)\frac{\partial \mathrm{U}}{\partial \psi} = \mathrm{S}' \qquad [5.41]$$

Equation [5.41] can be rewritten as a one-dimensional equation in the direction of the secant plane:

$$\frac{\partial \mathrm{U}}{\partial t} + \mathrm{A}_\xi \frac{\partial \mathrm{U}}{\partial \xi} = \mathrm{S}_\xi \qquad [5.42]$$

where A_ξ and A_ψ are defined as

$$\left.\begin{aligned} A_\xi &= \cos\theta\ A_x + \sin\theta\ A_y \\ S_\xi &= S' + (\sin\theta\ A_x - \cos\theta\ A_y)\frac{\partial U}{\partial \psi} \end{aligned}\right\} \qquad [5.43]$$

If A_ξ has m real, distinct eigenvalues for all θ, the system [5.42] is hyperbolic regardless of the orientation of the secant plane. The matrix A_ξ is a linear combination of the matrices A_x and A_y. It spans all the possible linear combinations of A_x and A_y as θ spans the interval $[0, 2\pi]$. The definition of a hyperbolic system as given in section 5.1.2 is justified as follows: system [5.14] is hyperbolic if its one-dimensional restriction [5.42] to all the possible secant planes is hyperbolic.

As in the one-dimensional case (see Chapter 2), equation [5.40] allows Riemann invariants to be defined in the secant plane. The vector W is defined in differential form as:

$$dW = K_\xi^{-1} dU \qquad [5.44]$$

where K_ξ is the matrix formed by the eigenvectors of A_ξ. Equation [5.42] is rewritten as:

$$\frac{\partial W}{\partial t} + \Lambda_\xi \frac{\partial W}{\partial \xi} = K_\xi^{-1} S_\xi \qquad [5.45]$$

where Λ_ξ is the diagonal matrix formed with the eigenvalues of A_ξ. Equation [5.45] is equivalent to:

$$\frac{dW_p}{dt} = K_\xi^{-1} S_\xi \quad \text{for } \frac{d\xi}{dt} = \lambda_\xi^{(p)} \qquad [5.46]$$

5.3.1.3. *Domain of influence, domain of dependence*

A hyperbolic system of conservation laws leads to several characteristic surfaces in the general case. The domain of influence of the solution is contained within the characteristic surface, the spatial extent of which is the largest. The domain of influence includes not only the characteristic surface but also all the points contained in the volume delineated by the surface. Such a surface is illustrated by Figure 5.4 in the phase space. For the sake of clarity, only one family of characteristic surfaces is sketched in the figure.

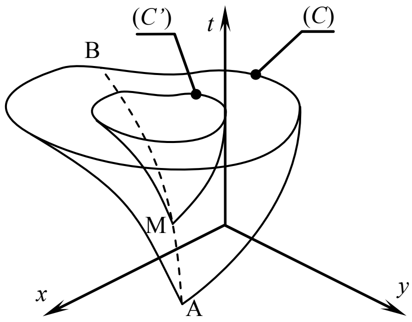

Figure 5.4. *Definition sketch for the domain of influence in the phase space*

When the equations are nonlinear the speeds of the various waves are variable in the phase space. This accounts for the curvature of the surfaces issued from A and M in Figure 5.4. Consider the characteristic surface issued from A. Point A influences all the points M of all the curves (AB) that belong to the surface. Several characteristic surfaces are issued from point B (for the sake of clarity, only the widest surface is drawn). The intersection between the characteristic surface issued from A and the plane ($t = t_B$) is a closed contour denoted by (C) in the figure. The intersection between the characteristic surface issued from M and the plane ($t = t_B$) is a closed contour denoted by (C') in the figure.

When M = A, (C') = (C). When M = B, (C') = B. The size of (C') decreases gradually as M spans all the possible locations from A to B along the line (AB). Consequently, the domain of influence of A includes the curve (C) and all its inner points. Since this is true for any time t_B, the domain of influence of the solution includes all the points in the volume delineated by the widest of all the existing characteristic surfaces issued from A.

A similar reasoning leads to the conclusion that the domain of dependence of the solution at the point B is made of all the points contained in the volume that is delimited by the widest of all the characteristic surfaces passing at B (Figure 5.5). For the sake of clarity, only the widest characteristic surface (C) is represented in Figure 5.5. The characteristic surface is formed by an infinity of generating curves (AB), where the points A are points in the plane ($t = t_A$), $t_A < t_B$. The intersection between the characteristic surface and the plane ($t = t_A$) is a closed curve denoted by (C).

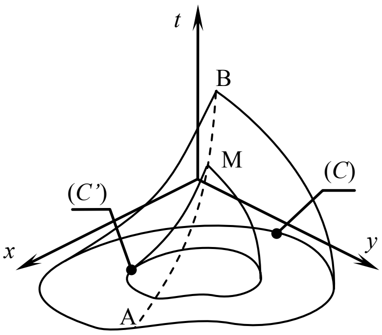

Figure 5.5. *Definition sketch for the domain of dependence in the phase space*

Consider a point M located along the curve (AB). Several characteristic surfaces pass at M. Only the widest one is sketched in the figure for the sake of clarity. The intersection between the surface and the plane ($t = t_A$) is a closed curve (C'). (C') = (C) for M = B, while (C') = A for M = A. The curve (C') spans all the inner points of the curve (C) as the point M spans all the possible locations along the curve (AB). Consequently, all the points enclosed in (C) influence at least one point along the curve (AB), thus influencing indirectly the solution at the point B. Since this is true for all possible values of t_A, the domain of dependence of the solution at B includes the most extended characteristic surface that passes at B as well as all the points contained in the volume delineated by the surface.

5.3.2. *Three-dimensional hyperbolic systems*

In the case of a three-dimensional hyperbolic system, the phase space is four-dimensional. The characteristic surface approach leads us to seek a hyper-surface defined as:

$$t = \phi(x, y, z) \tag{5.47}$$

from which the solution cannot be calculated at points that do not belong to the surface. Applying the same reasoning as in section 5.3.1.1, the following condition is derived for the function ϕ:

$$\left| \mathrm{I} - \frac{\partial \phi}{\partial x}\mathrm{A}_x - \frac{\partial \phi}{\partial y}\mathrm{A}_y - \frac{\partial \phi}{\partial z}\mathrm{A}_z \right| = 0 \tag{5.48}$$

If the secant plane approach is to be used, a local coordinate system (ξ, ψ, ζ) is defined, that is obtained from the original coordinate system by applying a first rotation of angle θ in the (x, y) plane, followed by a second rotation in the thus obtained (ξ, z) plane (Figure 5.6). The two coordinate systems are related by the following differentials:

$$\left.\begin{aligned} \mathrm{d}\xi &= \cos\varphi\cos\theta\ \mathrm{d}x + \cos\varphi\sin\theta\ \mathrm{d}y + \sin\varphi\ \mathrm{d}z \\ \mathrm{d}\psi &= -\sin\theta\ \mathrm{d}x + \cos\theta\ \mathrm{d}y \\ \mathrm{d}\zeta &= -\sin\varphi\cos\theta\ \mathrm{d}x - \sin\varphi\sin\theta\ \mathrm{d}y + \cos\varphi\ \mathrm{d}z \end{aligned}\right\} \qquad [5.49]$$

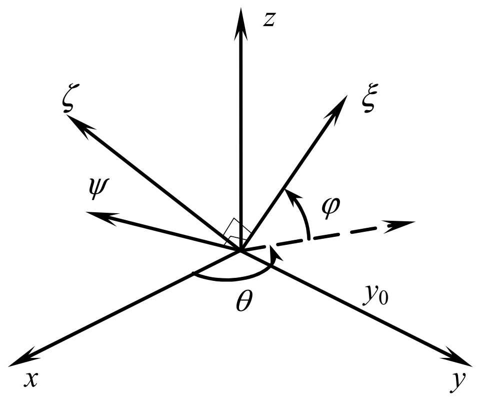

Figure 5.6. *Definition sketch for the local coordinate system in the secant plane approach as applied to three-dimensional hyperbolic systems*

Equation [5.19] can be rewritten in the form [5.42] by defining A_ξ and S_ξ as:

$$\left.\begin{aligned} A_\xi &= \cos\varphi\cos\theta\ A_x + \cos\varphi\sin\theta\ A_y + \sin\varphi\ A_z \\ S_\xi &= S' + (\sin\theta A_x - \cos\theta\ A_y)\frac{\partial U}{\partial \psi} \\ &\quad + [(\cos\theta\ A_x + \sin\theta\ A_y)\sin\varphi + \cos\varphi\ A_z]\frac{\partial U}{\partial \zeta} \end{aligned}\right\} \qquad [5.50]$$

The non-conservation form [5.45] and the characteristic form [5.46] are applicable in the local coordinate system.

5.4. Application: the two-dimensional shallow water equations

5.4.1. *Governing equations*

5.4.1.1. *Physical context – assumptions*

The two-dimensional free-surface flow equations, also called two-dimensional shallow water equations, can be viewed as a two-dimensional extension of the Saint Venant equations. They are often used in floodplain modeling studies or coastal modeling, where the classical, one-dimensional Saint Venant equations do not suffice to provide a correct description of the flow. This is the case in particular when sharp contrasts appear in the velocity field or in the water depths (Figure 5.7).

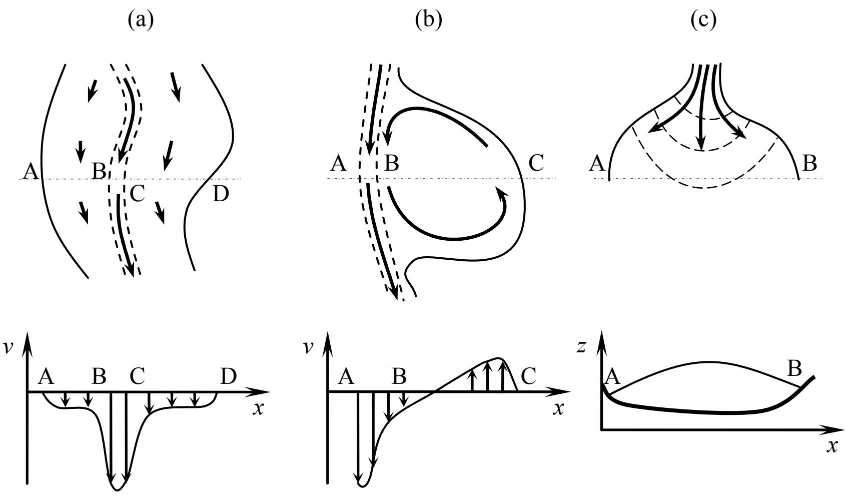

Figure 5.7. *Typical situations where the one-dimensional approximation is invalid. Plan view (top), cross-sectional view (bottom)*

Figure 5.7 illustrates three typical situations where the one-dimensional approach is invalid:

a) A wide floodplain, where the water depth is small compared to that in the main channel (Figure 5.7a), invalidates the one-dimensional assumption [CUN 80]. If a uniform water level is assumed over the cross-section, the difference between the depth in the main channel (indicated by a dashed line in Figure 5.7a) and the floodplain (solid lines) leads to different values for the flow velocity because the friction terms are larger in the floodplain than in the main channel. If a uniform velocity field is assumed across the entire section, the slope of the energy line cannot be the same in the floodplain and in the main channel because the depth is not the same. Therefore the water depth cannot be the same in the main channel and in the floodplain.

b) Presence of a storage pocket in the floodplain (Figure 5.7b). The pocket may be filled as a result of water stage increase during a flood. As a result, a swirling zone, also called a dead zone, appears in the pocket. The swirl appears as a consequence of lateral momentum diffusion due to viscosity and turbulent diffusion. The direction of the flow changes along the segment [BC] that is drawn across the pocket. Under steady state conditions, or near the peak flow, the amount of water stored in the pocket is nearly constant, which means that the average discharge in the pocket is zero. In other words, the pocket does not participate in the dynamics of the river. Most of the discharge is transferred via the main channel [AB]. Neglecting this by assuming a uniform flow velocity over the whole cross-section [AC] may lead to tremendous error in the assessment of the friction term and in the subsequently calculated discharge and water depth.

c) Propagation of a flood wave over rapidly varying geometries (Figure 5.7c). A transient propagating into a region where the channel geometry varies strongly is subjected to two-dimensional effects. As mentioned in sections 5.4.2 and 5.4.3, multidimensional waves tend to expand in the radial direction. The elevation of the free surface and the velocity profile are therefore not constant along a cross-section when a wave enters a sudden widening as sketched in Figure 5.7c.

The governing equations for two-dimensional free surface flow are derived from the following assumptions:

– Assumption (A1). The water is assumed to be incompressible in the range of ordinary pressure and water levels. The density is constant.

– Assumption (A2). The vertical acceleration of the water molecules is negligible compared to the horizontal acceleration. The pressure field is considered to be hydrostatic.

– Assumption (A3). The flow is turbulent. The head loss due to bottom friction is proportional to the square of the velocity.

– Assumption (A4). The diffusion of momentum due to turbulence and viscosity, the Coriolis effect and the shear stress due to the wind are neglected.

Note that Assumption (A4) is introduced only for the sake of simplicity in the treatment of the equations, the main purpose being to focus on the hyperbolic part of the equations. It should be kept in mind however that momentum diffusion, Coriolis forces and wind-induced forces are taken into account in most simulation packages nowadays.

5.4.1.2. *Continuity equation*

The continuity equation is derived as explained in section 5.2. The conserved variable is the mass per unit length, the fluxes are the mass discharges per unit width in the *x*- and *y*-direction and the source term is zero:

$$\left.\begin{aligned} U &= \rho h \\ F &= \rho h u \\ G &= \rho h v \\ S &= 0 \end{aligned}\right\} \qquad [5.51]$$

where h is the water depth, u and v are the *x*- and *y*-velocity respectively and ρ is the (constant) water density. Substituting definitions [5.51] into equation [5.1] and using Assumption (A1) leads to the following PDE:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \qquad [5.52]$$

5.4.1.3. *Equation for the momentum in the x-direction*

The momentum equation is derived using the *x*-momentum per unit surface as the conserved variable:

$$U = \rho h u \qquad [5.53]$$

The flux F in the *x*-direction results from two terms: (i) the inertial term, that accounts for the advection of the conserved variable U at a speed u in the *x*-direction, and (ii) the pressure force, expressed as the integral of the pressure between the bottom and the free surface (see sections 2.4.2.3 and 2.5.2.3 for the details of the proof):

$$F = \rho h u^2 + P \qquad [5.54]$$

Note that Assumption (A2) leads to a hydrostatic pressure field, which leads to:

$$P = \int_0^h p(\eta)\mathrm{d}\eta = \int_0^h \rho g \eta \mathrm{d}\eta = \rho g \frac{h^2}{2} \qquad [5.55]$$

where g is the gravitational acceleration and η is the distance to the free surface. Substituting equation [5.55] into equation [5.54] leads to:

$$F = \left(hu^2 + \frac{gh^2}{2} \right) \rho \tag{5.56}$$

The flux G in the y-direction accounts for the transport of the conserved variable $U = \rho hu$ at a speed v in the y-direction. G is expressed as:

$$G = \rho huv \tag{5.57}$$

The source term expresses the influence of two factors, namely the friction force $R_{f,x}$ against the bottom and the projection of the reaction of the bottom onto the x-axis. According to Assumption (A3), $R_{f,x}$ is proportional to the square of the intensity of the velocity vector; its orientation is opposite to that of the flow. As in the Saint Venant equations, the friction force is expressed so as to involve the friction slope $S_{f,x}$ in the x-direction:

$$R_{f,x} = \rho gh S_{f,x} \tag{5.58}$$

The friction slope is estimated using the two-dimensional generalization of the Chezy, Strickler or Manning formulations (see equations [2.107–108])

$$\begin{aligned}
S_{f,x} &= \frac{(u^2 + v^2)^{1/2} u}{C^2 R_H} && \text{(Chézy)} \\
S_{f,x} &= \frac{(u^2 + v^2)^{1/2} u}{K_{\mathrm{Str}}^2 R_H^{4/3}} && \text{(Strickler)} \\
S_{f,x} &= n_M^2 \frac{(u^2 + v^2)^{1/2} u}{R_H^{4/3}} && \text{(Manning)}
\end{aligned} \tag{5.59}$$

The projection of the reaction of the bottom onto the x-axis is derived exactly as in the Saint Venant equations (see section 2.5.2.3, equations [2.112–113]):

$$R_{0,x} = -\rho gh \frac{\partial z_b}{\partial x} = \rho gh S_{0,x} \tag{5.60}$$

Substituting equations [5.53], [5.56–58] and [5.60] into equation [5.1], dividing by the density yields the x-momentum equation:

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(hu^2 + g\frac{h^2}{2} \right) + \frac{\partial}{\partial y}(huv) = (S_{0,x} - S_{f,x}) gh \tag{5.61}$$

5.4.1.4. *Equation for the y-momentum*

The y-momentum equation is derived exactly in the same way as the x-momentum equation. Reproducing the reasoning of section 5.4.1.3 leads to the following equation:

$$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}\left(hv^2 + g\frac{h^2}{2}\right) = (S_{0,y} - S_{f,y})gh \qquad [5.62]$$

where the friction slope in the y-direction is given by:

$$\begin{aligned} S_{f,y} &= \frac{(u^2+v^2)^{1/2}v}{C^2 R_H} && \text{(Chezy)} \\ S_{f,y} &= \frac{(u^2+v^2)^{1/2}v}{K_{\mathrm{Str}}^2 R_H^{4/3}} && \text{(Strickler)} \\ S_{f,y} &= n_M^2 \frac{(u^2+v^2)^{1/2}v}{R_H^{4/3}} && \text{(Manning)} \end{aligned} \qquad [5.63]$$

5.4.1.5. *Vector form*

System [5.52], [5.61], [5.62] can be written in vector conservation form as in equation [5.12], recalled here:

$$\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} + \frac{\partial \mathrm{G}}{\partial y} = \mathrm{S}$$

with the following definitions for U, F, G and S:

$$\begin{aligned} \mathrm{U} &= \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, & \mathrm{F} &= \begin{bmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{bmatrix}, \\ \mathrm{G} &= \begin{bmatrix} hv \\ huv \\ hv^2 + gh^2/2 \end{bmatrix}, & \mathrm{S} &= \begin{bmatrix} 0 \\ (S_{0,x} - S_{f,x})gh \\ (S_{0,y} - S_{f,y})gh \end{bmatrix} \end{aligned} \qquad [5.64]$$

The non-conservation form [5.14], recalled here:

$$\frac{\partial U}{\partial t} + A_x \frac{\partial U}{\partial x} + A_y \frac{\partial U}{\partial y} = S'$$

is obtained with S = S' and:

$$\left.\begin{aligned} A_x &= \frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 & 0 \\ c^2 - u^2 & 2u & 0 \\ -uv & v & u \end{bmatrix} \\ A_y &= \frac{\partial G}{\partial U} = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ c^2 - v^2 & 0 & 2v \end{bmatrix} \end{aligned}\right\} \qquad [5.65]$$

where $c = (gh)^{1/2}$ is the speed of the waves in still water.

In what follows the characteristic surfaces and the Riemann invariants are derived using the secant plane approach. The bicharacteristic approach is described in detail in [DAU 67] and will not be detailed here.

5.4.2. *The secant plane approach*

5.4.2.1. *Characteristic surfaces*

As shown in section 5.3.1.2, the secant plane approach consists of using a projection of the equations onto a secant plane so as to obtain a one-dimensional equation in the form [5.42], recalled here:

$$\frac{\partial U}{\partial t} + A_\xi \frac{\partial U}{\partial \xi} = S_\xi$$

where A_ξ and S_ξ are defined as in equation [5.43], recalled here:

$$\left.\begin{aligned} A_\xi &= \cos\theta \; A_x + \sin\theta \; A_y \\ S_\xi &= S' - (\sin\theta \; A_x + \cos\theta \; A_y)\frac{\partial U}{\partial \psi} \end{aligned}\right\}$$

Substituting the definitions [5.65] into equation [5.43] yields:

$$\mathrm{A}_\xi = \begin{bmatrix} 0 & \cos\theta & \sin\theta \\ (c^2 - u^2)\cos\theta - uv\sin\theta & 2u\cos\theta + v\sin\theta & u\sin\theta \\ (c^2 - v^2)\sin\theta - uv\cos\theta & v\cos\theta & u\cos\theta + 2v\sin\theta \end{bmatrix}$$

[5.66]

The components of the velocity in the ξ- and ψ-direction are introduced:

$$\left.\begin{aligned} u_\xi &= u\cos\theta + v\sin\theta \\ u_\psi &= -u\sin\theta + v\cos\theta \end{aligned}\right\} \qquad [5.67]$$

Using definitions [5.67], the expression of A_ξ can be simplified into:

$$\mathrm{A}_\xi = \begin{bmatrix} 0 & \cos\theta & \sin\theta \\ c^2\cos\theta - uu_\xi & u\cos\theta + u_\xi & u\sin\theta \\ c^2\sin\theta - vu_\xi & v\cos\theta & v\sin\theta + u_\xi \end{bmatrix} \qquad [5.68]$$

The eigenvalues λ of A_ξ verify:

$$\left| A_\xi - \lambda I \right| = 0 \qquad [5.69]$$

which gives the characteristic equation:

$$(\lambda - u_\xi)\left[(\lambda - u_\xi)^2 - c^2\right] = 0 \qquad [5.70]$$

Equation [5.70] has the following solutions:

$$\left.\begin{aligned} \lambda^{(1)} &= u_\xi - c \\ \lambda^{(2)} &= u_\xi \\ \lambda^{(3)} &= u_\xi + c \end{aligned}\right\} \qquad [5.71]$$

The surface associated with the first eigenvalue is defined as:

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \lambda^{(1)} = u_\xi - c \qquad [5.72]$$

that is:

$$\cos\theta\frac{\mathrm{d}x}{\mathrm{d}t} + \sin\theta\frac{\mathrm{d}y}{\mathrm{d}t} = \cos\theta\ u + \sin\theta\ v - c \qquad [5.73]$$

The angle θ between the secant plane and the x-axis is eliminated from equation [5.73] by first differentiating equation [5.73] with respect to θ:

$$-\left(\frac{\mathrm{d}x}{\mathrm{d}t} - u\right)\sin\theta + \left(\frac{\mathrm{d}y}{\mathrm{d}t} - v\right)\cos\theta = 0 \qquad [5.74]$$

and raising equations [5.73] and [5.74] to the square. The difference between the resulting equations leads to:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t} - u\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t} - v\right)^2 = c^2 \qquad [5.75]$$

Equation [5.75] is the equation of a circle in the plane (x, y). The center of the circle moves at a speed (u, v) and the radius of the circle grows at a speed c (Figure 5.8). Note that this surface is identical to that associated with the eigenvalue $\lambda^{(3)}$ because changing $-c$ into $+c$ leads identically to equation [5.75]. Reasoning along the same line, the characteristic surface associated with the eigenvalue $\lambda^{(2)}$ is easily shown to be the straight line along which the center of the circle moves.

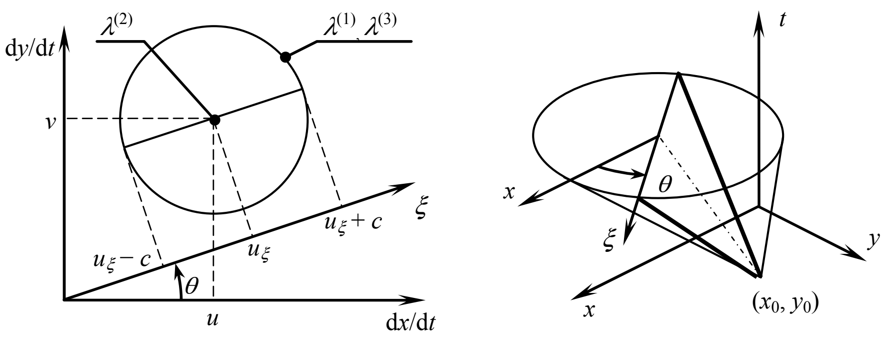

Figure 5.8. *Bicharacteristic surfaces as derived using the secant plane approach. Plan view (left), perspective view (right)*

5.4.2.2. *Derivation of the Riemann invariants*

The Riemann invariants are derived using equation [5.42]. The eigenvectors of A_ξ are:

$$K_\xi^{(1)} = \begin{bmatrix} 1 \\ u - c\cos\theta \\ v - c\sin\theta \end{bmatrix},\quad K_\xi^{(2)} = \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix},\quad K_\xi^{(3)} = \begin{bmatrix} 1 \\ u + c\cos\theta \\ v + c\sin\theta \end{bmatrix} \qquad [5.76]$$

The inverse of the matrix K formed by the eigenvectors is:

$$K_\xi^{-1} = \begin{bmatrix} \dfrac{u_\xi + c}{2c} & -\dfrac{\cos\theta}{2c} & -\dfrac{\sin\theta}{2c} \\ -u_\psi & -\sin\theta & \cos\theta \\ \dfrac{-u_\xi + c}{2c} & \dfrac{\cos\theta}{2c} & \dfrac{\sin\theta}{2c} \end{bmatrix} \qquad [5.77]$$

Consequently, the vector W is defined by the differential:

$$dW = K_\xi^{-1} dU = \frac{1}{2c} \begin{bmatrix} (u_\xi + c)dh - \cos\theta\ d(uh) - \sin\theta\ d(vh) \\ [-u_\psi dh - \sin\theta\ d(uh) + \cos\theta\ d(vh)]2c \\ (-u_\xi + c)dh + \cos\theta\ d(uh) + \sin\theta\ d(vh) \end{bmatrix} \qquad [5.78]$$

Equation [5.78] is simplified using the reciprocal of equations [5.67]:

$$\left.\begin{aligned} \cos\theta\ du + \sin\theta\ dv &= du_\xi \\ -\sin\theta\ du + \cos\theta\ dv &= du_\psi \end{aligned}\right\} \qquad [5.79]$$

Substituting equations [5.79] into equation [5.78] and differentiating the equality $h = c^2/g$ into $dh = 2c\ dc/g$ leads to:

$$dW = K_\xi^{-1} dU = \frac{1}{2c} \begin{bmatrix} (u_\xi + c)dh - d(hu_\xi) \\ -u_\psi dh + d(hu_\psi) \\ (-u_\xi + c)dh + d(hu_\xi) \end{bmatrix} = \frac{1}{2cg} \begin{bmatrix} -c^2 d(u_\xi - 2c) \\ 2c^2 du_\psi \\ c^2 d(u_\xi + 2c) \end{bmatrix} \qquad [5.80]$$

The source term S_ξ is given by the second equation [5.43], recalled here:

$$S_\xi = S' + (\sin\theta\, A_x - \cos\theta\, A_y)\frac{\partial U}{\partial \psi}$$

Expanding the terms d(uh), d(vh), using d$h = c^2/g$, substituting equations [5.67] and rewriting equations [5.39] as:

$$\left.\begin{aligned} \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} &= \frac{\partial}{\partial \xi} \\ -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y} &= \frac{\partial}{\partial \psi} \end{aligned}\right\} \qquad [5.81]$$

leads to:

$$S_\xi = \frac{1}{2c}\begin{bmatrix} \begin{array}{c} -\dfrac{2c^2}{g}u_\psi\dfrac{\partial c}{\partial \psi} - \dfrac{c^3}{g}\dfrac{\partial u_\psi}{\partial \psi} \\ +\dfrac{c^2}{g}u_\psi\dfrac{\partial u_\xi}{\partial \psi} + (S_{0,\xi} - S_{f,\xi})c^2 \end{array} \\ \\ \dfrac{4c^3}{g}\dfrac{\partial c}{\partial \psi} + \dfrac{2c^2}{g}u_\psi\dfrac{\partial u_\psi}{\partial \psi} - 2(S_{0,\psi} - S_{f,\psi})c^2 \\ \\ \begin{array}{c} \dfrac{2c^2}{g}u_\psi\dfrac{\partial c}{\partial \psi} + \dfrac{c^3}{g}\dfrac{\partial u_\psi}{\partial \psi} \\ -\dfrac{c^2}{g}u_\psi\dfrac{\partial u_\xi}{\partial \psi} - (S_{0,\xi} - S_{f,\xi})c^2 \end{array} \end{bmatrix} \qquad [5.82]$$

where $S_{0,\xi}$ and $S_{f,\xi}$ are respectively the bottom and friction slope in the ξ-direction, and $S_{0,\psi}$ and $S_{f,\psi}$ are respectively the bottom and friction slope in the ψ-direction. Substituting equations [5.80] and [5.82] into equation [5.42] yields the following differential relationships:

$$\left.\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(u_\xi - 2c) = S_{\xi,1} && \text{for } \frac{\mathrm{d}\xi}{\mathrm{d}t} = u_\xi - c\\
&\frac{\mathrm{d}u_\psi}{\mathrm{d}t} = S_{\xi,2} && \text{for } \frac{\mathrm{d}\xi}{\mathrm{d}t} = u_\xi\\
&\frac{\mathrm{d}}{\mathrm{d}t}(u_\xi - 2c) = S_{\xi,3} && \text{for } \frac{\mathrm{d}\xi}{\mathrm{d}t} = u_\xi + c
\end{aligned}\right\} \qquad [5.83]$$

where the components of the source term S_ξ are given by:

$$\left.\begin{aligned}
&S_{\xi,1} = S_{\xi,3} = -u_\psi \frac{\partial u_\xi}{\partial \psi} + c\frac{\partial u_\psi}{\partial \psi} + 2u_\psi \frac{\partial c}{\partial \psi} - (S_{0,\xi} - S_{f,\xi})g\\
&S_{\xi,2} = u_\psi \frac{\partial u_\psi}{\partial \psi} + 2c\frac{\partial c}{\partial \psi} - (S_{0,\psi} - S_{f,\psi})g
\end{aligned}\right\} \qquad [5.84]$$

Note that the coordinate system (ξ, ψ) attached to the secant plane coincides with the global coordinate system (x, y) for $\theta = 0$. Then u_ξ and u_ψ coincide with u and v respectively and the first and third Riemann invariants coincide with the classical invariants $u - 2c$ and $u + 2c$ derived in section 2.5.3.3 for the one-dimensional flow equations in rectangular channels.

5.4.3. *Interpretation – determination of the solution*

5.4.3.1. *Domain of influence, domain of dependence*

The secant plane approach leads to the same result as the bicharacteristic approach developed in [DAU 67]. There are two characteristic surfaces in the phase space. The first surface is generated by a circle, the center of which moves at the speed of the flow and the radius of which expands at the propagation speed of the waves in still water. The second surface is the line generated by the successive locations of the center of the circle. The wave speeds $u - c$ and $u + c$ and the Riemann invariants $u - 2c$ and $u + 2c$ are recovered when u is redefined as the speed in the local coordinate ξ attached to the secant plane. This is because the shallow water equations are invariant by rotation. For the sake of clarity, u, v and c are considered constant hereafter. However, the reasoning remains valid for non-constant flow variables.

The domain of influence of the solution is located inside the first characteristic surface. It includes both the surface and the region of the phase space delimited by the surface. Indeed, a characteristic surface may be drawn from each point M located on a generating line (AB) of the surface issued from A (Figure 5.9). The segment

[MB] belongs to both characteristic surfaces. The intersection of the first surface with the plane $t = t_B$ is a circle (C), the intersection of the second surface with the plane is a circle (C'). The point A influences the point M, thus influencing indirectly all the points on the circle (C'). The circle (C') spans the set of the internal points of the circle (C) as the point M spans all the possible locations along the segment [AB]. Consequently point A influences not only the points located on (C) but also all its internal points. The domain of influence of A includes all the points located inside the first characteristic surface.

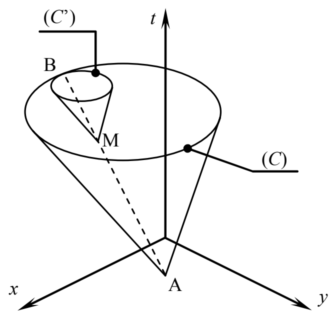

Figure 5.9. *Definition sketch for the domain of influence in the phase space*

The domain of dependence is determined by extending the reasoning to negative time intervals, that is, by travelling backward in time. Consider the characteristic surface passing at B (Figure 5.10). At any point M located on a generating curve, a characteristic surface can be drawn that passes at M. The intersection of the characteristic surface passing at M with the plane $t = t_A$ is a circle denoted by (C'). The intersection between the characteristic surface passing at B and the plane $t = t_A$ is denoted by (C).

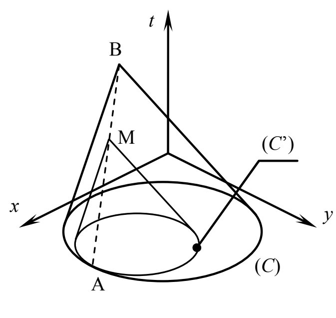

Figure 5.10. *Definition sketch for the domain of dependence in the phase space*

The circle (C') spans the complete set of interior points of the circle (C) as the point M spans all the possible locations along the line (AB). Consequently, the point B is influenced directly or indirectly by all the points located inside the characteristic surface passing at B.

5.4.3.2. *Calculation of the solution*

In spite of their apparent simplicity, the Riemann invariants derived in section 5.4.2 cannot be used to compute analytical solutions in a straightforward manner. Indeed, equation [5.84] includes derivatives in the direction normal to the secant plane. In the case of a genuinely two-dimensional flow field, such derivatives are non-zero and no analytical expression can be found for them, unless the flow configuration is very simple (e.g. radial symmetry). The characteristic form must then be approximated. Two possible approaches have been reported in the literature:

1) The first approach consists of selecting three bicharacteristic lines [A_1M], [A_2M] and [A_3M] passing at the point M of interest and writing the characteristic relationships [5.83–84] along them. The transverse derivatives are estimated from the known solution at points A_1, A_2 and A_3 (see Figure 5.11). The three characteristic relationships allow the three flow variables to be determined uniquely at M. The approach was applied to gas dynamics and magnetohydrodynamics by Sauerwein [SAU 66, SAU 67] and to the two-dimensional shallow water equations by Daubert and Graffe [DAU 67], Katopodes [KAT 77], Katopodes and Strelkoff [KAT 79] and Gerritsen [GER 82]. The question remains however of the optimal choice of points A_1, A_2 and A_3. In fact, each particular choice for these points leads to a different solution.

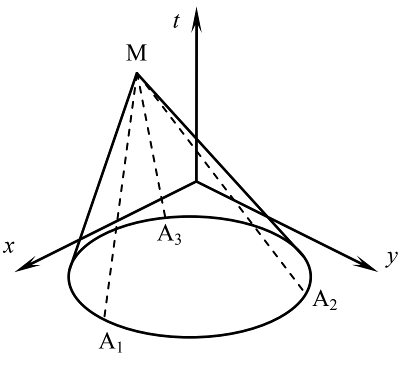

Figure 5.11. *Calculating the solution at point M using three bicharacteristics*

2) In another approach, proposed by the author [GUI 03b], the proper selection of the points A_1, A_2 and A_3 is less crucial because the characteristic relationships are

integrated over the entire domain of dependence. Integrating equations [5.83–84] over the three surfaces [A_1A_2M], [A_2A_3M] and [A_3A_1M] allows the three independent flow variables to be determined uniquely at M, while taking into account the details of the variations of the initial condition along the curve ($A_1A_2A_3$).

5.5. Summary

5.5.1. *What you should remember*

An $m \times m$ multidimensional system of conservation laws is said to be hyperbolic if any linear combination of the Jacobian matrices of the fluxes with respect to the conserved variable has m real, distinct eigenvalues.

The notion of characteristic curve defined for one-dimensional problems may be extended to characteristic surfaces for two-dimensional problems and to characteristic volumes for three-dimensional problems.

The characteristic surfaces may be derived using the bicharacteristic approach presented in section 5.3.1.1 and the secant plane approach presented in section 5.3.1.2 indifferently.

In the case of a two-dimensional problem the characteristic surfaces are conical envelopes or curved lines in the phase space (x, y, t).

5.5.2. *Application exercises*

5.5.2.1. *Exercise 5.1: the Doppler effect*

Consider a mobile sound source that moves at a speed u smaller than the speed of sound. The frequency N of the sound is constant. Using the secant plane approach, show that the frequency N' of the sound as heard by an immobile observer is given by:

$$N' = \frac{N}{(1 - M\cos\theta)} \qquad [5.85]$$

where M is the Mach number and θ is the angle between the velocity vector of the source and the direction of the straight line drawn from the observer to the source (Figure 5.12).

This phenomenon is known as the Doppler effect.

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

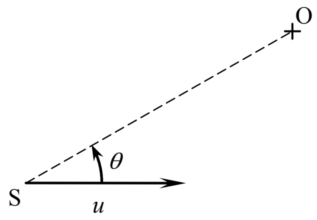

Figure 5.12. *Mobile sound source*

5.5.2.2. *Exercise 5.2: visual assessment of the Mach number*

Airplanes entering high moisture regions at supersonic speeds sometimes generate condensation patterns that develop next to the convex part of the wings and the hull. A condensation pattern indicates a sudden pressure drop, which is an indication that a shock wave is present (Figure 5.13).

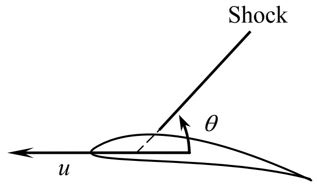

Figure 5.13. *Condensation zone developing along the shock wave for a plane in supersonic flight*

Show that the angle between the shock and the velocity vector of the airplane is given by:

$$M = \frac{1}{\sin\theta} \qquad [5.86]$$

which allows the Mach number M to be determined visually.

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.