Algebra

* The principle of Counting

If an act could be performed by n ways, another act could be performed by m ways, then the two acts can be performed together by m × n ways.

Example 1 :
A student could go to his school by 3 ways and go back to his home by 2 ways by How many ways could he go and back ?

Sol:
The n Number of ways = 3 ×2 = 6 ways.

Example 2 :
A school gives 3 prizes for the gifted pupils in Tennis ,Swimming and football. If the Number of Competitors are respectively 8, 7 and 12.
By how many ways the 3 prizes can be Distributed ?

Sol:
The n Number of ways = 8 ×7×12 = 672

* Permutations

Def : if we have a set of elements, any arrangement by taking some or all of these elements in a definite order is called a permutation.
I.e. if the Number of elements is n , taken r of them in a definite order, this could be written as No of permutations $= {}^{n}P_{r}$, $n \ge r$, $n, r \in z^{+}$.

* Important Rules

1) $^{n}P_{r} = n\ (n-1)\ (n-2) \ldots\ldots\ldots\ldots\ (n-r+1)$

2) $^{n}P_{n} = n\ (n-1)\ (n-2) \ldots\ldots \ldots \times 3 \times 2 \times 1 = \underline{|n}$,

$\underline{|n}$ is called Factorial $\underline{n}$

3) $\underline{|n} = n\ \underline{|n-1} = n\ (n-1)\ \underline{|n-2} = \ldots\ldots$

4) $$^{n}P_{r} = \frac{\underline{|n}}{\underline{|n-r}}$$

5) $^{n}P_{0} = 1$

6) $\underline{|1} = 1$

7) $\underline{|0} = 1$

Example 3 :

How many permutations can be Performod from 5 kinds of fruit, each of them consists of 2 kinds with out repetition

Sol :

Number of permutations $= {}^{5}P_{2} = 5 \times 4 = 20$

Example 4 :

From the digits 1,2,3,4,5 How many Numbers could be performed from it without repetition such that the unit digit is not 4 and the tenth digit is not 5.

Sol :

No of ways for writing $= 5\times4\times3\times2\times1 = 120$

No of ways to write the unit digit is 4 and the tenth digit is 5

$= 1\times1\times3\times2\times1 = 6$

$\because$ The No of required ways

$= 120 - 6 = 114.$

Example 5 :

if $n = \underline{|120}$, find$^{\text{٧}}$ $^{n}P_{\text{٤}}$

Sol :

$\because\ n = \underline{|120}$ $\quad \therefore\ n = 5$

$^{2n}P_4 = {}^{10}P_4 = 10 \times 9 \times 8 \times 7 = 5040.$

Example 6 :

Prove that $\underline{|2n} = 1 \times 3 \times 5 \times \ldots \times (2n-1) \times 2^n \times \underline{|n}$

Sol :

$L.H.S = \underline{|2n}$

$= 2n\ (2n-1)\ (2n-2)\ (2n-3) \ldots \times 4 \times 3 \times 2 \times 1$

$= (2n)(2n-2) \ldots \times 4 \times 2 \times (2n-1) \times (2n-3) \times \ldots \times 3 \times 1$

$= 2^n \times (n \times (n-1) \times \ldots \times 2 \times 1) \times (2n-1) \times (2n-3) \times \ldots \times 3 \times 1$

$= 2^n \times \underline{|n} \times (2n-1) \times (2n-3) \times \ldots \times 3 \times 1 = R.H.S.$

Example 7 :

Find the value of $^{y}P_{x}$ if

$^{x+y}P_3 = 210$, $\underline{|x-y} = 6$

sol :

$\therefore\ ^{x+y}P_3 = 210 \qquad \therefore\ x + y = 7 \longrightarrow 1$

$\because\ \underline{|x-y} = 3 \times 2 \times 1 \quad \therefore x - y = 3 \longrightarrow 2$

Add 1 , 2

$\therefore 2x = 10 \Rightarrow x = 5 \Rightarrow y = 2$.

Example 8:

if $^{7}P_3 = 2 \times {}^{7}P_{r-1}$, find the value of

$$\frac{\underline{|r+1}}{\underline{|r+2}} + \frac{\underline{|r}}{\underline{|r+1}} + \frac{\underline{|r-2}}{\underline{|r-1}}$$

sol :

$$^7P_r = 2 \times {}^7P_{r-1}$$

$$\therefore \frac{^7P_r}{^7P_{r-1}} = 2$$

$$\frac{\lfloor 7}{\lfloor 7-r} \times \frac{\lfloor 8-r}{\lfloor 7} = 2$$

$$\therefore \frac{\lfloor 7 \times (8-r) \lfloor 7-r}{\lfloor 7-r \times \lfloor 7} = 2$$

$$\therefore 8 - r = 2 \Rightarrow r = 6$$

$$\therefore \frac{\lfloor r+1}{\lfloor r+2} + \frac{\lfloor r}{\lfloor r+1} + \frac{\lfloor r-2}{\lfloor r-1}$$

$$= \frac{1}{r+2} + \frac{1}{r+1} + \frac{1}{r-1} =$$

$$= \frac{1}{8} + \frac{1}{7} + \frac{1}{5} = \frac{131}{280}$$

Example 9 :

Find the Number of the elements of the set A If:

$X = \{ x : X \in Z, -2 \leq x \leq 6 \}$

$A = \{ (a,b,c) , a,b,c \in X , a \neq b \neq c \{$

Sol :

N (X) = 9

No of element of A = $^9P_3 = 9 \times 8 \times 7 = 504$

Example 10 :

if $3 \times {}^nP_{r-1} = 5 \times {}^{n-1}P_{r-1}$, $^{n+1}P_r = 2 \times {}^nP_r$. Find n , r

sol :

$$\frac{^{n}P_{r-1}}{^{n-1}P_{r-1}} = \frac{5}{3}$$

$$\therefore \frac{\lfloor n}{\lfloor n+1-r} \times \frac{\lfloor n-r}{\lfloor n-1} = \frac{5}{3}$$

$$\therefore \frac{n}{n+1-r} = \frac{5}{3}$$

$$3n = 5n + 5 - 5r$$

$$5r = 2n + 5 \longrightarrow \quad 1$$

$$\frac{^{n+1}P_{r}}{^{n}P_{r}} = 2$$

$$\frac{\lfloor n+1}{\lfloor n+1-r} \times \frac{\lfloor n-r}{\lfloor n} = 2$$

$$\frac{n+1}{n+1-r} = 2$$

$$\therefore n+1 = 2n+2-2r$$

$$\therefore 2r = n+1 \longrightarrow \quad 2$$

eq 1 ÷ eq 2

$$\frac{5}{2} = \frac{2n+5}{n+1}$$

$$\therefore 4n + 10 = 5n + 5$$

$$n = 5 \Rightarrow r = 3$$

* Combination

Definition :

If we have a set of elements ,any arrangement by taking all or some of these elements -without consideration of the order of these elements -is called a combination .

I.e. …If the number of elements is n ,taken r of them each time -without consideration of the order of its elements -then ,the number of combinations is symbolized by $^{n}C_{r}$, $n \geq r$, $n, r \in z^{+}$

* Important rules

1) $^nC_r = \dfrac{^nP_r}{\lfloor r}$

2) $^nC_r = \dfrac{\lfloor n}{\lfloor r \; \lfloor n-r}$

3) $^nC_r = {^nC_{n-r}}$

4) $^nC_n = 1$

5) $^nC_{zero} = 1$

6) If $^nC_r = {^nC_d} \longrightarrow r=d$ or $r+d=n$

7) $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

Example 1:

Find the value of n if $^nC_2 = 435$

Sol :

$^nC_2 = \dfrac{^nP_2}{\lfloor 2} = 435 \longrightarrow {^nP_2} = 870 \longrightarrow n=30$

Example 2:

If : $^nC_{r^2+2r} = {^nC_{2r+5}}$, $^nC_3 = 120$

find the value of $^nC_{7r+3}$

Sol :

$^nC_3 = \dfrac{^nP_3}{\lfloor 3} = 120 \therefore {^nP_3} = 120\times3\times2\times1 = 720$

$= 10\times9\times8 \qquad n=10$

$^{10}C_{r^2+2r} = {^{10}C_{2r+5}}$

$\therefore r^2+2r = 2r+5$ **or** $r^2+2r+2r+5=10$

$\therefore r^2 = 5$ $\qquad \therefore r^2+4r-5 = 0$

$r = \mp\sqrt{5}$ refused

$(r+5)(r-1)=0$

$r=-5$ refused or $r=1$

${}^{n}C_{7r+3} = {}^{10}C_{10} = 1$

Example 3:

Prove that ${}^{n}C_r + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

Then find the value of

I) $\dfrac{{}^{17}C_6 + {}^{17}C_5}{{}^{18}C_5}$

II) Prove that ${}^{n}C_{r+1} + {}^{n+1}C_r + {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_{r+1}$

Sol :

I) L.H.S. $= {}^{n}C_r + {}^{n}C_{r+1} =$

$$\frac{\lfloor n}{\lfloor r \; \lfloor n-r} + \frac{\lfloor n}{\lfloor r+1 \; \lfloor n-r-1}$$

$$= \frac{(r+1)\lfloor n + (n-r)\lfloor n}{\lfloor r+1 \; \lfloor n-r}$$

$$= \frac{(n+1)\lfloor n}{\lfloor r+1 \; \lfloor n-r} = \frac{\lfloor n+1}{\lfloor r+1 \; \lfloor n-r}$$

$$= {}^{n+1}C_{r+1} = R.H.S$$

$$\frac{{}^{17}C_6 + {}^{17}C_5}{{}^{18}C_5} = \frac{{}^{18}C_6}{{}^{18}C_5} = \frac{18-6+1}{6} = \frac{13}{6}$$

II) L.H.S $= ({}^{n}C_{r+1} + {}^{n+1}C_r) + ({}^{n}C_r + {}^{n}C_{r-1})$

$= {}^{n+1}C_{r+1} + {}^{n+1}C_r = {}^{n+2}C_{r+1} = R.H.S$

Example 4:

By how many ways ,you can choose 7 persons from a group of 9 girls ,5 boys such that the chosen group contains 3 boys

SOL :

No. of ways $= {}^{9}C_4 \times {}^{5}C_3 = 126 \times 10 = 1260$

Example 5 :

Find the expansion of ${}^{m+n}C_n$,then prove that $\lfloor\underline{17}$ is divisible by $\lfloor\underline{12}\ \lfloor\underline{5}$

Sol :

$$ {}^{m+n}C_n = \frac{\lfloor\underline{m+n}}{\lfloor\underline{n}\ \lfloor\underline{m}} = a \ , \ a \in Z^+ $$

$\therefore \quad \lfloor\underline{m+n} = a\ \lfloor\underline{n}\ \lfloor\underline{m}$

$\therefore \lfloor\underline{m+n}$ is Divisible by $\lfloor\underline{n}\ \lfloor\underline{m}$

Let $n = 12$, $\quad m = 5$

$\therefore \quad \lfloor\underline{17} = a \lfloor\underline{12}\ \lfloor\underline{5}$

$\therefore \quad \lfloor\underline{17}$ is Divisible by $\lfloor\underline{12}\ \lfloor\underline{5}$

Example 6:

If ${}^{n}P_4 = 360$, $\lfloor\underline{r} = 24$,
Find the value of ${}^{2n}C_r$

SOL :

${}^{n}P_4 = 360 \dashrightarrow n = 6$, $\quad \lfloor\underline{r} = 24 \dashrightarrow \quad r = 4$

${}^{2n}C_r = {}^{12}C_4 = 495$

Example 7:

If ${}^{m}P_3 = 210$,${}^{m+n}P_3 = 720$
Find the value of ${}^{m}C_n$.

SOL :

${}^{m}P_3 = 210 \dashrightarrow \quad m = 7$

${}^{m+n}P_3 = 720 \dashrightarrow \quad m+n = 10 \dashrightarrow \quad n = 3$

$$ {}^{m}C_n = {}^{7}C_3 = \frac{7\times6\times5}{3\times2\times1} = 35 $$

Example 8:

$\lfloor\underline{n} = 720$, $\quad {}^{n+1}C_r : {}^{n+1}C_{r-1} = 3/5$

Find the value of ${}^{n+1}C_{r-2}$

SOL :

$\lfloor\underline{n} = 720 \dashrightarrow \qquad n=6$

$$\frac{{}^{n+1}C_r}{{}^{n+1}C_{r-1}} = \frac{(n+1)-r+1}{r} = \frac{3}{5}$$

$$\frac{8-r}{r} = \frac{3}{5} \dashrightarrow \qquad r=5$$

${}^{n+1}C_{r-2} = {}^{7}C_3 = 35$

Example 9:

If ${}^5P_r = 60$, $\lfloor\underline{n} = 120$,

Find ${}^nC_r$

SOL :

${}^5P_r = 60 \dashrightarrow \qquad r = 3$

$\lfloor\underline{n} = 120 \dashrightarrow \qquad n = 5$

$\therefore {}^nC_r = {}^5C_3 = 10$

Example 10:

If $5 \times {}^nC_6 = 12 \times {}^nC_4$

Find the value of ${}^{n-3}P_3$

SOL :

$$\frac{{}^nC_6}{{}^nC_4} = \frac{12}{5} \rightarrow \frac{{}^nC_6}{{}^nC_5} \times \frac{{}^nC_5}{{}^nC_4} = \frac{12}{5}$$

$$\frac{n-5}{6} \times \frac{n-4}{5} = \frac{12}{5} \qquad (\times 30)$$

$(n-5)(n-4) = 72 = 8 \times 9$

$\therefore n-5 = 8 \dashrightarrow \qquad n=13$

${}^{n-3}P_3 = {}^{10}P_3 = 10 \times 9 \times 8 = 720$

Example 11 :

If ${}^{x}C_2 = 6$, ${}^{10}P_y = 720$

Find the value of $\lfloor\underline{y-x+5}$

SOL :

${}^{x}C_2 = 6 \longrightarrow \quad x = 4$

${}^{10}P_y = 720 \longrightarrow \quad y = 3$

$\lfloor\underline{y-x+5} = \lfloor\underline{3-4+5} = \lfloor\underline{4} = 24$

Example12:

If ${}^{6}P_r = 120$, ${}^{n+1}C_r : {}^{n+1}C_{r-1} = 2$

Find the value of $\lfloor\underline{n-r}$

Sol :

${}^{6}P_r = 120 \longrightarrow \quad r = 3$

$${}^{n+1}C_r : {}^{n+1}C_{r-1} = \frac{(n+1) - r + 1}{r} = 2$$

$\frac{n-1}{3} = 2 \longrightarrow \quad n = 7$

$\lfloor\underline{n-r} = \lfloor\underline{7-3} = \lfloor\underline{4} = 24$

Example13:

Prove that $\frac{{}^{n}C_r - {}^{n-1}C_{r-1}}{{}^{n}C_r} = \frac{n-r}{n}$

Hence find the value of $\frac{{}^{1000}C_{100} - {}^{999}C_{99}}{{}^{1000}C_{100}}$

SOL :

$$\text{L.H.S} = \frac{{}^{n}C_r}{{}^{n}C_r} - \frac{{}^{n-1}C_{r-1}}{{}^{n}C_r}$$

$$= 1 - \frac{\lfloor\underline{n-1}}{\lfloor\underline{r-1}\,\lfloor\underline{n-r}} \times \frac{\lfloor\underline{r}\,\lfloor\underline{n-r}}{\lfloor\underline{n}}$$

$$= 1 - \frac{r}{n} = \frac{n-r}{n} = \text{R.H.S.}$$

$$\frac{{}^{1000}C_{100} - {}^{999}C_{99}}{{}^{1000}C_{100}} = \frac{1000-100}{1000} = \frac{900}{1000} = 0.9$$

The Binomial Theorem With a (+ve) integer power *

$(a+b)^1=a+b$
$(a+b)^2=a^2+2ab+b^2$
$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

Theorem:

$$(a+b)^n = {}^nc_0a^n + {}^nc_1a^{a-1}b + {}^nc_2a^{n-2}b^2 + \ldots\ldots. + {}^nc_nb^n.$$

Results : 1) $(1+x)^n = 1+{}^nc_1x+{}^nc_2x^2 + \ldots\ldots\ldots + x^n.$

2) $(1-x)^n = 1-{}^nc_1x+{}^nc_2x^2 - \ldots\ldots\ldots + (-x)^n.$

The general term in the expansion of $(x+a)^n$.

$T_{r+1}={}^nc_r \times (2^{nd})^r \times (1^{st})^{n-r}.$, $r = 0, 1, 2, \ldots\ldots, n$

Example(1):

Find the expansion of $(a+b)^4$.

Sol :

$(a+b)^4 = a^4+{}^4c_1a^3b+{}^4c_2a^2b^2+{}^4c_3ab^3 + b^4$

$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Example(2) :

Find the expansion of $(1-3x)^5$

Sol :

$(1-3x)^5 = 1-{}^5c_1 3x+{}^5c_2(3x)^2-{}^5c_3(3x)^3+{}^5c_4(3x)^4-{}^5c_5(3x)^5$

$\therefore (1-3x)^5 = 1 - 15x + 90x^2 - 270x^3 + 405x^4 - 243x^5$

Example(3) :

Find T_4 in the expansion of

$$\left(\frac{2}{\sqrt{x}} - \frac{\sqrt{x}}{2}\right)^{11}$$

sol :

$T_{r+1} = {}^nc_r \times (2^{nd})^r \times (1^{st})^{n-r}$

$T_4 = {}^{11}c_3 \times (\frac{-\sqrt{x}}{2})^3 \times (\frac{2}{\sqrt{x}})^8$

$$= 165 \times \frac{-(\sqrt{X})^3}{(2)^3} \times \frac{(2)^8}{\sqrt{(x)^8}}$$

$$= \frac{-5280}{x^2\sqrt{x}}$$

Example(4) :

Find the coefficient of T_6 in the expansion of $(2x-3)^8$

Sol :

$T_{r+1} = {}^nc_r \times (2^{nd})^r \times (1^{st})^{n-r}$

Coeff of $T_6 = {}^8c_5 \times (-3)^5 \times (2\times 1)^3$

$= 56 \times (-243) \times 8 = -108864$

Example(5) :

Find the coeff. of the r^{th} term in the expansion

of $(x + \frac{1}{x})^{2n}$

Sol :

Coeff . of T_r

$= {}^{2n}c_{r-1} \times (\frac{1}{1})^{r-1} \times (1)^{2n+1-r}$

$$= {}^{2n}C_{r-1} = \frac{\underline{|2n}}{\underline{|r-1}\ \ \underline{|2n+1-r}}$$

Example(6):

Find the value of $(1+x)^7 - (1-x)^7$

Sol :

Exp. $=2(T_2 + T_4 + T_6 + T_8)$

$=2({}^7C_1\ x + {}^7C_3\ x^3 + {}^7C_5\ x^5 + {}^7C_7\ x^7)$

$=2(7x + 35x^3 + 21x^5 + x^7)$

$=14x + 70x^3 + 42x^5 + 2x^7$

Example(7) :

Without using calculator ,find the value of :
$(1.01)^5+(0.99)^5$, approximate your result to 3 decimal places

Sol :

$(1.01)^5+(0.99)^5 = (1+0.01)^5 +(1-0.01)^5$
$=2 (T_1 + T_3 + T_5)$
$=2 [1+ {}^5C_2 \times (0.01)^2 + {}^5C_4 \times (0.01)^4]$
$=2 [1+10 \times (0.01)^2 + 5 \times (0.01)^4]$
$=2+ 20 \times (0.01)^2 + 10 \times (0.01)^4$
$=2.002$

* The middle term in the expansion of$(x+a)^n$

(1) If n is odd:

The No of terms = n+1 (even)

$\therefore$ There are 2 middle terms of order $\frac{n+1}{2}$, $\frac{n+3}{2}$

(2) If n is even:

The No of terms =n+1 (odd)

$\therefore$ There is only one middle term of order $\frac{n}{2}+1$

Example (8):

Find the middle term in the expansion of

$(2x^2+\frac{1}{x})^{10}$

Sol :

$\because$ N = 10 the No of terms = 11
$\therefore$ There is only one middle term which is T_6

$T_6 = {}^{10}C_5 \times (x^{-1})^5 \times (2x^2)^5$
$= {}^{10}C_5 \times x^{-5} \times 2^5 \times x^{10}$
$= 32 \times {}^{10}C_5 \times x^5$

Example (9) :

Find the middle terms in the expansion of

$(\frac{x}{3} - \frac{2}{y})^7$

Sol :

$\because$ N=7 the No of terms =8

$\therefore$ There are 2 middle terms which are

T_4 , T_5

$T_4 = {}^7C_3 \times (\frac{-2}{y})^3 \times (\frac{x}{3})^4$

$= -\frac{8}{81} \times {}^7C_3 \times \frac{x^4}{y^3}$

$T_5 = {}^7C_4 \times (\frac{-2}{Y})^4 \times (\frac{X}{3})^3$

$= \frac{16}{27} \times {}^7C_3 \times \frac{X^3}{Y^4}$

Example (10) :

If a ,b are the2 middle terms in the expansion of

$(x-\frac{1}{x})^{15}$ according to the descending power of X Prove that $a+bx^2 = 0$

Sol :

$T_8 = a$, $T_9 = b$

L.H.S = $a+bx^2 = T_8 + X^2.T_9$

$= {}^{15}C_7 \times (\frac{-1}{X})^7 \times X^8 + X^2 \times {}^{15}C_8 \times (\frac{-1}{X})^8 \times X^7$

$= {}^{15}C_7 \times (-X) + {}^{15}C_7 \times X = 0$ =R.H.S

Example (11) :

Find the coeff. Of x^9 in the expansion of

$(x^3 - \frac{1}{x^4})^{10}$

Sol:

$T_{r+1} = {}^{10}C_r \times (\frac{-1}{x^4})^r \times (x^3)^{10-r}$

$= {}^{10}C_r \times (-1)^r \times x^{-4r} \times x^{30-3r}$

$= {}^{10}C_r \times (-1)^r \times x^{30-7r}$

$\because$ 30-7r =9 --→ r=3

$\therefore$ The term which contains x^9 is T_4

Coeff . of $T_4 = {}^{10}C_3 \times (-1)^3 = -120$

Example (12):

Find the term free of x in the expansion of

$(x - \frac{1}{2x^2})^9$

Sol:

$T_{r+1} = {}^9C_r \times (\frac{-1}{2x^2})^r \times (x)^{9-r}$

$= {}^9C_r \times (\frac{-1}{2})^r \times x^{-2r} \times x^{9-r}$

$= {}^9C_r \times (\frac{-1}{2})^r \times x^{9-3r}$

$\therefore$ 9-3r =0----→ r= 3

$\therefore$ the term free of x is T_4

$T_4 = {}^9C_3 \times (\frac{-1}{2})^3 = \frac{-21}{2}$

Example (13):

Prove that there is no term free of x in the expansion of $(2x^3 - \frac{3}{x})^9$

Sol:

$T_{r+1} = {}^9C_r \times (\frac{-3}{x})^r \times (2x^3)^{9-r}$

$= {}^9C_r \times (-3)^r \times x^{-r} \times (2)^{9-r} \times x^{27-3r}$

$= {}^9C_r \times (-3)^r \times (2)^{9-r} \times x^{27-4r}$

$\therefore 27-4r = 0 \longrightarrow$

$r = \frac{27}{4} \notin Z^+ \cup \{0\}$

$\therefore$ There is no term free of x in this expansion

Example (14):

Prove that in the expansion of $(x^2 + \frac{1}{x})^n$ there exists a term free of x if n. is a multiple of 3 ,and find this term when n. =12

Sol :

$T_{r+1} = {}^nC_r \times (x^{-1})^r \times (x^2)^{n-r}$

$= {}^nC_r \times x^{2n-3r}$

$\therefore 2n-3r = 0 \longrightarrow r = \frac{2n}{3} \in Z^+ \cup \{0\}$

if n is divisible by 3

(a multiple of 3)

<u>at n=12</u>

$\therefore \quad r = \frac{2 \times 12}{3} = 8$

$T_9 = {}^{12}C_8 = {}^{12}C_4 = 495$

*** The ratio between any term and its precedent in the expansion of$(x+a)^n$**

$$\frac{T_{r+1}}{T_r} = \frac{n - r + 1}{r} \times \frac{2^{nd}}{1^{st}}$$

Example (15) :

If the middle terms in the expansion of $(2x+3)^{17}$ are equal .find the value of X

Sol :

The middle terms are T_9 ,T_{10}

$$\because T_9 = T_{10} \quad \therefore \frac{T_{10}}{T_9} = 1$$

$$\therefore \frac{17 - 9 + 1}{9} \times \frac{3}{2x} = 1$$

$$\therefore 2x = 3$$

$$\therefore x = \frac{3}{2}$$

Example (16):

T_2 ,T_3 ,T_4 in the expansion of $(x+y)^n$ are respectively 240, 720,1080

Find the values of x,y and n.

Sol :

$$\frac{T_4}{T_3}=\frac{n-2}{3}\times\frac{y}{x}=\frac{1080}{720}=\frac{3}{2} \qquad (1)$$

$$\frac{T_3}{T_2}=\frac{n-1}{2}\times\frac{y}{x}=\frac{720}{240}=3 \qquad (2)$$

$(1)\div(2)$

$$\therefore \frac{n-2}{3}\times\frac{2}{n-1}=\frac{1}{2} \qquad (\times\frac{3}{2})$$

$$\frac{n-2}{n-1}=\frac{3}{4}$$

$\therefore 4n-8=3n-3 \qquad \therefore n=5 \ (3)$

from (3) in (2)

$$\therefore 1\times(\frac{y}{x})=\frac{3}{2}$$

$$\therefore \frac{y}{x}=\frac{3}{2} \qquad \therefore y=\frac{3x}{2} \qquad (4)$$

$\because T_2=240$

$\therefore {}^n c_1 \times y\times x^{n-1}=240$

$$\therefore {}^5 c_1\times\frac{3x}{2}\times x^4=240$$

$\therefore x^5=32=2^5$

$\therefore x=2 \qquad y=3$

Example 17:

The coefficients of 3 consecutive terms are respectively 20,190,1140 . Find the value of n, and the order of these terms in the exp. of $(1+x)^n$

Sol :

Let these terms are T_r , T_{r+1} , T_{r+2}

Coefficient of $T_{r+2} = {}^nc_{r+1} = 1140 \rightarrow$ (1)

Coefficient of $T_{r+1} = {}^nc_r = 190 \rightarrow$ (2)

Coefficient of $T_r = {}^nc_{r-1} = 20 \rightarrow$ (3)

Eq.(1) ÷ eq. (2)

$$\frac{{}^nC_{r+1}}{{}^nC_r}=\frac{n-r}{r+1}=6$$

n-r=6r+6

$n = 7r+6 \rightarrow$ (4)

eq.(2) ÷ eq.(3)

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} = \frac{19}{2}$$

$$\therefore \frac{7r+6-r+1}{r} = \frac{19}{2}$$

$$\frac{6r+7}{r} = \frac{19}{2}$$

$$\therefore 19r = 12r+14 \rightarrow \qquad r = 2 \qquad \therefore \textit{terms are } T_2, T_3, T_4$$

$$\therefore n = 7\times 2+6 \rightarrow \qquad n = 20$$

Example (18) :

In the expansion of $(1 + x)^n$ according to the powers of x in ascending order, If $T_4 = \frac{25}{3} T_2$ and if $T_5 = T_6$, Find the values of n , x

Sol :

$$\frac{T_4}{T_2} = \frac{25}{3}$$

$$\therefore \frac{T_4}{T_3} \times \frac{T_3}{T_2} = \frac{25}{3}$$

$$\therefore \frac{n-2}{3} \times \frac{n-1}{2} \times x^2 = \frac{25}{3} \qquad (\times 6) \qquad (\times 6)$$

$$(n-2)(n-1)x^2 = 50 \qquad (1)$$

$$\frac{T_6}{T_5} = 1$$

$$\frac{n-4}{5} \times x = 1 \qquad (\times 5)$$

$$(n-4)\times x = 5 \qquad \text{by squaring}$$

$(n-4)^2 \times x^2 = 25 \qquad (2)$

eq (1) ÷ eq (2)

$$\frac{n^2 - 3n + 2}{n^2 - 8n + 16} = \frac{2}{1}$$

$\therefore 2n^2 - 16n + 32 = n^2 - 3n + 2 \qquad \therefore n^2 - 13n + 30 = 0$

$(n-10)(n-3) = 0$

$\therefore n = 10 \qquad n = 3$

(refused because number of terms ≠ only 4

from (3) in (2)

$$\therefore x^2 = \frac{25}{36}$$

$$\therefore x = \frac{5}{6} \quad \text{or } x = \frac{-5}{6}$$

(refused because T_5, T_6 have the same)

Example 19 :

In the expansion of $(X^2 + \frac{1}{X})^9$ according to the powers of X In descending order ,

I) find the term free of X

II)If the ratio between the term free of x and the 6th term is 2:3 , find the value of X

Sol :

I)$T_{r+1} = {}^9C_r \times (X^{-1})^r \times (X^2)^{9-r}$

$= {}^9C_r \times X^{-r} \times X^{18-2r}$

$= {}^9C_r \times X^{18-3r}$

$\therefore 18-3r = 0$ ---→ r= 6

$\therefore$ The term free of X is $T_7 = {}^9C_6 = {}^9C_3$

= 84

ii) $\frac{T_7}{T_6} = \frac{2}{3}$

$\frac{9-5}{6} \times \frac{1}{x^3} = \frac{2}{3}$ $\quad (\times \frac{3}{2})$

$\therefore x^3 = 1 \quad \therefore x = 1$

Example 20 :

In the expansion of $(x^2 - \frac{1}{x^2})^{12}$ according to the power of x in descending order, find the term contains x^4, and find the Ratio between the coefficient of this term and the middle term

Sol :

$T_{r+1} = {}^{12}C_r \times (\frac{-1}{x^2})^r \times (x^2)^{12-r}$

$= {}^{12}C_r \times (-1)^r \times (x)^{-2r} \times (x)^{24-2r}$

$= {}^{12}C_r \times (-1)^r \times (x)^{24-4r}$

$\therefore 24-4r = 4 \rightarrow \quad r=5$

$\therefore$ The term contains x^4 is T_6

$T_6 = {}^{12}C_5\, x^4 \quad T_6 = -{}^{12}C_5\, X^4$

T_7 is the middle term

$\frac{Coff.T_6}{T_7} = \frac{6}{12-5} \times \frac{(1)}{(-1)} = \frac{-6}{7}$

Example 21 :

If the coefficient of x^{16} in the expansion of $(x^3 + \frac{a}{x})^8$ is equal to the term free of x in this expansion , find the value of a .

Sol :

In $(x^3 + \frac{a}{x})^8$

$T_{r+1} = {}^8C_r \times (\frac{a}{x})^r \times (x^3)^{8-r}$

$= {}^8C_r \times a^r \times x^{-r} \times x^{24-3r}$

$= {}^8C_r \times a^r \times x^{24-4r}$

$\therefore 24 - 4r = 16 \rightarrow \quad r=2$

$\therefore$ Coeff .of x^{16} = Coeff. of T_3

$= {}^8C_2 \times a^2 = 28 a^2$

To find the term free of x

$\therefore 24-4r = 0 \longrightarrow \quad r = 6$

T_7 is the term free of x

$T_7 = {}^8C_6 \times a^6 = {}^8C_2 \times a^6 = 28 a^6$

$\therefore 28a^6 = 28 a^2 \quad (\div 28)$

$a^2(a^4 - 1) = 0 \longrightarrow \quad \therefore a = \pm 1$

Example 22:

Find the value of the term free of x in the expansion of

$(9x^2 + \frac{1}{3x})^9$. Then prove that the middle terms are equal when $x = \frac{1}{3}$.

Sol :

$T_{r+1} = {}^9C_r \times (\frac{1}{3x})^r \times (9x^2)^{9-r}$

$= {}^9C_r \times (\frac{1}{3})^r \times x^{-r} \times 3^{18-2r} \times x^{18-2r}$

$= {}^9C_r \times 3^{18-3r} \times x^{18-3r}$

$\therefore 18-3r = 0 \longrightarrow \quad r = 6$

The term free of x is T_7

$T_7 = {}^9C_6 = {}^9C_3 = 84$

The 2 middle terms are T_5 , T_6

$$\frac{T_6}{T_5} = \frac{9-4}{5} \times \frac{(\frac{1}{3x})}{9x^2} = 1$$

$$\therefore \frac{1}{27x^3} = 1 \qquad \therefore x = \frac{1}{3}$$

Example 23 :

In the expansion of $(a\,x+\frac{1}{bx})^{10}$ according to the powers of x in descending order , if the term free of x is equal to the coefficient of the 7^{th} term , prove that 6 a b = 5

Sol :

$T_{r+1} = {}^{10}C_r \times (\frac{1}{bx})^r \times (a\,x)^{10-r}$

$= {}^{10}C_r \times (\frac{1}{b})^r \times x^{-r} \times (a)^{10-r} \times (x)^{10-r}$

$= {}^{10}C_r \times (\frac{1}{b})^r \times (a)^{10-r} \times (x)^{10-2r}$

$\therefore 10 - 2r = 0 \rightarrow r=5$

$\therefore$ The term free of x is T_6

$T_6 = {}^{10}C_5 \times (\frac{1}{b})^5 \times (a)^5$

$\therefore T_6 = {}^{10}C_5 \times \frac{a^5}{b^5}$

Coeff. of $T_7 = {}^{10}C_6 \times (\frac{1}{b})^6 \times (a)^4$

$= {}^{10}C_6 \times \frac{a^4}{b^6}$

$\therefore {}^{10}C_5 \times \frac{a^5}{b^5}$

$= {}^{10}C_6 \times \frac{a^4}{b^6} \qquad (\times \frac{b^6}{a^4})$

$\therefore {}^{10}C_5 \times ab = {}^{10}C_6 \qquad (\div {}^{10}C_5)$

$\therefore ab = \frac{10-6+1}{6} = \frac{5}{6}$

$\therefore 6ab = 5$

COMPLEX NUMBERS

* Introduction

If the S. S = R

Solve the eq $x^2 + 1 = 0$

Sol :

$x^2 = -1$

$x = \mp \sqrt{-1}$

If S. S. is R $\therefore$ the solution set is Φ

and if we consider $i = \sqrt{-1}$

$\therefore$ S. S. = {i, -i}

Where, $i = \sqrt{-1}$

$\therefore$ $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

In general:

$i^{4n} = 1, i^{4n+1} = i, i^{4n+2} = i^2 = -1, i^{4n+3} = i^3 = -i \quad, n \in z$

* The set of complex Numbers (C)

C = {x + yi, x, y Î R, $i^2 = -1$}

The complex number is symbolized by Z = x + yi

x is called the real part, y is called the imaginary part.

And if x = 0, y ≠ 0 ⇒ Z is a pure imaginary.

And if y = 0, x ≠ 0 ⇒ Z is a pure rule.

* Operations on complex numbers

1- The 2 complex numbers

$Z_1 = x_1 + y_1 i$, $Z_2 = x_2 + y_2 i$

Are equal iff $x_1 = x_2$, $y_1 = y_2$

2- If $Z_1 = x_1 + y_1 i$, $Z_2 = x_2 + y_2 i$,

Then $Z_1 + Z_2 = (x_1 + x_2) + (y_1 + y_2)i$

$Z_1 Z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y2 + x_2 y_1)$

* Properties of addition and subtraction of complex Numbers

1- Commutative: Verified in addition, multiplication.

2- Associative: verified in addition, multiplication.

3- Identity: a- In addition is zero.

b- In multiplication is one.

4- Inverse: a- In addition: -Z is the inverse of Z.

b- In multiplication: $\frac{1}{Z}$ is the inverse of Z.

ex. if $Z = x + yi$

$-Z = -x - yi$

$$\frac{1}{Z} = \frac{x}{x^2 + y^2} - \frac{yi}{x^2 + y^2}$$

5- Distribution of multiplication over addition

$Z_1 (Z_2 + Z_3) = Z_1 Z_2 + Z_1 Z_3$

* The conjugate of a complex number

If $Z = x + yi$, then $\bar{Z} = x - yi$ is the conjugate of the complex number Z.

Properties of the conjugate

1) $Z + \overline{Z} = 2x$ (pure real)

2) $Z - \overline{Z} = 2yi$ (pure imaginary).

3) $Z . \overline{Z} = x^2 + y^2$

4) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

5) $\overline{Z_1 Z_2} = \overline{Z_1} . \overline{Z_2}$

6) If $Z = x + yi$ is a root of an eq., its coeff. $\in$ R then

$\overline{Z} = x - yi$ is another root for the same eq.

Example 1

Write in the simplest form

i^{83}, i^{-62}, i^{-15}, i^{12n+7}

Sol :

$i^{83} = i^3 = -i$, $i^{-62} = i^2 = -1$

$i^{-15} = i$, $i^{12n+7} = i^3 = -i$

Example 2

Find the value of $(3 - 2\sqrt{-2})(5 + \sqrt{-2})$

Sol:

Exp = $(3 - 2\sqrt{2}\,i)(5 + \sqrt{2}\,i)$ = (15 + 4)

$i\,(3\sqrt{2} - 10\sqrt{2})$

= 19 - $7\sqrt{2}\,i$

Example 3

Put in the form of x + yi the number $\frac{(3+i)(3-i)}{3-4i}$ by 2 methods

Sol :

$$\frac{10}{3-4i} \times \frac{3+4i}{3+4i} = \frac{(3+i)(3-i)}{3-4i} = \frac{10(3+4i)}{25} = \frac{6}{5}+\frac{8}{5}i$$

Example 4

If $x = \frac{26}{5-i}$, $y = \frac{2(3+2i)}{1+i}$.

Prove that x, y are conjugate, also find the value of $x^2 + xy + y^2$

Sol :

$$x = \frac{26}{5-i} \times \frac{5+i}{5+i} = \frac{26(5+i)}{26} = 5 + i \qquad (1)$$

$$y = \frac{2(3+2i)}{1+i} \times \frac{1-i}{1-i} = \frac{2(3+2i)(1-i)}{2}$$

$$\therefore\ y = \frac{2(5-i)}{2} = 5 - i \qquad (2)$$

From (1), (2) $\therefore$ x, y are conjugate.

$$x^2 + xy + y^2 = (5 + i)^2 + (5 + i)(5 - i) + (5 - i)^2$$

$$= (24 + 10i) + 26 + (24 - 10i) = 74$$

Example 5

Find the values of $x, y \in R$ if

x + yi = $\left(1 - i\sqrt{2}\right)^5$

Sol :

x + yi = $\left(1 - i\sqrt{2}\right)\left[\left(1 - i\sqrt{2}\right)^2\right]^2$

= $\left(1 - i\sqrt{2}\right)\left[1 - 2 - 2\sqrt{2}\,i\right]^2$

= $\left(1 - i\sqrt{2}\right)\left(-1 - 2\sqrt{2}\,i\right)^2$

= $\left(1 - i\sqrt{2}\right)\left(1 + 4\sqrt{2}\,i - 8\right)$

= $\left(1 - i\sqrt{2}\right)\left(-7 + 4\sqrt{2}\,i\right)$

= $-7 + 4\sqrt{2}\,i + 7\sqrt{2}\,i + 8$

= $1 + 11\sqrt{2}\,i$ $\Rightarrow$ $x = 1,\ y = 11\sqrt{2}$

Example 6

If (a + bi) (1 - i) = 2 + i

Prove that $2(a^3 + b^3) = 7$

Sol :

a + bi = $\frac{2+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+3i}{2} = \frac{1}{2} + \frac{3}{2}i$

$\therefore\ a = \frac{1}{2}$, $b = \frac{3}{2}$

L.H.S = $2(a^3 + b^3) = 2\left(\frac{1}{8} + \frac{27}{8}\right) = \frac{2 \times 28}{8} = 7$ = R.H.S.

Example 7

Find the S. S. of

$2x - {}^2x + 3 = 0$ if $x \in C$

Sol :

$$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} \qquad a = 2,\ b = -1,\ c = 3$$

$$x = \frac{1 \mp \sqrt{1 - (4 \times 2 \times 3)}}{4}$$

$$\therefore\ x = \frac{1 \mp \sqrt{-23}}{4} = \frac{1 \mp \sqrt{23}\,i}{4}$$

$$S.\ S. = \left\{ \frac{1 + \sqrt{23}\,i}{4}, \frac{1 - \sqrt{23}\,i}{4} \right\}$$

Example 8

If $Z \in C$, find the S. S. of

$4Z + 7\overline{Z} = 8$

Sol :

Let $Z = x + iy$

$\therefore\ 4(x + iy) + 7(x - iy) = 8$

$4x + 4yi + 7x - 7yi = 8$

$\therefore\ 11x = 8 \Rightarrow x = \frac{8}{11}$, $\quad y = 0 \Rightarrow Z = \frac{8}{11}$

Example 9

If (-1) is a root of the equation

$x^3 - x^2 + 2 = 0$,

and Prove theis find the other 2 roots are conjugate

Sol :

∵ -1 is a root of the given equation

∴ $x + 1$ is a factor for the expression

$x^3 - x^2 + 2$

and to find the other factor, we use long division

$$
\begin{array}{r|l}
x + 1 & x^3 - x^2 + 2 \\
\hline
x^2 - 2x + 2 & x^3 + x^2 \\
 & \overline{- 2x^2 \quad + 2} \\
 & - 2x^2 - 2x \\
 & \overline{2x + 2} \\
 & \underline{2x + 2} \\
 & 0 \quad 0
\end{array}
$$

∴ $x^2 - 2x + 2 = 0$, by the use of formula

$$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \mp \sqrt{4 - (4 \times 1 \times 2)}}{2 \times 1}$$

$$x = \frac{2 \mp \sqrt{-4}}{2}$$

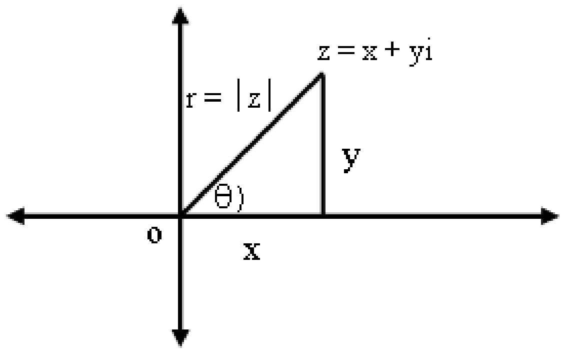
$$x = \frac{2 \mp 2i}{2} = 1 \mp i$$

The other 2 roots are:

$1 + i$, $1 - i$ which are conjugate

* Graphical representation of complex numbers

The complex number Z = x + yi is represented by a point in the plane where:



x - axis represents the real part,

y - axis represents the imaginary part.

$r = |Z| = \sqrt{x^2 + y^2}$ is called the modulus of the complex number.

$\theta = \tan^{-1} \frac{y}{x}$ is called the amplitude (Argument) of the complex number.,

$0° \leq \theta < 360°$

$\cos \theta = \frac{x}{r}$, $\sin \theta = \frac{y}{r}$

$\therefore$ Z = r (cos θ+ i sin θ)

Is called the trigonometric form a complex number.

Example 10

Find the modulus and the principle amplitude of each of the following complex numbers and represent each of them on Argand Diagrams.

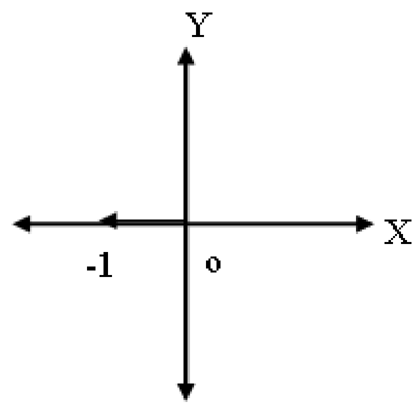
a) -1 b) $1 - \sqrt{3}i$ c) $-\sqrt{3}+i$

Sol :

a) Z = -1

$|Z| = 1$, $\theta = 180°$

$\therefore$ Z = cos 180° + i sin 180°



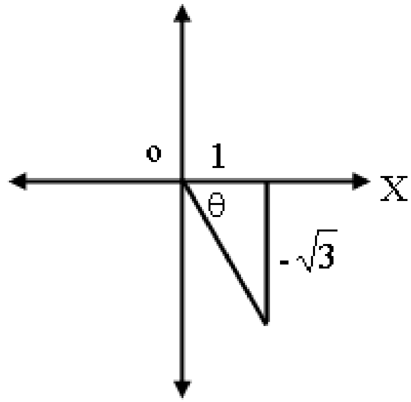
b) $Z = 1 - \sqrt{3}i$

$r = |Z| = \sqrt{1+3} = 2$

$\theta \in 4^{th}$ quad.

$\theta = 360° - 60° = 300°$

$\therefore$ Z = 2 (Cos 300° + i sin 300°)



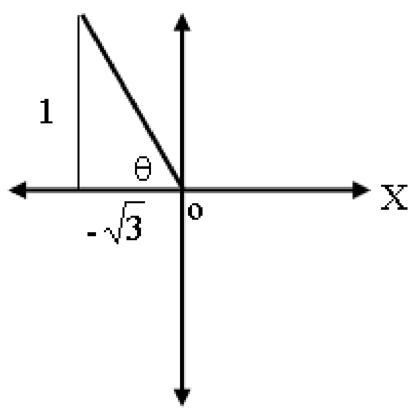
c) $Z = -\sqrt{3} + i$

$r = |Z| = \sqrt{1+3} = 2$

$\theta \in 2^{nd}$ quad.

$\theta = 180^{\circ} - 30^{\circ} = 150^{\circ}$

$\therefore\ Z = 2\ (\cos 150^{\circ} + i \sin 150^{\circ})$



Example 11

Write in the Alg form each of:

a) $Z = 7\ (\cos 60^{\circ} + i \sin 60^{\circ})$

b) $Z = 4\ [\cos (-150^{\circ}) + i \sin (-150^{\circ})]$

Sol :

a) $Z = 7 (\cos 60° + i \sin 60°)$

$= 7 (\frac{1}{2} + \frac{\sqrt{3}}{2} i)$

$= \frac{7}{2} + \frac{7\sqrt{3}}{2} i$

b) $Z = 4 [\cos (-150°) + i \sin (-150°)]$

$= 4 [\cos 210° + i \sin 210°]$

$= 4 [-\frac{\sqrt{3}}{2} - \frac{1}{2} i] = -2\sqrt{3} - 2i$

Example 12

If $Z = r (\cos \theta + i \sin \theta)$. Find each

of $-Z$, $\frac{1}{Z}$ in Trig. form.

Sol :

$-Z = r (-\cos \theta - i \sin \theta)$

$= r [\cos (180 + \theta) + i \sin (180 + \theta)]$

$\frac{1}{Z} = \frac{1}{r (\cos \theta + i \sin \theta)}$

$= \frac{1}{r} \times \frac{1}{\cos \theta + i \sin \theta} \times \frac{\cos \theta - i \sin \theta}{\cos \theta - i \sin \theta}$

$= \frac{1}{r} [\cos \theta - i \sin \theta]$

$= \frac{1}{r}[\mathrm{Cos}\,(360 - \theta) + i\,\mathrm{Sin}\,(360 - \theta)]$

Remember

1- Sin (A $\mp$ B) = Sin A Cos B $\mp$ Cos A Sin B

2- Cos (A $\mp$ B) = Cos A Cos B $\pm$ Sin A Sin B

3- Sin 2A = 2 Sin A Cos A.

4- Cos 2A = Cos^2 A - Sin^2 A

= 2 Cos^2 A - 1

= 1 - 2 Sin^2 A.

* The modulus and amplitude of product and quotient of 2 complex numbers

First: Multiplication

$|Z_1 Z_2| = |Z_1||Z_2|$

Amp (Z_1Z_2) = Amp (Z_1) + Amp (Z_2)

Second: Division

$$\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$$

Amp $\left(\frac{Z_1}{Z_2}\right)$ = Amp (Z_1) - Amp (Z_2)

Results: If Z = r (Cos θ + i Sin θ),

Then:i) $Z^n = r^n$ (Cos n θ + i Sin n θ)

ii) $\frac{1}{Z} = Z^{-1} = \frac{1}{r}$ [Cos (-θ) + i Sin (-θ)]

Example 13

Find the trig form for each of Z_1Z_2, $\frac{Z_1}{Z_2}$ where;

$Z_1 = 6\,[\text{Cos}\ \frac{3\pi}{8} + i\ \text{Sin}\ \frac{3\pi}{8}]$

$Z_2 = 2\,[\text{Cos}\ \frac{\pi}{8} + i\ \text{Sin}\ \frac{\pi}{8}]$

Sol :

$Z_1Z_2 = 12\,[\text{Cos}\ \frac{\pi}{2} + i\ \text{Sin}\ \frac{\pi}{2}]$

$\frac{Z_1}{Z_2} = 3\,[\text{Cos}\ \frac{\pi}{4} + i\ \text{Sin}\ \frac{\pi}{4}]$

Example 14

Find the trig form for each of Z_2 , $\frac{1}{Z}$

where Z = 4 (Sin a - i Cos a).

Sol :

$\because \quad x \in R^{+} \quad , \quad y \in R^{-}$

$\therefore$ Z Î 4^{th} quad.

$\therefore\ Z = 4\,[\text{Cos}\,(270^\circ + a) + i\ \text{Sin}\,(270^\circ + a)]$

$Z^2 = 16[\text{Cos}(180^\circ + 2a) + i\ \text{Sin}\,(180^\circ + 2a)]$

$\frac{1}{Z} = \frac{1}{4}\,[\text{Cos}\,(90^\circ - a) + i\ \text{Sin}\,(90^\circ - a)]$

Example 15

Write down the trig form for each

$Z_1 = \frac{1}{2}(-1 + \sqrt{3}\,i)$, $Z_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2} i$, also

prove that $Z_1^3 = Z_2^3 = 1$

Sol :

$Z_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$

$x_1 = \frac{-1}{2}$, $y_1 = \frac{\sqrt{3}}{2}$

$|Z_1| = r_1 = \sqrt{x_1^2 + y_1^2} = 1$

$q_1 \in 2^{nd}$ quad $q_1 = 120^o$

$Z_1 = \text{Cos } 120^o + i \text{ Sin } 120^o$

$Z^3{}_1 = \text{Cos } 0° + i \text{ Sin } 0° = 1 \rightarrow (1)$

$Z_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2} i$ $x_2 = \frac{-1}{2}$,

$y2 = -\frac{\sqrt{3}}{2}$

$|Z_2| r_2 = \sqrt{x_2^2 + y_2^2} = 1$ $\text{Cos } \theta_2 = \frac{-1}{2}$

, $\text{Sin } \theta_2 = -\frac{\sqrt{3}}{2}$

$\theta_2 \in 3^{rd}$ quad $\Rightarrow \theta_2 = 240°$

$\therefore Z_2 = \text{Cos } 240° + i \text{ Sin } 240°$

Z_2^3 = Cos 0° + i Sin 0° = 1→ (2)

From (1), (2)

∴ $Z_1^3 = Z_2^3 = 1$

Example 16

If Z_1 = 13 (Cos θ + i Sin θ), Z_2 = Sin 2θ + i Cos 2θ)

Where $\theta \in \left]0, \frac{\pi}{2}\right[$ $\tan\theta = \frac{5}{12}$

Find Z_1Z_2

Sol :

Z_1= 13 (Cos θ + i sin θ)

Z_2= Cos(90° - 2 θ) + i Sin (90° - 2 θ)

Z_1Z_2= 13[Cos(90° - θ)+ i Sin (90° - θ)]

= 13 [Sin θ + iCos θ]

$= 13\left[\frac{5}{13} + \frac{12}{13}i\right]$

= 5 + 12i

* De Moiver's theorem

Example 17

Find the cubic roots of the complex number

Z = 8 (Cos π + i Sin π)

Sol :

$Z \quad = 8 (\cos \pi + i \sin \pi)$

$Z^{1/3} \quad = 8^{1/3} [\cos \pi + i \sin \pi]^{1/3}$

$= 2 \left[\cos \frac{\pi + 2 m \pi}{3} + i \sin \frac{\pi + 2 m \pi}{3}\right]$

m = 0, 1, 2

When: m = 0

$Z^{1/3} = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) = 1 + \sqrt{3}\, i$

When: m = 1

$Z^{1/3} = 2 (\cos \pi + i \sin \pi) = -2$

When: m = 2

$Z^{1/3} = 2 \left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right) = 1 - \sqrt{3}\, i$

Example 18

Use De Moiver's theorem to find the square roots of the complex number

$Z = 1 + \sqrt{3}\, i$

Sol :

x = 1 **y =** $\sqrt{3}$

$\therefore \quad r = 2$

$\cos \theta = \frac{x}{r} = \frac{1}{2}$

Sin θ = $\frac{y}{r} = \frac{\sqrt{3}}{2}$

∴ θ = 60°

∴ Z = 2 (Cos 60° + i Sin 60°)

$Z^{\frac{1}{2}} = \sqrt{2}\left(\text{Cos}\,\frac{60^o + 2\,m\,\pi}{2} + i\,\text{Sin}\,\frac{60^o + 2\,m\,\pi}{2}\right)$ m = 0, 1

at m = 0

$\therefore Z^{\frac{1}{2}} = \sqrt{2}\left(\text{Cos}\,30^o + i\,\text{Sin}\,30^o\right) = \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}\,i$ at m=1

$z^{\frac{1}{2}} = \sqrt{2}\left(\cos 210^o + i \sin 210^o\right) = \frac{-\sqrt{6}}{2} - \frac{\sqrt{2}}{2}\,i$

Example 19

Find the square roots of the complex number Z = 3 + 4i. Without transforming it into trig form

Sol :

Let $Z^2 = 3 + 4i$

∴ $(x + yi)^2 = 3 + 4i$

∴ $(x^2 - y^2) + 2\,xyi = 3 + 4i$

∴ $x^2 - y^2 = 3$ -----⇒ (1)

2xy = 4

∴ $y = \frac{2}{x}$ ----⇒ (2)

From (2) in (1)

∴ $x^2 - \frac{4}{x^2} = 3$ $(x\ x^2)$

$\therefore \quad x^4 - 3x^2 - 4 = 0$

$(x^2 - 4)(x^2 + 1) = 0$

$x^2 = 4 \Rightarrow x = \mp 2$

or $x^2 = -1$ refused

When $x = 2 \Rightarrow y = 1$

When $x = -2 \Rightarrow y = -1$

$\therefore \quad Z = 2 + i$ or $Z = -2 - i$

$\therefore$ The square roots are 2+i , -2-I

Example 20

If a = 8- 6i

Find $a^{\frac{3}{2}}$

Sol :

Let $(x + yi)^2 = 8 - 6i$

$\therefore \quad x^2 - y^2 = 8$ ....(1)

$2xy = -6$

$\therefore \quad y = -\frac{3}{x}$ ....(2)

From (2) in (1)

$\therefore \quad x^2 - \frac{9}{x^2} = 8$ (´ x^2)

$x^4 - 8x^2 - 9 = 0$

$(x^2 - 9)(x^2 + 1) = 0$

$\therefore \quad x^2 - 9 = 0 \Rightarrow \quad x^2 = 9 \Rightarrow x = \pm 3$

$x^2 + 1 = 0 \quad \dashrightarrow \quad x^2 = -1$ refused

when:

$x = 3 \dashrightarrow \; y = -1 \dashrightarrow \; a^{\frac{1}{2}} = 3 - i$

Or when :

$x = -3 \dashrightarrow \; y = 1 \dashrightarrow \; a^{\frac{1}{2}} = -3 + i$

$\therefore \quad a^{\frac{3}{2}} = (3 - i)^3$

$= (8 - 6i)(3 - i)$

$= 24 - 6 - 18i - 8i$

$\therefore a^{\frac{3}{2}} = 18 - 26i$

or $\quad a^{\frac{3}{2}} = (-3 + i)^3 = -(3 - i)^3$

$\therefore \quad a^{\frac{3}{2}} = -18 + 26I$

Example 21

Use De Moiver's Theorem in finding cos 2θ, sin 2θ in terms of cos θ , sin θ

Sol :

$(\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) +$

$2i\cos\theta\sin\theta$

$= (\cos 2\theta) + (\sin 2\theta)\, i$

$\therefore \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2 \sin\theta \cos\theta$

Example 22

Find the s.s. of

$(1+i)\, x^2 - (1+3i)\, x + 2\,(2-3i) = 0$

Sol :

$$x^2 - \left(\frac{1+3i}{1+i}\right) x + \frac{2\,(2-3i)}{1+i} = 0$$

$$\therefore\ x^2 - \left(\frac{1+3i}{1+i} \times \frac{1-i}{1-i}\right) x + \frac{2\,(2-3i)\ (1-i)}{(1+i)\ (1-i)} =$$

$$\therefore x^2 - \left(\frac{4+2i}{2}\right) x + (-1-5i) = 0$$

$$\therefore\ x^2 - (2+i)\, x + (-1-5i) = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{(2+i) \pm \sqrt{3+4i - 4(1)\,(-1-5i)}}{2}$$

$$x = \frac{2+i \pm \sqrt{7+24\,i}}{2}$$

To find $\sqrt{7 + 24i}$

Let $(a + bi)^2 = 7 + 24i$

$(a^2 - b^2) + 2a\,bi = 7 + 24i$

$a^2 - b^2 = 7$ .... (1)

$2\,ab = 24 \rightarrow b = \frac{12}{a}$ .... (2)

From (2) in (1)

$a^2 - \frac{144}{a^2} = 7$ $(\times a^2)$

$a^4 - 7a^2 - 144 = 0$

$(a^2 - 16)\ (a^2 + 9) = 0$

$a^2 - 16 = 0 \rightarrow a^2 = 16 \rightarrow$

$a = \pm 4 \rightarrow b = \pm 3$

$a^2 + 9 = 0 \rightarrow a^2 = -9$ (refused)

$\therefore \sqrt{7 + 24i} = 4 + 3i$

$\therefore x = \frac{2 + i \pm (4 + 3i)}{2}$

$\therefore x = 3 + 2i$ or $x = -1 - i$

* The exponential form of a complex number

If $Z \in C$, its modulus = r, and its amplitude is θ

- $Z = re^{\theta i}$ is called the exponential form of a complex number.

Operations on complex numbers in exp. form

(1) $r_1 e^{\theta_1 i} \times r_2 e^{\theta_2 i} = r_1 r_2\, e^{(\theta_1 + \theta_2) i}$

(2) $\dfrac{r_1\, e^{\theta_1 i}}{r_2\, e^{\theta_2 i}} = \dfrac{r_1}{r_2}\; e^{(\theta_1 - \theta_2(i}$

(3) $(r\, e^{\theta i})^n = r^n\; e^{n\theta i}$

(4) $\sqrt[n]{r\, e^{\theta_i}} = \sqrt[n]{r}\;,\; e^{\frac{\theta + 2m\pi}{n} i}$

$m = 0, 1, 2, \ldots\ldots., \quad m-1$

Example 23

Write down in the exp. form $\left(\dfrac{2\,(1+i)}{(1-i)}\right)^7$

Sol :

$$\left(\frac{2\,(1+i)}{(1-i)}\right)^7 = \left(\frac{2\,(1+i)\,(1+i)}{(1-i)\,(1+i)}\right)^7$$

$$= \left(\frac{2\,(1+i)^2}{2}\right)^7$$

$$= [(1+i)^2]^7 = (1+2i-1)^7 = (2i)^7$$

$$= (2 \times e^{\frac{\pi}{2} i})^7 = 128\; e^{\frac{3\pi}{2} i}$$

Example 24

If $z = 2\,(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$

Express each of z, $\frac{1}{z}$ in the exp. Form

Sol :

$r = 2 \qquad \theta = \frac{\pi}{3}$

$z = r\,e^{\theta i}$

$z = 2 \times e^{\frac{\pi}{3}i}$

$\frac{1}{z} = \frac{1}{2} \times e^{-\frac{\pi}{3}i}$

$\therefore \quad \frac{1}{z} = \frac{1}{2} \times e^{\frac{5\pi}{3}i}$

Example 25

If z = 1+i , find z^6 in the exp. form.

Sol :

$z = 1 + i$

$r = \sqrt{2} \qquad \theta = 45° = \frac{\pi}{4}$

$z = (\sqrt{2}) \times e^{\frac{\pi}{4}i}$

$z^6 = (\sqrt{2})^6 \times e^{\frac{6\pi}{4}i} = 8 \times e^{\frac{3\pi}{2}i}$

Example 26

If $z_1 = 2 \times e^{\frac{\pi}{6}i}$,

$z_2 = \sqrt{2} \times e^{\frac{\pi}{4}i}$

Find each of z_1z_2 , $\frac{z_1}{z_2}$, z_1^4 , $\sqrt[3]{z_1}$, $\sqrt[4]{z_2}$

Sol :

$$z_1z_2 = 2\sqrt{2} \times e^{\frac{5\pi}{12}i}$$

$$\frac{z_1}{z_2} = \sqrt{2} \times e^{\frac{23\pi}{12}i}$$

$$z_1^4 = 16 \times e^{\frac{2\pi}{3}i}$$

$$\sqrt[3]{z_1} = \sqrt[3]{2} \times e^{\frac{\frac{\pi}{6}+2m\pi}{3}i} \quad , \qquad m = 0, 1, 2$$

$$\sqrt[4]{z_2} = \sqrt[8]{2} \times e^{\frac{\frac{\pi}{4}+2m\pi}{4}i} , \ m = 0, 1, 2,3$$

Example 27

Put $\left(\frac{4}{\sqrt{3}+i}\right)^2$ on exp. form

Sol :

$$z = \left(\frac{4}{\sqrt{3}+i}\right)^2 = \left(\frac{4}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}\right)^2$$

$$\therefore \quad z = \left(\frac{4(\sqrt{3}-i)}{4}\right)^2 = (\sqrt{3} - i)^2$$

$$\therefore \quad = 2 - 2\sqrt{3}\ i$$

$r = 4$ $\qquad\qquad$ $\theta = 300° = \frac{5\pi}{3}$

$\therefore \quad z = 4 \times e^{\frac{5\pi}{3}i}$

Example 28

Put $1-\sqrt{3}\,i$ on trig. form, then find its roots on exp. form

Sol :

$z = 1 - \sqrt{3}\,i \qquad\qquad r = 2$

$\therefore \quad z = 2$ ´ $e^{\frac{5\pi}{3}i}$

$\theta = 300° = \frac{5\pi}{3}$

$\therefore \quad z = 2\,(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3})$

$z^{\frac{1}{2}} = \sqrt{2} \times e^{\frac{\frac{5\pi}{3} + 2m\pi}{2} i} \quad , \quad m = 0, 1$

when m = 0 : $\quad z^{\frac{1}{2}} = \sqrt{2} \times e^{\frac{5\pi}{6}i}$

when m = 1 : $\quad z^{\frac{1}{2}} = \sqrt{2} \times e^{\frac{11\pi}{6}i}$

Example 29

Put $\frac{2\,(5 - 3\sqrt{3}\,i)}{1 + 2\sqrt{3}\,i}$ in the exp. form, and find its square roots in exp. form.

Sol :

$$z = \frac{2(5-3\sqrt{3}\,i)}{1+2\sqrt{3}\,i} \times \frac{1-2\sqrt{3}\,i}{1-2\sqrt{3}\,i}$$

$$z = \frac{2(5-3\sqrt{3}\,i)(1-2\sqrt{3}\,i)}{13} = -2 - 2\sqrt{3}\,i$$

$x = -2$ $\qquad$ $y = -2\sqrt{3}$

$r = 4$ $\qquad$ $\theta = 240° = \frac{4\pi}{3}$

$z = 4\,(\cos 240° + i \sin 240°)$

$= 4 \times e^{\frac{4\pi}{3}i}$

$$z^{\frac{1}{2}} = 2\left(\cos\frac{240^{o} + 2m\pi}{2} + i\sin\frac{240^{o} + 2m\pi}{2}\right), \; m = 0, 1$$

at $m = 0$

$z^{\frac{1}{2}} = 2\,(\cos 120° + i \sin 120°)$

$= 2 \times e^{\frac{2\pi}{3}i}$

at $m = 1$

$z^{\frac{1}{2}} = 2\,(\cos 300° + i \sin 300°)$

$= 2 \times e^{\frac{5\pi}{3}i}$

Example 30

Put $\left(\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\,i\right)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ in the exp.form .

Sol :

$z_1 = \cos 30° - i \sin 30°$

$= \cos 330° + i \sin 330°$

$z_2 = \cos 45° + i \sin 45°$

$z_1 z_2 = \cos 15° + i \sin 15°$

$r = 1$

$\therefore \quad z_1 z_2 = e^{\frac{\pi}{12} i} \qquad \theta = \frac{\pi}{12}$

Example 31

Put the complex number $z = \frac{\sqrt{3}+i}{\sqrt{3}-i}$ on exp. form and find its cubic roots

Sol :

$$z = \frac{\sqrt{3}+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i} = \frac{2+2\sqrt{3}i}{4}$$

$$\therefore z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$x = \frac{1}{2} \qquad y = \frac{\sqrt{3}}{2} \qquad r = 1 \quad \theta = \frac{\pi}{3}$

$$\therefore \quad z = e^{\frac{\pi}{3} i}, \qquad z^{\frac{1}{3}} = e^{\frac{\frac{\pi}{3} + 2m\pi}{3} i},$$

$m = 0, 1, 2$

when $\quad m = 0 : \quad z^{\frac{1}{3}} = e^{\frac{\pi}{9}i}$

when $\quad m = 1 : \quad z^{\frac{1}{3}} = e^{\frac{7\pi}{9}i}$

when $\quad m = 2 : \quad z^{\frac{1}{3}} = e^{\frac{13\pi}{9}i}$

* The cubic roots of unity

If $\quad x^3 = 1, \quad$ find the values of x , $x \in C$

Sol :

$x = (1)^{\frac{1}{3}} = (\cos 0^\circ + i \sin 0^\circ)^{\frac{1}{3}}$

$x = \cos \frac{0+2m\pi}{3} + i \sin \frac{0+2m\pi}{3}$

m = 0, 1, 2

when m = 0:

$\therefore x = \cos 0^\circ + i \sin 0^\circ = 1$

when m = 1:

$x = \cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

when m = 2 : $\quad \therefore x = \cos 240^\circ +$

$i \sin 240^\circ = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$\therefore$ The cubic roots of unity are

$1 \ , \ -\frac{1}{2} + \frac{\sqrt{3}}{2} i \ , -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

which called $1, \omega, \omega^2$

* The properties of cubic roots of unity

1)One of them is real and the others are complex.

2)The 2 complex numbers are conjugate.

3)The modulus of each of the roots is one.

4)The square of any of the 2 complex numbers equals the other.

5)$1 + \omega + \omega^2 = 0$

6)$1 \times \omega \times \omega^2 = \omega^3 = 1$

7)$\omega^m = \omega^n$ where **<u>n</u>** is the rest of dividing m by 3.

8)$\omega - \omega^2 = \pm \sqrt{3}\, i$

Example (1)

Prove that

$(1 - \omega + \omega^2)\ (1 + \omega - \omega^2) = 4$

Sol :

L.H.S $= (-2\omega)\ (-2\omega^2) = 4\omega^3 = 4$

= R.H.S

Example (2)

Prove that

$(1 - \omega + \omega^2)\ (1 - \omega^2 + \omega^4)\ (1 - \omega^4 + \omega^8) \times$ ..... to 2n

terms equals 2^{2n}

Sol :

L.H.S= $(-2\omega)\ (-2\omega^2)\ (-2\omega)$ .... to 2n terms

$= (-2\omega)^n\ (-2\omega^2)^n$

$= (4\omega^3)^n = 4^n = (2^2)^n = 2^{2n}$ = R.H.S

Example (3)

If $x = \dfrac{-1+\sqrt{3}i}{2}$

Prove that $x^8 + x^4 + 1 = 0$

Sol :

$x = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\ i$ Let $x = \omega$

L.H.S $= x^8 + \omega^4 + 1 = \omega^8 + \omega^4 + 1$

$= \omega^2 + \omega + 1 = 0$ = R.H.S

Example (4)

If $x = a + b$

$y = a\omega + b\omega^2$

$z = a\omega^2 + b\omega$

Prove that 1st $xyz = a^3 + b^3$

2nd $x^2 + y^2 + z^2 = 6ab$

Sol :

1st : $xyz = (a+b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$= (a+b)(a^2 + ab\omega^2 + ab\omega + b^2)$

$= (a+b)(a^2 - ab + b^2) = a^3 + b^3$

2nd: $x^2 + y^2 + z^2 = (a+b)^2 +$

$(a\omega + b\omega^2)^2 + (a\omega^2 + b\omega)^2$

$= a^2 + 2ab + b^2 + a^2\omega^2 + 2ab + b^2\omega + a^2\omega + 2ab +$

$b^2\omega^2$

$= 6ab + a^2(1 + \omega^2 + \omega) + b^2(1 + \omega + \omega^2)$

$= 6ab + 0 + 0 = 6ab$

Example (6)

Prove that $\frac{1}{4 + 3\omega + 2\omega^2} + \frac{1}{5 + 3\omega + 4\omega^2} = 1$

Sol :

L.H.S= $\frac{1}{2\omega^2 + 2\omega + 2 + \omega + 2}$

$+ \frac{1}{3\omega + 3\omega^2 + 3 + \omega^2 + 2}$

$= \frac{1}{\omega + 2} + \frac{1}{\omega^2 + 2}$

$= \frac{\omega^2 + 2 + \omega + 2}{(\omega + 2)(\omega^2 + 2)} = \frac{3}{\omega^3 + 2\omega^2 + 2\omega + 4}$

$= \frac{3}{1 + 4 - 2} = \frac{3}{3} = 1$ = R.H.S

Example (7)

Find the value of $\left(\frac{5-3\omega^2}{5\omega-3}-\frac{2-7\omega}{2\omega^2-7}\right)^2$

Sol :

Exp $= \left(\frac{\omega^2\,(5\omega-3)}{(5\omega-3)}-\frac{\omega\,(2\omega^2-7)}{(2\omega^2-7)}\right)^2$

$= (\omega^2 - \omega)^2 = (\pm\sqrt{3}\ i)^2 = -3$

Example (8)

Find the eq. whose roots are

$(1+\omega-\omega^2)^3 \quad , \quad (1-\omega+\omega^2)^3$

Sol :

1^{st} root $=$ $(1+\omega-\omega^2)^3 = (-2\omega^2)^3 = -8$

2^{nd} root $=$ $(1-\omega+\omega^2)^3 = (-2\omega^2)^3 = -8$

$\therefore$ The eq. : x^2 -(sum of roots)x + product of roots = 0

$\therefore$ $x^2 + 16x + 64 = 0$

Example (9)

Prove that:

$(1+\frac{1}{\omega}+i)\ (1+\frac{1}{\omega^2}+i) = i$

Sol :

L.H.S.$= (1+\omega^2+i)\,(1+\omega+i)$

$= (-\omega+i)\,(-\omega^2+i)$

$= 1 - \omega i - \omega^2 i - 1$

$= -i(\omega + \omega^2) = i = R.H.S$

Example (10)

Prove that:

$$(2+7\omega + 2\omega^2)(2+7\omega^2 + 2\omega^4) = 25$$

Sol :

$L.H.S. = (2 + 7\omega + 2\omega^2)(2+7\omega^2 + 2\omega)$

$= (7\omega - 2\omega)(7\omega^2 - 2\omega^2)$

$= (5\omega)(5\omega^2) = 25\omega^3 = 25$

Example (11)

If $x = \omega + \frac{1}{\omega}$,

$y = -1 - \frac{1}{\omega^2}$

$z = \frac{7 + 5\omega}{5 + 7\omega^2}$, prove that $x^2 + y + z = 0$

Sol :

$x^2 + y + z = (\omega + \omega^2)^2 + (-1 - \omega)$

$+ \frac{\omega(7\omega^2 + 5)}{5 + 7\omega^2}$

$= 1 + \omega^2 + \omega = 0 = R.H.S$

Determinants

Definition *

A determinant of degree **n** (consists of **n** rows and n columns) arises from .eliminating **(n-1)** variables from a system of n linear equations

Example (1)

Write the determinant arises from each of the following
.systems of equations

a) $2x = -6$
$x + 3 = 0$

:Sol

$2x = -6$
$x = -3$

$$\therefore \Delta = \begin{vmatrix} 2 & -6 \\ 1 & -3 \end{vmatrix}$$

b) $x + y = 3$
$x - y = 1$
$2x + 3y = 7$

: Sol

$$\therefore \Delta = \begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 2 & 3 & 7 \end{vmatrix}$$

Remark

The value of a determinant of degree two is the product of the elements of .the main diagonal minus the product of the elements of the other diagonal

The co-factors of the elements of adeterminant *

Let a determinant of degree 3, the element a_{ij}
.(which lies in i^{th} row, j^{th} column)
If we cancel the i^{th} row and j^{th} column, and multiply the resulting determinant by $(-1)^{i+j}$, the resulting determinant is called the cofactor of a_{ij}
. and denoted by A_{ij}

Example (2)

:Find the value of each of the following

(a $\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 2 \\ 5 & 3 & 4 \end{vmatrix}$$ (b

:Sol

$\Delta = 2 \times 1 - 3 \times 5 = -13$ a)

By the use of the element of 1st row b)

$$\therefore \Delta = 1 \times \begin{vmatrix} 7 & 2 \\ 3 & 4 \end{vmatrix} - 2 \times \begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix} + 3 \times \begin{vmatrix} 4 & 7 \\ 5 & 3 \end{vmatrix}$$

(69-) + 12 - 22 =

Example (3)

:Solve the equation

$$\begin{vmatrix} x & 0 & 0 \\ 1 & x & x \\ 5 & 2 & x \end{vmatrix} = 3x$$

: Sol

By using of the element of the 1st row

$\therefore$ $x (x^2 - 2x) = 3x$

$\therefore$ $= x (x^2 - 2x - 3)$

$\therefore$ $x (x - 3) (x + 1) = 0$

$\therefore$ $x = 0$ & $x = 3$ & $x = -1$

Example (4)

Find the value of **k** which makes (x-1) one of the .factors of the following determinant

$$\begin{vmatrix} -3 & 1 & -1 \\ 2 & 1 & 5 \\ 2 & x+1 & x+1+k \end{vmatrix}$$

: Sol

x-1 is a factor for the determinant

.x = 1 is a root for the resulting eq

$$\begin{vmatrix} x-3 & 1 & -1 \\ 2 & 1 & 5 \\ 2 & x+1 & x+1+k \end{vmatrix} = 0$$

$$\begin{vmatrix} -2 & 1 & -1 \\ 2 & 1 & 5 \\ 2 & 2 & k+2 \end{vmatrix} = 0$$

by the use of elements of 1^{st} row

$0 = (4 - 2) - (2k + 4 - 10) - (k + 2 - 10)$ 2- ∴

$2 = 0 - (2k - 6) - (k - 8)$ 2- ∴

$2k + 16 - 2k + 6 - 2 = 0$- ∴

k = 5 $4k + 20 = 0$ Þ- ∴

The Binomial Theorem With a (+ve) integer * power

$(a+b)^1=a+b$

$(a+b)^2=a^2+2ab+b^2$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

Theorem: $(a+b)^n = {}^nc_0a^n + {}^nc_1a^{a-1}b + {}^nc_2a^{n-2}b^2 + \dots\dots + {}^nc_nb^n$.

Results : 1) $(1 + x)^n = 1 + {}^nc_1x + {}^nc_2x^2 + \dots\dots\dots + x^n$.

2) $(1 - x)^n = 1 - {}^nc_1x + {}^nc_2x^2 - \dots\dots\dots + (-x)^n$.

<u>The general term in the expansion of $(x+a)^n$.</u>

$T_{r+1} = {}^nc_r \times (2^{nd})^r \times (1^{st})^{n-r}$.

Example(1)

Find the expansion of $(a+b)^4$.

Sol :

$(a + b)^4 = a^4 + {}^4c_1a^3b + {}^4c_2a^2b^2 + {}^4c_3ab^3 + b^4$

$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Example(2)

Find the expansion of $(1-3x)^5$

Sol :

$(1-3x)^5 = 1 - {}^5c_1 3x + {}^5c_2 (3x)^2 - {}^5c_3 (3x)^3 + {}^5c_4 (3x)^4 - {}^5c_5 (3x)^5$

$\therefore (1-3x)^5 = 1 - 15x + 90x^2 - 270x^3 + 405x^4 - 243x^5$

Example(3)

Find T_4 in the expansion of

$$\left(\frac{2}{\sqrt{x}} - \frac{\sqrt{x}}{2}\right)^{11}$$

sol :

$T_{r-1} = {}^nc_r \times (2^{nd})^r \times (1^{st})^{n-r}$

$$T_4 = {}^{11}c_3 \times \left(\frac{-\sqrt{x}}{2}\right)^3 \times \left(\frac{2}{\sqrt{x}}\right)^8$$

$$= 165 \times \frac{-(\sqrt{X})^3}{(2)^3} \times \frac{(2)^8}{\sqrt{(x)^8}}$$

$$= \frac{-5280}{x^2\sqrt{x}}$$

Example(4)

Find the coefficient of T_6 in the expansion of $(2x-3)^8$

Sol :

$T_{r+1} = {}^nc_r \times (2^{nd})^r \times (1^{st})^{n-r}$

Coeff of $T_6 = {}^8c_5 \times (-3)^5 \times (2\times1)^3$

$= 56 \times (-243) \times 8 = -108864$

Example(5)

Find the coeff. of the r^{th} term in the expansion of $\left(x + \frac{1}{x}\right)^{2n}$

Sol :

Coeff . of T_r

$$= {}^{2n}c_{r-1} \times \left(\frac{1}{1}\right)^{r-1} \times (1)^{2n+1-r}$$

$$= {}^{2n}C_{r-1} = \frac{\lfloor 2n}{\lfloor r-1 \quad \lfloor 2n+1-r}$$

Example(6)

Find the value of $(1+x)^{7} - (1-x)^{7}$

Sol :

Exp. $=2(T_2 + T_4 + T_6 + T_8)$

$=2({}^{7}C_1 \times x + {}^{7}C_3 \times x^3 + {}^{7}C_5 \times x^5 + {}^{7}C_7 \times x^7)$

$=2(7x + 35x^3 + 21x^5 + x^7)$

$=14x + 70x^3 + 42x^5 + 2x^7$

Example(7)

Without using calculator ,find the value of : $(1.01)^5 + (0.99)^5$, approximate your result to 3 decimal places

Sol :

$(1.01)^5 + (0.99)^5 = (1+ 0.01)^5 + (1- 0.01)^5$

$=2 (T_1 + T_3 + T_5)$

$=2 [1+ {}^{5}C_2 \times (0.01)^2 + {}^{5}C_4 \times (0.01)^4]$

$=2 [1+10 \times (0.01)^2 + 5 \times (0.01)^4]$

$=2+ 20 \times (0.01)^2 + 10 \times (0.01)^4$

$=2.002$

Properties of determinants *

In any determinant if the rows replace the columns and -1 the columns replace the rows in the same order the value of .the determinant is unchanged

The value of a determinant is not changed by evaluating it -2 .in terms of the elements of any of its rows (columns)

In any determinant if the positions of 2 rows (columns) are -3 interchanged, the value of the resulting determinant is equal .to the value of the original determinant multiplied by (-1)

If the corresponding elements in 2 rows (columns) of any -4 .determinant are equal, the value of the determinant is zero

If there is a common factor in all the elements of any row -5 (column) in a determinant, then this factor can be taken .outside the determinant

If all the elements of any row (column) in a determinant -6 .are zero, then the value of the determinant is zero

In a determinant if all the elements of any row (column) -7 are written as the sum of 2 elements, the value of the .determinant can be written as the sum of 2 determinants

In a determinant, if we add to all the elements of any row -8 (column) a multiple of the elements of another row (column), .the value of the determinant is unchanged

If we multiply the elements of any row (column) of a -9 determinant by the cofactors of the corresponding elements .in another row (column), the sum is zero

The value of the determinant-10

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & 0 & a_{33} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \quad \text{is} \quad a_{11}\, a_{22}\, a_{33}$$

The determinant of this form is called triangular and a_{11} , a_{22} , a_{33} are called .the elements of the principal diagonal

Example (5)

Without evaluating the determinant, prove that the value of each of the following determinants is zero, and mention the used property.

a) $\begin{vmatrix} 3 & 5 & -1 \\ 3 & 5 & -1 \end{vmatrix} = 0$ because $r_1 = r_2$

b) $\begin{vmatrix} 2 & 7 \\ 0 & 0 \end{vmatrix} = 0$ because the elements of r_2 are zeros

c) $\begin{vmatrix} 3 & 1 \\ 9 & 3 \end{vmatrix} = 0$ because after taking **3** common from r_2 then the 2 rows are equal

Example (6)

Without evaluating the determinant $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$ prove that its value $= (x+2a)(x-a)^2$

Sol:

$c_1 + (c_3 + c_2)$

$$\therefore \Delta = \begin{vmatrix} x+2a & a & a \\ x+2a & x & a \\ x+2a & a & x \end{vmatrix}$$

$$= (x+2a) \begin{vmatrix} 1 & a & a \\ 1 & x & a \\ 1 & a & x \end{vmatrix}$$

$r_2 - r_1 \quad , \quad r_3 - r_1$

$$\therefore \Delta = (x+2\alpha) \begin{vmatrix} 1 & a & a \\ 0 & x-a & 0 \\ 0 & 0 & x-a \end{vmatrix}$$

$$\therefore \Delta = (x+2a)\,(x-a)^2$$

Example (7)

:By the use of the properties of determinants, prove that

$$0 = \begin{vmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

: Sol

$$\text{,Let} \quad \begin{vmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{vmatrix} = x$$

When replacing columns by rows and rows by columns by the same order, we get

$$x = \begin{vmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{vmatrix} (-1)^3$$

$\therefore x = -x$

$\therefore x = 0$

$\therefore$ The value of the determinant is zero.

* The middle term in the expansion of $(x+a)^n$

(1) If n is odd:

The No of terms = n+1 (even)

$\therefore$ There are 2 middle terms of order $\frac{n+1}{2}, \frac{n+3}{2}$

(2) If n is even:

The No of terms =n+1 (odd)

$\therefore$ There is only one middle term of order $\frac{n}{2}+1$

Example (8)

Find the middle term in the expansion of $(2x^2 + \frac{1}{x})^{10}$

Sol :

$\because N = 10$ the No of terms = 11

$\therefore$ There is only one middle term which is T_6

$T_6 = {}^{10}C_5 \times (x^{-1})^5 \times (2x^2)^5$

$= {}^{10}C_5 \times x^{-5} \times 2^5 \times x^{10}$

$= 32 \times {}^{10}C_5 \times x^5$

Example (9)

Find the middle terms in the expansion of $\left(\frac{x}{3} - \frac{2}{y}\right)^7$

Sol :

 N=7 the No of terms =8

$\therefore$ There are 2 middle terms which are T_4 , T_5

$T_4 = {}^7C_3 \times \left(\frac{-2}{y}\right)^3 \times \left(\frac{x}{3}\right)^4$

$= -\frac{8}{81} \times {}^7C_3 \times \frac{x^4}{y^3}$

$T_5 = {}^7C_4 \times \left(\frac{-2}{Y}\right)^4 \times \left(\frac{X}{3}\right)^3$

$= \frac{16}{27} \times {}^7C_3 \times \frac{X^3}{Y^4}$

Example (10)

If a ,b are the2 middle terms in the expansion of $(x-\frac{1}{x})^{15}$ according to the descending power of X

Prove that $a+bx^2 = 0$

Sol :

$T_8 = a$, $T_9 = b$

L.H.S $= a+bx^2 = T_8 + X^2.T_9$

$= {}^{15}C_7 \times \left(\frac{-1}{X}\right)^7 \times X^8 + X^2 \times {}^{15}C_8 \times \left(\frac{-1}{X}\right)^8 \times X^7$

$= {}^{15}C_7 \times (-X) + {}^{15}C_7 \times X = 0 =$ R.H.S

Example (11)

Find the coeff. Of x^9 in the expansion of $\left(x^3 - \frac{1}{x^4}\right)^{10}$

Sol:

$T_{r+1} = {}^{10}C_r \times \left(\frac{-1}{x^4}\right)^r \times (x^3)^{10-r}$

$= {}^{10}C_r \times (-1)^r \times x^{-4r} \times x^{30-3r}$

$= {}^{10}C_r \times (-1)^r \times x^{30-7r}$

$\therefore$ 30-7r =9 ---→ r=3

$\therefore$ The term which contains x^9 is T_4

Coeff . of $T_4 = {}^{10}C_3 \times (-1)^3 = -120$

Example (12)

Find the term free of x in the expansion of $\left(x - \frac{1}{2x^2}\right)^9$

Sol:

$T_{r+1} = {}^{9}C_r \times \left(\frac{-1}{2x^2}\right)^r \times (x)^{9-r}$

$= {}^{9}C_r \times \left(\frac{-1}{2}\right)^r \times x^{-2r} \times x^{9-r}$

$= {}^{9}C_r \times \left(\frac{-1}{2}\right)^r \times x^{9-3r}$

$\therefore$ 9-3r =0----→ r= 3

$\therefore$ the term free of x is T_4

$T_4 = {}^{9}C_3 \times \left(\frac{-1}{2}\right)^3 = \frac{-21}{2}$

Example (13)

Prove that there is no term free of x in the expansion of $\left(2x^3 - \frac{3}{x}\right)^9$

Sol:

$T_{r+1} = {}^{9}C_r \times \left(\frac{-3}{x}\right)^r \times (2x^3)^{9-r}$

$= {}^{9}C_r \times (-3)^r \times x^{-r} \times (2)^{9-r} \times x^{27-3r}$

$= {}^{9}C_r \times (-3)^r \times (2)^{9-r} \times x^{27-4r}$

$\therefore$ 27-4r =0 ----→

$r = \frac{27}{4} \notin \mathbb{Z}^+ \cup \{0\}$

$\therefore$ There is no term free of x in this expansion

Example (14)

Prove that in the expansion of

$(x^2 + \frac{1}{x})^n$ there exists a term free of x if n. is a multiple of 3 ,and find this term when n. =12

Sol :

$T_{r+1} = {}^nC_r \times (x^{-1})^r \times (x^2)^{n-r}$

$= {}^nC_r \times x^{2n-3r}$

$\therefore$ 2n-3r =0 ---> $r = \frac{2n}{3} \in Z^+ \cup \{0\}$

if n is divisible by 3

(a multiple of 3)

<u>at n=12</u>

$$\therefore \quad r = \frac{2 \times 12}{3} = 8$$

$T_9 = {}^{12}C_8 = {}^{12}C_4 = 495$

* The ratio between any term and its precedent in the expansion of(x+a)^n

$$\frac{T_{r+1}}{T_r} = \frac{n - r + 1}{r} \times \frac{2^{nd}}{1^{st}}$$

Example (15)

If the middle terms in the expansion of $(2x+3)^{17}$ are equal .find the value of X

Sol :

The middle terms are T_9 ,T_{10}

$T_9 = T_{10}$ $\quad \therefore \frac{T_{10}}{T_9} = 1$

$$\therefore \frac{17 - 9 + 1}{9} \times \frac{3}{2x} = 1$$

$$\therefore 2x = 3$$

$$\therefore x = \frac{3}{2}$$

Example (16)

T_2 ,T_3 ,T_4 in the expansion of $(x+y)^n$ are respectively 240, 720,1080

Find the values of x,y and n.

Sol :

$$\frac{T_4}{T_3} = \frac{n-2}{3} \times \frac{y}{x} = \frac{1080}{720} = \frac{3}{2} \qquad (1)$$

$$\frac{T_3}{T_2} = \frac{n-1}{2} \times \frac{y}{x} = \frac{720}{240} = 3 \qquad (2)$$

$(1) \div (2)$

$$\therefore \frac{n-2}{3} \times \frac{2}{n-1} = \frac{1}{2} \qquad (\times \frac{3}{2})$$

$$\frac{n-2}{n-1} = \frac{3}{4}$$

$$\therefore 4n - 8 = 3n - 3 \qquad \therefore n = 5 \ (3)$$

from (3) in (2)

$$\therefore 1 \times (\frac{y}{x}) = \frac{3}{2}$$

$$\therefore \frac{y}{x} = \frac{3}{2} \qquad \therefore y = \frac{3x}{2} \qquad (4)$$

$$\because T_2 = 240$$

$$\therefore {}^n c_1 \times y \times x^{n-1} = 240$$

$$\therefore {}^5 c_1 \times \frac{3x}{2} \times x^4 = 240$$

$$\therefore x^5 = 32 = 2^5$$

$$\therefore x = 2 \qquad y = 3$$

Example 17

The coefficients of 3 consecutive terms are respectively 20,190,1140 . Find the value of n, and the order of these terms in the exp. of $(1+x)^n$

Sol :

Let these terms are T_r , T_{r+1} , T_{r+2}

Coefficient of T_{r+2} = ${}^n c_{r+1}$ =1140 -→ (1)

Coefficient of T_{r+1} = ${}^n c_r$ = 190→ (2)

Coefficient of $T_r = {}^nc_{r-1} = 20$ -→ (3)

Eq.(1) ÷ eq. (2)

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1} = 6$$

n-r=6r+6

n = 7r+6 --→ (4)

eq.(2) ÷ eq.(3)

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} = \frac{19}{2}$$

$$\therefore \frac{7r+6-r+1}{r} = \frac{19}{2}$$

$$\frac{6r+7}{r} = \frac{19}{2}$$

$$\therefore 19r = 12r + 14 \rightarrow \qquad r = 2$$

$$\therefore n = 7 \times 2 + 6 \rightarrow \qquad n = 20$$

Example (18)

In the expansion of $(1 + x)^n$ according to the powers of x in ascending order, If $T_4 = \frac{25}{3} T_2$ and if $T_5=T_6$, Find the values of n , x

Sol :

$$\frac{T_4}{T_2} = \frac{25}{3}$$

$$\therefore \frac{T_4}{T_3} \times \frac{T_3}{T_2} = \frac{25}{3}$$

$$\therefore \frac{n-2}{3} \times \frac{n-1}{2} \times x^2 = \frac{25}{3} \qquad (\times 6) \qquad (\times\ 6\ )$$

$$(n-2)(n-1)x^2 = 50 \qquad (1)$$

$$\frac{T_6}{T_1} = 1$$

$$\frac{n-4}{5} \times x = 1 \qquad (\times 5)$$

$$(\text{n - 4}) \times \text{x} = 5 \qquad \text{by squaring}$$

$$(\text{n - 4})^2 \times x^2 = 25 \qquad (2)$$

$$\text{eq}\,(1)\ \div\ \text{eq}\,(2)$$

$$\frac{\text{n}^2 - 3n + 2}{n^2 - 8n + 16} = \frac{2}{1}$$

$$\therefore 2n^2 - 16n + 32 = n^2 - 3n + 2n^2 - 13n + 30 = 0$$

$$(n-10)(n-3) = 0$$

$$\therefore n = 10 \qquad \text{n} = 3$$

(refused because number of terms $\neq$ only 4

from (3) in (2)

$$\therefore\ \text{x}^2 = \frac{25}{36}$$

$$\therefore x = \frac{5}{6} \quad \text{or x} = \frac{-5}{6}$$

(refused because $T_{5,}$ T_6 of different signs)

Example 19

In the expansion of $(X^2 + \frac{1}{X})^9$
according to the powers of X In descending order ,
I) find the term free of X
II)If the ratio between the term free
of x and the 6th term is 2:3 , find the value of X

Sol :

I)$T_{r+1} = {}^9C_r \times (X^{-1})^r \times (X^2)^{9-r}$

$= {}^9C_r \times X^{-r} \times X^{18-2r}$

$= {}^9C_r \times X^{18-3r}$

$\therefore$ 18-3r = 0 ---→ r= 6

$\therefore$ The term free of X is $T_7 = {}^9C_6 = {}^9C_3$

= 84

ii) $\frac{T_7}{T_6} = \frac{2}{3}$

$\frac{9-5}{6} \times \frac{1}{x^3} = \frac{2}{3}$ $(\times \frac{3}{2})$

$\therefore x^3 = 1$ $\therefore x = 1$

Example 20

In the expansion of $(x^2 - \frac{1}{x^2})^{12}$ according to the power of x in descending order, find the term contains x^4 , and find the Ratio between the coefficient of this term and the middle term

Sol :

$T_{r+1} = {}^{12}C_r \times (\frac{-1}{x^2})^r \times (x^2)^{12-r}$

$= {}^{12}C_r \times (-1)^r \times (x)^{-2r} \times (x)^{24-2r}$

$= {}^{12}C_r \times (-1)^r \times (x)^{24-4r}$

$\therefore$ 24-4r= 4 --→ r=5

$\therefore$ The term contains x^4 is T_6

$T_6 = {}^{12}C_5 \, x^4$

T_7 is the middle term

$\frac{\text{Coff.} T_6}{T_7} = \frac{6}{12-5} \times \frac{(1)}{(-1)} = \frac{-6}{7}$

Example 21

If the coefficient of x^{16} in the expansion of $(x^3 + \frac{a}{x})^8$ is equal to the term free of x in this expansion , find the value of a .

Sol :

In $\left(x^3 + \frac{a}{x}\right)^8$

$T_{r+1} = {}^8C_r \times \left(\frac{a}{x}\right)^r \times (x^3)^{8-r}$

$= {}^8C_r \times a^r \times x^{-r} \times x^{24-3r}$

$= {}^8C_r \times a^r \times x^{24-4r}$

$\therefore$ 24 - 4r = 16 --→ r=2

$\therefore$ Coeff .of x^{16} = Coeff. of T_3

$= {}^8C_2 \times a^2 = 28\, a^2$

To find the term free of x

$\therefore$ 24-4r = 0 ---→ r =6

T_7 is the term free of x

$T_7 = {}^8C_6 \times a^6 = {}^8C_2 \times a^6 = 28\, a^6$

$\therefore$ $28a^6 = 28\, a^2$ $(\div 28)$

$a^2(a^4 - 1) = 0$ --→ $\therefore$ $a = \pm 1$

Example 22

Find the value of the term free of x in the expansion of

$\left(9x^2 + \frac{1}{3x}\right)^9$. Then prove that the middle terms are equal when $x = \frac{1}{3}$.

Sol :

$T_{r+1} = {}^9C_r \times \left(\frac{1}{3x}\right)^r \times (9x^2)^{9-r}$

$= {}^9C_r \times \left(\frac{1}{3}\right)^r \times x^{-r} \times 3^{18-2r} \times x^{18-2r}$

$= {}^9C_r \times 3^{18-3r} \times x^{18-3r}$

$\therefore$ 18-3r = 0 ---→ r = 6

The term free of x is T_7

$T_7 = {}^9C_6 = {}^9C_3 = 84$

The 2 middle terms are T_5 , T_6

$$\frac{T_6}{T_5} = \frac{9-4}{5} \times \frac{\left(\frac{1}{3x}\right)}{9x^2} = 1$$

$$\therefore \frac{1}{27x^3} = 1 \qquad \therefore x = \frac{1}{3}$$

Example 23

In the expansion of $(a x + \frac{1}{bx})^{10}$ according to the powers of x in descending order , if the term free of x is equal to the coefficient of the 7^{th} term , prove that 6 a b = 5

Sol :

$T_{r+1} = {}^{10}C_r \times (\frac{1}{bx})^r \times (a x)^{10-r}$

$= {}^{10}C_r \times (\frac{1}{b})^r \times x^{-r} \times (a)^{10-r} \times (x)^{10-r}$

$= {}^{10}C_r \times (\frac{1}{b})^r \times (a)^{10-r} \times (x)^{10-2r}$

$\therefore 10 - 2r = 0 \rightarrow r=5$

$\therefore$ The term free of x is T_6

$T_6 = {}^{10}C_5 \times (\frac{1}{b})^5 \times (a)^5$

$\therefore T_6 = {}^{10}C_5 \times \frac{a^5}{b^5}$

Coeff. of $T_7 = {}^{10}C_6 \times (\frac{1}{b})^6 \times (a)^4$

$= {}^{10}C_6 \times \frac{a^4}{b^4}$

$\therefore {}^{10}C_5 \times \frac{a^5}{b^5}$

$= {}^{10}C_6 \times \frac{a^4}{b^6} \quad (\times \frac{b^6}{a^4})$

$\therefore {}^{10}C_5 \times ab = {}^{10}C_6 \quad (\div {}^{10}C_5)$

$\therefore ab = \frac{10-6+1}{6} = \frac{5}{6}$

$\therefore 6ab = 5$