Mini-projects for the Broadening Course Assessment

Cornelia Druţu

November 12, 2018

1 Virtual solvability is not a feature of the large scale geometry, while virtual nilpotency is

A consequence of M. Gromov's theorem that a group is virtually nilpotent if and only if it has polynomial growth is that virtual nilpotency is invariant by quasi-isometries.

The corresponding question about virtual solvability has been answered in the negative by Anna Erschler, who provided examples establishing the failure of quasi-isometry invariance of the class of (virtually) solvable groups.

The project could overview the case of nilpotent groups and detail Erschler's examples, or focus on Gromov's theorem, either with the proof given in the book or with any other proof mentioned in the introduction.

Reference: Introduction, Section 14.5 and Chapter 16 of Druţu-Kapovich and references therein.

2 Geometry of the Lamplighter Group. Horocyclic products of trees.

The Cayley graph of the Lamplighter group with respect to a certain generating set can be identified with a level set of a function called the *Busemann function* in a product of two trees. This is a particular case of a construction that yields a particular type of graphs called the Diestel-Lieder graphs. Their geometric, algebraic and analytic properties have been investigated in the paper mentioned below. The project could focus on one or several aspects of such graphs, in general or for the particular case of a lamplighter group.

Reference: L. Bartholdi, M. Neuhauser, W. Woess, Horocyclic products of trees, JEMS 10, p. 771–816.

3 Amenable groups and Følner sets

J. von Neumann introduced the notion of amenable group in an attempt to understand the Banach-Tarski paradox, and its occurence in dimension 3 but not 2. Solvable groups are amenable groups. While von Neumann's initial's definition has focused on a notion of invarianr mean and of invariant (finitely additive) measure, there is an equivalent definition using a special family of sets, called the Følner sets.

Solvable groups are amenable, and for some of them the shape of the possible Følner sets is interesting in itself.

The project could overview the topic of amenable groups and then focus on some examples of solvable groups, and their Følner sets, or focus entirely on amenable groups and some key results related to them.

Reference: Chapter 18 of Druţu-Kapovich and references therein.

4 Selberg's Lemma

A key property of finitely generated linear groups is that they are virtually torsion-free. This is known as "Selberg's Lemma" even though it is in fact an important Theorem.

In Chapter 26 of Druţu-Kapovich a proof of this is given, based on a variant of residual finiteness. In section 26.7 other approaches are quoted.

The project could outline any of the existing proofs, evetually adding details illustrating the importance of Selberg's Lemma.

Reference: Chapter 26 of Druţu-Kapovich and references therein.

5 Approximate subgroups of nilpotent, solvable or linear groups

Given a group G and a constant $K \geq 1$, a finite subset A of G is a K-approximate group if $1 \in A, A - 1 = A$, and there exists a symmetric subset X of AA of size at most K such that $AA \subset XA$.

The project could investigate the existing results about approximate subgroups of either nilpotent groups and/or solvable groups and/or linear groups.

References:

E. Breuillard; B. Green, *Approximate groups. I: The torsion-free nilpotent case.* J. Inst. Math. Jussieu 10 (2011), no. 1, 37–57.

E. Breuillard; B. Green, *Approximate groups, II: The solvable linear case.* Q. J. Math. 62 (2011), no. 3, 513–521.

E. Breuillard, B. Green, T. Tao, *Approximate subgroups of linear groups.* Geom. Funct. Anal. 21 (2011), no. 4, 774–819.

6 Virtually free pro-p groups whose torsion elements have finite centralizer

The project could focus on the results of P. Zalesskii and co-authors on the structure of virtually free pro-p groups such that every non-trivial torsion element has finite centraliser (references provided below). Any other topic on free pro-p groups that relates to the course is likewise acceptable.

W. Herfort, P. A. Zalesskii, *Virtually free pro-p groups whose torsion elements have finite centralizer*. Bull. Lond. Math. Soc. 40 (2008), no. 6, 929–936.

J. W. MacQuarrie, P.A. Zalesskii, *Second countable virtually free pro-p groups whose torsion elements have finite centralizer.* Selecta Math. (N.S.) 23 (2017), no. 1, 101–115.

7 Quasi-isometric rigidity of fundamental groups of non-geometric Haken manifolds with zero Euler characteristic

A remarkable theorem of Kapovich-Leeb states: if a finitely generated group with a word metric is quasi-isometric to the fundamental groups of a non-geometric Haken manifold with zero Euler characteristic, then, up to taing a quotients by a finite normal subgroup and taking a finite index subgroup, the group is isomorphic to the fundamental group of a Haken manifold of the same kind. This has been done by Kapovich-Leeb in the non-geometric case, in the geometric case it follows from work of several authors.

The project could focus on the work of Kapovich-Leeb, or on the contrary on the geometric case, pick some (or all) of the eight geometries and explain how a similar result follows from the literature.

M. Kapovich, B. Leeb, *Quasi-isometries preserve the geometric decomposition of Haken manifolds.* Invent. Math. 128 (1997), no. 2, 393–416.

Chapter 25 of Druţu-Kapovich and references therein.

8 Any other topic related to the course

Any topic related to the course that you would be interested in investigating further, either in the book Druţu-Kapovich, or in another book, or in a published paper. Let me know, and (eventually with some adjustments) we'll formulate a project.