Course 311: Commutative Algebra and Algebraic Geometry Problems
Academic year 2007–8

1. (a) Show that the cubic curve $\{(t, t^2, t^3) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ is an algebraic set.

   (b) Show that the cone $\{(s \cos t, s \sin t, s) \in \mathbb{A}^3(\mathbb{R}) : s, t \in \mathbb{R}\}$ is an algebraic set.

   (c) Show that the unit sphere $\{(z, w) \in \mathbb{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\}$ in $\mathbb{A}^2(\mathbb{C})$ is not an algebraic set.

   (d) Show that the curve $\{(t \cos t, t \sin t, t) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ is not an algebraic set.

2. Let K be a field, and let $\mathbb{A}^n$ denote n-dimensional affine space over the field K.

   Let V and W be algebraic sets in $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively. Show that the Cartesian product $V \times W$ of V and W is an algebraic set in $\mathbb{A}^{m+n}$, where

$$\begin{aligned} V \times W &= \{(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) \in \mathbb{A}^{m+n} : \\ &\qquad (x_1, x_2, \ldots, x_m) \in V \text{ and } (y_1, y_2, \ldots, y_n) \in W\}. \end{aligned}$$

3. Give an example of a proper ideal I in $\mathbb{R}[X]$ with the property that $V[I] = \emptyset$. [Hint: consider quadratic polynomials in X.]

4. Show that the ideal I of $K[X, Y, Z]$ generated by the polynomials $X^2 + Y^2 + Z^2$ and $XY + YZ + ZX$ is not a radical ideal.

5. Prove that a topological space Z is irreducible if and only if every non-empty open set in Z is connected.

6. Let K be a field, and let $\mathbb{A}^n$ denote n-dimensional affine space over the field K.

   (a) Consider the algebraic set

$$\{(x, y, z) \in \mathbb{A}^3 : xy = yz = zx = 0\}.$$

   Is this set irreducible? Is it connected (with respect to the Zariski topology)?

(b) Consider the algebraic set

$$\{(x, y) \in \mathbb{A}^2(K) : (y - x)(y - x^2) = 0\},$$

where K is a field with at least 3 elements. Is this set irreducible? Is it connected (with respect to the Zariski topology)?