Chapter 9

Treatment of Source Terms

9.1. Introduction

The numerical techniques presented in Chapters 6 to 8 mostly deal with hyperbolic conservation laws and hyperbolic systems without source terms. The equations or systems of equations found in practical engineering applications, however, contain source terms arising from variations in the geometry. Examples are the water hammer equations in pipes with variable cross-sections, the Saint Venant equations in non-prismatic channels and/or variable bottom slope, or the one-dimensional Euler equations in domains of variable cross-sectional areas. A careless discretization of the source terms may induce artificial perturbations in the computed profiles, if not solution instability. The need for source term discretization techniques that preserve equilibrium conditions without introducing spurious oscillations in the computed variables has led to the general notion of well-balanced schemes.

Giving a complete and exhaustive description of numerical techniques for source term discretization is beyond the scope of this book. The subject would actually deserve a book in itself. The purpose of this chapter is to present the broad lines of the main families of numerical techniques introduced over the past two decades to deal with source terms in hyperbolic systems of conservation laws. The water hammer equations (the simplest possible hyperbolic system of conservation laws) and the shallow water equations (a subject of intensive research over the past two decades, see e.g. [HER 07, TOR 07]) are used as illustrative examples.

Section 9.2 introduces the issue of geometric source terms discretization with two examples. The key notion of *C*–property is then introduced. Section 9.3 deals

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with the upwind approach to source term discretization. Section 9.4 presents the quasi-steady wave algorithm, and section 9.5 is an introduction to well-balancing techniques.

9.2. Problem position

9.2.1. *Example 1: the water hammer equations*

The issue of source term discretization is illustrated using the water hammer equations introduced in section 2.3. The water hammer equations form the simplest possible 2×2 hyperbolic system of conservation laws, with constant, opposite wave speeds. This system is linear. The conservation form [2.2] of the equations, recalled here:

$$\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} = \mathrm{S}$$

is obtained by defining the conserved variable U, the flux F and the source term S as in equation [2.68], recalled hereafter:

$$\mathrm{U} = \begin{bmatrix} \rho A \\ \rho Q \end{bmatrix}, \mathrm{F} = \begin{bmatrix} \rho Q \\ Ap \end{bmatrix}, \mathrm{S} = \begin{bmatrix} 0 \\ \dfrac{\partial A}{\partial x} p - \rho g A \sin\theta - k|u|u \end{bmatrix}$$

where A is the cross-sectional area of the pipe, k is the friction coefficient, p is the pressure, Q is the liquid discharge, u is the flow velocity, θ is the angle between the axis of the pipe and the horizontal, and ρ is the density of the fluid. Recall that the variations in the mass per unit length ρA and the pressure force Ap obey relationship [2.44], that can be rewritten as:

$$\mathrm{d}(Ap) = c^2\ \mathrm{d}(\rho A) \qquad [9.1]$$

where c is the (assumed uniform) sound speed. Equation [9.1] is to be understood for a fixed value of the abscissa x along the pipe. It is also recalled that c is constant in time, regardless of the value of the pressure p.

Consider the simple configuration where the pipe is frictionless, horizontal, with a variable cross-sectional area A (see Figure 9.1). The source term S simplifies to

$$\mathrm{S} = \begin{bmatrix} 0 \\ \dfrac{\partial A}{\partial x} p \end{bmatrix} \qquad [9.2]$$

Assume that the initial and boundary conditions in the pipe are such that static equilibrium is verified, that is:

$$\left.\begin{array}{l}\left.\begin{array}{l}p(x,0)=p_0\\u(x,0)=0\end{array}\right\}\quad\forall x\in[0,L]\\p(0,t)=p(L,t)=p_0\qquad\forall t>0\end{array}\right\}\qquad[9.3]$$

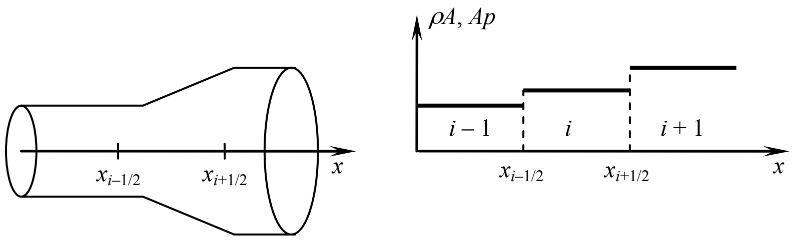

Figure 9.1. *Definition sketch for the water hammer equations with variable pipe cross-section. Left: geometry of the pipe. Right: resulting initial, steady-state profile for the conserved variable ρA and the pressure force Ap*

Substituting these conditions into equation [2.2] with definition [2.68] and simplification [9.2] leads to:

$$\left.\begin{array}{l}\dfrac{\partial}{\partial t}(\rho A)=-\dfrac{\partial}{\partial x}(\rho Q)=-\dfrac{\partial}{\partial x}(\rho uA)=0\\\dfrac{\partial}{\partial t}(\rho Q)=-\dfrac{\partial}{\partial x}(Ap)+p\dfrac{\partial A}{\partial x}=-A\dfrac{\partial p}{\partial x}=0\end{array}\right\}\quad\forall\begin{cases}x\in[0,L]\\t>0\end{cases}\qquad[9.4]$$

In other words, static equilibrium is preserved at later times at all points in the pipe.

Assume now that the water hammer equations are to be solved using the finite volume discretization [7.3], recalled here:

$$\mathrm{U}_i^{n+1}=\mathrm{U}_i^n+\frac{\Delta t}{\Delta x}\left(\mathrm{F}_{i-1/2}^{n+1/2}-\mathrm{F}_{i+1/2}^{n+1/2}\right)+\Delta t\,\mathrm{S}_i^{n+1/2}$$

Assume that a classical approximate Riemann solver, such as Roe's, Lax-Friedrichs' or the HLL solver (see Appendix C for more details), is used to compute the flux at the interfaces between the computational cells. The water hammer equations being linear and c being assumed uniform, all three solvers lead to the same formula:

$$\begin{aligned}\begin{bmatrix}\rho Q \\ Ap\end{bmatrix}_{i-1/2}^{n+1/2} &= \frac{1}{2}\begin{bmatrix}(\rho Q)_{i-1}^{n} + (\rho Q)_{i}^{n} \\ (Ap)_{i-1}^{n} + (Ap)_{i}^{n}\end{bmatrix} + \frac{c}{2}\begin{bmatrix}(\rho A)_{i-1}^{n} - (\rho A)_{i}^{n} \\ (\rho Q)_{i-1}^{n} - (\rho Q)_{i}^{n}\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix}(\rho Q)_{i-1}^{n} + (\rho Q)_{i}^{n} \\ (Ap)_{i-1}^{n} + (Ap)_{i}^{n}\end{bmatrix} + \frac{1}{2c}\begin{bmatrix}(Ap)_{i-1}^{n} - (Ap)_{i}^{n} \\ c^2(\rho Q)_{i-1}^{n} - c^2(\rho Q)_{i}^{n}\end{bmatrix}\end{aligned} \qquad [9.5]$$

If the initial state verifies static equilibrium at time level n, Q is zero and p is equal to p_0 within all the cells. Equation [9.5] becomes:

$$\begin{bmatrix}\rho Q \\ Ap\end{bmatrix}_{i-1/2}^{n+1/2} = \begin{bmatrix}\dfrac{A_{i-1} - A_i}{2c} p_0 \\ \dfrac{A_{i-1} + A_i}{2} p_0\end{bmatrix} \qquad [9.6]$$

The first component of the vector equation [9.6] indicates that, if $A_{i-1} \neq A_i$, an artificial, non-zero mass discharge is computed at the interface $i - 1/2$. This non-zero discharge triggers pressure waves that propagate into the computational domain, thus destroying the steady-state character of the numerical solution.

This simple example shows that the source terms in hyperbolic systems should not be discretized independently of the conservation part, otherwise violating simple equilibrium requirements. Moreover, the presence of the source term in the momentum equation may lead us to revise the discretization of the continuity equation.

9.2.2. *Example 2: the shallow water equations*

The one-dimensional shallow water equations are obtained by restricting the two-dimensional shallow water equations (see section 5.4) to their one-dimensional projection. The one-dimensional shallow water equations can be written in the form [2.2] by defining U, F and S as in equation [7.83], recalled here:

$$\mathrm{U} = \begin{bmatrix}h \\ q\end{bmatrix} = \begin{bmatrix}h \\ hu\end{bmatrix}, \quad \mathrm{F} = \begin{bmatrix}q \\ M\end{bmatrix} = \begin{bmatrix}hu \\ hu^2 + gh^2/2\end{bmatrix}, \quad \mathrm{S} = \begin{bmatrix}0 \\ (S_{0,x} - S_{f,x})gh\end{bmatrix}$$

where g is the gravitational acceleration, h is the water depth, u is the flow velocity, $S_{0,x}$ and $S_{f,x}$ are respectively the bottom and friction slope in the x-direction. q and M are respectively the unit discharge and specific force.

Assume as in the previous section that the shallow water equations are discretized using the finite volume technique [7.3], and that the HLL Riemann solver is used in the calculation of the fluxes (see Appendix C for more details). Consider the situation, shown in Figure 9.2, where the water is initially at rest ($\zeta = h + z_b$ = Const, $u = 0$ everywhere).

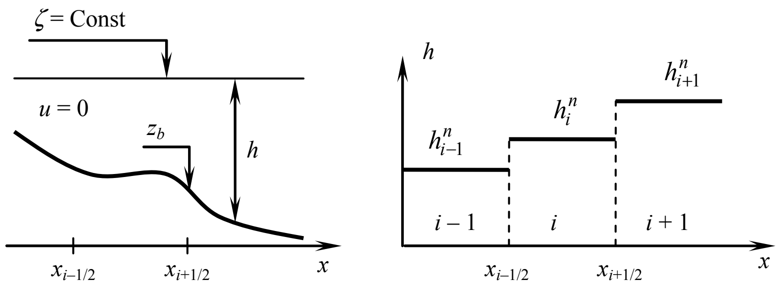

Figure 9.2. *Definition sketch for the one-dimensional shallow water equations with variable bottom level: left: bottom geometry and steady-state equilibrium conditions; right: resulting discretized, initial state*

The formula for the HLL Riemann solver is given by the second equation [C.3], recalled here:

$$F^* = \frac{\lambda^+ F_L - \lambda^- F_R}{\lambda^+ - \lambda^-} - \lambda^- \lambda^+ (U_L - U_R)$$

Applying this formula at the interface between the cells $i - 1$ and i with $u = 0$ yields the following expression for the unit discharge q:

$$q_{i-1/2} = -\frac{\lambda^- \lambda^+}{\lambda^+ - \lambda^-}(h_{i-1}^n - h_i^n) \qquad [9.7]$$

where λ^- and λ^+ are respectively estimates of the minimum wave speed $u - c$ and the maximum wave speed $u + c$. Since the water is at rest, applying e.g. Davis' formula [C.5] for λ^- and λ^+ leads to:

$$\lambda^+ = -\lambda^- = \left[g \max(h_{i-1}^n, h_i^n)\right]^{1/2} \qquad [9.8]$$

Substituting equation [9.8] into the flux formula [9.7] gives:

$$(hu)_{i-1/2} = \frac{\left[g\ \max(h_{i-1}^n, h_i^n)\right]^{1/2}}{2}(h_{i-1}^n - h_i^n) \qquad [9.9]$$

If the bottom is horizontal, $h_{i-1}^n = h_i^n$ and the unit discharge remains equal to zero. But if the bottom is not horizontal, $h_{i-1}^n \neq h_i^n$. As a consequence, a non-zero discharge is computed even though the initial conditions correspond to water at rest under static equilibrium conditions. This generates artificial waves that propagate throughout the computational domain, eventually destroying the static character of the solution.

9.2.3. *Stationary solution and C–property*

In the above two examples, the artificial fluxes are diffusive fluxes stemming from second-order truncation errors (see Appendix B for detailed considerations on truncation errors and consistency aspects). As shown in [VAZ 99, CHA 03], these artificial fluxes can be eliminated if the discretization of the source terms can be made second-order accurate with respect to space.

In what follows, the source term is assumed to take the form:

$$\mathrm{S} = \mathrm{f}(\mathrm{U})\frac{\partial \varphi}{\partial x} \qquad [9.10]$$

where f is a known vector function of U and φ is a parameter. For instance, in the case of the water hammer equations, $\varphi = A$ and $\mathrm{f} = [0, p]^T$. In the case of the one-dimensional shallow water equations, $\varphi = z_b$ and $\mathrm{f} = [0, gh]^T$.

First-order consistency condition imposes that if φ and U are identical in the cells i and $i + 1$, the discretized source term must be zero:

$$\left.\begin{array}{l} \mathrm{U}_{i-1} = \mathrm{U}_i \\ \varphi_{i-1} = \varphi_i \end{array}\right\} \Rightarrow \mathrm{S}(\varphi_{i-1}, \varphi_i, \mathrm{U}_{i-1}, \mathrm{U}_i) = 0 \qquad [9.11]$$

However, the first-order consistency condition is not sufficient to guarantee a satisfactory discretization of the source term. As an example, in sections 9.2.1 and 9.2.2, condition [9.11] is satisfied. Nevertheless, steady-state conditions are not preserved for arbitrary geometries. It is thus necessary to define so-called “enhanced consistency" conditions [CHA 03], that allow the so-called *C*–property to be verified.

A flow field that verifies steady-state (or stationary) conditions at time level n should satisfy static equilibrium conditions at time level $n + 1$. This is true only if the difference between the fluxes is balanced exactly by the source term. For a finite volume scheme, this condition may be written as:

$$\frac{\partial \mathrm{U}}{\partial t} = 0 \;\Rightarrow\; \mathrm{F}_{i-1/2}^{n+1/2} - \mathrm{F}_{i+1/2}^{n+1/2} + \Delta x\left(\mathrm{S}_{i-1/2}^{+} + \mathrm{S}_{i+1/2}^{-}\right) = 0 \qquad [9.12]$$

In [VAZ 99], the so-called *C*–property is defined as follows:

– the discretization is said to satisfy the exact *C*–property if the steady-state condition [9.12] is satisfied exactly;

– the discretization is said to satisfy the approximate *C*–property if the steady-state condition [9.12] is satisfied with at least second-order accuracy.

As in [VAZ 99], source terms involving sums of functions in the form [9.10] can be broken into several elementary source terms to be discretized independently from each other. The *C*–property was originally written for static conditions, that is, for a fluid at rest [VAZ 99]. However, more general steady-state preserving discretizations have been proposed by a number of authors, see for instance [HUB 00], [BUR 04].

9.3. Source term upwinding techniques

9.3.1. *Principle*

Source term upwinding has been applied to the calculation of the one-dimensional shallow water equations [BER 94] in channels of constant width, to the solution of the two-dimensional shallow water equations [BER 98, BRU 02] and to the solution of the Saint Venant equations in channels with variable width [VAZ 99, GAR 00]. Applications of the technique to higher-order TVD schemes can be found in [BUR 01]. Extensions of the method to various numerical techniques are provided in [CHA 03], where the source term is split into centered and upwind parts, which are discretized so as to preserve stationary solutions.

The technique is particularly adapted to discretizations where wave speeds and their propagation directions are well-identified and used explicitly in the discretization of the fluxes. This is the case in particular with finite volume methods (see Chapter 7) that use approximate Riemann solvers based on approximations of the wave speeds (see Appendix C for examples). Consider a hyperbolic system discretized as in equation [7.3] (a slightly different writing is used hereafter):

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}\left(F_{i-1/2}^{n+1/2} - F_{i+1/2}^{n+1/2} + \Delta x\, S_i^{n+1/2}\right) \qquad [9.13]$$

where the source term can be written as in equation [9.10].

It is first recalled that in finite volume techniques using Riemann solvers, the solution at the interface between two adjacent cells is classically approximated as a succession of discontinuities separating regions of constant state (Figure 9.3).

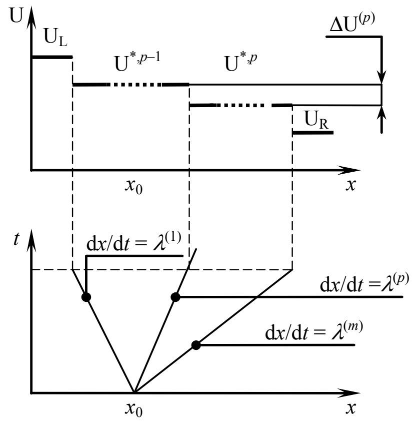

Figure 9.3. *Source term upwinding across the waves in the solution of the Riemann problem. Definition sketch in the physical space (top) and in the phase space (bottom)*

Denoting by $U^{*,p-1}$ and $U^{*,p}$ the solution on the left- and right-hand sides of the pth wave, the jumps in U and F across the pth wave may be written as:

$$\left.\begin{aligned} U^{*,p} - U^{*,p-1} &= \alpha^{(p)} K^{(p)} \\ F^{*,p} - F^{*,p-1} &= \lambda^{(p)}(U^{*,p} - U^{*,p-1}) = \lambda^{(p)}\alpha^{(p)} K^{(p)} \end{aligned}\right\} \qquad [9.14]$$

where the coefficients $\alpha^{(p)}$ are the wave strengths and the vectors $K^{(p)}$ are the eigenvectors of the Jacobian matrix A of F with respect to U. Note that summing the jumps across all the waves leads to:

$$\left.\begin{aligned} U_R - U_L &= K\alpha \\ F_R - F_L &= K\Lambda\alpha \end{aligned}\right\} \qquad [9.15]$$

where K and a are respectively the matrix of eigenvectors of A and the vector formed by the wave strengths. Inverting the first equation [9.15] yields directly the expression of the wave strength vector α:

$$\alpha = \mathrm{K}^{-1}(\mathrm{U}_\mathrm{R} - \mathrm{U}_\mathrm{L}) \qquad [9.16]$$

Source term upwinding consists of discretizing the source term in the same way as the space derivative of the flux in upwind schemes:

$$\left.\begin{aligned} \Delta x\, \mathrm{S}_i^{n+1/2} &= \sum_p \beta^{(p)} \mathrm{K}^{(p)} = \mathrm{K}\beta \\ \beta &= \mathrm{K}^{-1} \Delta x\, \mathrm{S}_i^{n+1/2} \end{aligned}\right\} \qquad [9.17]$$

where β is a vector formed by the components $\beta^{(p)}$ of the wave strengths of the source term. The total source term [9.17] is split into two parts:

– the part that corresponds to negative wave speeds, $\lambda^{(p)} < 0$, is assigned to the cell on the left-hand side of the initial discontinuity,

– the part that corresponds to positive wave speeds, $\lambda^{(p)} > 0$, is assigned to the cell on the right-hand side of the initial discontinuity.

The discretization [9.13] is rewritten in the form:

$$\mathrm{U}_i^{n+1} = \mathrm{U}_i^n + \frac{\Delta t}{\Delta x}\left[\mathrm{F}_{i-1/2}^{n+1/2} - \mathrm{F}_{i+1/2}^{n+1/2} + \Delta x\left(\mathrm{S}_{i-1/2}^{+} + \mathrm{S}_{i+1/2}^{-}\right)\right] \qquad [9.18]$$

where $\mathrm{S}_{i-1/2}^{+}$ is the contribution to the cell i of the source term between the cells $i-1$ and i, and $\mathrm{S}_{i+1/2}^{-}$ is the contribution to the cell i of the source term between the cells i and $i+1$. The source term $\mathrm{S}_{i-1/2}^{+}$ arises from the contribution of positive wave speeds, while the term $\mathrm{S}_{i+1/2}^{-}$ arises from the contribution of the negative wave speeds:

$$\left.\begin{aligned} \mathrm{S}_{i-1/2}^{+} &= \sum_{\lambda^{(p)}\geq 0} \left[\beta^{(p)}\mathrm{K}^{(p)}\right]_{i-1/2}^{n+1/2} \\ \mathrm{S}_{i+1/2}^{-} &= \sum_{\lambda^{(p)}< 0} \left[\beta^{(p)}\mathrm{K}^{(p)}\right]_{i+1/2}^{n+1/2} \end{aligned}\right\} \qquad [9.19]$$

The contributions $S^{+}_{i-1/2}$ and $S^{-}_{i+1/2}$ are known if the wave strengths $\beta^{(p)}$ can be determined and if an estimate can be provided for the total source term. The wave strengths are obtained from the second equation [9.17]. The estimate of the source term is obtained by integrating equation [9.10] across each interface:

$$\left.\begin{aligned} \beta^{n+1/2}_{i-1/2} &= K^{-1}\Delta x\, S^{n+1/2}_{i-1/2} \approx (K^{-1}\ f)^{n+1/2}_{i-1/2}\,(\varphi_i - \varphi_{i-1}) \\ \beta^{n+1/2}_{i+1/2} &= K^{-1}\Delta x\, S^{n+1/2}_{i+1/2} \approx (K^{-1}\ f)^{n+1/2}_{i+1/2}\,(\varphi_{i+1} - \varphi_i) \end{aligned}\right\} \qquad [9.20]$$

where $f^{n+1/2}_{i-1/2}$ is an "average" value of f between the cells $i-1$ and i. Its expression for f_{LR} must be devised in such a way that the exact or approximate *C*–property defined in section 9.2.3 is satisfied.

9.3.2. *Application example 1: the water hammer equations*

The source term upwinding technique is applied to the water hammer equations. The discretization is assumed to be given by equation [7.3], with formula [9.5] for the calculation of the fluxes. Equation [9.5] is recalled:

$$\begin{bmatrix} \rho Q \\ Ap \end{bmatrix}^{n+1/2}_{i-1/2} = \frac{1}{2}\begin{bmatrix} (\rho Q)^n_{i-1} + (\rho Q)^n_i \\ (Ap)^n_{i-1} + (Ap)^n_i \end{bmatrix} + \begin{bmatrix} \dfrac{(Ap)^n_{i-1} - (Ap)^n_i}{2c} \\ \dfrac{(\rho Q)^n_{i-1} - (\rho Q)^n_i}{2} c \end{bmatrix}$$

The eigenvalues and eigenvectors of the water hammer equations (see section 2.3) are also recalled:

$$K = \begin{bmatrix} 1 & 1 \\ -c & c \end{bmatrix}, \qquad K^{-1} = \frac{1}{2c}\begin{bmatrix} c & -1 \\ c & 1 \end{bmatrix} \qquad [9.21]$$

The source term is discretized as (here at interface $i - 1/2$):

$$\Delta x\, S^{n+1/2}_{i-1/2} \approx \begin{bmatrix} 0 \\ (A_i - A_{i-1})\,p_{i-1/2} \end{bmatrix} \qquad [9.22]$$

where the estimate $p_{i-1/2}$ must be determined so as to satisfy the *C*–property (this point is examined at the end of the section). The vector β of wave strengths is given by:

$$\beta = K^{-1}\Delta x\, S^{n+1/2}_{i-1/2} = \frac{1}{2c}\begin{bmatrix} (A_{i-1} - A_i)\,p_{i-1/2} \\ (A_i - A_{i-1})\,p_{i-1/2} \end{bmatrix} \qquad [9.23]$$

The waves with speeds – c and + c are respectively directed to the cells $i - 1$ and i. This leads to the following contributions for the source term at the interface $i - 1/2$:

$$\left.\begin{aligned} S_{i-1/2}^{-} &= \beta^{(1)} K^{(1)} = \begin{bmatrix} \dfrac{(A_{i-1} - A_i)\, p_{i-1/2}}{2c} \\ (A_i - A_{i-1})\, p_{i-1/2} \end{bmatrix} \\ S_{i-1/2}^{+} &= \beta^{(2)} K^{(2)} = \begin{bmatrix} \dfrac{(A_i - A_{i-1})\, p_{i-1/2}}{2c} \\ (A_i - A_{i-1})\, p_{i-1/2} \end{bmatrix} \end{aligned}\right\} \qquad [9.24]$$

The cell i receives on the left-hand interface $i - 1/2$ a total contribution formed by the sum of the flux $\mathrm{F}_{i-1/2}^{n+1/2}$ and the contribution $\mathrm{S}_{i-1/2}^{+}$ of the source term:

$$\mathrm{F}_{i-1/2}^{n+1/2} + \mathrm{S}_{i-1/2}^{+} = \frac{1}{2} \begin{bmatrix} (\rho Q)_{i-1}^{n} + (\rho Q)_{i}^{n} + \dfrac{(Ap)_{i-1}^{n} - (Ap)_{i}^{n}}{c} \\ \qquad + \dfrac{(A_i - A_{i-1})\, p_{i-1/2}}{c} \\ (Ap)_{i-1}^{n} + (Ap)_{i}^{n} + [(\rho Q)_{i-1}^{n} - (\rho Q)_{i}^{n}]c \\ \qquad + (A_i - A_{i-1})\, p_{i-1/2} \end{bmatrix} \qquad [9.25]$$

The expression for $p_{i-1/2}$ is now examined. For an initial, steady-state configuration, with a uniform pressure p_0, discharge Q_0 and density ρ_0, equation [9.25] simplifies to:

$$\mathrm{F}_{i-1/2}^{n+1/2} + \mathrm{S}_{i-1/2}^{+} = \frac{1}{2} \begin{bmatrix} \rho_0 Q_0 + \dfrac{A_{i-1} - A_i}{2c} \left(p_0 - p_{i-1/2}\right) \\ \dfrac{A_{i-1} + A_i}{2}\, p_0 + (A_i - A_{i-1})\, p_{i-1/2} \end{bmatrix} \qquad [9.26]$$

Any estimate for $p_{i-1/2}$ that satisfies the consistency condition:

$$\left.\begin{aligned} p_{i-1/2} &= f(p_{i-1}^{n}, p_{i}^{n}) \\ f(U,U) &= U \end{aligned}\right\} \qquad [9.27]$$

satisfies the *C*–property exactly. This is the case with the following two estimates:

$$\left.\begin{aligned} p_{i-1/2} &= \frac{p_{i-1}^n + p_i^n}{2} \\ p_{i-1/2} &= \frac{A_{i-1} p_{i-1}^n + A_i p_i^n}{A_{i-1} + A_i} \end{aligned}\right\} \qquad [9.28]$$

In both cases, $p_{i-1/2} = p_0$ and equation [9.26] becomes:

$$\mathrm{F}_{i-1/2}^{n+1/2} + \mathrm{S}_{i-1/2}^{+} = \frac{1}{2}\begin{bmatrix} \rho_0 Q_0 \\ A_i p_0 \end{bmatrix} \qquad [9.29]$$

It is easy to check that:

$$\mathrm{F}_{i+1/2}^{n+1/2} + \mathrm{S}_{i+1/2}^{-} = \frac{1}{2}\begin{bmatrix} \rho_0 Q_0 \\ A_i p_0 \end{bmatrix} \qquad [9.30]$$

The mass discharge is equal to the uniform-steady-state value $\rho_0 Q_0$ at all cell interfaces and the equilibrium condition [9.12] is satisfied.

9.3.3. *Application example 2: the shallow water equations with HLL solver*

Applications of the source term upwinding technique to the shallow water equations in conjunction with Roe's solver, Van Leer's Q-scheme and flux splitting techniques can be found in [BER 94, BER 98, GAR 00, VAZ 99]. In this section, the HLL solver is applied.

For the sake of simplicity, the frictionless restriction of the one-dimensional shallow water equations is considered:

$$\begin{aligned} &\mathrm{U} = \begin{bmatrix} h \\ q \end{bmatrix} = \begin{bmatrix} h \\ hu \end{bmatrix}, \qquad \mathrm{F} = \begin{bmatrix} q \\ M \end{bmatrix} = \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix}, \\ &\mathrm{S} = \begin{bmatrix} 0 \\ S_{0,x}\, gh \end{bmatrix} = \begin{bmatrix} 0 \\ -gh\ \partial z_b / \partial x \end{bmatrix} \end{aligned} \qquad [9.31]$$

where M and q are respectively the specific force and unit discharge. The eigenvalues and eigenvectors are obtained as particular cases of the Saint Venant equations in prismatic, rectangular channels:

$$\lambda^{(1)} = u - c, \qquad \lambda^{(2)} = u + c,$$
$$\mathrm{K}^{(1)} = \begin{bmatrix} 1 \\ \lambda^{(1)} \end{bmatrix} \qquad \mathrm{K}^{(2)} = \begin{bmatrix} 1 \\ \lambda^{(2)} \end{bmatrix}, \qquad \mathrm{K}^{-1} = \frac{1}{\lambda^{(2)} - \lambda^{(1)}} \begin{bmatrix} \lambda^{(2)} & -1 \\ -\lambda^{(1)} & 1 \end{bmatrix} \tag{9.32}$$

The HLL solver is used for the estimate of the flux:

$$\left.\begin{aligned} q_{i-1/2} &= \frac{\lambda^+ q_{i-1}^n - \lambda^- q_i^n}{\lambda^+ - \lambda^-} - \frac{\lambda^- \lambda^+}{\lambda^+ - \lambda^-}(h_{i-1}^n - h_i^n) \\ M_{i-1/2} &= \frac{\lambda^+ M_{i-1}^n - \lambda^- M_i^n}{\lambda^+ - \lambda^-} - \frac{\lambda^- \lambda^+}{\lambda^+ - \lambda^-}(q_{i-1}^n - q_i^n) \end{aligned}\right\} \tag{9.33}$$

where λ^- and λ^+ are estimated as in equation [C.5]:

$$\left.\begin{aligned} \lambda^- &= \min\left[(u-c)_{i-1}^n, (u-c)_i^n, 0\right] \\ \lambda^+ &= \max\left[(u+c)_{i-1}^n, (u+c)_i^n, 0\right] \end{aligned}\right\} \tag{9.34}$$

The wave strengths of the source term are obtained from equation [9.20]:

$$\begin{aligned} \beta_{i-1/2}^{n+1/2} &= \mathrm{K}^{-1} \Delta x \, \mathrm{S}_{i-1/2}^{n+1/2} \\ &= \frac{1}{\lambda^{(2)} - \lambda^{(1)}} \begin{bmatrix} \lambda^{(2)} & -1 \\ -\lambda^{(1)} & 1 \end{bmatrix}_{i-1/2}^n \begin{bmatrix} 0 \\ g h_{i-1/2}^{n+1/2} (z_{b,i-1} - z_{b,i}) \end{bmatrix} \\ &= \frac{g h_{i-1/2}^{n+1/2}}{\lambda^{(2)} - \lambda^{(1)}} (z_{b,i-1} - z_{b,i}) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned} \tag{9.35}$$

The source term at the interface $i - 1/2$ is split into two terms according to equation [9.19]:

$$\left.\begin{aligned} \mathrm{S}_{i-1/2}^- &= \begin{cases} 0 & \text{if } \lambda^- \geq 0 \\ \beta^{(1)} K^{(1)} & \text{if } \lambda^- < 0 \text{ and } \lambda^+ \geq 0 \\ \beta^{(1)} K^{(1)} + \beta^{(2)} K^{(2)} & \text{if } \lambda^+ < 0 \end{cases} \\ \mathrm{S}_{i-1/2}^+ &= \begin{cases} \beta^{(1)} K^{(1)} + \beta^{(2)} K^{(2)} & \text{if } \lambda^- \geq 0 \\ \beta^{(2)} K^{(2)} & \text{if } \lambda^- < 0 \text{ and } \lambda^+ \geq 0 \\ 0 & \text{if } \lambda^+ < 0 \end{cases} \end{aligned}\right\} \tag{9.36}$$

with:

$$\left.\begin{aligned}\beta^{(1)}\mathrm{K}^{(1)} &= \frac{gh_{i-1/2}^{n+1/2}}{\lambda^{(2)}-\lambda^{(1)}}(z_{b,i-1}-z_{b,i})\begin{bmatrix}-1\\-\lambda^{(1)}\end{bmatrix}\\ \beta^{(2)}\mathrm{K}^{(2)} &= \frac{gh_{i-1/2}^{n+1/2}}{\lambda^{(2)}-\lambda^{(1)}}(z_{b,i-1}-z_{b,i})\begin{bmatrix}1\\\lambda^{(2)}\end{bmatrix}\end{aligned}\right\} \quad [9.37]$$

The terms $\mathrm{S}_{i-1/2}^{-}$ and $\mathrm{S}_{i-1/2}^{+}$ are used in equation [9.18]. They contribute respectively to the cell $i-1$ and i. The second components of $\beta^{(1)}\mathrm{K}^{(1)}$ and $\beta^{(2)}\mathrm{K}^{(2)}$ in equation [9.37] indicate that the source term arising from the topography is split proportionally into the wave speeds $\lambda^{(1)}$ and $\lambda^{(2)}$.

A straightforward estimate for the wave speeds is:

$$\left.\begin{aligned}\lambda^{(1)}{}_{i-1/2}^{n+1/2} &= \lambda^{-}\\ \lambda^{(1)}{}_{i-1/2}^{n+1/2} &= \lambda^{+}\end{aligned}\right\} \quad [9.38]$$

where λ^{-} and λ^{+} are given by equation [9.34]. The expression for $h_{i-1/2}^{n}$ is obtained by writing the requirements imposed by the *C*–property. Consider the case where the water is initially at rest. Then, $\lambda^{+} = -\lambda^{-} = \max(c_{i-1}^{n}, c_{i}^{n})$ and the following estimates are obtained:

$$\left.\begin{aligned}q_{i-1/2}^{n+1/2} &= -\frac{\lambda^{-}\lambda^{+}}{\lambda^{+}-\lambda^{-}}(h_{i-1}^{n}-h_{i}^{n}) = \frac{-\lambda}{2}(h_{i-1}^{n}-h_{i}^{n}) = \frac{\lambda^{+}}{2}(h_{i-1}^{n}-h_{i}^{n})\\ M_{i-1/2}^{n+1/2} &= \frac{\lambda^{+}M_{i-1}^{n}-\lambda^{-}M_{i}^{n}}{\lambda^{+}-\lambda^{-}} = \frac{M_{i-1}^{n}+M_{i}^{n}}{2} = \frac{g}{2}[(h_{i-1}^{n})^{2}-(h_{i}^{n})^{2}]\\ \beta^{(1)}\mathrm{K}^{(1)} &= \frac{gh_{i-1/2}^{n+1/2}}{2\lambda^{+}}(z_{b,i-1}-z_{b,i})\begin{bmatrix}-1\\\lambda^{+}\end{bmatrix}\\ \beta^{(2)}\mathrm{K}^{(2)} &= \frac{gh_{i-1/2}^{n+1/2}}{2\lambda^{+}}(z_{b,i-1}-z_{b,i})\begin{bmatrix}1\\\lambda^{+}\end{bmatrix}\end{aligned}\right\} \quad [9.39]$$

Noting that $z_{b,i-1} - z_{b,i} = h_i^n - h_{i-1}^n$ under equilibrium conditions leads to:

$$\left.\begin{aligned} \beta^{(1)}\mathrm{K}^{(1)} &= \frac{g h_{i-1}^{n+1/2}}{2\lambda^+}(h_i^n - h_{i-1}^n)\begin{bmatrix} -1 \\ \lambda^+ \end{bmatrix} \\ \beta^{(2)}\mathrm{K}^{(2)} &= \frac{g h_{i-1}^{n+1/2}}{2\lambda^+}(h_i^n - h_{i-1}^n)\begin{bmatrix} 1 \\ \lambda^+ \end{bmatrix} \end{aligned}\right\} \qquad [9.40]$$

The contribution of the interface $i - 1/2$ to the cell i is thus given by (a similar reasoning may be made for the contribution of the interface to the cell $i - 1$):

$$\mathrm{F}_{i-1/2}^{n+1/2} + \beta^{(2)}\mathrm{K}^{(2)} = \begin{bmatrix} \dfrac{\lambda^+}{2}(h_{i-1}^n - h_i^n) + \dfrac{g h_{i-1/2}^C}{2\lambda^+}(h_i^n - h_{i-1}^n) \\ \dfrac{g}{2}[(h_{i-1}^n)^2 - (h_i^n)^2] + \lambda^+ g h_{i-1/2}^M \dfrac{h_i^n - h_{i-1}^n}{2\lambda^+} \end{bmatrix} \qquad [9.41]$$

The C–property is satisfied if the vector $\mathrm{F}_{i-1/2}^{n+1/2} + \beta^{(2)}\mathrm{K}^{(2)}$ is zero. It is worth noting that two different estimates are introduced for $h_{i-1/2}^{n+1/2}$ in equation [9.41]: $h_{i-1/2}^C$ is used in the continuity equation, while $h_{i-1/2}^M$ is used in the momentum equation. The reason is that it is not possible to find a single expression for $h_{i-1/2}^{n+1/2}$ that satisfies the C–property in both the continuity and momentum equations. It is easy to check that the following estimates allow the C–property to be verified:

$$\left.\begin{aligned} h_{i-1/2}^C &= \frac{{\lambda^+}^2 + {\lambda^-}^2}{2g} \\ h_{i-1/2}^M &= \frac{h_{i-1}^n + h_i^n}{2} \end{aligned}\right\} \qquad [9.42]$$

The expression for $h_{i-1/2}^M$ coincides with the formula given in [BER 98] for Van Leer's Q–scheme. [BER 98] also give the C–preserving formula for Roe's solver (see Appendix C for a brief description of the solver):

$$h_{i-1/2}^{n+1/2} = (h_{i-1}^n h_i^n)^{1/2} \qquad [9.43]$$

9.4. The quasi-steady wave algorithm

9.4.1. *Principle*

The quasi-steady wave algorithm [LEV 98] was introduced for finite volume discretizations. In this method, the parameter φ in the source term [9.10] is assumed piecewise constant, with a discontinuity located in the middle of the computational cells (Figure 9.4).

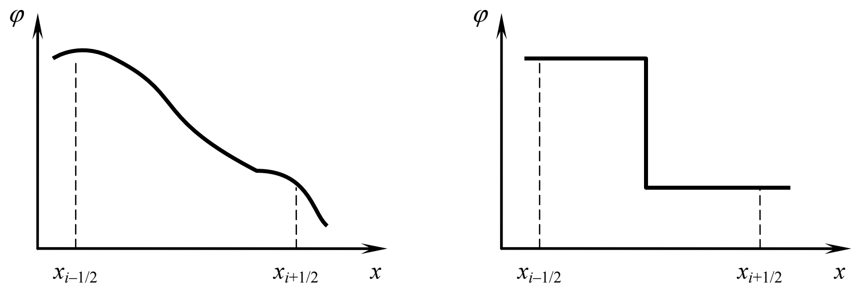

Figure 9.4. *Quasi-steady wave algorithm. Left: variations in φ. Right: discretized parameter in the cell i*

The discontinuity in φ in the middle of the cell generates an additional Riemann problem, but the additional waves triggered by this Riemann problem do not reach the interfaces of the cell if the computational time step is kept small enough. Under these conditions, the additional Riemann problem does not influence the balance at the cell interfaces and the classical balance equation [7.3] can be used. The source term is assigned to the cell i:

$$\left.\begin{aligned} \mathrm{U}_i^{n+1} &= \mathrm{U}_i^n + \frac{\Delta t}{\Delta x}\left(\mathrm{F}_{i-1/2}^{n+1/2} - \mathrm{F}_{i+1/2}^{n+1/2} + \Delta x\,\mathrm{S}_i^{n+1/2}\right) \\ \Delta x\,\mathrm{S}_i^{n+1/2} &= \mathrm{S}(\mathrm{U}_i^n, \varphi_{i-1/2}, \varphi_{i+1/2}) \end{aligned}\right\} \qquad [9.44]$$

In equation [9.44], an explicit estimate is proposed for the source term, but other estimates may be proposed. For definition [9.10] of the source term S, we have:

$$\Delta x\,\mathrm{S}_i^{n+1/2} = \int_{x_{i-1/2}}^{x_{i+1/2}} \mathrm{f}(x)\frac{\partial \varphi}{\partial x}\,\mathrm{d}x \approx [\varphi(x_{i+1/2}) - \varphi(x_{i-1/2})]\,\mathrm{f}_i^{n+1/2} \qquad [9.45]$$

where $f_i^{n+1/2}$ is an average value of the vector function f over the cell *i*. If an explicit scheme is retained, f is defined from the known values at time level *n*. It must satisfy the consistency condition and the *C*–property presented in section 9.2.3:

(1) consistency condition:

$$\varphi_{i-1/2} = \varphi_{i+1/2} \Rightarrow \mathrm{S}(\varphi_{i-1/2}, \varphi_{i+1/2}, \mathrm{U}_i) = 0 \qquad [9.46]$$

(2) steady-state condition: an initial-steady state must yield the following equality:

$$\frac{\partial \mathrm{U}}{\partial t} = 0 \quad \Rightarrow \qquad \mathrm{F}_{i-1/2}^{n+1/2} - \mathrm{F}_{i+1/2}^{n+1/2} + \Delta x\, \mathrm{S}_i^{n+1} = 0 \qquad [9.47]$$

9.4.2. *Application to the water hammer equations*

Consider the water hammer equations in a frictionless, horizontal pipe with a variable cross-sectional area *A*. The fluxes are computed as in equation [9.26], with the difference that *A* is constant across a given interface but may vary from one interface to the next. For interface *i* – 1/2, the formula for the flux becomes:

$$\mathrm{F}_{i-1/2}^{n+1/2} = \begin{bmatrix} \rho Q \\ Ap \end{bmatrix}_{i-1/2}^{n+1/2} = \begin{bmatrix} \dfrac{(\rho Q)_{i-1}^n + (\rho Q)_i^n}{2} + A_{i-1/2} \dfrac{p_{i-1}^n - p_i^n}{2c} \\ A_{i-1/2} \dfrac{p_{i-1}^n + p_i^n}{2} + \dfrac{(\rho Q)_{i-1}^n - (\rho Q)_i^n}{2} c \end{bmatrix} \qquad [9.48]$$

while the source term is computed as:

$$\Delta x\, S_i^{n+1/2} = (A_{i+1/2} - A_{i-1/2}) p_i^n \qquad [9.49]$$

It is easy to check that equations [9.48–49] verify the consistency condition [9.46] and the *C*–property [9.47].

9.4.3. *Application to the one-dimensional shallow water equations*

The definition [9.33] of U, F and S for the one-dimensional shallow water equations is recalled:

$$\mathrm{U} = \begin{bmatrix} h \\ q \end{bmatrix} = \begin{bmatrix} h \\ hu \end{bmatrix}, \qquad \mathrm{F} = \begin{bmatrix} q \\ M \end{bmatrix} = \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix},$$

$$\mathrm{S} = \begin{bmatrix} 0 \\ S_{0,x}\, gh \end{bmatrix} = \begin{bmatrix} 0 \\ -gh\ \partial z_b / \partial x \end{bmatrix}$$

In the present application example, the bottom level z_b is reconstructed using a piecewise constant function. The discontinuity in the bed level is assumed to be located at the center of the cell i. The average bottom level over the cell is thus:

$$z_{b,i} = \frac{z_{b,i-1/2} + z_{b,i+1/2}}{2} \tag{9.50}$$

The water depths $h_{i-1/2,\mathrm{L}}$ and $h_{i-1/2,\mathrm{R}}$ on the left- and right-hand sides of the interface $i - 1/2$ are given by:

$$\left.\begin{aligned} h_{i-1/2,\mathrm{L}} &= h_{i-1}^n + z_{b,i-1} - z_{b,i-1/2} = h_{i-1}^n + \frac{z_{b,i-3/2} - z_{b,i-1/2}}{2} \\ h_{i-1/2,\mathrm{R}} &= h_i^n + z_{b,i} - z_{b,i-1/2} = h_i^n + \frac{z_{b,i+1/2} - z_{b,i-1/2}}{2} \end{aligned}\right\} \tag{9.51}$$

Moreover, steady state is assumed over the cells. This means that:

$$\begin{aligned} q_{i-1/2,\mathrm{L}} &= q_{i-1}^n \\ q_{i-1/2,\mathrm{R}} &= q_i^n \end{aligned} \tag{9.52}$$

The flux $\mathrm{F}_{i+1/2}^{n+1/2}$ is computed by solving a Riemann problem with left and right states $[h_{i-1/2,\mathrm{L}}, q_{i-1/2,\mathrm{L}}]^T$ and $[h_{i-1/2,\mathrm{L}}, q_{i-1/2,\mathrm{L}}]^T$. For instance, if the HLL solver is used, we have:

$$\begin{aligned} \mathrm{F}_{i-1/2}^{n+1/2} = {} & \frac{\lambda^+}{\lambda^+ - \lambda^-} \begin{bmatrix} q \\ M \end{bmatrix}_{i-1/2,\mathrm{L}}^n - \frac{\lambda^-}{\lambda^+ - \lambda^-} \begin{bmatrix} q \\ M \end{bmatrix}_{i-1/2,\mathrm{R}}^n \\ & - \frac{\lambda^- \lambda^+}{\lambda^+ - \lambda^-} \left(\begin{bmatrix} h \\ q \end{bmatrix}_{i-1/2,\mathrm{L}}^n - \begin{bmatrix} h \\ q \end{bmatrix}_{i-1/2,\mathrm{R}}^n \right) \end{aligned} \tag{9.53}$$

The source term for the continuity equation is zero. The expression of the source term in the momentum equation is derived from the force exerted by the bottom discontinuity on the volume of water contained in the cell i. To do so, a control volume containing the discontinuity is defined (Figure 9.5). Remember that the pressure below the free surface obeys the following equation (see section 2.5.2.3):

$$\frac{p(z)}{\rho} = (\zeta - z)g \qquad [9.54]$$

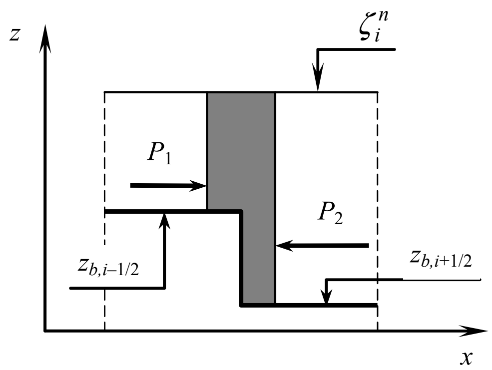

Figure 9.5. *Quasi-steady wave algorithm for the shallow water equations. Balancing the forces exerted on a control volume that contains the bottom step*

Integrating equation [9.54] between $z_{b,i-1/2}$ and ζ_i^n yields the expression for the pressure force per unit width P_1 on the left-hand side of the control volume:

$$\frac{P_1}{\rho} = \frac{g}{2}\left(h_i^n + \frac{z_{b,i+1/2} - z_{b,i-1/2}}{2}\right)^2 \qquad [9.55]$$

Remember that in the Saint Venant and shallow water equations, the momentum equations are divided by the (constant) water density ρ. Conversely, the pressure force per unit width P_2 verifies:

$$\frac{P_2}{\rho} = -\frac{g}{2}\left(h_i^n + \frac{z_{b,i-1/2} - z_{b,i+1/2}}{2}\right)^2 \qquad [9.56]$$

The reaction R exerted by the bottom step is obtained by integrating equation [9.53] between $z_{b,i+1/2}$ and $z_{b,i-1/2}$ (the reaction is in the direction of positive x if $z_{b,i-1/2} > z_{b,i+1/2}$):

$$\frac{R}{\rho} = gh_i^n (z_{b,i-1/2} - z_{b,i+1/2}) \qquad [9.57]$$

By definition, R/ρ is the integral of the momentum source term over the cell i. Consequently, the source term is given by:

$$S = \begin{bmatrix} 0 \\ R/\rho \end{bmatrix} = \begin{bmatrix} 0 \\ gh_i^n (z_{b,i-1/2} - z_{b,i+1/2}) \end{bmatrix} \qquad [9.58]$$

It is easy to check that expressions [9.53] and [9.58] verify the consistency condition and the *C*–property.

9.5. Balancing techniques

9.5.1. *Well-balancing*

9.5.1.1. *Principle*

In the traditional well-balanced approach, the parameter φ in the source term is discontinuous across the cell interfaces (Figure 9.6). This generates a Riemann problem with a discontinuity in the flux function. The solution of such a Riemann problem is not trivial. In [GRE 96], a solution is proposed for a scalar conservation law: the discontinuity in φ is taken as the limit case of a piecewise linear function.

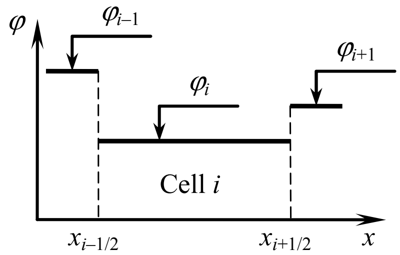

Figure 9.6. *Discretization of the parameter* φ

When hyperbolic systems of conservation laws are dealt with, the discontinuity in the flux function may induce extra waves in the solution of the Riemann problem compared to a Riemann problem with continuous fluxes. This means that extra, unknown intermediate regions of constant state may appear in the solution, thus requiring additional relationships to close the problem. As an example, exact solutions to the Riemann problem for the shallow water equations with a bottom step are presented in [ALC 01]. At least 20 different possible wave patterns are identified, against 4 possible patterns for the shallow water equations on a flat bottom. Entropy considerations are used to connect the states on the left- and right-hand sides of the step.

The flux function being discontinuous, the classical finite volume formula [7.3] cannot be used. It must be modified into (here for an explicit formula):

$$\mathrm{U}_i^{n+1} = \mathrm{U}_i^n + \frac{\Delta t}{\Delta x_i}\left[\mathrm{F}_{i-1/2,\mathrm{R}}^{n+1/2} - \mathrm{F}_{i+1/2,\mathrm{L}}^{n+1/2} + \Delta x\,\mathrm{S}(\varphi_{i+k}, U_{i+k}^n)\right] \quad [9.59]$$

where $\mathrm{F}_{i-1/2,\mathrm{L}}^{n+1/2}$ and $\mathrm{F}_{i-1/2,\mathrm{R}}^{n+1/2}$ are respectively the fluxes computed on the left- and right-hand side of the interface $i - 1/2$. These fluxes are computed by solving the Riemann problem with left and right states:

$$\left.\begin{array}{l} \dfrac{\partial \mathrm{U}}{\partial t} + \dfrac{\partial \mathrm{F}(\mathrm{U},\varphi)}{\partial x} = 0 \\ \left.\begin{array}{l} \mathrm{U}(x,t^n) = \mathrm{U}_\mathrm{L} \\ \varphi(x) = \varphi_\mathrm{L} \end{array}\right\} \quad \text{for } x < x_{i-1/2} \\ \left.\begin{array}{l} \mathrm{U}(x,t^n) = \mathrm{U}_\mathrm{R} \\ \varphi(x) = \varphi_\mathrm{R} \end{array}\right\} \quad \text{for } x > x_{i-1/2} \end{array}\right\} \quad [9.60]$$

Solving this problem generates a flux F_L and F_R on the left- and right-hand sides of the interface.

In [CHA 03], the discretization [9.59] is applied to the shallow water equations for a number of flux splitting-based numerical techniques such as Roe’s, Van Leer’s, Lax-Friedrichs and Lax-Wendroff’s schemes.

9.5.1.2. *Application example: the water hammer equations*

In the case of the water hammer equations, an analytical solution can be found to the Riemann problem [9.60]. The parameter φ of concern is the cross-sectional area A of the pipe, that is equal respectively to A_L and A_R on the left- and right-hand sides of the discontinuity:

$$\left.\begin{aligned} &\frac{\partial \mathrm{U}}{\partial t}+\frac{\partial \mathrm{F}(\mathrm{U},A)}{\partial x}=0 \\ &\mathrm{U}(x,0)=\begin{cases}\mathrm{U_L} & \text{for } x<x_0 \\ \mathrm{U_R} & \text{for } x>x_0\end{cases} \\ &A(x)=\begin{cases}A_\mathrm{L} & \text{for } x<x_0 \\ A_\mathrm{R} & \text{for } x>x_0\end{cases}\end{aligned}\right\} \quad [9.61]$$

where U and F are given by [2.68]:

$$\mathrm{U}=\begin{bmatrix}\rho A \\ \rho Q\end{bmatrix}, \qquad \mathrm{F}=\begin{bmatrix}\rho Q \\ Ap\end{bmatrix} \quad [9.62]$$

and $\mathrm{d}(Ap) = c^2\,\mathrm{d}(\rho A)$.

When the cross-sectional area of the pipe is constant, the solution of the Riemann problem for the water hammer equations (see Chapter 4) is made of two contact discontinuities separating the left and right states from an intermediate region of constant state (Figure 9.7a). In the case of a piecewise constant cross-sectional area, there are two intermediate regions of constant state (Figure 9.7b).

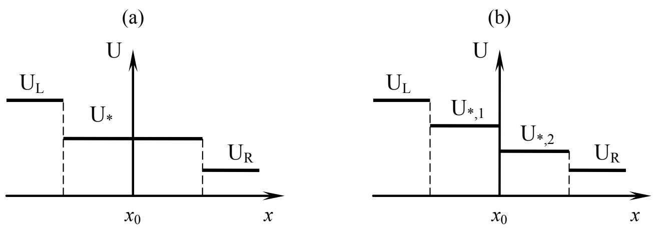

Figure 9.7. *Solution of the Riemann problem for the water hammer equations: (a) constant cross-sectional area; (b) piecewise constant cross-sectional area*

Applying the characteristic form [2.82] leads to:

$$\left.\begin{aligned} p_{*,1}+\rho c u_{*,1} &= p_\mathrm{L}+\rho c u_\mathrm{L} \\ p_{*,2}-\rho c u_{*,2} &= p_\mathrm{R}-\rho c u_\mathrm{R}\end{aligned}\right\} \quad [9.63]$$

where subscripts *,1 and *,2 indicate respectively the intermediate regions of constant state on the left- and right-hand sides of the discontinuity. Moreover, mass and momentum conservation impose the extra conditions:

$$\left.\begin{aligned}(\rho Q)_{*,1} &= (\rho Q)_{*,2}\\ p_{*,1} &= p_{*,2}\end{aligned}\right\} \qquad [9.64]$$

with $Q_{*,1} = A_L u_{*,1}$ and $Q_{*,2} = A_R u_{*,2}$. System [9.63−64] can be solved uniquely for p and Q:

$$\left.\begin{aligned}p_{*,1} = p_{*,2} &= \frac{A_L p_L + A_R p_R}{A_L + A_R} + \rho c \frac{Q_L - Q_R}{A_L + A_R}\\ (\rho Q)_{*,1} = (\rho Q)_{*,2} &= \frac{A_R Q_L + A_L Q_R}{A_L + A_R} + \frac{A_L A_R}{A_L + A_R}\frac{p_L - p_R}{c}\end{aligned}\right\} \qquad [9.65]$$

The flux and source terms at the interface are given by:

$$\begin{aligned}&F_{i-1/2,L}^{n+1/2} = \begin{bmatrix}(\rho Q)_{*,1}\\ A_L p_{*,1}\end{bmatrix}, \qquad F_{i-1/2,R}^{n+1/2} = \begin{bmatrix}(\rho Q)_{*,1}\\ A_R p_{*,1}\end{bmatrix},\\ &S_{i-1/2,L}^{n+1/2} = \begin{bmatrix}0\\ \min(A_R - A_L, 0)p_{*,1}\end{bmatrix}, \qquad S_{i-1/2,R}^{n+1/2} = \begin{bmatrix}(\rho Q)_{*,1}\\ \max(A_R - A_L, 0)p_{*,1}\end{bmatrix}\end{aligned} \qquad [9.66]$$

With this discretization, the C–property is satisfied exactly.

9.5.2. *Hydrostatic pressure reconstruction for free surface flow*

The hydrostatic reconstruction method is presented in [AUD 05] for shallow water flows. The bottom is assumed constant over each cell, therefore bottom level discontinuities occur only at the interfaces between the cells (Figure 9.8). The method is described for a first-order finite volume scheme hereafter, but a second-order extension can be found in [AUD 05].

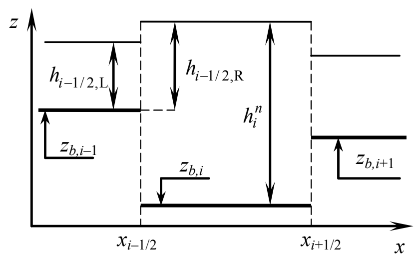

Figure 9.8. *Definition sketch for the hydrostatic reconstruction method*

In contrast with the approach presented in the previous section, the flux function is continuous across the interfaces, and the source term is exerted at each interface. The equations are discretized in the form:

$$\mathrm{U}_i^{n+1} = \mathrm{U}_i^n + \frac{\Delta t}{\Delta x_i}\left[\mathrm{F}_{i-1/2}^{n+1/2} - \mathrm{F}_{i+1/2}^{n+1/2} + \Delta x\left(\mathrm{S}_{i-1/2,\mathrm{R}}^{n+1/2} + \mathrm{S}_{i+1/2,\mathrm{L}}^{n+1/2}\right)\right] \quad [9.67]$$

where the source terms $\mathrm{S}_{i-1/2,\mathrm{L}}^{n+1/2}$ and $\mathrm{S}_{i-1/2,\mathrm{R}}^{n+1/2}$ are respectively the reaction of the bottom step onto the cell located on the left-hand side of the interface $i - 1/2$ (that is, cell $i - 1$) and the cell located on the right-hand side of interface $i - 1/2$ (that is, cell i). The reaction is assigned to the cell with the lower bottom level.

The flux at the interface $i - 1/2$ is computed by solving an equivalent Riemann problem with modified left and right states $\mathrm{U}_{i-1/2,\mathrm{L}}$ and $\mathrm{U}_{i-1/2,\mathrm{R}}$ defined as:

$$\mathrm{U}_{i-1/2,\mathrm{L}} = \begin{bmatrix} h_{i-1/2,\mathrm{L}} \\ h_{i-1/2,\mathrm{L}} u_{i-1}^n \end{bmatrix}, \quad \mathrm{U}_{i-1/2,\mathrm{R}} = \begin{bmatrix} h_{i-1/2,\mathrm{R}} \\ h_{i-1/2,\mathrm{R}} u_i^n \end{bmatrix} \quad [9.68]$$

where u is the flow velocity and the reconstructed water depths $h_{i-1/2,\mathrm{L}}$ and $h_{i-1/2,\mathrm{R}}$ on the left- and right-hand sides of the interface are given by:

$$\left.\begin{aligned} h_{i-1/2,\mathrm{L}} &= \max(\zeta_{i-1}^n - z_{b,i-1/2}, 0) = \max(h_{i-1}^n + z_{b,i-1} - z_{b,i-1/2}, 0) \\ h_{i-1/2,\mathrm{R}} &= \max(\zeta_i^n - z_{b,i-1/2}, 0) = \max(h_i^n + z_{b,i} - z_{b,i-1/2}, 0) \end{aligned}\right\} \quad [9.69]$$

where $z_{b,i-1/2}$ is the higher of the two bottom levels:

$$z_{b,i-1/2} = \max(z_{b,i-1}, z_{b,i}) \quad [9.70]$$

The source term at the interface is given by the integral of the pressure force on the bottom step:

$$\left.\begin{aligned} \Delta x\,\mathrm{S}_{i-1/2,\mathrm{L}}^{n+1/2} &= \begin{bmatrix} 0 \\ \frac{g}{2}\left(h_{i-1/2,\mathrm{L}}^2 - (h_{i-1}^n)^2\right) \end{bmatrix} \\ \Delta x\,\mathrm{S}_{i-1/2,\mathrm{R}}^{n+1/2} &= \begin{bmatrix} 0 \\ \frac{g}{2}\left((h_i^n)^2 - h_{i-1/2,\mathrm{R}}^2\right) \end{bmatrix} \end{aligned}\right\} \quad [9.71]$$

The *C*–property is satisfied exactly.

9.5.3. *Auxiliary variable-based balancing*

9.5.3.1. *Introduction*

As shown by the introductory examples in Section 9.2, flux and source term balancing problems arise from the fact that upwind numerical methods such as flux splitting and Riemann solver-based techniques use a flux function, part of which is proportional to the gradient in the conserved variable. As observed in [BUR 04], when geometry-induced source terms are present in the equations, the gradient in the conserved variable incorporates a geometric term that does not vanish under steady-state conditions. Source term upwinding allows the geometric term to be balanced with the flux gradient at least under static conditions (the *C*–property). The quasi-steady wave algorithm eliminates this drawback because the geometric parameter does not vary at the interface where the source term is computed. Hydrostatic reconstruction-based schemes also eliminate the problem by using the same value of the geometric parameter on both sides of the interface between the computational cells (the maximum of the two bottom levels in the case of the shallow water equations). Alternative balancing techniques have been explored in the literature:

(1) A first possible option proposed in the literature consists of modifying the system of equations to be solved. In [GAL 03], the shallow water equations on irregular topography are augmented by a third equation involving the bottom level and various alternative variables, such as the hydraulic head, are considered in the discretization of the equations.

(2) A second option consists of reconstructing the variations in the geometric parameter in such a way that steady-state conditions are preserved in the balancing of flux gradients and source terms. This is the case in [KES 10] for example, where the topographic source terms are discretized in Discontinuous Galerkin (DG) techniques (see Chapter 8) by projecting the topographic data onto the space of reconstruction functions.

(3) A third approach consists of estimating the fluxes and source terms in a coupled way. The influence of the source terms is accounted for in the calculation of the fluxes, thereby ensuring the balance between the gradient of the fluxes and the source terms. This is the approach followed by [LHO 07] and [FIN 10] in their derivation of approximate-state solvers for the water hammer equations, the shallow water equations and the shallow water equations with porosity.

(4) A fourth path, that is well-suited to finite volume techniques, consists of defining "auxiliary" variables to be used in the estimates of the gradients for the calculation of the fluxes. Such auxiliary variables do not need to be conserved variables. They must be defined so as to allow the *C*–property and/or steady-state conditions to be preserved. As an example, [NUJ 95] proposes that the free surface elevation ζ be used instead of the water depth h in the calculation of the mass flux

for the shallow water equations. The authors of [ZHO 01] note that reconstructing the free surface elevation instead of the water depth enhances the stable character of higher-order schemes and allows spurious oscillations to be eliminated from the numerical solutions of the shallow water equations. [LIA 09] proposes a free surface elevation-based discretization of the pressure terms. Although not justified theoretically in Nujic's original publication in 1995, the use of the free surface elevation instead of the water depth for the shallow water equations can be justified as follows: since z_b is constant, the free surface elevation $\zeta = h + z_b$ verifies $\partial \zeta / \partial t = \partial h / \partial t$. Therefore, ζ may be used as the first component of the conserved variable U instead of h. The free surface elevation is also used instead of the conserved variable ϕh in the solution of the two-dimensional shallow water equations with a variable porosity ϕ [GUI 06]. In [BUR 04], the open channel equations are solved by expressing the variation in the cross-sectional area as a function of momentum balance. A similar formula is proposed in [LEE 10] for the HLL and Roe's Riemann solver (see Appendix C). This formula coincides with those of an approximate-state Riemann solver for the shallow water equation [LHO 07, FIN 10]. This leads us to wonder whether a general methodology can be used to define auxiliary flow variables that preserve the C–property and steady-state flow conditions more efficiently than the conserved variables do. This option is explored in the following sections. The approach is called "auxiliary variable-based balancing" for the sake of convenience.

9.5.3.2. *Principle of Auxiliary Variable-based (AV) balancing*

As mentioned in Appendix C and underlined by a number of authors, flux estimates based on classical Riemann solvers such as the HLL/HLLC, Roe's, the Lax-Friedrichs solver, etc. can be expressed as the sum of a centered flux (that is unconditionally unstable when used with first- or second-order time stepping [VIC 82]) and a diffusive flux (that contributes to stabilize the numerical solution):

$$\mathrm{F}(\mathrm{U_L}, \mathrm{U_R}) = a\mathrm{F_L} + (1-a)\mathrm{F_R} + \mathrm{D}(\mathrm{U_L} - \mathrm{U_R}) \qquad [9.72]$$

where a is a coefficient between 0 and 1, and D is a diffusion matrix that depends on the flux formula used. For instance, in the discretization [9.5] provided for the water hammer equations (section 9.2.1), $a = 1/2$ and D is a diagonal matrix equal to $c/2$ times the identity matrix:

$$\mathrm{D} = \frac{c}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{c}{2}\mathrm{I} \qquad [9.73]$$

In the HLL Riemann solver used for the shallow water equations example of section 9.2.2:

$$\left.\begin{aligned} a &= \frac{\lambda^+}{\lambda^+ - \lambda^-} \\ \mathrm{D} &= -\frac{\lambda^-\lambda^+}{\lambda^+ - \lambda^-}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = -\frac{\lambda^-\lambda^+}{\lambda^+ - \lambda^-}\mathrm{I} \end{aligned}\right\} \qquad [9.74]$$

Although playing a crucial role to solution stability in the absence of source terms, the diffusive flux fails to preserve the *C*–property, as shown in the examples of section 9.2. The question then arises whether the diffusive flux may be formulated in a slightly different way, such that the *C*–property is preserved:

$$\mathrm{F}(\mathrm{U}_\mathrm{L}, \mathrm{U}_\mathrm{R}) = a\mathrm{F}_\mathrm{L} + (1-a)\mathrm{F}_\mathrm{R} + \mathrm{D}_\mathrm{V}(\mathrm{V}_\mathrm{L} - \mathrm{V}_\mathrm{R}) \qquad [9.75]$$

where D_V is a diffusion matrix, not necessarily equal to D, and V_L and V_R are appropriately defined auxiliary variables, functions of U and φ:

$$\left.\begin{aligned} \mathrm{V}_\mathrm{L} &= \mathrm{V}(\mathrm{U}_\mathrm{L}, \varphi_\mathrm{L}) \\ \mathrm{V}_\mathrm{R} &= \mathrm{V}(\mathrm{U}_\mathrm{R}, \varphi_\mathrm{R}) \end{aligned}\right\} \qquad [9.76]$$

V may also be defined in differential form as:

$$\mathrm{dV} = \mathrm{dV}(\mathrm{dU}, \varphi) = \left(\frac{\partial \mathrm{V}}{\partial \mathrm{U}}\right)_{x=\mathrm{Const}} \mathrm{dU} + \left(\frac{\partial \mathrm{V}}{\partial \varphi}\right)_{\mathrm{U}=\mathrm{Const}} \frac{\partial \varphi}{\partial \mathrm{x}} \mathrm{d}x \qquad [9.77]$$

The term $(\partial \mathrm{V}/\partial \mathrm{U})_{x=\mathrm{Const}}\,\mathrm{dU}$ accounts for the variations in the conserved variable U due to the gradients in the flow conditions, while the term $(\partial \mathrm{V}/\partial x)_{\mathrm{U}=\mathrm{Const}}\,\partial\varphi/\partial x\,\mathrm{d}\varphi$ accounts for the influence of the geometric source terms. The differential definition [9.77] is more convenient than the definition [9.76] when the definition of V contains terms that cannot be recast in the form of the differential of a variable (as it is the case with source terms). Assuming that the variable V is defined appropriately, the diffusion matrix D_V must be determined. A desirable property for this matrix is that it does not change the stability constraints of the original discretization. This means that the strength of the diffusion term $D_V (V_L - V_R)$ should be identical to that of the original diffusion term $D (U_L - U_R)$. The diffusive term in equation [9.72] is rewritten as (subscripts x = Const and U = Const are omitted for the sake of clarity):

$$\left.\begin{aligned} D(U_L - U_R) &\approx -\Delta x\, D\frac{\partial U}{\partial x} + O(\Delta x^2) \\ D_V(V_L - V_R) &= -\Delta x\, D_V \frac{\partial V}{\partial U}\frac{\partial U}{\partial x} - \Delta x\, D_V \frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial x} + O(\Delta x^2) \end{aligned}\right\} \quad [9.78]$$

The stability properties of the scheme are related to the first-order derivative. In the absence of source terms, equation [9.78] reduces to:

$$\left.\begin{aligned} D(U_L - U_R) &\approx -\Delta x\, D\frac{\partial U}{\partial x} + O(\Delta x^2) \\ D_V(V_L - V_R) &= -\Delta x\, D_V \frac{\partial V}{\partial U}\frac{\partial U}{\partial x} + O(\Delta x^2) \end{aligned}\right\} \quad [9.79]$$

The diffusion terms in equation [9.79] are equivalent provided that:

$$D_V \frac{\partial V}{\partial U} = D \qquad \Leftrightarrow \qquad D_V = D\left(\frac{\partial V}{\partial U}\right)^{-1} \quad [9.80]$$

9.5.3.3. *Discretization of the momentum source term*

Estimate [9.75] satisfies steady-state conditions in the continuity equation. Indeed, under steady-state conditions, the first components $F_{1,L}$ and $F_{1,R}$ of F_L and F_R are identical, $F_{1,L} = F_{1,R} = q$. If V is chosen appropriately, $V_L = V_R$ for steady flow configurations, and the first component of equation [9.75] becomes:

$$F_1 = aF_{1,L} + (1-a)F_{1,R} = q \quad [9.81]$$

In contrast with the source term upwinding technique, no extra source term needs to be added to the continuity equation. The issue remains, however, of the discretization of the momentum source term. Numerical experiments (see section 9.6) show that using the source term upwinding technique only for the momentum source term provides good results.

9.5.3.4. *Application to the water hammer equations*

Consider the water hammer equations in a non-horizontal pipe with variable cross-sectional area. The angle of the pipe with the horizontal is denoted by θ and the friction coefficient is denoted by k. The discretization [9.5] is used, with D given by equation [9.75]. The conserved variable is $U = [\rho A, \rho Q]^T$. A possible choice for the auxiliary variable V is:

$$\mathrm{dV}=\begin{bmatrix}\mathrm{d}p+(\rho g\sin\theta+k|u|u/A)\,\mathrm{d}x\\ \mathrm{d}(\rho Q)\end{bmatrix} \qquad [9.82]$$

This choice is motivated by the fact that under steady-state conditions, the vector dV is zero. Indeed, the first component expresses steady-state energy conservation (the slope of the energy line is equal to the head loss per unit length) while the second component accounts for mass conservation. Therefore, definition [9.82] not only verifies the *C*-property exactly, but it also verifies steady-state conditions. The following estimate is obtained at the interface $i-1/2$:

$$\begin{aligned}\Delta x_{i-1/2}\left(\frac{\partial \mathrm{V}}{\partial x}\right)_{i-1/2}&=\Delta x_{i-1/2}\begin{bmatrix}\dfrac{\partial p}{\partial x}+\rho g\sin\theta+\dfrac{k|u|u}{A}\\ \dfrac{\partial}{\partial x}(\rho Q)\end{bmatrix}_{i-1/2}\\ &\approx\begin{bmatrix}p_i^n-p_{i-1}^n+\rho g(z_i-z_{i-1})+\Delta x_{i-1/2}\left(\dfrac{k|u|u}{A}\right)_{i-1/2}^n\\ (\rho Q)_i^n-(\rho Q)_{i-1}^n\end{bmatrix}\end{aligned} \qquad [9.83]$$

where z_i is the elevation of the cell i and $\Delta x_{i-1/2}$ is the distance between the centers of the cells $i-1$ and i. The Jacobian matrix of V with respect to U is given by (the second equation [2.74] is used):

$$\frac{\partial \mathrm{V}}{\partial \mathrm{U}}=\begin{bmatrix}\partial p/\partial(\rho A) & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}c^2\partial p/\partial(Ap) & 0\\ 0 & 1\end{bmatrix}\approx\begin{bmatrix}c^2/A & 0\\ 0 & 1\end{bmatrix} \qquad [9.84]$$

Substituting equations [9.73] and [9.84] into equation [9.80] leads to the following expression for the matrix $\mathrm{D_V}$:

$$\mathrm{D_V}=\frac{1}{2}\begin{bmatrix}A/c & 0\\ 0 & c\end{bmatrix} \qquad [9.85]$$

and the final discretization becomes:

$$\mathrm{F}_{i-1/2}^{n+1/2}=\frac{1}{2}\begin{bmatrix}(\rho Q)_{i-1}^n+(\rho Q)_i^n\\ (Ap)_{i-1}^n+(Ap)_i^n\end{bmatrix}+\frac{1}{2}\begin{bmatrix}\dfrac{A_{i-1/2}}{c}\Delta p_{i-1/2}^n\\ c\left[(\rho Q)_{i-1}^n-(\rho Q)_i^n\right]\end{bmatrix} \qquad [9.86]$$

where $A_{i-1/2}$ is an average value of A between the cells i–1 and i, and the variation Δp is estimated as:

$$\Delta p_{i-1/2}^{n} = p_{i-1}^{n} - p_{i}^{n} + \rho g(z_{i-1} - z_{i}) - \frac{\Delta x_{i-1/2}}{2}\left[\left(\frac{k|u|u}{A}\right)_{i-1}^{n} + \left(\frac{k|u|u}{A}\right)_{i}^{n}\right] \quad [9.87]$$

Examples of possible estimates for $A_{i-1/2}$ are:

$$\left.\begin{aligned} A_{i-1/2} &= \frac{A_{i-1/2} + A_{i-1/2}}{2} \\ A_{i-1/2} &= \min(A_{i-1/2}, A_{i-1/2}) \\ A_{i-1/2} &= (A_{i-1/2} A_{i-1/2})^{1/2} \end{aligned}\right\} \quad [9.88]$$

These are not the only possible estimates. Any estimate satisfying the consistency condition [9.46] may be used.

9.5.3.5. *Application to the shallow water equations*

Consider discretization [9.35] presented in section 9.3.3 for the shallow water equations discretized with the HLL Riemann solver. The components of the flux are given by:

$$\left.\begin{aligned} q_{i-1/2} &= \frac{\lambda^{+} q_{i-1}^{n} - \lambda^{-} q_{i}^{n}}{\lambda^{+} - \lambda^{-}} - \frac{\lambda^{-}\lambda^{+}}{\lambda^{+} - \lambda^{-}} \Delta h_{i-1/2} \\ M_{i-1/2} &= \frac{\lambda^{+} M_{i-1}^{n} - \lambda^{-} M_{i}^{n}}{\lambda^{+} - \lambda^{-}} - \frac{\lambda^{-}\lambda^{+}}{\lambda^{+} - \lambda^{-}} (q_{i-1}^{n} - q_{i}^{n}) \end{aligned}\right\} \quad [9.89]$$

where $\Delta h_{i-1/2}$ is an approximation of the difference in h that satisfies the C–property. The source term contributions are changed to:

$$\left.\begin{aligned} \beta^{(1)}\mathrm{K}^{(1)} &= \frac{g h_{i-1}^{n+1/2}}{2\lambda^{+}} (h_{i}^{n} - h_{i-1}^{n}) \begin{bmatrix} 0 \\ \lambda^{+} \end{bmatrix} \\ \beta^{(2)}\mathrm{K}^{(2)} &= \frac{g h_{i-1}^{n+1/2}}{2\lambda^{+}} (h_{i}^{n} - h_{i-1}^{n}) \begin{bmatrix} 0 \\ \lambda^{+} \end{bmatrix} \end{aligned}\right\} \quad [9.90]$$

The only remaining problem is to define a suitable estimate for $\Delta h_{i-1/2}$ in the first equation [9.90]. Note that the classical estimate:

$$\Delta h_{i-1/2}^{(0)} = h_{i-1}^n - h_i^n \qquad [9.91]$$

does not satisfy the C–property.

(1) A first option is Nujic's variable transformation [NUJ 95], $\zeta = h + z_b$:

$$\mathrm{V} = \begin{bmatrix} h + z_b \\ q \end{bmatrix}, \qquad \frac{\partial \mathrm{V}}{\partial \mathrm{U}} = \mathrm{I}, \ \mathrm{D_V} = \mathrm{D} \qquad [9.92]$$

This leads to the following estimate for Δh in equation [9.89]:

$$\Delta h_{i-1/2}^{(1)} = \zeta_{i-1}^n - \zeta_i^n \qquad [9.93]$$

The C–property is verified. However, arbitrary steady-state configurations are not preserved. Consider for instance uniform flow conditions over a non-zero bottom slope. Then both the water depth h and the unit discharge q are constant. Since the slope is non-zero, ζ is not constant and the first equation [9.88] yields a flux that is different from the average cell value.

(2) A second possible choice for V is proposed in [BUR 04] and [LEE 10]:

$$\mathrm{dV} = \begin{bmatrix} \mathrm{d}M - (S_0 - S_f) gh \,\mathrm{d}x \\ \mathrm{d}q \end{bmatrix}, \qquad \frac{\partial \mathrm{V}}{\partial \mathrm{U}} = \begin{bmatrix} c^2 - u^2 & 2u \\ 0 & 1 \end{bmatrix} \qquad [9.94]$$

Note that the first row of the Jacobian matrix $\partial \mathrm{V} / \partial \mathrm{U}$ is obtained using the relationship $\mathrm{d}M = (c^2 - u^2)\,\mathrm{d}h + 2u\,\mathrm{d}q$ and by excluding the integral in the differentiation of V. This can be justified by the fact that the integral is a function of x and that its derivative with respect to x is non-zero even if V is constant. Equation [9.94] leads to the following estimate for Δh:

$$\begin{aligned}\Delta h_{i-1/2}^{(2)} &= \frac{M_{i-1}^n - M_i^n - 2u_{i-1/2}(q_{i-1}^n - q_i^n) + (z_{b,i-1} - z_{b,i} - \Delta x\, S_f)}{(c^2 - u^2)_{i-1/2}} \\ &= -\frac{M_{i-1}^n - M_i^n - (\lambda^- + \lambda^+)(q_{i-1}^n - q_i^n) + (z_{b,i-1} - z_{b,i} - \Delta x\, S_f)}{\lambda^- \lambda^+}\end{aligned} \qquad [9.95]$$

This formula coincides with that of the approximate-state Riemann solver presented in [LHO 07]. [BUR 04] and [LHO 07] acknowledge the need for a specific treatment of critical points, that correspond to $\lambda^- = 0$ or $\lambda^+ = 0$. This yields a division by zero in formula [9.95]. [BUR 04] proposed that the final estimate for Δh be the minmod of the estimates given by equations [9.90] and [9.95]. [LHO 07] and [FIN 10] proposed that the estimate of the flux be based on a characteristics-based estimate of the flow variables under critical conditions.

When combined with the HLL or Roe's solver, formula [9.95] is seen to introduce a downwinding effect on the unit discharge in the neighborhood of critical points [FIN 10]. This can be avoided by defining dV as:

$$\mathrm{dV} = \begin{bmatrix} \mathrm{d}M - 2u\,\mathrm{d}q - (S_0 - S_f)gh\,\mathrm{d}x \\ \mathrm{d}q \end{bmatrix}, \qquad \frac{\partial \mathrm{V}}{\partial \mathrm{U}} = \begin{bmatrix} c^2 - u^2 & 0 \\ 0 & 1 \end{bmatrix} \qquad [9.96]$$

thus eliminating the difference $q_{i-1}^n - q_i^n$ from the numerator in equation [9.95]. The following estimate is obtained:

$$\Delta h_{i-1/2}^{(3)} = -\frac{M_{i-1}^n - M_i^n + (z_{b,i-1} - z_{b,i} - \Delta x\, S_f)}{\lambda^- \lambda^+} \qquad [9.97]$$

(3) A third possibility is to use the hydraulic head $H = \zeta + u^2/(2g)$ instead of M in the first component of V:

$$\mathrm{dV} = \begin{bmatrix} \mathrm{d}H + S_f\ \mathrm{d}x \\ \mathrm{d}q \end{bmatrix}, \qquad \frac{\partial \mathrm{V}}{\partial \mathrm{U}} = \begin{bmatrix} c^2 - u^2 & u \\ 0 & 1 \end{bmatrix} \qquad [9.98]$$

The first row in the Jacobian matrix is obtained using the relationship $\mathrm{d}H = (1 - u^2/c^2)\,\mathrm{d}h + u\,\mathrm{d}q + \mathrm{d}z_b$. The following estimate is obtained for Δh:

$$\Delta h_{i-1/2}^{(4)} = -c_{i-1/2}^2 \frac{H_{i-1}^n - H_i^n - z_{b,i-1} + z_{b,i} + \dfrac{\lambda^- + \lambda^+}{2}(q_i^n - q_{i-1}^n) + \Delta x\, S_f}{\lambda^- \lambda^+}$$

[9.99]

For the same reason as in option (2), the following simplification is proposed for transcritical flow:

$$\mathrm{dV} = \begin{bmatrix} \mathrm{d}H - u\,\mathrm{d}q + S_f\ \mathrm{d}x \\ \mathrm{d}q \end{bmatrix}, \qquad \frac{\partial \mathrm{V}}{\partial \mathrm{U}} = \begin{bmatrix} c^2 - u^2 & 0 \\ 0 & 1 \end{bmatrix} \qquad [9.100]$$

thus leading to:

$$\Delta h_{i-1/2}^{(4)} = -c_{i-1/2}^{2} \frac{H_{i-1}^{n} - H_{i}^{n} - z_{b,i-1} + z_{b,i} + \Delta x\, S_f}{\lambda^{-} \lambda^{+}} \quad [9.101]$$

9.6. Computational example

The performance of the various source term discretization techniques presented in the previous sections is illustrated by an application to the one-dimensional shallow water equations. The following steady-state configuration is considered: water flows with a constant unit discharge in a channel with a non-horizontal bottom (Figure 9.9). The bottom level forms a bump. It is obtained from the following equation:

$$z_b(x) = Z \exp\left[-\left(\frac{x - x_0}{X} \right)^2 \right] \quad [9.102]$$

The water is initially at rest, with a constant free surface elevation and a zero discharge at all points of the computational domain. A constant unit discharge and water level are prescribed at the upstream and downstream end of the channel respectively.

The numerical solution is examined after a sufficiently long simulated time, so that the transient regime vanishes and steady state is reached. The constant unit discharge and downstream water level are chosen such that the flow regime is subcritical upstream of the bump, supercritical on the top part of the bump and a hydraulic jump appears on the downstream side of the bump.

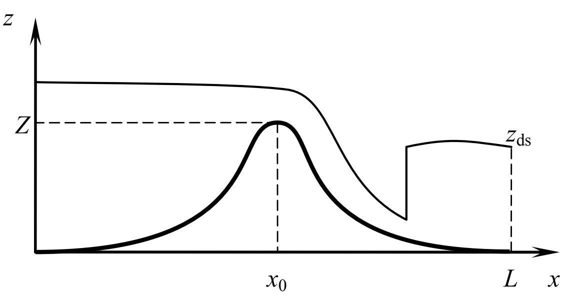

Figure 9.9. *Steady-state flow configuration. Bold line: bottom level. Thin line: free surface flow elevation*

The parameters of the test case are given in Table 9.1. The numerical solution is computed using the first-order Godunov scheme, with left and right states of the Riemann problems at the cell interfaces taken equal to the average values of the variables over the computational cells. The HLL solver presented in Appendix C is used in the computation of the fluxes at the interfaces between the computational cells.

Figure 9.10 shows the longitudinal profiles for the water level and unit discharge computed at $t = 10^4$ seconds using the source term upwinding approach. Under steady-state conditions, the unit discharge should be expected to be uniform over the entire domain. This is not the case with the numerical solution: the transition from subcritical to supercritical conditions is observed to induce strong variations in the cell average of the unit discharge.

Note that this should not be attributed to continuity issues, because the finite volume technique is intrinsically conservative. The unit discharge profile plotted in Figure 9.10 is the cell average of the quantity $q = hu$, that is, the momentum in the computational cells, and not the value of the mass flux at the interfaces between the cells. Under steady-state conditions, the unit discharge at the interfaces between the computational cells is uniform all over the computational domain.

Symbol	Meaning	Value
g	Gravitational acceleration	9.81 m/s^2
L	Length of the computational domain	400 m
n_M	Manning's friction coefficient	0.01
q_{us}	Upstream discharge	1 m^2/s
X	Characteristic size of the bump (equation [9.102])	50 m
x_0	Abscissa of the center of the bump	200 m
Z	Height of the bump	1 m
z_{ds}	Water level at the downstream boundary	1 m
Δx	Computational cell size	2 m
ζ	Initial water level	1.01 m

Table 9.1. *Parameters of the test case*

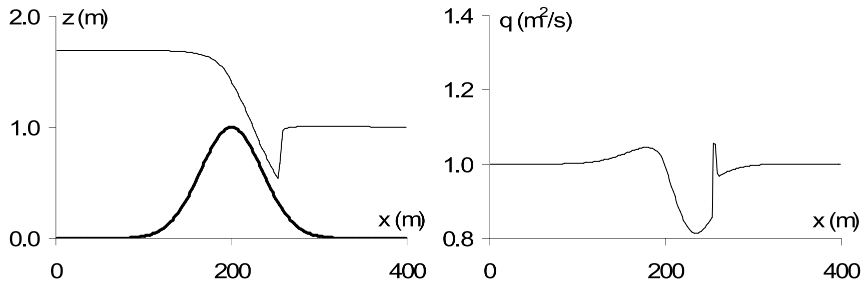

Figure 9.10. *Steady-state water level and unit discharge profiles computed by the source term upwinding method. Bold line (left): bottom level*

Also note that the computed discharge profile exhibits a peak at the location of the jump. This is because the water depth varies abruptly across the jump. Since the unit discharge at the cell interfaces is given by an equation in the form [9.72], q_L and q_R cannot be identical to the unit discharge at the interface if $h_L \neq h_R$. A higher-order reconstruction technique would be needed to reconstruct the left and right states at each interface more accurately and obtain a constant discharge.

Figure 9.11 shows the water level and unit discharge profiles computed using the quasi steady wave propagation algorithm. Both the extent and amplitude of the zone of non-uniform unit discharge are reduced compared to the source term upwinding method, except for the peak that corresponds to the hydraulic jump.

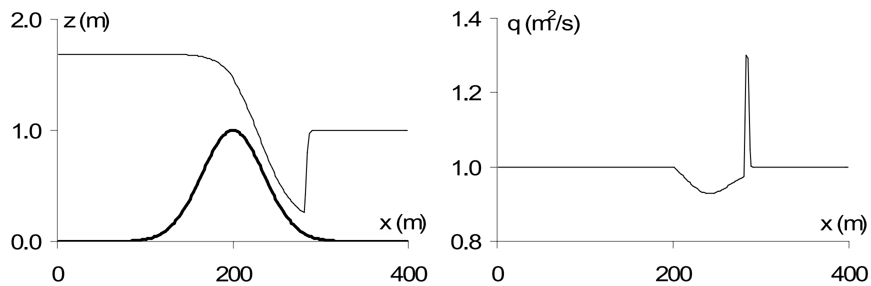

Figure 9.11. *Steady-state water level and unit discharge profiles computed by the quasi-steady wave propagation method. Bold line (left): bottom level*

Figure 9.12 shows the free surface and unit discharge profiles obtained using the hydrostatic reconstruction method. The amplitude of the discharge peak at the location of the hydraulic jump is similar to that in Figure 9.11, but the amplitude of the non-uniform discharge zone upstream of the jump is smaller in Figure 9.12.

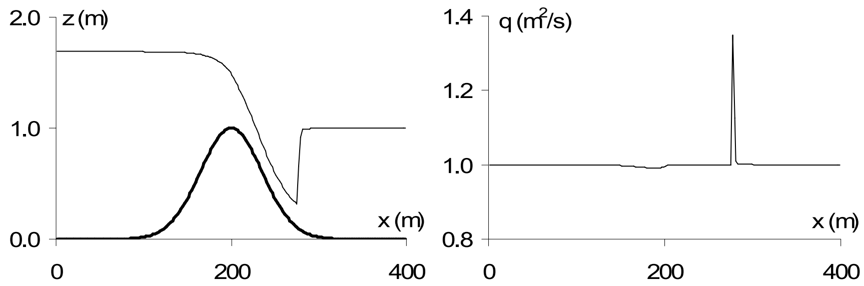

Figure 9.12. *Steady-state water level and unit discharge profiles computed by the hydrostatic reconstruction method*

Figures 9.13 and 9.14 show the profiles obtained using the auxiliary variable method.

Figure 9.13a shows the result given by Nujic's method [NUJ 95], equation [9.93]. The quality of the numerical solution is similar to that of the hydrostatic reconstruction method, but the discharge peak is smaller.

Figure 9.13b shows the results obtained from the momentum-based method proposed in [BUR 04]. As in [BUR 04], $\Delta h_{i-1/2}$ is computed as the minmod of the estimates [9.91] and [9.95]. The amplitude of the peak is reduced compared to that in Figure 9.13a, but a small variation in the unit discharge is still visible upstream of the bump.

This variation is eliminated when $\Delta h_{i-1/2}$ is computed as the minmod of [9.91] and [9.97] (see Figure 9.13c). This, however, is achieved at the expense of a sharper oscillation at the location of the hydraulic jump. An optimally accurate method should therefore use estimate [9.95] at the shock and [9.97] at other points.

Figure 9.14a shows the profiles obtained using the minmod of [9.91] and the hydraulic head-based estimate [9.99]. Figure 9.14b shows the profiles obtained from the minmod of [9.91] and the hydraulic head-based estimate [9.101]. Both methods give similar results in the neighborhood of the shock. However, estimate [9.101] is better upstream of the bump.

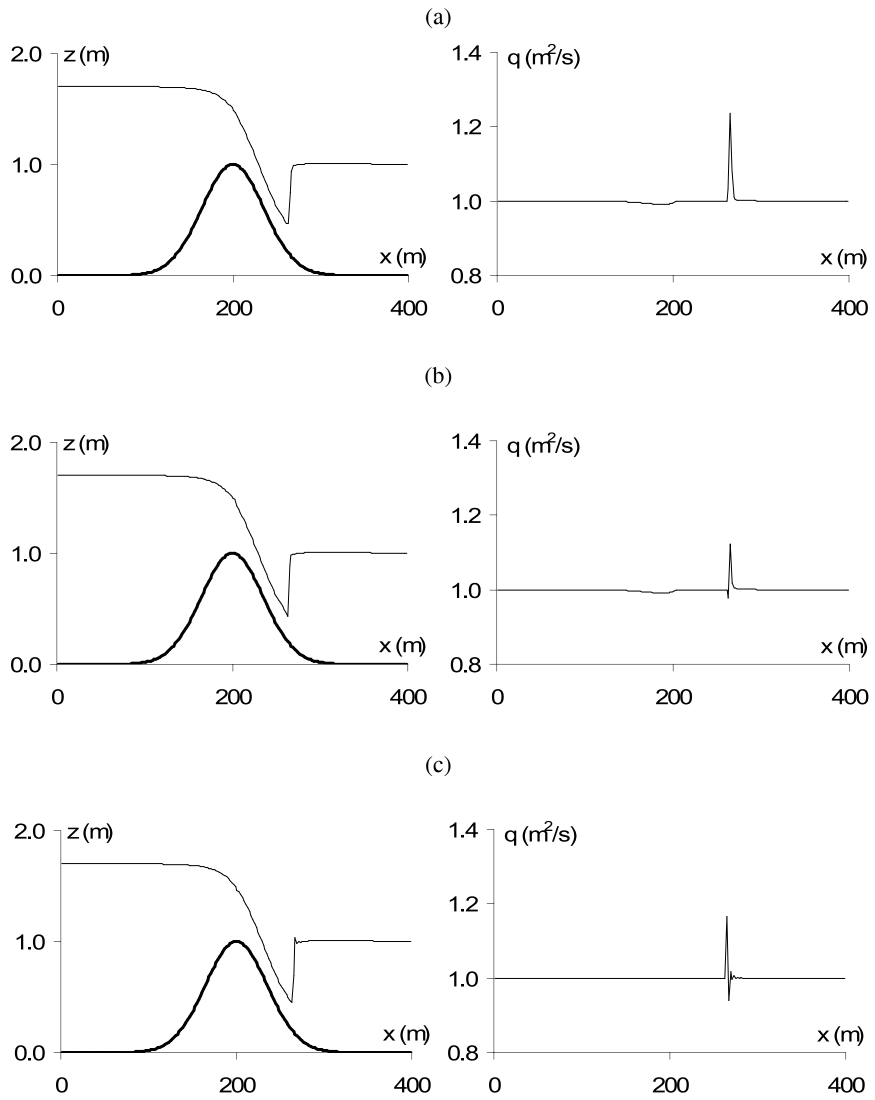

Figure 9.13. *Steady-state water level and unit discharge profiles computed by the auxiliary variable-based method. (a)* $\Delta h_{i-1/2} = \Delta h^{(1)}_{i-1/2}$ *,(b)* $\Delta h_{i-1/2} = \min\bmod[\Delta h^{(0)}_{i-1/2}, \Delta h^{(2)}_{i-1/2}]$ *,*
(c) $\Delta h_{i-1/2} = \min\bmod[\Delta h^{(0)}_{i-1/2}, \Delta h^{(3)}_{i-1/2}]$

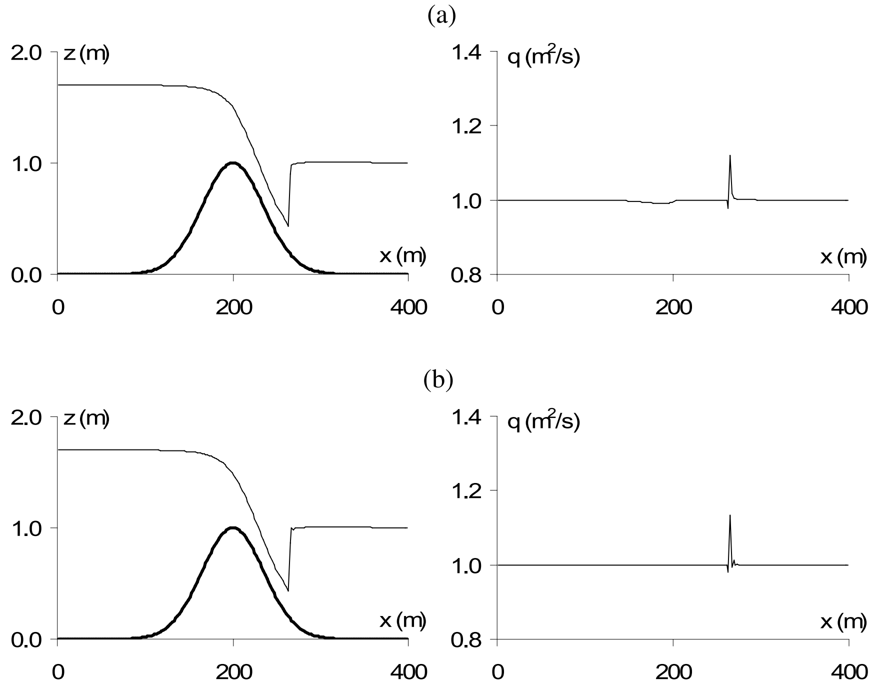

Figure 9.14. *Steady-state water level and unit discharge profiles computed by the auxiliary variable-based method. (a)* $\Delta h_{i-1/2} = \min mod[\Delta h^{(0)}_{i-1/2}, \Delta h^{(4)}_{i-1/2}]$ *,*
(b) $\Delta h_{i-1/2} = \min mod[\Delta h^{(0)}_{i-1/2}, \Delta h^{(5)}_{i-1/2}]$

9.7. Summary

Real-world applications of computational hydraulics or computational fluid dynamics involve the discretization of geometry-induced source terms. Discretizing such source terms independently from the fluxes may induce stability problems. This is due to a lack of balance between the second-order (diffusion-like) terms in the discretization of both fluxes and source terms.

The notion of *C*–property plays an essential role in the definition of well-balanced schemes. The *C*–property states that any initial condition verifying static equilibrium conditions should yield a static solution at later times for arbitrary geometries. A discretization of the flow equations may satisfy the *C*–property exactly or approximately. The *C*–property may be extended to steady-state flow conditions (of which static equilibrium is only a particular case).

A number of methods are presented in this chapter for geometric source term discretization in finite volume techniques:

– Source term upwinding (section 9.3) uses a decomposition of the source term in the base of eigenvectors of the Jacobian matrix A. The source term is thus broken into as many components as there are waves. Each component is assigned to the computational cell into which the wave originating from the cell interface travels. Some of the flow variables must be estimated at the interfaces between the computational cells. The estimates must be derived in such a way that the *C*–property be verified.

– The quasi-steady wave algorithm (section 9.4) uses a particular reconstruction of the geometry. The variations in the geometric parameter are lumped in the middle of the computational cell. The parameter is continuous at the cell interfaces, which allows a standard Riemann problem (that is, with a continuous flux function and no source term) to be defined and solved.

– In well-balancing techniques (section 9.5.1), a solution is sought for the Riemann problem with a discontinuous flux function and source term. In the original publication of the method, the discontinuity is achieved as the limit case of a continuous, ramp-shaped function.

– Balancing techniques such as the hydrostatic reconstruction (section 9.5.2) consist of modifying the states of the Riemann problem at the interface between two cells. The cross-sectional areas (or water depth) on both sides of the interface are modified using the water level and the higher of the bottom levels on both sides of the interface. The unit discharge is re-computed as the product of the flow velocity in the cell and the modified water depth.

– The auxiliary variable-based balancing technique (section 9.5.3) consists of redefining the flow variables used in the diffusive part of the flux estimate. These so-called auxiliary variables are defined so as to achieve a zero gradient under steady-state conditions, which allows the *C*– and extended *C*–property to be satisfied.