Metric Geometry and Geometric Group Theory

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Contents

CHAPTER 1

Introduction

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1.1. Introduction – Geometric group theory

We study the connection between geometric and algebraic properties of groups and the spaces they act on.

1.1.1. Example: Groups of polynoimal growth. Let M be a compact Riemannian manifold, $\tilde{M}$ its universal cover. Riemannian balls will be denoted $B(x,r)$, and vol will denote Riemannian volume on M and counting measure on $\Gamma = \pi_1(M)$. We would like to understand the algebraic structure of Γ.

- Assume M has non-negative Ricci curvature. This local property carries over to $\tilde{M}$.
- Then ("local to global"; Bishop-Gromov inequality) $\tilde{M}$ has *polynomial volume growth*:
$$\exists C, d : \forall x : \operatorname{vol} B_{\tilde{M}}(x,r) \le Cr^d \,.$$
- Then ("Quasi-isometry"; Milnor-Švarc Lemma) $\Gamma = \pi_1(M)$ has polynomial volume growth: with respect to some set of generators,
$$\exists C' : \operatorname{vol} B_{\Gamma}(x,r) \le C' r^d \,.$$
- Then (geometry to algebra; Gromov's Theorem) There exists a finite index subgroup $\Gamma' \subset \Gamma$ which is *nilpotent*.
- Then (topological corollary) M has a finite cover with a nilpotent fundamental group.

1.1.2. Example: Rigidity.

THEOREM 1. *(Margulis Superrigidity; special case) Let $\varphi\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_m(\mathbb{Z})$ be a group homomorphism with Zariski-dense image and $n,m \ge 2$. Then φ extends to a group homomorphism $\varphi\colon \mathrm{SL}_n(\mathbb{R}) \to \mathrm{SL}_m(\mathbb{R})$.*

1.2. Additional examples

1.2.1. Examples of Metric spaces. $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$, $\mathbb{R}^n$, Hilbert space.
$\mathbb{H}^2$, $\mathbb{H}^n$, $\mathbb{H}^\infty$
Riemannian manifolds
Banach spaces, function spaces
Graphs and trees

"Outer space"

1.2.2. Examples of Groups. $\mathbb{Z}$ and $\mathbb{Z}^d$, D_∞, Heisenberg groups, upper-triangular group.
$\mathrm{SL}_n(\mathbb{Z})$, congruence subgroups, lattices in real Lie groups, in $\mathfrak{p}$-adic Lie groups.
All these groups are *linear* (subgps of $\mathrm{GL}_n(F)$ for some field F).
$\pi_1(M^2)$: either $\mathbb{Z}^2$ or a lattice in $\mathrm{SL}_2(\mathbb{R}) = \mathrm{SO}(2,1)$.
$\pi_1(M^3)$: more complicated. Includes lattices in $\mathrm{SO}(3,1)$.
$\pi_1(M^4)$: any finitely-presented group [**5**].

1.2.3. Example: Random Groups. We shall consider presentations of the form $\Gamma = \langle S|R\rangle$ where S is fixed and R is chosen at random.

Let $S = \left\{a_i^{\pm}\right\}_{i=1}^{k}$, $F = \langle S\rangle$ the free group on k generators, and fix a parameter $0 < d < 1$ ("density") and two real numbers $0 < a < b$. For an integer l let $S_l \subset F$ denote the set of reduced words of length l. Then $\#S_l = 2k(2k-1)^{l-1} \sim k^l$. Choose N_l such that $ak^{dl} \leq N_l \leq bk^{dl}$. We shall make our group by choosing N_l relators at random from S_l. In other words, we set $\mathcal{A}_l = \binom{S_l}{N_l}$. Given $R \in \mathcal{A}_l$ we set $\Gamma_R = \langle S|R\rangle$ and think of it as a "group-valued random variable".

DEFINITION 2. Let $\mathcal{P}$ be a property of groups. We say that $\Gamma_R \in P$ *asymptotically almost surely* (a.a.s.) if

$$\lim_{l\to\infty} \frac{\#\{R \in \mathcal{A}_l \,:\, \Gamma_R \in \mathcal{P}\}}{\#\mathcal{A}_l} = 1\,.$$

Note that we have suppressed the dependence on d.

DEFINITION 3. Assume that $\mathcal{P}$ passes to quotients. Then if $\mathcal{P}$ holds a.a.s. at density d, it also holds a.a.s. at any density $d' > d$ and we can set:

$$d^*(\mathcal{P}) = \inf\left\{d' | \mathcal{P} \text{ holds a.a.s. at density } d'\right\}\,.$$

Examples:

LEMMA 4. *(Birthday paradox) Let N be a large set. If we choose $N^{1/2+\varepsilon}$ elements at random then a.a.s. we have chosen the same element twice.*

PROPOSITION 5. *Assume $d > \frac{1}{2}$. Then a.a.s. $|\Gamma_R| \leq 2$.*

PROOF. With high probability R contains many pairs of the form ws, ws' with distinct $s, s' \in S$. Thus with high probability we have $s' = s'$ in Γ_R. Hence Γ_R is a quotient of $\left\langle a_1 | a_1 = a_1^{-1}\right\rangle \simeq C_2$. If l is even this is the group we get, if l is odd then we get the trivial group. □

THEOREM 6. *(Gromov) If $d < \frac{1}{2}$ then a.a.s. $|\Gamma_R|$ is infinite.*

The proof is based on studying the properties of Γ_R as a *metric space*.

1.2.4. Example: Property (T) [4].

DEFINITION 7. Let G be a localy compact group. We say that G has *Kazhdan Property (T)* if any action of G by (affine) isometries on a Hilbert space has a (global) fixed point.

EXAMPLE 8. A compact group has property (T) by averaging. An abelian group has property (T) iff it is compact by Pontrjagin duality.

THEOREM 9. *(Kazhdan et. al.)*

(1) *A group with property (T) is compactly generated.*
(2) *Let $\Gamma < G$ be a lattice. Then G has property (T) iff Γ has it.*
(3) *Any simple Lie group of rank≥ 2 has property (T) (both real and $\mathfrak{p}$-adic).*

COROLLARY 10. *Let Γ be a lattice in a Lie group of higher rank. Then Γ is finitely genrated and has finite abelianization.*

THEOREM 11. *(Margulis) Let G be a higher-rank center-free Lie group, $\Gamma < G$ a latice, $N \lhd \Gamma$ a normal subgroup. Then Γ/N is finite.*

Part 1

Basic constructions

We will mainly care about“large scale” properties of metric spaces. For this we need a category where “small-scale” effects don’t matter. For example, on a very large scale ,the strip $\mathbb{R}\times[0,1]$ and the cylinder $\mathbb{R}\times S^1$ look more-or-less the same as the line $\mathbb{R}$. On a very large scale, all bounded metric spaces are no different from a single point.

In algebraic toopology, it is common to work in the “homotopy category”, where $\mathbb{R}\times[0,1]$ can be shunk to $\mathbb{R}$. We would like to do the same in the metric sense. Quasi-isometry is the key word.

1.3. Quasi-isometries

Let (X,d_X), (Y,d_Y) be metric spaces.

DEFINITION 12. Let $f\colon X\to Y$.

(1) Say that f is *Lipschitz* if $\exists L>0\forall x,x'\in X : d_Y(f(x),f(x'))\le Ld_X(x,x')$.
(2) Say that f is *bi-Lipschitz* if $\exists L>0\forall x,x'\in X : L^{-1}d_X(x,x')\le d_Y(f(x),f(x'))\le Ld_X(x,x')$.

These are “smooth” notions. They presrve the local structure. We shall be interested in a more “large scale” version of this notion which completely ignores small-scale information. There is a strong analogy to the category of homotopy equivalence classes of maps between topological spaces.

DEFINITION 13. Let $f,f'\colon X\to Y$.

(1) Say that f is a *quasi-isometry* if $\exists L,D>0$ such that for any $x,x'\in X$,
$$\frac{1}{L}d_X(x,x')-D\le d_Y\left(f(x),f(x')\right)\le Ld_X(x,x')+D.$$
(2) Say f and f' are *at finite distance* if $\exists R>0\forall x\in X : d_Y(f(x),f'(x))\le R$. This is clearly an equivalence relation that wil be denoted $f\sim f'$.

LEMMA 14. *Let (Z,d_Z) be a third metric space, and let $f,f',f''\colon X\to Y$, $g,g'\colon Y\to Z$.*

(1) Assume f and g are quasi-isometries. Then so is $g\circ f$.
(2) Assume, in addition that $f\sim f'$ and $g\sim g'$. Then f' is a quasi-isometry and $g\circ f\sim g'\circ f'$.

PROOF. Denote the constants by L_f, D_f etc. Then for any $x,x'\in X$ we have:
$$\begin{aligned} d_Z\left(g(f(x)),g(f(x'))\right) &\le L_g d_Y\left(f(x),f(x')\right)+D_g \\ &\le L_gL_f d_X\left(x,x'\right)+\left(L_gD_f+D_g\right). \end{aligned}$$
$$\begin{aligned} d_Z\left(g(f(x)),g(f(x'))\right) &\ge \frac{1}{L_g}d_Y\left(f(x),f(x')\right)-D_g \\ &\le \frac{1}{L_gL_f}d_X\left(x,x'\right)-\left(\frac{D_f}{L_g}+D_g\right). \end{aligned}$$
For the second part, we first estimate
$$\begin{aligned} d_Y\left(f'(x),f'(x'))\right) &\le 2R_f+d_Y\left(f(x),f(x')\right) \\ &\le L_f d_X\left(x,x'\right)+(2R+D_g) \end{aligned}$$
and in similar fashion $d_Y(f'(x),f'(x')))\ge L_f^{-1}d_X(x,x')-(2R+D_g)$.

Finally, for any $x \in X$ we have:

$$\begin{aligned} d_Z\left(g(f(x)), g'(f'(x))\right) &\leq d_Z\left(g(f(x)), g'(f(x))\right) + d_Z\left(g'(f(x)), g'(f'(x))\right) \\ &\leq R_g + L_{g'} R_f + D_{g'}\,. \end{aligned}$$

□

DEFINITION 15. Let $\mathcal{MQI}$ be the category whose objects are all metric spaces, and such that its arrows are equivalence clase of quasi-isometries.

LEMMA 16. *Let $f\colon X \to Y$ be a quasi-isometry and let $[f]$ be its equivalence class, though of as arrow of $\mathcal{MQI}$.*

(1) $[f]$ is a monomorphism. In other words, if $f \circ g \sim f \circ g'$ then $g \circ g'$.
(2) $[f]$ is an epimorphism iff for some $R > 0$, $f(X)$ is R-dense: $\sup\{d_Y(y, f(X))\}_{y \in Y} \leq R$.

DEFINITION 17. In the second case we say f is a quasi-isometric equivalence, and that X and Y are *quasi-isometric*.

EXAMPLE 18. Let $\mathbb{R}$ have its usual metric. Then the inclusion $\mathbb{Z} \subset \mathbb{R}$ is such an equivalence.

Proposition 49 below generalizes this observation.

EXAMPLE 19. Every metric space is quasi-isometric to a discrete one. This is based on the following useful observation.

DEFINITION 20. Let (X, d) be a metric space, $A \subset X$. Say that A is ε*-separated* if $d(A, A) \subset \{0\} \cup [\varepsilon, \infty)$, ε*-dense* $d(A, X) \leq \varepsilon$, that is if every point of X is ε-close to A. Say it is an ε*-net* if it satisfies both properties.

LEMMA 21. *An (inclusion-)maximal ε-separated set is an ε-net. An ε-dense in X is quasi-isometric to X.*

PROOF. If there exists point at distance at least ε from an ε-separated set then it can be added to the set keeping its separation. Maximal separated sets exist by Zorn's lemma. An inclusion map is an isometric embedding, in particular a quasi-isometric embedding. □

1.4. Geodesics & Lengths of curves

Let (X, d_X) be a metric space.

DEFINITION 22. We say (X, d_X)has:

(1) *rough midpoints,* if for some $D > 0$ and all $x, x' \in X$ there exists $m \in X$ such that $d(x, m), d(x', m) \leq \frac{1}{2} d(x, x') + D$;
(2) *approximate midpoints,* if for every $x, x' \in X$ and $\varepsilon > 0$ there exists $m \in X$ such that $d(x, m), d(x', m) \leq \frac{1}{2} d(x, x') + \varepsilon$;
(3) *(exact) midpoints,* if for every $x, x' \in X$ there exists $m \in X$ such that $d(x, m) = d(x', m) = \frac{1}{2} d(x, x')$.
(4) unique midpoints, if for every $x, x' \in X$ there exists a unique exact midpoint $m \in X$.

DEFINITION 23. For a continuous $\gamma\colon [a, b] \to X$ we set

$$l(\gamma) = \sup\left\{ \sum_{i=1}^{n} d_X(x_i, x_{i-1}) \mid a = x_0 \leq x_1 \leq \cdots \leq x_n = b \right\}.$$

If $l(\gamma) < \infty$ we say γ is *rectifiable* and $l(\gamma)$ is its *length*.

LEMMA 24. *$l(\gamma) \geq d_X(\gamma(a), \gamma(b))$. If γ is a* concatenation *$\gamma_1 \vee \gamma_2$ then $l(\gamma) = l(\gamma_1) + l(\gamma_2)$.*

DEFINITION 25. For $x, x' \in X$ set $d_X^*(x, x') = \inf\{l(\gamma) \mid \gamma(a) = x,\ \gamma(b) = x'\}$ ($\inf \emptyset = \infty$ by convention). Call this the *length metric associated to d_X*.

LEMMA 26. *$d_X^* \geq d_X$ pointwise. Furthermore, curves have the same length under d_X and d_X^*. In particular, $(d_X^*)^* = d_X^*$.*

PROOF. The first claim follows from the first claim of Lemma 26. It implies $l_{d_X^*}(\gamma) \geq l_{d_X}(\gamma)$ for any curve γ, and we need to prove the reverse. For this let $\gamma \in C([a,b] \to X)$, and let $a \leq t_0 \leq \cdots \leq t_n \leq b$ be any partition of $[a,b]$. By definition of d_X^* we have $d_X^*(\gamma(t_i), \gamma(t_{i+1})) \leq l_{d_X}(\gamma \restriction_{[t_i, t_{i+1}]})$ (the distance is the infimum over the length of all curves connecting the two points). It follows that:

$$\sum_{i=0}^{n-1} d_X^*(\gamma(t_i), \gamma(t_{i+1})) \leq \sum_{i=0}^{n-1} l_{d_X}(\gamma \restriction_{[t_i, t_{i+1}]}) = l_{d_X}(\gamma)\,.$$

□

DEFINITION 27. We say d_X is a *length metric* and that (X, d_X) is a *length space* if $d_X^* = d_X$. We say d_X is *geodesic* and that (X, d_X) is a *geodesic space* if the infimum is a minimum, that is if for every $x, x' \in X$ there exists a continuous curve γ connecting them with $l(\gamma) = d_X(x, x')$. Such a distance-minimizing curve is called a *geodesic curve* of simply a *geodesic*.

REMARK 28. Given a geodesic curve $\gamma\colon [a,b] \to X$ connecting x and x', the function $s(t) = d_X(x, \gamma(t))$ is monotone non-decreasing and continuous. It is then easy to check that $\tilde{\gamma}(s) = \gamma\left(\min S^{-1}(s)\right)$ is also a geodesic connecting x, x', in fact an isometry $[0, d_X(x, x')] \to X$. We call two geodesics *equivalent* if they give rise to the same isometry, and say (X, d_X) is *uniquely geodesic* if for every $x, x' \in X$ there exists a unique isometry $\gamma\colon [0, d_X(x, x')] \to X$ mapping the endpoints of the interval to x, x' respectively.

NOTATION 29. An equivalence class of geodesics contains a unique constant-speed representative with domain $[0,1]$. We usually denote it $t \mapsto [x, x']_t$, with $[x, x']$ denoting both the image and the function. The notation hides the fact that space may not be uniquely geodesic — $[x, x']$ will generally denote the choice of *some* geodesic connecting x, x'.

LEMMA 30. *Let (X, d_X) be a complete metric space.*

(1) d_X is a length space iff it has approximate midpoints.
(2) d_X is geodesic iff it has exact midpoints.
(3) d_X is uniquely geodesic iff it has unique midpoints.

One case where midpoints are unique is the case of a *convex* metric

DEFINITION 31. Call the geodesic metric d_X *strictly convex* if for $p, x, y \in X$ with $x \neq y$ every midpoint $m = [x, y]_{\frac{1}{2}}$ satisfies $d_X(p, m) < \max\{d_X(p, x), d_X(p, y)\}$.

LEMMA 32. *A strictly convex metric has unique midpoints and is, in particular, uniquely geodesic.*

PROOF. Let m_1, m_2 be distinct midpoints for $x, y \in X$, where $d_X(x, y) = 2d$. Let m be a midpoint of m_1, m_2. Then $d(x, m), d(y, m) < d$ (both x and y are at distance d to m_1, m_2). This contradicts our definition of d. □

DEFINITION 33. (X, d_X) is *locally compact* if for every $x \in X$ some closed ball $B_X(x, r)$ is compact. (X, d_X) is *proper* if all balls are compact, that is if subsets are compact iff they are closed and bounded.

REMARK 34. It is usefuly to note that a space where every ball of a fixed radius R is compact is complete, since every Cauchy sequence is eventually contained in such a ball.

THEOREM 35. *(Hopf-Rinow) A complete and locally compact length space is geodesic and proper.*

REMARK 36. It is useful to note that a space where every ball of a fixed radius R is compact is complete, since every Cauchy sequence is eventually contained in such a ball.

DEFINITION 37. Say that the metric d_X is *geodesically complete* if every geodesic segment is contained in a two-sided infinite geodesic. In other words, every isometry $[a, b] \to X$ extends to an isometry $\mathbb{R} \to X$. Note that this extension need not be unique even if X is uniquely geodesic.

LEMMA 38. *A Riemannian manifold is complete (in the sense of Riemannian geometry) iff it is geodesically complete.*

Problem Set 1

Length spaces and geodesics

1. Prove Lemma 30 in the notes.
Hint: Given $x, x' \in X$ construct your curve first on the dyadic rationals $\mathbb{Z}\left[\frac{1}{2}\right] \cap [0,1]$. You will need the axiom of choice.

LEMMA. *Let* (X, d_X) *be a complete metric space.*
(1) d_X *is a length space iff it has approximate midpoints.*
(2) d_X *is geodesic iff it has exact midpoints.*
(3) d_X *is uniquely geodesic iff it has unique midpoints.*

2. Prove Theorem 35 in the notes.

THEOREM. *(Hopf-Rinow) A complete and locally compact length space is geodesic and proper.*

Vector spaces

3. Let V be a normed space satisfying the *CAT(0)* inequality: for every $p, x, y \in V$ and $m = \frac{1}{2}x + \frac{1}{2}y$ one has:
$$\|m-p\|^2 \leq \frac{1}{2}\|x-p\|^2 + \frac{1}{2}\|y-p\|^2 - \frac{1}{4}\|x-y\|^2 .$$
Prove that V is isometric to a Hilbert space.

The Gromov-Hausdorff Metric

4. Let (X, d) be a compact metric space. Let $\mathcal{C}_X$ be the set of non-empty closed subsets of X. The *Hausdorff metric* on $\mathcal{C}_X$ is defined as follows: for $A, B \in \mathcal{C}_X$ we set
$$d_{\mathrm{H}}(A,B) = \sup_{b\in B}\inf_{a\in A} d(a,b) + \sup_{a\in A}\inf_{b\in B} d(a,b).$$
(a) Show that d_{H} is a metric.
(b) Show that $(\mathcal{C}_X, d_{\mathrm{H}})$ is a compact metric space.

5. Fix a compact metric space (K, d_K), and let $\mathcal{M}_K$ be the class of all triples (X, d_X, f) where (X, d_X) is a compact metric space and $f\colon K \to X$ is an isometric embedding. We define the *Gromov-Hausdorff* metric $d_{\mathrm{GH}}(X,Y)$ of (X, d_X, f_X), $(Y, d_Y, f_Y) \in \mathcal{M}_K$ to be the infimum over all Hausdorff distances $d_{\mathrm{H}}(F(X), G(Y))$ where $(Z, d_Z, f_Z) \in \mathcal{M}_K$ and $F\colon X \to Z$, $G\colon Y \to Z$ are isometric embeddings such that $F \circ f_X = G \circ f_Y = f_Z$.
(a) Show that d_{GH} is a metric on $\mathcal{M}_K$, and that $d_{\mathrm{GH}}(X,Y) \leq \mathrm{diam}(X) + \mathrm{diam}(Y)$ (the *diameter* of a metric space X is $\sup\{d(x,y) \mid x, y \in X\}$).
(b) $\{(X_n, d_n, f_n)\}_{n=1}^{\infty} \subset \mathcal{M}_K$ converges to $(X_\infty, d_\infty, f_\infty) \in \mathcal{M}_K$ in the Gromov-Hausdorff metric. Show that $\lim_{n\to\infty} \mathrm{diam}(X_n) = \mathrm{diam}(X_\infty)$.
(c) ($(\mathcal{M}_K, d_{\mathrm{GH}})$ is complete) Let $\{(X_n, d_n, f_n)\}_{n=1}^{\infty} \subset \mathcal{M}_K$ be a Cauchy sequence. Show that it converges.

6. $((\mathcal{M}_K, d_{\mathrm{GH}})$ is not compact) Let St_n be the n-pointed star, that is the metric realization of the graph on $n+1$ vertices $\{s\} \cup \{v_i\}_{i=1}^n$ with edges $[s, v_i]$ of unit length. Let $B_n \subset \mathbb{R}^n$ be the unit ball with the induced L^2 metric. Think of both as elements of $\mathcal{M}_\emptyset$.
   (a) Show that $d_{\mathrm{GH}}(\mathrm{St}_n, \mathrm{St}_m) = \delta_{n,m}$.
   (b) Show that $d_{\mathrm{GH}}(B_n, B_m) = \delta_{n,m}$.

REMARK. (Non-compact spaces) Let (X, d_X, x) and (Y, d_Y, y) be two pointed proper metric spaces. We can set $d_{\mathrm{GH}}(X,Y) = \sum_{n=1}^\infty 2^{-n} d_{\mathrm{GH}}((B_X(x,n), d_X, x), (B_Y(y,n), d_Y, y))$ (the factor 2^{-n} was simply chosen to make the series converge, using the diameter bound from 4(a) above). Convergence in this metric is equivalent to Gromov-Hausdorff convergence of every ball of finite radius. This notion of convergence preserves the properties of being a length space or a geodesic space. This will be proved in the next problem set using ultrafilters.

Hint for 5(c): passing to a rapidly converging subsequence, choose $F_n \colon X_n \to X_{n+1}$ which does not chance distances additively by more than ε_n, where $\sum_n \varepsilon_n < \infty$. Use this to define a notion of a Cauchy sequence for $\{x_n\}_{n=1}^\infty$ with $x_n \in X_n$. Define a limiting pseudo-metric on the space of such sequences. Finally, identify equivalent sequences and show that the resulting space is a limit.

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1.5. Vector spaces

1.5.1. Affine geometry.

DEFINITION 39. An *affine vector space* A over a field F is a principal homogenous space for a vector space V over F. Note that for any $a,b \in A$ there is a well-defined vector $b-a \in V$.

$\mathbb{A}^n(F)$ will denote affine n-space over F.

Note that for $\{a_i\}_{i=1}^n \subset A$ and $\{t_i\}_{i=1}^n \subset F$ such that $\sum_{i=1}^n t_i = 1$, the element $\sum_i t_i a_i$ defined by identifying A with V by a choice of origin does not depend on the choice of origin. We call this element an *affine combination* of the a_i.

DEFINITION 40. If $\mathrm{char}(F) \neq 2$ (which we assume henceforth) all this is equivalent to giving an *affine structure* on A. That is a map $A \times A \times F \to A$, denoted $(a,b,t) \mapsto [a,b]_t$, such that for some (equivalently, every) $z \in A$ the maps:

$$t \cdot_z a \stackrel{\text{def}}{=} [z,a]_t$$

and

$$a +_z b \stackrel{\text{def}}{=} 2 \cdot_z [a,b]_{\frac{1}{2}}$$

satisfy the axioms for a vector space over F (translation by $z' - z$ gives an isomorphism of the vector space structures associate to z, z').

A map of affine spaces over F is an *affine map* if it preserves the affine structure.

Fix an affine space A.

LEMMA 41. *Let* $\mathrm{Aff}(A)$ *denote the group of invertible affine maps from A to itself. Then* $\mathrm{Aff}(A) \simeq V \rtimes \mathrm{GL}(V)$ *where V, the underlying vector space, acts by translation and* $\mathrm{GL}(V)$ *acts by linear maps around a fixed origin. The isomoprhism is given by the choice of that origin.*

DEFINITION 42. The *affine hull* $\mathrm{aff}(S)$ of a subset $S \subset A$ is the intersection of all affine subspaces containing S. Clearly, an affine map on $\mathrm{aff}(S)$ is uniquely defined by its values on S. A finite set S is said to be *in general position* if $\mathrm{aff}(S)$ is isomorphic to affine $(\#S-1)$-space.

1.5.2. Banach spaces. Let F be $\mathbb{R}$ or $\mathbb{C}$, V be a vector space over F. A *norm* on V is a map $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ such that for $v,w \in V$ and $\alpha \in F$, $\|\alpha v\| = |\alpha| \, \|v\|$, $\|v\| = 0$ iff $v = 0$, and $\|v+w\| \leq \|v\| + \|w\|$. This defines a metric on any affine space over V by $d(x,y) = \|x-y\|$. This metric is always geodesic since the map $t \mapsto (1-t)v + tw$ is a constant-speed geodesic. It is also geodesically complete.

1.5.3. Euclidean space. Let $\mathbb{E}^n$ denote affine $\mathbb{R}^n$ with the metric $d(\underline{x},\underline{y}) = \left\|\underline{x}-\underline{y}\right\|_2$.

LEMMA 43. *This metr*

PROOF. It is clear that the map $t \mapsto [x,y]_t$ for $t \in [0,1]$ is a geodesic connecting x,y. The metric is convex:

It is clear that any geodesic segment can be extended to an isometry $\mathbb{R} \hookrightarrow \mathbb{E}^n$ via $t \mapsto [a,b]_t$. $\square$

The main pleasant feature of Euclidean space is high degree of symmetry.

LEMMA 44. *Let* $A \subset \mathbb{E}^n$ *be an affine subspace. Then A is isometric to* $\mathbb{E}^k$ *for some k.*

PROOF. We may choose the origin to lie on A. Then the claim amounts to choosing an orthonormal basis for A and extending it to $\mathbb{E}^n$. $\square$

1.6. Manifolds

DEFINITION 45. A *model Riemannian manifold* is a connected open subset $U \subset \mathbb{R}^n$ together with map $g \in C^1(U, M_n(\mathbb{R}))$ such that for any $x \in U$, $g(x)$ is a positive-definite symmetric matrix. The *Riemannian length* of a curve $\gamma \in C^1([a,b],U)$ is

$$l_{\mathrm{R}}(\gamma) = \int_a^b \sqrt{\langle g(\gamma(t)) \cdot \gamma'(t), \gamma'(t) \rangle} dt\,.$$

The *Riemannian metric* $d_{\mathrm{R}}(x,x')$ on (U,g) is given by the infimum of $l_{\mathrm{R}}(\gamma)$ on all continuously differentiable curves connecting x and x'.

A *Riemannian manifold* (Y,d_Y) is a connected second countable geodesic metric space which is locally isometric to a model Riemannian manifold.

EXAMPLE 46. The hyperbolic plane *is the model Riemannian manifold over the open set* $\mathbb{H} = \{x+iy | y>0\}$ *with the metric* $ds^2 = \frac{dx^2+dy^2}{y^2}$.

1.7. Groups and Cayley graphs

Let Γ be a discrete group, $S \subset \Gamma$ a finite symmetric ($S = S^{-1}$) generating set. We define a graph $\mathrm{Cay}(\Gamma;S)$ as folows: its vertex set is Γ, and the edge set is the set of pairs (x,xs) where $x \in \Gamma$ and $s \in S$. This graph is undirected and connected (a restatement of the fact that S is symmetric generates Γ). The left regular action of Γ on itself preserves this graph structure.

We let d_S denote the graph metric on $\mathrm{Cay}(\Gamma;S)$, thought of as a metric on Γ. This is a left-invariant metric and we have $d_S(x,y) = d_S(x^{-1}y,1)$.

LEMMA 47. *Let S' be another such set. Then the identity map is a quasi-isometric equivalence of d_S and $d_{S'}$.*

PROOF. It suffices to check one side of the inequality, and only for distances from 1. Assume that any $s' \in S'$ satisfies $d_S(s',1) \le L$. Writing any $x \in \Gamma$ as a word in the element of S' of length $d_{S'}(x,1)$ and expanding each element in terms of S we indeed see that

$$d_S(x,1) \le L \cdot d_{S'}(x,1)\,.$$

□

EXAMPLE 48. Any Cayley graph has rough midpoints. The *geometric realization*

PROPOSITION 49. *(Milnor-Švarc) Let Y be a proper geodesic metric space, and let Γ act discretely and co-compactly by isometries on Y. Then:*

(1) Γ is finitely generated.
(2) For some (any) generating set S, (Γ,d_S) is quasi-isometric to (Y,d_Y).

PROOF. Fix a basepoint $y_0 \in Y$. First of all, there exists a closed ball $B(y_0,R)$ which maps surjectively on the quotient $\bar{Y}$. Otherwise for each n let $\bar{y}_n$ not lie in the image of $B(y_0,R_n)$ in $\bar{Y}$ where $R_n \to \infty$. Passing to a subsequence we may assume $\bar{y}_n \to \bar{y}_\infty$ in $\bar{Y}$, and let $y_\infty \in Y$ be any preimage. Then for N large enough, $B(y_0,R_N)$ contains a neighbourhood of y_∞ and hence its image in $\bar{Y}$ contains all $\bar{y}_n$ for after some point, a contradiction. Fixing R, it follows that $Y = \cup_{\gamma\in\Gamma} B(\gamma y_0,R)$, that is that for any $y' \in Y$ there exists $\gamma \in \Gamma$ such that $d_Y(y',\gamma y_0) \le R$.

Next, let $S = \{\gamma \in \Gamma | d(y_0, \gamma y_0) \le 10R\}$. This is a finite set by definition and is symmetric since $d(y_0, \gamma^{-1} y_0) = d(\gamma y_0, \gamma\gamma^{-1} y_0)$. The bound $d_Y(\gamma y_0, y_0) \le 10R \cdot d_S(\gamma, 1)$ follows by induction on $d_S(\gamma, 1)$ using the isometry of the action.

The non-trivial part is the lower bound

$$d_S(\gamma, 1) \le \frac{1}{5R} d_Y(\gamma y_0, y_0) + 1$$

which also demonstrates that S generates Γ. For this let $c : [0, D] \to Y$ be the geodesic connecting y_0 and γy_0. For $0 \le i \le \lfloor \frac{D}{5R} \rfloor = I$ let $y_i = c(5iR)$ and let γ_i be such that $d(\gamma_i y_0, y_i) \le R$. Also set $\gamma_{I+1} = \gamma$. Then $d(\gamma_i y_0, \gamma_{i+1} y_0) \le 7R$ and hence $\gamma_{i+1} = \gamma_i s_i$ for some $s_i \in S$, which gives the desired bound. It follows that for any γ, γ':

$$\frac{1}{10R} d_Y(\gamma y_0, \gamma' y_0) \le d_S(\gamma, \gamma') \le \frac{1}{5R} d_Y(\gamma y_0, y_0) + 1\,,$$

that is that $\gamma \mapsto \gamma y_0$ is a quasi-isometric equivalence. □

1.8. Ultralimits

Fix $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$. Call the elements of $\mathcal{F}$ *majorities*.

DEFINITION 50. Let Y be a topological space, $\{a_n\}_{n=1}^{\infty} \subset Y$. Say that a_n *converges to* $A \in Y$ *along* $\mathcal{F}$ (denoted $A = \lim_{\mathcal{F}} a_n$). If for every neighbourhood U of A, $\{n \in \mathbb{N} \mid a_n \in U\}$ is a majority.

What do we need to assume about $\mathcal{F}$ for this to make sense?

DEFINITION 51. Call $\mathcal{F} \subset \mathcal{P}(I)$ a *filter on I* if it is closed under intersection and the taking of supersets and does not contain the empty set.

EXAMPLE 52. The *co-finite filter* is $\mathcal{F}_{\mathrm{c}} = \{I \setminus F | F \text{ finite}\}$. A *principal filter* (or a "dictatorship") is one of the form $\{M \subset I | i \in M\}$ for some fixed $i \in I$.

PROPOSITION 53. *Let $\mathcal{F}$ be a filter on an index set I. Let $a\colon I \to Y$ be a sequence and assume $A = \lim_{\mathcal{F}} a_i$.*

(1) If Y is Hausdorff then A is unique.
(2) If Y' is another Hausdorff space and $\{b_i\}_{i \in I} \subset Y'$ converges to B then $\{(a_i, b_i)\}_{i \in I} \subset Y \times Y'$ converges to (A, B).
(3) If $f\colon Y \to Z$ is continuous then $f(A) = \lim_{\mathcal{F}} f(a_i)$.

PROOF. Let $A' \in Y$ be distinct from A and let $U, U' \subset Y$ be disjoint neighbourhoods of A, A' respectively. Then $a^{-1}(U), a^{-1}(U')$ are disjoint subsets of I and cannot both belong to $\mathcal{F}$.

Any neighbourhood of (A, B) contains one of the form $U \times U'$. Then $(a \times b)^{-1}(U \times U') = a^{-1}(U) \cap b^{-1}(U')$.

Finally, for any neighbourhood U of $f(A)$, $f^{-1}(U)$ is a neighbourhood of A and hence $(f \circ a)^{-1}(U) \in \mathcal{F}$. □

COROLLARY 54. *(Arithmetic of limits) Let $\mathcal{F}$ be a filter, $\{a_i\}_{i \in I}, \{b_i\}_{i \in I} \subset \mathbb{R}$, and assume $A = \lim_{\mathcal{F}} a_i$, $B = \lim_{\mathcal{F}} b_i$. Then:*

(1) $-a_i$ converges to $-A$ along $\mathcal{F}$.
(2) $a_i + b_i$ converges to $A + B$ along $\mathcal{F}$.
(3) If $A \neq 0$, $\frac{1}{a_i}$ converges to $1/A$ along $\mathcal{F}$.

(4) $a_i b_i$ converges to AB along $\mathcal{F}$.

DEFINITION 55. An *ultrafilter* is a filter ω such that for any $J \subset I$ either $J \in \omega$ or $\neg J \in \omega$.

EXAMPLE 56. Every principal filter is an ultrafilter.

LEMMA 57. *Let ω be an ultrafilter and let $A = \cup_{k=1}^{K} A_k \subset I$ be a finite union. Then $A \in \omega$ iff $A_k \in \omega$ for some k.*

An ultrafilter is non-principal iff it contains the co-finite filter.

PROOF. If none of the A_k belong to ω then $\neg A = \cap_k \neg A_k \in \omega$. If $A_k \in \omega$ then $A \in \omega$ by definition of filter. Every finite set is a finite union of singletons. □

PROPOSITION 58. *A filter is an ultrafilter iff it is maximal w.r.t. inclusion. Every filter is contained in an ultrafilter.*

PROOF. For a filter $\mathcal{F}$ on I and a set $J \in \mathcal{P}(I)$ let

$$\mathcal{F}[J] = \{(A \cap J) \cup B \mid A \in \mathcal{F}, B \in \mathcal{P}(I)\} .$$

This set is closed under intersection and the taking of supersets. It is a filter iff $\neg J \notin \mathcal{F}$, showing the first claim. The second claim then follows by Zorn's lemma. □

From now on fix a non-principal ultrafilter ω on $\mathbb{N}$.

PROPOSITION 59. *(Bolzano-Weierstraß theorem) Let (K,d) be a compact metric space, $\{a_n\}_{n=1}^{\infty} \subset K$ a sequence. Then $\lim_\omega a_n$ exists.*

PROOF. For each $\varepsilon > 0$ we can cover K with finitely many balls of radius ε: $K = \cup_{j=1}^{m} B(x_j, \varepsilon)$. Let $A_j = \left\{n \mid a_n \in B(x_j, \varepsilon)\right\}$. Then $\mathbb{N} = \cup_{j=1}^{m} A_j$ and by Lemma 57 we can find j such that $A_j \in \omega$. In that case set $B_\varepsilon = B(x_j, \varepsilon)$, $A_\varepsilon = a^{-1}(B_\varepsilon)$. Then the set $\{B_\varepsilon\}_{\varepsilon > 0}$ has the finite intersection property, since its inverse image $\{A_\varepsilon\}_{\varepsilon>0} \subset \omega$ has it. Let $\{x\} = \cap_{\varepsilon>0} B_\varepsilon$. Then for any $\varepsilon > 0$, $a^{-1}(B(x,\varepsilon)) \supset a^{-1}(B_{\varepsilon/2}) \in \omega$ so $\lim_\omega a_n = x$. □

COROLLARY 60. $\lim_\omega$ *defines a bounded linear functional $\ell^\infty \to \mathbb{C}$ which is an algebra homomorphism.*

DEFINITION 61. For each $n \in \mathbb{N}$ assume we are given a pointed metric space (X_n, d_n, p_n). We shall let $\tilde{X}$ denote the space of *bounded sequences*

$$\tilde{X} = \left\{ \underline{x} \in \prod_n X_n \middle| \exists R \forall n : d_n(x_n, p_n) \le R \right\} .$$

For $\underline{x}, \underline{y} \in \tilde{X}$, the sequence $\{d_n(x_n, y_n)\}_{n=1}^{\infty}$ is bounded by the triangle inequality and we set:

$$\tilde{d}_\omega(\underline{x}, \underline{y}) = \lim_\omega d_n(x_n, y_n) .$$

LEMMA 62. *The function $\tilde{d}_\omega$ is a pseudometric.*

PROOF. Symmetry, non-negativity and the triangle inequality hold pointwise, hence at the limit. □

DEFINITION 63. The *ultralimit* (or *limit of* (X_n, d_n, p_n) *along* ω), denoted

$$\lim_{\omega}(X_n, d_n, p_n)\ ,$$

is the quotient $\left(\tilde{X}, \tilde{d}_\omega, \underline{p}\right)$ where points at $\tilde{d}_\omega$-distance zero are identified. $\tilde{d}_\omega$ descends to a metric d_ω on this space.

The completion of this space will be called the *completed ultralimit*, and will be denoted

$$\hat{\lim_{\omega}}(X_n, d_n, p_n)\ .$$

PROPOSITION 64. *(ultralimits of functions) Let* (X_n, d_n, p_n), (Y_n, d'_n, p'_n) *be two sequences of metric spaces. Let* $f_n\colon X_n \to Y_n$ *be a sequence of* (L_n, D_n)*-quasi-isometries, where we assume* $L_n \le L$, $D_n \le D$*. Assume that* $\underline{f}(\underline{p}) \in \tilde{Y}$*, for the product function*

$$\underline{f}\colon \prod_n X_n \to \prod_n Y_n\,.$$

Then the image $\underline{f}(\tilde{X})$ *lies in* $\tilde{Y}$ *and* $\underline{f}$ *is the pull-back of an* $(\lim_\omega L_n, \lim_\omega D_n)$*-QI* $\lim_\omega f\colon \lim_\omega X_n \to \lim_\omega Y_n$*. When* $\lim_\omega D_n = 0$*, the function* $\lim_\omega f$ *is uniquely defined.*

PROOF. For each n we have

$$d'_n(f_n(x_n), p'_n)) \le L_n d_n(x_n, p_n) + D_n + d'_n(f_n(p_n), p'_n)\,,$$

where all terms on the RHS are uniformly bounded. Now for equivalence class $[\underline{x}] \in \lim_\omega X_n$ we choose a representative $\underline{x} \in \tilde{X}$ and arbitrarily set $\lim_\omega f([\underline{x}]) = [\underline{f}(\underline{x})]$. That $\lim_\omega f$ is a QI as advertized is clear. It is also clear that $d'_\omega(\underline{f}(\underline{x}), \underline{f}(\underline{z})) \le L d_\omega(\underline{x}, \underline{z}) + \lim_\omega D_n$. In particular, if $\lim_\omega D_n = 0$ then $\left[\underline{f}(\underline{x})\right]$ is independent of the choice of representative $\underline{x} \in [\underline{x}]$. □

Examples: The asymptotic cone and the tangent cone. Let (X, d) be a metric space, and let $p \in X$.

DEFINITION 65. Let $L_i \to \infty$, and let ω be a non-principal ultrafilter on $\mathbb{N}$.

(1) The *trangent cone* $T^\omega_p X$ associated to this data is the ultralimit

$$\lim_{\omega}(X, L_i d, p)\ .$$

(2) The *asymptotic cone* $C^\omega_p X$ is the ultralimit

$$\lim_{\omega}\left(X, \frac{1}{L_i} d, p\right)\ .$$

EXAMPLE 66. Let G be a graph with the graph metric. Then every asymptotic cone of X is geodesic.

Problem set 2

Neccessity of Ultrafilters

1. Let $L\colon \ell^\infty(\mathbb{N}) \to \mathbb{C}$ be positive (map sequence with non-negative elements to non-negative reals), non-zero, and respect arithmetic of limits. Then L is of the form $\lim_\omega$ for some ultrafilter ω.

Ultralimits and Gromov-Hausdorff limits

2. Let $\{(X_n, d_n)\}_{n=1}^\infty$ be a family of metric spaces of uniformly bounded diameters. For a fixed non-principal ultrafilter ω, show that the isometry class of the ultralimit $\lim_\omega (X_n, d_n, x_n)$ is independent of the choice of basepoints x_n.

3. Let $\{(X_n, d_n, f_n)\}_{n=1}^\infty \subset \mathcal{M}_K$ be a Cauchy sequence with respect to the Gromov-Hausdorff metric d_{GH}. Show that for any non-principal ultrafilter ω, the limit $\lim_\omega (X_n, d_n)$ belongs to $\mathcal{M}_K$ and is a limit of the sequence. Conclude that $(\mathcal{M}_K, d_{\mathrm{GH}})$ is complete.

4. Let (X_n, d_n, x_n) be a sequence of pointed spaces, ω a non-principal ultrafilter.
   (a) If every X_n is a length space then so is $\lim_\omega (X_n, d_n, x_n)$.
   (b) If every X_n is a geodesic space then so is $\lim_\omega (X_n, d_n, x_n)$.
   (c) Conclude that Gromov-Hausdorff limits also preserve these properties.

Tangent cones

Recall the definition of the tangent cone: $T_p^\omega = \lim_\omega (Y, n \cdot d_Y, p)$. For a model Riemannian manifold you may use the following definition of the isometry class of the tangent cone in the sense of differential geometry: If $U \subset \mathbb{R}^n$ with metric g then $T_p^{\mathrm{DG}} U$ is isometric to $\mathbb{R}^n$ with the L^2-norm associated to $g(p)$.

5. (locality) Let U be an neighbourhood of p. Prove that the inclusion map $U \hookrightarrow Y$ gives rise to an isometry $T_p^\omega U \simeq T_p^\omega Y$.

6. Let G be a locally finite graph, $Y = |G|$ its geometric realization. Calculate $T_p Y$ for any $p \in Y$.

7. Let (Y, d_Y) be a Riemannian manifold, and let $p \in Y$. Prove that the tangent cone $T_p^\omega Y$ is naturally isometric to $T_p^{\mathrm{DG}} Y$, the tangent space in the sense of differential geometry.
   Hint: Local geodesics at p give a map $T_p^{\mathrm{DG}} Y \to T_p^\omega Y$. For the reverse map use the compactness of the sphere $S^{n-1} \subset \mathbb{R}^n$.

Asymptotic cones

8. Let G be a graph, and let Y be its vertex set with the graph metric. Show that every asymptotic cone of Y is geodesic.

Part 2

Groups of polynomial growth

1.9. Group Theory

1.9.1. Free groups.

PROPOSITION 67. *Let G be a connected graph. Then:*

1. *The* geometric realization $|G|$ *is homotopic to a join ("bouquet") of circles.*
2. *If G is finite then the number of circles is* $\#E(G) - \#V(G) + 1$.
3. *Every covering space of* $|G|$ *is of the form* $|H|$ *where H is a graph.*
4. *If the covering is n-sheetd then* $\#E(H) = n \cdot \#E(G)$, $\#V(H) = n \cdot \#V(G)$.

COROLLARY 68. *Let G be a connected graph.*

1. $\pi_1(G)$ *is free.*
2. *If G is finite then* $\pi_1(G) \simeq F_r$ *wheer* $r = |E(G)| - |V(G)| + 1$.
3. *Let* $\Gamma < F_X$*. Then* Γ *is free.*
4. *If X is finite and* $[F_X : \Gamma] = n$ *then* $\text{rk}(\Gamma) = n(\#X - 1) + 1$.

COROLLARY 69. *cLet* Γ *be f.g.,* $\Gamma_1 < \Gamma$ *a subgroup of finite index. Then* Γ_1 *is finitely generated.*

PROPOSITION 70. *Let* Γ *be f.g. Then* $\{\Gamma_1 < \Gamma \mid [\Gamma : \Gamma_1] = n\} < \infty$.

PROOF. Say $S = \left\{s_i^{\pm}\right\}_{i=1}^{k} < \Gamma$ is a generating set. Let $X \subset S_{[n]}^{k}$ be the set of ordered sequences which generate a transitive subgroup. Let $X_1 < X$ be the set of sequences $\{\sigma_i\}_{i=1}^{k} \in X$ such that the map $s_i \mapsto \sigma_i$ extends to group hom. Then the set under consideration injects into X_1. □

COROLLARY 71. *Let* Γ *be f.g.,* Γ_1 *a subgroup of f.i. Then there exists a characteristic subgroup of finite index contained in* Γ_1 *(the intersection of all subgroups of the same index).*

1.9.2. Finitely generated abelian groups.

LEMMA 72. *Every subgroup of* $\mathbb{Z}^d$ *is finitely generated.*

PROOF. The first claim is easy. For the second, fix $A < \mathbb{Z}^d$. Then $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathbb{Q}^d$ is a subspace. Choose a basis B for this subspace, and extend it to a basis B' of $\mathbb{Q}^d$. Clearing denominators, we may assume $B' \subset \mathbb{Z}^d$ and $B \subset A$. Since B' spans $\mathbb{Z}^d$ over $\mathbb{Q}$, there exists $N \in \mathbb{N}$ such that $\mathbb{Z}^d \subset \oplus_{b \in B'} \frac{1}{N}\mathbb{Z}b$. Let $A_1 = \langle B \rangle = \oplus_{b \in B} \mathbb{Z}b$. Then both A_1 and $A/A_1 \hookrightarrow \mathbb{Z}^d/(\frac{1}{N}\mathbb{Z})^d \simeq (\mathbb{Z}/N\mathbb{Z})^d$ are finitely generated. □

COROLLARY 73. *Let A be a finitely generated Abelian group.*

1. *Every subgroup of A is finitely generated.*
2. *Let* $A_{tors} \subset A$ *be the subgroup of elements of finite order. Then* A_{tors} *is finite (generated by finitely many elements of finite order), and is the direct product of its Sylow subgroups.*

FACT 74. *(Structure theorem for finitely generated Abelian groups) Let A be a finitely generated Abelian group. Then there exist a finite sequence of prime powers* $\left\{q_i = p_i^{e_i}\right\}_{i=1}^{r}$ *such that*

$$A \simeq \mathbb{Z}^d \bigoplus \oplus_{i=1}^{r} \mathbb{Z}/q_i\mathbb{Z}\,.$$

In particular, A is infinite iff $d \geq 1$.

1.9.3. Solvable groups. Let Γ be a group. For $x,y\in\Gamma$ set $[x,y]=xyx^{-1}y^{-1}$. For $A,B\subset\Gamma$ let $[A,B]$ denote the *subgroup* generated by $\{[a,b]\}_{a\in A,b\in B}$.

DEFINITION 75. The *derived subgroup* $\Gamma^{(1)}=\Gamma'$ is $[\Gamma,\Gamma]$.

LEMMA 76. *Γ' is the smallest normal subgroup with an Abelian quotient. In particular it is characteristic.*

DEFINITION 77. The *derived series* is the series of subgroups given by $\Gamma^{(0)}=\Gamma$ and $\Gamma^{(i+1)}=\left[\Gamma^{(i)},\Gamma^{(i)}\right]$. Say Γ is *solvable* if $\Gamma^{(i)}=\{1\}$ for some i. In that case call the smallest such i the *degree of solvability.*

LEMMA 78. *Γ is solvable iff there exists a chain of subgroups $\Gamma=\Gamma^0\supset\Gamma^1\supset\cdots\supset\Gamma^n=\{1\}$ such that for $0\le i\le n-1$, $\Gamma_{i+1}\lhd\Gamma_i$ and Γ_i/Γ_{i+1} is Abelian.*

PROOF. Since the derived series is such a chain, necessity is clear. For sufficiency, given such a chain it follows by induction that $\Gamma^n\supset\Gamma^{(n)}$. □

LEMMA 79. *Let Γ be solvable. Then so are every subgroup and quotient of Γ. Conversely, if $N\lhd\Gamma$ and both N and Γ/N are solvable then so is Γ.*

EXAMPLE 80. Let $B_n\subset\mathrm{GL}_n$ be the subgroup of upper-triangular matrices, R a commutative ring. Then $B_n(R)$ is solvable.

PROOF. Let $N_n\subset B_n$ the the subgroup of unipotent matrices. Then $B_n(R)/N_n(R)\simeq(R^\times)^n$ is Abelian. We will see below that N_n is solvable. □

1.9.4. Nilpotent groups.

DEFINITION 81. The *lower central series* is the series of subgroups given by $\Gamma_0=\Gamma$ and $\Gamma_{i+1}=[\Gamma,\Gamma_i]$. Say Γ is *nilpotent* if $\Gamma_i=\{1\}$ for some i. In that case call the smallest such i the *degree of nilpotence.*

LEMMA 82. *The Γ_i are clearly characteristic.*

EXAMPLE 83. $N_n\subset\mathrm{GL}_n$ is nilpotent.

PROPOSITION 84. *Let Γ be nilpotent and finitely generated. The the Γ_i are all finitely generated.*

PROOF. By induction the degree of nilpotence n, the case $n=1$ being clear. Let Γ be a group of degree $n+1$. Then Γ_n is central. For $1\le i\le n-1$ fix $S_i\subset\Gamma_i$ such that their images generate $\Gamma_i/\Gamma_n=(\Gamma/\Gamma_n)_i$. If $S_n\subset\Gamma_n$ is a generating set of Γ_n, then $S_i\cup S_n$ generate Γ_i. It thus remains to show that Γ_n is finitely generated. We first note that, if $a\equiv a'\pmod{Z_\Gamma}$, $b\equiv b'\pmod{Z_\Gamma}$ then $[a,b]=[a',b']$. It follows that Γ_n is generated by the commutators $[\gamma,\gamma_{n-1}]$ where $\gamma\in\langle S_0\rangle$, $\gamma_{n-1}\in\langle S_{n-1}\rangle$.

We now use the identities:

$$[ab,c]=abcb^{-1}a^{-1}c^{-1}=a[b,c]a^{-1}[a,c],$$

$$[a_1a_2,b_1b_2]=[a_2,b_1]^{a_1}[a_2,b_2]^{a_1b_1}[a_1,b_1][a_1,b_2]^{b_1},$$

which holds in every group ($x^g \stackrel{\text{def}}{=} gxg^{-1}$). Returning to our nilpotent group of degree $n+1$, if $a_\alpha \in \Gamma$ and $b_\beta \in \Gamma_{n-1}$ ($\alpha,\beta \in \{1,2\}$), we have $[a_\alpha, b_\beta] \in \Gamma_n$ which is central. In that case,

$$[a_1a_2, b_1b_2] = \prod_{\alpha,\beta=1}^{2} \left[a_\alpha, b_\beta\right] .$$

Applying this inductively, we see that every element of $\Gamma_n = [\Gamma, \Gamma_{n-1}]$ is a product elements of the finite set

$$S_n = \{[\gamma_0, \gamma_{n-1}] \mid \gamma_0 \in S_0,\, \gamma_{n-1} \in S_{n-1}\} .$$

□

COROLLARY 85. *Every subgroup of a finitely generated nilpotent group is finitely generated.*

PROOF. Again by induction. For $n=0$ there's nothing to prove. Say Γ has degree $n+1$ and let $\Delta < \Gamma$. Then $\Delta \cap \Gamma_n$ is a subgroup of a finitely generated abelian group, hence finitely generated. Also, $\Delta/\Delta\cap\Gamma_n$ injects into the group Γ/Γ_n which is nilpotent of degree n. □

PROPOSITION 86. *Let Γ be a f.g. nilpotent group. Then it has polynomial growth.*

PROOF. By induction on the degree of nilpotence, the case $n=1$ being clear. Say Γ has degree $n+1$, let $S_i = \left\{s_{ij}\right\} \subset \Gamma_i$ generate Γ_i/Γ_{i+1} and let $S_{[k]} = \cup_{i=k}^{n} S_i$, which generates Γ_k (also write $S = S_{[0]}$). For any $s \in S$ and $s_{ij} \in S_i$, we have $[s, s_i] \in \Gamma_{i+1}$ by definition. Fix C such that for any s, i, j this has length at most C in $S_{[i+1]}$. We shall show that element in $B_\Gamma^S(r)$ can also be written as a a word ab, where $a = \prod_k s_{0k}^{e_k}$with $\sum_k e_k \le r$ and a $b \in \Gamma^{(1)}$ has of length $O(r^k)$ in $S_{[1]}$.

Let $w = \prod_\alpha s_{i_\alpha j_\alpha} \in B_S(r)$. We now produce a sequence of identities $w = a_t b_t$ where $a_t = \prod_{k \le K} s_{0k}^{e_k(t)}$, and b_t is a word in $S_{[1]} \cup \{s_{0k}\}_{k \ge K}$. Initially set $a_0 = 1$, $b_0 = w$ and $K = 0$. For each t, if $b_t \notin S_{[1]}^*$, let $k' \ge K$ be minimal such that b_t contains the letter $s_{0k'}$Then

$$b_t = xsy$$

where $x = \prod_l x_l$ is a word in $S_{[1]} \cup \{s_{0k}\}_{k > k'}$, $s = s_{0k'}^{\varepsilon}$ ($\varepsilon \in \{\pm 1\}$), and y a word in $S_{[1]} \cup \{s_{0k}\}_{k \ge k'}$. We then set $a_{t+1} = a_t s$ and b_{t+1} be the word

$$b_{t+1} = \left(\prod_l \widetilde{[s^{-1}, x_l]} \cdot x_l\right) y$$

where the tilde indicates replacing the commutator its shortest representing word in the alphabet $S_{[i+1]}$ if $x_l \in S_i$. Note that we chose C so that each such "replacement word" has at most C letters.

To see that the process must terminate after at most r steps, set $E_k(0) = 0$ for all k, and set $E_k(t+1) = \begin{cases} E_k(t)+1 & b_t = xsy;\, s = s_{0k} \\ E_k(t) & \text{otherwise} \end{cases}$ .It is then clear that $\sum_k E_k(t)$ is increasing in t and bounded above by the number of letters of S_0 appearing in w, which is at most r. Also, $|e_k(t)| \le E_k(t)$ for all t. Say that process terminates after $T \le r$ steps. We thus have $w = a_T b_T$ where $a_T = \prod_k s_{0k}^{e_k(T)}$, $b_T \in S_{[1]}^*$. To estimate the word length of b_T we consider the directed forest whose vertices are given by letters of all the words b_t and a letter x_l in b_t is connected to the letters in b_{t+1} which replaced $[s^{-1}, x_l]$. A vertex of the forest is a root iff it can be thought of as one of the "original" letters of b_0, and hence there are at most $rT \le r^2$ roots. The degree of every vertex is at most C by definition, and each path has length at most n. It follows that there are at most $C^n r^2$ leaves in the tree, and hence that $|b_T| \leq C^{n+1} r^2$.

This construction gives an injective map

$$B_S(r) \to B_{\mathbb{Z}^{\#S_0}}(r) \times B_{\mathrm{Cay}(\Gamma^{(1)};S_{[1]})}(C^{n+1} r^2)\,.$$

Now $\mathbb{Z}^{\#S_0}$ and $\Gamma^{(1)}$ have polynomial growth (the second by induction) so we are done. □

Problem Set 3

Measures for non-analysts

NOTATION. For a locally compact Hasudorff space X, we write $C_c(X) \subset C_0(X) \subset C_b(X) \subset C(X)$ for the spaces of compactly supported continuous functions on X, the space of continuous functions decaying at infinity, the space of bounded continuous functions, and the space of all continuous functions. When X is compact these spaces are all equal to the space $C(X)$ of continuous functions. $C_b(X)$ is a Banach space w.r.t. the supremum norm in which $C_0(X)$ is a closed subspace, in fact the close of $C_c(X)$. If U is a relatively compact open subset of X there is a natural norm-preserving embedding $C_0(U) \hookrightarrow C_c(X)$ given by extending each $f \in C_0(U)$ to be zero on $X \setminus C$ (check this!).

DEFINITION. Let X be a locally compact space. A *finite measure* on X will mean a bounded linear functional on $C_0(X)$, that is a linear functional $\mu \colon C_0(X) \to \mathbb{C}$ with a constant M such that for all $f \in C_0(X)$, $|\mu(f)| \leq M \|f\|_\infty$. A *Radon measure* on X will mean a linear functional $\mu \colon C_c(X) \to \mathbb{C}$ such that, for each relatively compact open subset $U \subset X$, the restriction of μ to $C_0(U)$ is a finite measure (why can't we use the restrictions to compact sets instead?). We give each space of measures the *weak-* topology*: we say that $\mu = \lim_{n\to\infty} \mu_n$ if, for each $f \in C_c(X)$, $\mu(f) = \lim_{n\to\infty} \mu_n(f)$.

For a measure μ and a function f we sometimes write $\int f d\mu$ for $\mu(f)$. Given a Radon measure μ on X and $1 \leq p < \infty$, we let $L^p(\mu)$ denote the closure of $C_c(X)$ in the norm $\left(\int |f|^p \, d\mu\right)^{1/p}$.

1. (a special case of the Banach-Alaoglu Theorem) Let X be a locally compact space. Show that the spaces $\mathcal{M}(X)$ of probability measures on X is compact in the weak-* topology.
   Hint: Embed the space of measures in a product of compact balls.

Haar measure

Let G be a first countable locally compact group. In other words, G is a locally compact space endowed with a continuous map $G \times G \to G$ $(g,h) \mapsto g^{-1}h$ satisfying the group axioms, and there is a nested sequence of open sets $U_1 \supset U_2 \cdots \supset U_n \supset \cdots$ such that any open neighbourhood of the identity contains one of the U_n.

2. Let $f, f' \in C_c(X)$ be non-negative, and let $U \subset G$ be open. Set
$$(f \colon U) = \inf \left\{ \sum_{i=1}^{n} \alpha_i \mid \alpha_i \geq 0, f \leq \sum_{i=1}^{n} \alpha_i \cdot 1_{g_i U} \right\}.$$
   Show that $0 \leq (f : U) < \infty$. Assuming $f' \neq 0$ show that $(f \colon U) \leq (f' : U)(f : f')$ for an appropriately defined $(f : f')$ which is independent of U.

3. Fix $f_0 \in C_c(X)$ which is non-negative and non-zero. For a non-principal ultrafilter ω on the integers, show that $\mu(f) \stackrel{\text{def}}{=} \lim_\omega \frac{(f \colon U_n)}{(f_0 \colon U_n)}$ is a G-invariant positive Radon measure on G. Such μ is called a (left) *Haar measure* on G.

4. Let $\mathcal{N}$ be the set of open neighbourhood of the identity in G. For any $U \in \mathcal{N}$ set $F_U = \{V \in \mathcal{N} \mid V \subset U\}$. Show that $\mathcal{F} = \{S \subset \mathcal{N} \mid \exists U : S \supset F_U\}$ is a filter. Show how to use an ultrafilter extending this filter to remove the countability assumption.

5. Show that μ extends to a finite measure on G iff G is compact.

FACT. *If μ' is any other left Haar measure on G then $\mu' = c\mu$ for some $c \in \mathbb{R}_{>0}$.*

Amenability

DEFINITION. Let G be a topological group, X a topological space. A *continuous action* of G on X is a continuous map $G \times X \to X$ satisfying the usual axioms for a group action.

From now on all group actions will be assumed continuous.

5. Let X be a compact G-space. Show that $(g \cdot f)(x) = f(g^{-1}x)$ defines a continuous linear action of G on $C(X)$. Conclude that G acts on the space of measures on X.

6. Let G be a locally compact group. Show that the following are equivalent:
   (1) G has a *left-invariant mean*, that is a positive linear map $m\colon C_{\mathrm{b}}(G) \to \mathbb{C}$ such that $m(1_G) = 1$ and $m(g \cdot f) = m(f)$ for all $g \in G$ and $f \in C_{\mathrm{b}}(G)$.
   (2) Whenever G acts on a compact space, G fixes a probability measure on X.
   Hint: As in ex. 1 above, the space of bounded positive functionals on $C_{\mathrm{b}}(G)$ is compact.

   DEFINITION. Call G *amenable* if it satisfies these equivalent properties.

7. Show that every compact group is amenable.

8. Let $N \lhd G$ be a closed normal subgroup.
   (a) Assume that G is amenable and show that G/N is amenable.
   (b) Assume that both N and G/N are amenable and show that G is amenable.
   Hint: consider the space of N-invariant measures.

   REMARK. We will later show that any closed subgroup of an amenable group is amenable.

9. Let G be discrete, and assume every finitely-generated subgroup of G is amenable. Show that G is amenable as well.

10. Show that every discrete abelian group is amenable.
    Hint: start with $\mathbb{Z}$.

11. Show that every discrete nilpotent group is amenable.

1.10. Volume growth in groups

Program (Gromov): classification of groups up to quasi-isometry.

We fix a group Γ and a finite symetric genrating set S. Let $X = \mathrm{Cay}(\Gamma;S)$, thought of as a metric space with the graph metric (the word metric w.r.t. S). This is a transitive metric space.

DEFINITION 87. The (volume) *growth* of Γ is the function[1] $N(r) = \#B_X(1,r)$. Note that $N(r) \leq |S|^r$ and that this depends on the choice of the generating set S.

(1) Say Γ has *polynomial growth* if $N(r) \ll r^d$ for some $d > 0$.
(2) Say Γ has *exponential growth* if $N(r) \gg c^r$ for some $c > 1$.
(3) Otherwise, say Γ has *intermediate growth*.

When Γ has polynomial growth, we set $d(\Gamma) = \limsup \frac{\log N_\Gamma(r)}{\log r}$ be the growth exponent of the group – this is the infimum of d such that $N_\Gamma(r) \leq Cr^d$ for some C. The results of this Part imply that if $d(\Gamma) < \infty$ it is an integer.

LEMMA 88. *Properties (1),(2) are QI-invariant. In fact, the number $d(\Gamma)$ is a QI-invariant. In particular they are independent of the choice of S and agree for commensurable groups.*

Free groups, as well as non-elementary hyperbolic groups (see the next chapter) have exponential growth. It is a non-trivial fact that there exist groups of intermediate growth. The first examples, realized as subgroups of the automorphism group of the rooted infinite binary tree, is due to Grigorchuk [].

1.11. Groups of polynomial growth: Algebra

Let Γ be a f.g. group of polynomial growth. Let $X = \mathrm{Cay}(\Gamma;S)$.

LEMMA 89. *Let $\varphi\colon \Gamma \to \mathbb{Z}$ be surjective. Then $\ker\varphi$ is finitely generated.*

PROOF. Let $\Delta = \ker\varphi$ and choose a generating set S for Γ of the form $s_0^{\pm}, \dots, s_k^{\pm}$ with $\varphi(s_0) = 1$, $\varphi(s_i) = 0$ for $1 \leq i \leq k$. Let

$$S_m = \left\{ s_0^t s_i s_0^{-t} \mid |t| \leq m,\, 1 \leq i \leq k \right\}$$

and set

$$\Delta_m = \langle S_m \rangle \subset \Delta.$$

Assume this sequence does not stabilize. Then for every m we can find $\alpha_m = s_0^{t_m} s_{i_m} s_0^{-t_m} \in S_m \setminus \Delta_{m-1}$. Let

$$B_m = \left\{ \prod_{i=1}^{m} \alpha_i^{\varepsilon_i} \mid \underline{\varepsilon} \in \{0,1\}^m \right\}.$$

Then $|B_m| = 2^m$ but $B_m \subset B_\Gamma(m(2m+1))$, a contradiction. ☐

LEMMA 90. *Let $\Gamma_1 < \Gamma$ be finitely generated and of infinite index. Then $d(\Gamma') \leq d(\Gamma) + 1$.*

[1] It's somewhat better to indentify putative growth functions f_1, f_2 if $cf_1(ar) \leq f_2(r) \leq Cf_1(br)$ for some $a,b,c,C > 0$ and all $r > 0$. Among polynomial functions the equivalence classes for this relation are given by the degree of the polynomial.

PROOF. Let $\bar{e} = \overline{x_0} \sim \overline{x_1} \sim \cdots \sim \overline{x_j} \sim \cdots \bar{x}_r$ be a path of length r in the connected infinite graph $\Gamma_1 \backslash X$, and assume the edges are labelled by s_j. It then follows that the subsets $B_0 = B_{\Gamma_1}(r)$, $B_1 = B_{\Gamma_1}(r)s_1$, $B_r = B_{\Gamma_1}(r)s_1 s_2 \cdots s_r$ of Γ are all disjoint, where thw balls $B_{\Gamma_1}(r)$ are given in terms of a fixed a set of generators $S_1 \subset \Gamma_1$. It follows that $N_\Gamma((R+1)r) \geq rN_{\Gamma_1}(r)$ where $S_1 \subset B_\Gamma(R)$. □

LEMMA 91. *Let Λ be a free abelian group, $\alpha \in \mathrm{Aut}(\Lambda)$, thought of as an element of $\mathrm{GL}(\Lambda \otimes \mathbb{C})$.*

(1) There exists a non-trivial α-stable sublattice $\Lambda' < \Lambda$ such that $\alpha \restriction \Lambda' \otimes \mathbb{C}$ is diagonable.
(2) If α is diagonable and all its eigenvalues lie in S^1, then α has finite order.
(3) If α has an eigenvalue of absolute value > 1, then we can find $x \in \Lambda$ and $e \in \mathbb{N}$ such that the elements

$$\left\{ \sum_{i=0}^{m} \varepsilon_i \alpha^{ei}(x) \middle| \underline{\varepsilon} \in \{0,1\}^m \times \{1\} \right\}$$

are all distinct.

PROOF. Let λ_j be the spectrum of α. For every $0 \leq k \leq \mathrm{rk}(\Lambda)$ let $P_k(\alpha) = \prod_j (\alpha - \lambda_j)^k \in \mathbb{Z}[\alpha] \subset \mathrm{End}(\Lambda \otimes \mathbb{Q})$ and let $V_k = \ker P_k(\alpha)$, $\Lambda_k = \Lambda \cap V_k$. For the maximal k such that $V_k \neq \{0\}$, $V_k \cap \Lambda$ works.

If all the eigenvalues are of absolute value 1 then orbits of α on $\Lambda \otimes \mathbb{C}$ are all bounded.

Otherwise, choose e such that α^e has an eigenvalue λ of absolute value at least 2, let $\beta \in \mathrm{Hom}(\Lambda \otimes \mathbb{C}, \mathbb{C})$ be non-zero such that $\beta \circ \alpha^e = \lambda \beta$, and let $x \in \Lambda$ be such that $\beta(x) \neq 0$. Then

$$\beta \left(\sum_{i=0}^{m} \varepsilon_i \alpha^{ei}(x) \right) = \left(\sum_{i=0}^{m+1} \varepsilon_i \lambda^i \right) \beta(x).$$

□

LEMMA 92. *(Inductive step) Let $\varphi\colon \Gamma \to \mathbb{Z}$ be surjective, and assume $\ker \varphi$ is virtually nilpotent. Then Γ is virtually nilpotent.*

PROOF. Let $\Delta < \ker \varphi$ be a maximal normal nilpotent subgroup, and let $z \in \varphi^{-1}(1)$. Then $\langle \Delta, z \rangle$ is of finite index in Γ. Then z normalizes Δ; in particular it normalizes its characteristic subgroups $\Delta^{(i)}$. We can refine this into a z-normalized central series $\Delta = \Delta_0 \triangleright \Delta_1 \triangleright \cdots \triangleright \Delta_r = \{1\}$ such that every Δ_{i-1}/Δ_i is either finite or a finitely generated free abelian group on which z_i acts via a semi-simple automorhpism α_i. If α_i is not of finite order then Δ/Δ_i does not have polynomial growth, since the previous Lemma constructs 2^m elements in a ball of radius $O(m)$. It follows that some power z^T centralizes each quotient. Then $\langle \Delta, z^T \rangle$ is nilpotent and of finite index in Γ. □

COROLLARY 93. *Let Γ be a virtually solvable group. Then either Γ is virtually nilpotent or it has exponential growth.*

THEOREM 94. *(Gromov) Let Γ be a group of polynomial growth. Then Γ has a finite index subgroup that surjects onto $\mathbb{Z}$.*

For the proof see Section 1.14.

THEOREM 95. *(Gromov) Every group of polynomial growth is virtually nilpotent.*

PROOF. By induction on $\lfloor d(\Gamma) \rfloor$, noting that $d(\Gamma) < 1$ implies that Γ is finite.

Let Γ be a group of polynomial growth, $\varphi\colon \Gamma_1 \to \mathbb{Z}$ be surjective with Γ_1 of finite index in Γ. Let $\Delta = \ker \varphi$. By Lemma 89, Δ is finitely generated. Lemma 90 then shows that $d(\Delta) \leq d(\Gamma) - 1$. By the inductive hypothesis, Δ is virtually nilpotent. By Lemma 92, so is Γ. □

1.12. Solvability of amenable linear groups in characteristic zero (following Shalom [11])

LEMMA 96. *Let F be a local field, $\Gamma \subset \mathrm{GL}_n(F)$ be amenable and have semi-simple Zariski closure. Then its topological closure is compact.*

PROOF. We may assume $\Gamma < \mathrm{GL}_n(\mathbb{C})$. Let G be its Zariski closure; $R = \mathrm{Rad}(G)$ (a solvable group), $H = G/R$ a semisimple group with finite center. Since $\Gamma \cap R$ is solvable, it suffices to show that the image $\Gamma R/R \subset H$ is finite. Dividing out by the center of H, we may thus assume wlg that Γ is a Zariski-dense amenable subgroup of the center-free semisimple group $H \subset \mathrm{GL}_m(\mathbb{C})$. Then for every automorphism $\varphi \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, $\varphi(\Gamma)$ is an amenable subgroup with semisimple Zariski-dense closure $\varphi(H)$. Since for every φ, the eigenvalues of all elements of $\varphi(\Gamma)$ are of modulus 1, it follows that these eigenvalues are all algebraic (in fact, roots of unity).

Now [Zi, 6.1.7] shows that there exists an embedding $\rho\colon H \to \mathrm{GL}_r(\mathbb{C})$ such that $\rho(\Gamma) \subset \mathrm{GL}_r(K)$ for a number field K. Let $\mathbb{H} \subset \mathrm{GL}_r$ be the K-subgroup which is the Zariski closure of $\rho(\Gamma)$. For each place $v \in |K|$, $\rho(\Gamma) \subset \mathrm{GL}_r(K_v)$ is amenable and its Zariski closure is the semisimple group $\mathbb{H}(K_v)$. Let L_v be its topological closure, a compact subgroup by the Lemma.

We have thus established that the image of $\rho(\Gamma)$ in $\mathrm{GL}_r(\mathbb{A}_K)$ is contained both in the discrete subgroup $\mathrm{GL}_r(K)$ and in the compact subgroup $\prod_v L_v$. It follows that $\rho(\Gamma)$ is finite. □

1.13. Facts about Lie groups

Let G be a Lie group with finitely many components, $\Gamma < G$ a finitely generated group.

THEOREM 97. *(Jordan) There exists $q = q(G) < \infty$ such that if Γ is finite then Γ has an abelian subgroup of index at most q.*

THEOREM 98. *(Tits alternative) Either Γ contains a free subgroup or it is virtually solvable.*

COROLLARY 99. *Assume Γ is infinite. Then either Γ has exponential growth, or it has a finite index subgroup that surjects onto $\mathbb{Z}$.*

PROOF. It's enough to consider the case of virtually solvable Γ, where shall repeatedly use Corollary 69. First, we may assume w.l.g. that Γ is solvable. Next, as long as Γ^{ab} is finite and non-trivial we may replace Γ with Γ', which is also finitely generated. After finitely many steps we must have Γ^{ab} infinite; otherwise Γ would have a composition series consisting of finite groups and hence be finite. Now Γ^{ab} is an infinite, finitely generated abelian group. By the classificatio theorem of such groups (Fact 74), Γ^{ab} (hence Γ) surjects onto $\mathbb{Z}$. □

1.14. Metric geometry – proof of theorem 94

Fix a group Γ of polynomial growth, S a finite generating set. Let $l(r) = \log N(r)$.

DEFINITION 100. Say r is *n-regular* if for $0 \le j \le n$, $l(2^{-j}r) \ge l(2r) - (j+1)(d+1)\log 2$.

LEMMA 101. *(existence of good scales) For each n we can find arbitrarily large n-regular scales r.*

PROOF. Consider the radii $r_k = 2^k$ and assume that, from some point onward, they are all not n-regular. Then for each k large enough we can find $1 \le j \le n+1$ such that $N(r_k) \ge 2^{j(d+1)}N(r_{k-j})$. It follows by induction that $N(r_k) \gg 2^{(d+1)k} = r_k^{d+1}$. □

From now on we fix an increasing sequence $\{r_n\}_{n=1}^{\infty}$ such that r_n is n-regular, a non-principal ultrafilter ω on $\mathbb{N}$. We then set:

$$(Y, d_\omega, \underline{e}) = \lim_{\omega} \left(X, \frac{1}{r_n} d_S, e \right).$$

PROPOSITION 102. *Y has the following properties:*

(1) It is transitive.
(2) It is a geodesic
(3) Every ball of radius 1 is compact.
(4) Y is of finite Hausdorff dimension.

PROOF. Let $\underline{x} \in \tilde{Y}$, and let $\gamma_n \in \Gamma$ satisfy $\gamma_n 1 = x_n$. By Proposition 64 $\lim_\omega \gamma_n$ is an isometry of Y mapping $\underline{e}$ to $\underline{x}$.

Since X is quasi-isometric with multiplicative distortion 1 to a locally geodesic space, Y is geodesic.

By transitivity it suffices to check the compactness of $B_Y(1)$; that would follow if, for each $j \geq 1$, we show that $B_Y(1)$ may be covered by a finite number of balls of radius 2^{-j+2}. For this let $n \geq j$ and let $T_n \subset B(r_n)$ be a maximal subset such that any two points are at distance greater than $2^{-j+1} r_n$. Then the balls $\left\{B_X(x, 2^{-j} r_n)\right\}_{x \in T_n}$ are all disjoint and contained in $B_X(e, (1+2^{-j}) r_n)$. We thus have:

$$|T_n| N(2^{-j} r_n) \leq N\left((1+2^{-j}) r_n\right) \leq N(2r_n).$$

By regularity we have

$$N(2^{-j} r_n) \geq 2^{-(j+1)(d+1)} N(2r_n),$$

and hence

$$|T_n| \leq 2^{(j+1)(d+1)}.$$

Set $I = 2^{(j+1)(d+1)}$. Repeating points, if necessary, we may assume that $T_n = \left\{t_n^i\right\}_{i=1}^{I}$ and set $\tilde{T} = \left\{\underline{t}^i\right\}_{i=1}^{T}$. Let $\underline{x} \in B_Y(1)$. By definition this implies that, for a majority of $n \in \mathbb{N}$, $d_S(x_n, e) \leq (1+2^{-j+1}) r_n$. For such n, let $x_n' \in B_X(r_n)$ lie on the shortest path connecting e and x_n and be as close as possible to x_n. Then $d_S(x_n, x_n') \leq 2^{-j+1} r_n$. For some $i = i(n)$ we have $d_S(x_n', t_n^i) \leq 2^{-j+1} r_n$ (otherwise we could add x_n' to T_n), and hence $r_n^{-1} d_S(x_n, t_n^i) \leq 2^{-j+2}$. Now for some $1 \leq i_0 \leq I$,

$$\left\{n \in \mathbb{N} \mid d_S(x_n, e) \leq (1+2^{-j+1}) r_n \,\text{and} i(n) = i_0\right\} \in \omega.$$

It follows that $d_\omega(\underline{x}, \underline{t}^{i_0}) \leq 2^{-j+2}$. In other words, $B_Y(1)$ is covered by the balls $\left\{B_Y(\underline{t}^i, 2^{-j+2})\right\}_{i=1}^{I}$.

Finally, since we can cover $B_Y(1)$ by at most $2^{(j+1)(d+1)}$ balls of radius 2^{-j+2}, it has covering dimension at most $d+1$; this also bounds the Hausdorff dimension. □

COROLLARY 103. *Y is a proper geodesic metric space of finite Hausdorff dimension.*

PROOF. By the transitivity, every ball of radius in Y is compact, and we may apply the Hopf-Rinow Theorem (Thm. 35). □

THEOREM 104. *(Montomery-Zippin, Gleason) Let $G = \mathrm{Isom}(Y)$ with Y as in the Corollary. Then G is a Lie group with finitely many connected components.*

Proof that Γ virtually surjects on $\mathbb{Z}$.

PROOF. Let Y be the asymptotic cone of $X = \mathrm{Cay}(\Gamma;S)$ constructed above, $G = \mathrm{Isom}(Y)$. The diagonal action of Γ on $\Gamma^{\mathbb{N}}$ gives a group homomorphism $\Gamma \to G$.

Case I Assume first that the image is infinite. It is then a finitely generated subgroup of polynomial growth in G and by Corollary 99 the image has a finite index subgroup which surjects onto $\mathbb{Z}$.

Otherwise, the image is finite. Let Δ_1 be the kernel of the of the homomorphism, a subgroup of finite index in Γ. Let Δ_2 be the intersection of all subgroups of index at most q of Δ_1 (q given by Jordan's Theorem 97). Let $S_1 \subset \Delta_1$ be a generating set. For $s \in S_1$ and $r \in R$ let $\delta_r(s) = \max\{d_S(x,sx) \mid x \in B_X(1,r)\}$, $\delta_r(S_1) = \max_{s\in S_1} \delta_r(s)$.

Case IIa Assume that $\sup_{r>0} \delta_r(S_1) < \infty$. Since $d_S(x,sx) = d_S(1,x^{-1}sx)$ it follows in this case that the conjugacy class of of each $s \in S_1$ is finite, and hence that the $Z_\Gamma(s)$ are of finite index in Γ. Their joint intersection with Δ_1, the center of Δ_1, is then of finite index in Δ_1.

Case IIb $\delta_r(S_1)$ is unbounded. We consider the action of Δ_1 on various asymptotic cones of X to show that Δ_2 has arbitarily large abelian quotients. It will follow that Δ_2^{ab} is infinite; since it's finitely generated it will surject on $\mathbb{Z}$.

Since Δ_1 acts trivially, $\lim_\omega r_n^{-1}\delta_{r_n}(S_1) = 0$. Given $\varepsilon > 0$, there exists a majority $A \in \omega$ such that $\delta_{r_n}(S_1) < \varepsilon r_n$ for $n \in A$. Now note that $\delta_{r+m}(S_1) \le \delta_r(S_1) + 2m$, while for $\gamma \in \Gamma$,

$$\delta_r(\gamma^{-1}S_1\gamma) \le \delta_{r+|\gamma|_S}(S_1) \le \delta_r(S_1) + 2\,|\gamma|_S\,.$$

By symmetry,

$$\left|\delta_r(S_1) - \delta_r(\gamma^{-1}S_1\gamma)\right| \le 2\,|\gamma|_S\,.$$

Since $\delta_r(S_1)$ is unbounded as $r \to \infty$, so is $\delta_r(\gamma^{-1}S_1\gamma)$ with r fixed and $\gamma \in \Gamma$ varying. For each $n \in A$ we can thus find γ such that $\delta_{r_n}(\gamma^{-1}S_1\gamma) > \varepsilon r_n$. Since $\delta_{r_n}(\gamma^{-1}S_1\gamma)$ can jump by at most 2 as we vary γ by one generator, there exists $\gamma_n \in \Gamma$ such that

$$\left|\delta_{r_n}(\gamma_n^{-1}S_1\gamma_n) - \varepsilon r_n\right| \le 2\,.$$

Consider now $Y_\varepsilon = \lim_\omega\left(X, \frac{1}{r_n}d_S, \gamma_n\cdot e\right)$. For $s \in S_1$ and $x_n \in B_X(\gamma_n e, r_n)$, $n \in A$ we have $d_S(sx_n,x_n) \le \varepsilon r_n + 2$. By the triangle inequality, every $\gamma \in \Delta_1$, though of as an element of $\mathrm{Isom}(X)$, satisfies (for $n \in A$)

$$\frac{1}{r_n}d_S\left(\gamma\cdot\gamma_n e, \gamma_n e\right) \le |\gamma|_{S_1}\left(\varepsilon + \frac{2}{r_n}\right).$$

Taking the limit we see that Δ_1 acts by isometries on Y_ε. Since X is transitive, Y_ε is isometric to Y and we have a homomorphism $\rho_\varepsilon\colon \Delta_1 \to G$. If $\rho(\Delta_1)$ is infinite we are back in case I so we may assume the image is finite.

We first check that it is non-trivial. For $n \in A$, $\delta_{r_n}(\gamma^{-1}S_1\gamma_n) \ge \varepsilon r_n - 2$. There thus exists $s_1 \in S_1$ and a majority $A_1 \in \omega$ contained in A such that for $n \in A_1$ there exists $x_n \in B_X(\gamma_n e, r_n)$ with $d_S(s_1x_n,x_n) \ge \varepsilon r_n - 2$. Taking the limit we see that $d_\omega\left(\rho_\varepsilon(s_0)\underline{x},\underline{x}\right) \ge \varepsilon$ and in particular that $\rho_\varepsilon(s_0) \neq 1$. On the other hand, the same limiting argument shows that $\rho_\varepsilon(s_0)$ is $(\varepsilon - B_Y(1))$-close to the identity of G. Using the exponential map it is clear that $\rho(s_0)$ must have order at least $\Omega(1/\varepsilon)$. We conclude that if Δ_1 only has finite images in $\mathrm{Isom}(Y)$ then these images have unbounded order.

Jordan's theorem implies that in each case $\rho_\varepsilon(\Delta_2)$ is abelian. This image has order at least

$$\frac{\#\rho(\Delta_1)}{[\Delta_1 : \Delta_2]} = \Omega(1/\varepsilon)\,.$$

□

Problem Set 4

1. Let Γ be quasi-isometric to $\mathbb{Z}$ (such groups are said to be *elementary*). Show that Γ is virtually isomorphic to $\mathbb{Z}$.

Growth Exponents

2. Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}_{\geq 0}$ be *sub-additive*, that is $a_{n+m} \leq a_n + a_m$. Show that $\lim_{n\to\infty} \frac{a_n}{n}$ exists. Let

   Γ be a group generated by the symmetric set S of size $2k$, $X = \mathrm{Cay}(\Gamma; S)$.

DEFINITION 105. $B(r)$ will denote the the ball of radius r in X. When r is *even* we also set $W(r) = \left\{ w \in S^l \mid w = 1 \text{ in } \Gamma \right\}$, $W'(r)$ the subset of *reduced* words (so that $W'(r)$ is empty iff Γ is freely generated by S). Note that there need not be any relations of odd length in Γ. We associate to Γ three exponents (depending on the choice of S, of course):

(1) The *growth exponent* is the number $g(\Gamma; S) = \lim_{r\to\infty} \frac{1}{r} \log_{2k-1} \#B(r)$.
(2) The *gross cogrowth exponent* is the number $\theta(\Gamma; S) = \lim_{\text{even } r\to\infty} \frac{1}{r} \log_{2k} \#W(r)$.
(3) The *cogrowth exponent* is the number $\eta(\Gamma; S) = \lim_{\text{even } r\to\infty} \frac{1}{r} \log_{2k-1} \#W'(r)$, with the proviso that $\eta = \frac{1}{2}$ for the free group.

3. Show that the limits above exist.

4. In fact, show that for any (resp. even) r, $\#B(r) \geq (2k-1)^{gr}$, $\#W(r) \geq (2k)^{\theta r}$ and $\#W'(r) \geq (2k-1)^{\eta r+2}$.

5. Show that Γ is freely generated by S iff $g(\Gamma; S) = 1$.

6. Let $g(\Gamma; S) = 0$. Show that Γ is amenable.
   Hint: an invariant mean can be found as a limit of averaging on balls.

Random Walks

Let $G = (V, E)$ be a connected locally finite graph (that is, E is a symmetric $\mathbb{Z}_{\geq 0}$-valued function on $V \times V$ which takes even values on the diagonal). For a vertex $u \in X$ let $d_G(u)$ denote the *degree* of u (that is the number of edges leaving u). Let $C(V)$ denote the space of ($\mathbb{C}$-valued) functions on V, and let and let $M\colon C(V) \to C(V)$ be the operator

$$(Mf)(x) = \frac{1}{d_G(u)} \sum_{(u,v)\in E} f(v).$$

Let ν_G be the measure on V assigning to u the weight $d_G(u)$.

7. Show that $\|M\|_{L^\infty(\nu_G)} = \|M\|_{L^1(\nu_G)} = 1$. Conclude that $\|M\|_{L^2(\nu_G)} \leq 1$. Show that M is self-adjoint on $L^2(\nu_G)$.

8. Show that $(M^t f)(x) = \sum_y p_t(x,y) f(y)$ where $p_t(x,y)$ is the standard random walk on X.

Clearly $p_{2t}(x,x) = \frac{\#W(2t)}{(2k)^{2t}}$ is the return probability of the random walk. Also, $\frac{\#W'(2t)}{2k(2k-1)^{2t-1}}$ is the return probability of the *non-backtracking* random walk on X.

9. Let $\lambda(\Gamma;S)$ denote the spectral radius of M. Show that $\lambda(\Gamma;S) = (2m)^{\theta-1}$.

10. Grigorchuk formula: $(2m)^{\theta} = (2m-1)^{\eta} + (2m-1)^{1-\eta}$ – hence we set $\eta = \frac{1}{2}$ for the free group.

Part 3

Hyperbolic groups

1.15. The hyperbolic plane

Let $\mathbb{H}^2$ be the model Riemannian manifold with underlying set $\mathbb{R} \times \mathbb{R}_{>0}$ (we shall denote the points by $z = x + iy$ with $y > 0$) and metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. This metric is conformal to the Euclidean metric and hence has the same angles.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}^2$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \stackrel{\text{def}}{=} \frac{az+b}{cz+d}.$$

Indeed, for $z = x+iy \in \mathbb{H}^2$, $a,b,c,d \in \mathbb{R}$ we have $cz+d \neq 0$ unless both $c,d = 0$, and

$$\Im\left(\frac{az+b}{cz+d}\right) = \frac{\Im(z)}{|cz+d|^2} > 0.$$

It is also easy to check that this is an action and that it preserves the metric. This action is transitive (use NAK decomposition) and the stabilizer of the identity is $K = \mathrm{SO}_2(\mathbb{R})$ (including the central element -1 which acts trivially).

In fact, $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on pairs at a fixed distance. Also, given three distances d_1, d_2, d_3 satisfying the triangle inequality fix two points x_2, x_3 at distance d_1 from each other. Then there exists either one or two points x_1 such that $d(x_1,x_2) = d_2$, $d(x_1,x_3) = d_3$.

Since the action of $\mathrm{PSL}_2(\mathbb{C})$ on the Riemann sphere $\hat{\mathbb{C}}$ preserves the class of lines and circles, the same holds for the action of $\mathrm{PSL}_2(\mathbb{R})$. Since the geodesic connecting iy_1, iy_2 is the imaginary axis it follows that geodesic rays meet the boundary $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ at right angles and are vertical lines or semicircular arcs with diameter on $\mathbb{R}$.

LEMMA 106. *Every triangle in $\mathbb{H}^2$ has area at most π.*

PROOF. Acting by an isometry we may assume our triangle has vertices iy_1, iy_2, z where $y_2 > y_1$, $\Re(z) \neq 0$, and $[iy_1, iy_2]$ is the longest side. Then $y_1 < \Im(z) < y_2$. Let $x_1, x_3 \in \mathbb{R}$ be such that $[iy_1, z] \subset [x_1, x_3]$. Then the ideal triangle x_1, ∞, x_3 contains the triangle iy_1, iy_2, z. Translating both triangles, we may assume that the ideal triangle has vertices $-R, \infty, R$. Its area is then at most:

$$\int_{|x|\leq R} dx \int_{x^2+y^2 \geq R^2} \frac{dy}{y^2} = \int_{-R}^{R} \frac{dx}{\sqrt{R^2-x^2}} = \pi.$$

□

1.16. δ-hyperbolic spaces

Let (X,d) be a proper geodesic metric space.

DEFINITION 107. X is called δ-hyperbolic if it satisfies the *slim triangles condition*: given any points $x,y,z \in X$ and any three geodesics $[x,y]$, $[y,z]$, $[z,x]$ connecting them, the image of the geodesic $[x,y]$ is contained in a δ-neighbourhood of the union of the images of the other two.

EXAMPLE 108. Any metric tree is 0-hyperbolic. A metric space is called an $\mathbb{R}$-tree if it is 0-hyperbolic.

PROPOSITION 109. *The hyperbolic plane $\mathbb{H}^2_\kappa$ is δ-hyperbolic where δ only depends on κ.*

PROOF. It clearly suffices to consider the case $\kappa = -1$. We shall exhibit a quasi-isometric equivalence of the hyperbolic plane with T_3, the 3-regular tree.

Alternatively, let x, y, z be three points in the hyperbolic plane, and let $q \in [y, z]$ be at least δ away from the union of $[x, y]$ and $[x, z]$. It follows that the convex hull of the three points contains the a semicircle of radius δ around q, and hence that 2π exceeds the area of a disk of radius δ. □

From now on assume X is δ-hyperbolic.

LEMMA 110. *Let $c\colon [0,1] \to X$ be a continuous rectifiable path in X parametrized proportional to arclength connecting $A = c(0)$ and $B = c(1)$. If $[A,B]$ is any geodesic segment connecting its endpoints and $x = [A,B]_t$ then $d(x, c([0,1])) \le \delta \log_2^+ l(c) + \delta + 1$.*

PROOF. If $l_2(c) \le 1$ then $d(A,B) \le 1$ and this is clear. Otherwise, let $C = c(1/2)$ and choose geodesic segments $[A,C]$ and $[C,B]$. Since X is δ-hyperbolic, x is within δ of a point x' on one of these segments, w.l.g. $x' = [A,C]_{t'}$. If $l(c) < 2$ then $d(A,C) < 1$. By induction it follows that x' is within 1 of the image of c, hence x is within

$$\delta + 1 + \delta \log_2^+ l(c)\,.$$

If $l(c) \ge 2$ it follows by induction that x' is within $\delta\,(\log_2 l(c) - 1) + \delta + 1$ of $c([0, \frac{1}{2}])$. □

LEMMA 111. *Let $\gamma\colon [a,b] \to X$ be an L,D-quasi-geodesic. Then there exists an $L,(4L+3D)$-quasigeodesic $\gamma'\colon [a,b] \to X$ with the same endpoints such that:*

(1) The Hausdorff distance between the images of γ, γ' is at most $2(L+D)$.
(2) γ' is an $(L+D)$-Lipschitz map. In particular, it is continuous and rectifiable.

PROOF. Let $V = \{a,b\} \bigcup \mathbb{Z} \cap [a,b]$, and let γ' be a concatenation of geodesic segements agreeing with γ on V. Since every segment has length at most $L+D$ this map is $(L+D)$-Lipschitz.

Given any $[t,t'] \subset [a,b]$ let $\lfloor t \rfloor$ be the point of V just below t, $\lceil t' \rceil$ the point above t'. Then $|t - \lfloor t \rfloor| \le 1$ and the same for t'. Then $d(\gamma(t), \gamma([t])) \le L + D$ by assumption, and $d(\gamma'(t), \gamma'([t])) \le L + D$ by the Lipschitz property. In particular, $d(\gamma(t), \gamma'(t)) \le 2(L+D)$.

This also implies:

$$\begin{aligned} d(\gamma'(t), \gamma'(t')) &\le d(\gamma([t]), \gamma([t'])) + 2(L+D) \\ &\le L\left|[t] - [t']\right| + D + 2(L+D) \\ &\le L\left|t - t'\right| + 2L + 2L + 3D\,. \end{aligned}$$

On the other side,

$$\begin{aligned} d(\gamma'(t), \gamma'(t')) &\ge d(\gamma([t]), \gamma([t'])) - 2(L+D) \\ &\ge \frac{1}{L}\left|[t] - [t']\right| - D - 2(L+D) \\ &\ge \frac{1}{L}\left|t - t'\right| - 2L - 2D\,. \end{aligned}$$

□

THEOREM 112. *Let $\gamma\colon [a,b] \to X$ be an L,D-quasigeodesic connecting A,B. Let $[A,B]$ be any geodesic segment connecting A,B. Then $d_H(\gamma([a,b]), [A,B]) \le R(\delta, L, D)$.*

PROOF. By the Lemma we may assume γ is Lipschitz. Let c parametrize $[A,B]$ according to arclength, and assume that $c(t)$ is at maximal distance R from the image of γ. Let $y=\gamma(s_1), z=\gamma(s_2)$ on the image of γ be at distance at most R from $A'=c(t-2R)$, $B'=c(t+2R)$. Let γ' be the restriction of γ to $[s_1,s_2]$

Since $d(y,z)\leq 6R$, $|s_1-s_2|\leq L(6R+2L+2D)$ and hence $l(\gamma')\leq L(L+R)(8L+2D)$. It follows that $[A',y]\cup\gamma'\cup[z,B']$ is a rectifiable curve of length at most k_1R+k_2 connecting A',B' and such that $c(t)$ lies on a geodesic connecting A',B' and is at distance at least R from the curve. By the previous Lemma, $R\leq\delta\max\{\log_2(k_1R+k_2),0\}+\delta+1$ hence $R\ll R_0(\delta,L,D)$.

Let $[a',b']\subset[a,b]$ be maximal such that $\gamma([a',b'])$ lies outside the R_0-neighbourhood of c. Every point of c is within R_0 of the image of γ, so we can find $w=c(t)$ such that $s\in[a,a']$ and $s'\in[b',b]$ such that $d(w,\gamma(s))\leq R_0$ and $d(w,\gamma(s'))\leq R_0$. Then $d(\gamma(s),\gamma(s'))\leq 2R_0$, so the length of $\gamma([a',b'])$ is bounded in terms of δ,L,D. □

DEFINITION 113. A path $c\colon[a,b]\to X$ is a *k-local geodesic* if $d(c(t),c(t'))=|t-t'|$ for $t,t'\in[a,b]$ with $|t-t'|\leq k$.

THEOREM 114. *Let c be a k-local geodesic with $k>8\delta$. Let γ be a geodesic connecccting $c(a)$ and $c(b)$. Then:*

(1) c lies in a 2δ-neighbourhood of γ.
(2) γ lies in a 3δ-neighbourhood of c.
(3) c is a quasi-gedoesic.

PROOF. Let $x'=c(t)$ be at maximal distance from the image of γ. Assume $t-a$, $b-t$ both greater than 4δ, and let $y'=c(a')$, $z'=c(b')$ such that $a'<t<b'$ is cenetered at t, of length between 8δ and k.

Say $y,z\in\gamma$ are closest to y',z'. Get quadrilateral y,y',z',y. Adding a diagonal shows that x' is 2δ-close to some w on a side other than c. If $w\in[y,y']$, then

$$\begin{aligned} d(x',y)-d(y,y') &\leq [d(x',w)+d(w,y)]-d(y,w)+d(w,y')] \\ &= d(x',w)-d(y',w) \\ &\leq d(x',w)-[d(y',x')-d(x',w)] \\ &\leq 2d(x',w)-d(x',y') \\ &< 4\delta-4\delta=0. \end{aligned}$$

Similarly $w\notin[z,z']$. It follows that $w\in[y,z]$, that is that any point of c is within 2δ of γ.

Now if $p=\gamma(t)$, □

1.17. Problem set 5

The Gromov product

Let (X,d) be a metric space.

DEFINITION 115. For $x,y,z \in X$ set

$$(y \cdot z)_x = \frac{1}{2}\left(d(y,z) + d(z,x) - d(y,z)\right).$$

Say that X is (δ)-hyperbolic if for every $x,y,z,w \in X$:

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (y \cdot z)_w\} - \delta. \tag{1.17.1}$$

This inequality is equivalent to the symmetric condition

$$d(x,w) + d(y,z) \leq \max\{d(x,y) + d(z,w),\, d(x,z) + d(y,w)\} + 2\delta.$$

Note that this makes sense even if X is not geodesic.

1. When X is a tree, verify that $(y \cdot z)_x$ is the distance from x to the geodesic segment $[y,z]$. If X is geodesic and δ-hyperbolic, verify that $|d(x,[y,z]),(y \cdot z)_x| \leq \delta$. Conclude that every δ-hyperbolic space is (δ)-hyperbolic.

For the converse see

Thin Triangles

Let X be a geodesic space, and let $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_0]$ be a geodesic triangle in X.

2. Show that there exist $a_i \in [x_{i-1},x_{i+1}]$ such that $d(x_i,a_{i+1}) = d(x_i,a_{i-1})$ ($i \pm 1$ calculated in $\mathbb{Z}/3\mathbb{Z}$).
3. Let X be δ-hyperbolic. Show that $d(a_i,a_{i+1}) \leq 4\delta$.
   Hint: a_i must be δ-close to a point $p \in [x_i,x_{i+1}] \cup [x_i,x_{i-1}]$. Say $p \in [x_i,x_{i+1}]$. Then the distance from x_{i+1} to p must be close to the distance between x_{i+1} and a_{i-1}.

A converse result also holds. See

Exponential divergence of geodesics

Let X be geodesic space.

DEFINITION 116. A map $e\colon \mathbb{N} \to \mathbb{R}$ is said to be a *divergence function* for X if for all $R,r \in \mathbb{N}$, $x \in X$, and any two geodesics $\gamma_i\colon [0,R+r] \to X$ ($i = 1,2$) issuing from x and parameterized according to length, the following condition holds:

If $d(\gamma_1(R),\gamma_2(R)) > e(0)$ then any path connecting $A_1 = \gamma_1(R+r)$ to $A_2 = \gamma_2(R+r)$ and lying outside the ball $B(x,R+r)$ has length at least $e(r)$.

4. Assume X is δ-hyperbolic. Show that $\max\left\{12\delta, 2^{(n-1)/\delta}\right\}$ is a divergence function.
   Hint: Consider a geodesic triangle with two sides given by $\gamma_i \restriction_{[0,R+r]}$. The their endpoints are A_1,A_2 with $d(A_1,A_2) = 2l$. Let m be the midpoint of the third side.

The Gromov Boundary

Part 4

Random Groups

1.18. Models for random groups

General scheme.

(1) Few-relators model
(2) Density models
(3) "Temperature" model
(4) Graph model
(5) Zuk's model

1.19. Local-to-Global

1.19.1. The Gromov-Papasoglu "Cartan-Hadamard" theorem.

DEFINITION 117. Let X be a complex of dimension 2.

(1) A *circle drawn in* X is a cycle in the 1-skeleton. A *disk drawn in* X is a cellular map from a complex isomorphic to a disk to X.
(2) Let f be a face of X. $L_{\mathrm{c}}(f) = |\partial f|$ will both denote the number of edges on the boundary of f. We also set the *combinatorial area* $A_{\mathrm{c}}(f)$ equal to this number. The *Euclidean area* of f is $A_{\mathrm{E}}(f) = |\partial f|^2$ [a regular n-gon in the plane has area $\sim \frac{n^2}{4\pi}$ if the sides have length 1].
(3) Let D be a disk drawn in X. $|\partial D|$ will denote the combinatorial length of its boundary, $A_{\mathrm{c}}(D)$ the total combinatorial area of its faces, $A_{\mathrm{E}}(D)$ the total Euclidean area, $A_{\mathrm{f}}(D)$ will denote the number of faces.

THEOREM 118. *(Short ...) Let* $\Gamma = \langle S|R\rangle$ *be a finite presentation,* $C > 0$*. Suppose that all minimal van-Kampen diagrams* D *w.r.t. this presentation satisfy*

$$|\partial D| \geq C \cdot A_c(D)\,.$$

Then Γ *is hyperbolic. In fact,* $X = \mathrm{Cay}(\Gamma;S)$ *is* $12\ell/C^2$*-hyperbolic where* ℓ *is the longest relation in* R*.*

The proof is discussed in [**8**, Prop. 7]

THEOREM 119. [**10**] *Assume that* X *is simply connected and* simplicial. *Let* $\mathcal{P}$ *be a property of disks in* X *such that any subdisk of a disk having* $\mathcal{P}$ *also has* $\mathcal{P}$*. Let* K *be an integer, and assume that any disk* D *drawn in* X *and having* $\mathcal{P}$ *satisfies:*

$$\frac{K^2}{2} \leq A_f(D) \leq 240K^2 \quad \Rightarrow \quad A_f(D) \leq \frac{1}{2\cdot 10^4}\,|\partial D|^2\,.$$

Then any disk D *having* $\mathcal{P}$ *and satisfies:*

$$K^2 \leq A_f(D) \quad \Rightarrow \quad A_f(D) \leq K\cdot|\partial D|\,.$$

PROOF. Let D be a disk of minimal boundary $> K^2$ such that $A_{\mathrm{f}}(D) > K\cdot|\partial D|$. Let f be a triangle in D with exactly one boundary edge. Then $D' = D - f$ has boundary length at most $|\partial D| + 1$. If $A_{\mathrm{f}}(D') = K^2$ then $A_{\mathrm{f}}(D) = K^2 + 1$, but then $A_{\mathrm{f}}(D) \leq 240K^2$ and by the assumptions of the Theorem, This implies $A_{\mathrm{f}}(D) \leq \frac{1}{5000}K^2$, a contradiction. It follows that $A_{\mathrm{c}}(D') > K^2$ as well, and hence $A_{\mathrm{c}}(D) - 1 = A_{\mathrm{c}}(D') \leq K(|\partial D| + 1)$. We conclude that

$$K\,|\partial D| < A_{\mathrm{c}}(D) \leq K\,|\partial D| + K + 1\,.$$

For similar reasons we may assume $L_{\mathrm{c}}(D) \geq 100K$: otherwise, $A_{\mathrm{c}}(D) \leq 100K^2 + K + 1$, and the same argument would imply $A_{\mathrm{f}}(D) \leq \frac{102^2}{2\cdot 10^4}K^2 < K^2$. Our goal now is to find an arc that will separate D into two pieces, one of which will violate the assumptions of the Theorem.

Let d_G be the graph metric on the 1-skeleton of D, and choose successive vertices $\{v_i\}_{i=1}^n$ on the boundary of D such that

$$d(v_i, v_{i+1}) = 20K$$

and $20K \leq d(v_n, v_1) < 40K$. Then $20K(n+1) \geq L_{\mathrm{c}}D$. If T is a connected subcomplex, we let $B_1(T)$ denote the the set of closed cells which intersect T, also a connected subcomplex. For a point $x \in D^0$ let , $B_0(x) = x$, $B_t(x) = B_1(B_{t-1}(x))$, $S_t(x) = B_t(P) - B_{t-1}(P)$.

Case 1: For any $i \neq j$, $B_{6K}(v_i) \cap B_{6K}(v_j) = \emptyset$ and for any i, $\mathrm{diam}(B_{6K}(v_i) \cap \partial D) \leq 20K$.

Then let $r < K$, let $C_r(v_i)$ be the closure of the connected component of $D - B_r(v_i)$ which contains the other v_j (they are connected along the boundary by assumption). Then $\gamma_r = C_r(v_i) \cap B_r(v_i)$ is an arc separating D into two simply connected parts, D_1 and D_2, with $v_i \in D_1$.

CLAIM 120. Let $l = |\partial D_1 \cap \partial D|$. Then $A_{\mathrm{f}}(D_1) \geq Kl - KL_{\mathrm{c}}(\gamma)$.

PROOF. $A_{\mathrm{f}}(D_1) \geq A_{\mathrm{f}}(D) - A_{\mathrm{f}}(D_2) > K|\partial D| - A_{\mathrm{f}}(D_2)$. By assumption, $A_{\mathrm{f}}(D_2) \leq K|\partial D_2| = K(|\partial D| - l + L_{\mathrm{c}}(\gamma))$. □

Case 1a: For some $1 \leq i \leq n$ and $K \leq r \leq 2K$, $l(\gamma) < K$.

Note that $D_1 \cap \partial D$ has length at least r on each side (when forming $B_r(v_i)$ we have to add a boudary edge at each step). It follows that

$$2K \leq |\partial D_1| \leq 21K\,.$$

By the Claim, $A_{\mathrm{f}}(D_1) \geq K^2$. We thus have $A_{\mathrm{f}}(D_1) \leq K|\partial D_1| \leq 21K^2$. Thus $A_{\mathrm{f}}(D_1) > \frac{1}{2\cdot 10^4}|\partial D_1|^2$ {but $\frac{1}{2} \leq A_{\mathrm{f}}(D_1)/K^2 \leq 240$.

Case 1b: We aren't in case 1a, and for some i and some $2K \leq r \leq 3K$, $L_{\mathrm{c}}(\gamma) < 80K$.

Then for $K \leq t \leq 2K-1$ we have $L_{\mathrm{c}}(\gamma_t) \geq K$. Each such γ_t contains at least K edges, and every triangle in $S_{t+1}(v_i)$ intersects at most two of them. It follows that $A_{\mathrm{c}}(D_1) \geq K^2/2$. Also, $|\partial D_1| < 80K + 20K = 100K$. It follows that $A_{\mathrm{c}}(D_1) < 100K^2$. But $K^2/2 > \frac{1}{2\cdot 10^4}|\partial D|^2$, a contradiction.

Case 1c: Otherwise, $l(\gamma_t) \geq 80K$ for $2K \leq t \leq 3K$ so $A_i = A_{\mathrm{c}}(B_{3K}(v_i)) \geq 40K^2$. Thus

$$A_{\mathrm{c}}(D) \geq \sum_i A_i \geq \left(\frac{|\partial D|}{20K} - 1\right)40K^2 > K|\partial D| + K + 1\,.$$

Case 2: Either $B_{6K}(v_i) \cap B_{6K}(v_j) \neq \emptyset$ for some $i \neq j$ or :diam$(B_{6K}(v_i) \cap \partial D) > 20K$ for some i. In either case, there exists an arc γ in D of length at most $12K$ which separates it into disks D_1, D_2 with $|\partial D_1 \cap \partial D|, |\partial D_2 \cap \partial D| \geq 20K$. We can assume that γ is an arc with this property and such that $|\partial D_1 \cap \partial D|$ is as small as possible. Then D_1 contains at least $\frac{|\partial D_1 \cap \partial D|}{20K} - 3$ of the points v_i, say $\{v_{i+k}\}_{k=1}^m$. Since γ is shortest possible, these points have $B_{3K}(v_{i+k}) \cap B_{3K}(v_{i+j}) = \emptyset$, and at most two of the $B_{3K}(v_{i+k})$ intersect γ. If $B_{3K}(v_{i+k})$ does not intersect γ then $\mathrm{diam}(B_{3K}(v_{i+k}) \cap \partial D) \leq |\partial D_1 \cap \partial D| \leq 20K$.

We now split into the same cases as above, but only using $v_i \in D_1$ such that $B_{3K}(v_i) \cap \gamma = \emptyset$. Cases a,b are the same. In case c, we get the inequality

$$A_{\mathrm{c}}(D_1) \geq \sum_s A_{\mathrm{c}}(B_{3K}(v_{i+s})) \geq \left(\frac{|\partial D_1|}{20K} - 6\right)40K^2\,.$$

By the minimality assumption, $A_c(D_1) \le K|\partial D_1|$ so $L_c(D_1) \le 240K$. Then $A_c(D_1) \le 240K^2$; by the claim we also have $A_c(D_1) \ge 8K^2$. Thus $A_c(D_1) > \frac{1}{2\cdot 10^4}|\partial D_1|^2$ which contradicts the assumptions. □

ALGORITHM 121. *(Papasoglu) To check if $\Gamma = \langle S|R\rangle$ is hyperbolic:*

(1) Convert $\langle S|R\rangle$ to a triangular presentation: replace the relation ab of length ≥ 4 with the two generator x and the relations $x = a$, $x^{-1} = b$ which are shorter, but of length ≥ 3.

(2) For every $K > 0$:

(a) Generate all van-Kampen diagrams D such that $A_f(D) \le 240K^2$. Determine which of the ones which also satisfy $A_f(D) \ge K^2/2$ are minimal.

(b) If all the minimal diagrams satisfy the hypothesis of the Theorem, then terminate.

PROOF. In the Cayley complex $\mathrm{Cay}(\Gamma;S)$, let $\mathcal{P}$ be the property of a disk being a minimal van-Kampen diagram for the boundary relation of the disk.

If the algorithm terminates, then by the Theorem every word has a small diagram, hence Γ is hyperbolic.

If Γ is hyperbolic then for some C, every word has a diagram which satisfies $A_f(D) \le C|w|_S$. In particular, for $K > 200C$ every minimal diagram with the appropriate area will satisfy the assumptions of the Theorem. □

COROLLARY 122. [**6**, Prop. 42]*Assume that X is simply connected, and that $|\partial f| \le \ell$ for every fact f of X. Let $\mathcal{P}$ be a property of disks in X such that any subdisk of a disk having $\mathcal{P}$ also has $\mathcal{P}$.*

Let $K \ge 10^{10}\ell$ be an integer, and assume that any disk D drawn in X and having $\mathcal{P}$ satisfies:

$$\frac{K^2}{10^3} \le A_E(D) \le 10^6 K^2 \quad \Rightarrow \quad |\partial D|^2 \ge 2\cdot 10^{14} A_E(D)\,.$$

Then any disk D having $\mathcal{P}$ satisfies:

$$|\partial D| \ge \frac{1}{10^4 K} A_E(D)\,.$$

PROOF. Note that a naive triangluation (divide an n-gon into $n-2$ triangles) won't work since we might get triangles with different sizes. Instead, intersect the regular Euclidean n-gon with edge length 1 (hence Euclidean area about $n^2/4\pi$) with the periodic triangulation of the plane into equilateral triangles of side 1. After sorting out the boundary we get a genuine triangulation with all sides between $1/10$ and 10, and area between $1/100$ and 100. Thus the distoration between the triangle metric and the Euclidean metric is at most 10. We have used about n^2 triangles.

Let Y be the simplical complex obtained from X by this triangulation, with combinatorial length L_{tr} and face-counting area A_{tr}. Let L, A be the *metric* length and area in Y where every face of X has the Euclidean metric from above. Then L_{tr}, L_c, L and A_{tr}, A_E, A are respectively uniformly equivalent by factors of at most 100.

Let B be a disk in Y with property $\mathcal{P}$ and area $1/2 \le A_{\mathrm{tr}}(B)/K^2 \le 240$. We shall verify that $L_{\mathrm{tr}}(B)^2 \ge 2\cdot 10^4 A_{\mathrm{tr}}(B)$.

If B comes from a disk D in X, then $L_{\mathrm{tr}}(B) \ge 10^{-2}|\partial D|$ while $A_{\mathrm{tr}}(B) \le 10^2 A_E(D)$. Otherwise, one approximates B by a disk from X by adding or removing the faces of X which are partially included in B, using isoperimetric inequalities for the unit disk of the Euclidean plane. □

PROPOSITION 123. *Assume every face $f \in X^2$ has $\ell_1 \le |\partial f| \le \ell_2$, and that for some $C > 0$ and an integer $K \ge 10^{24}(\ell_2/\ell_1)C^{-2}$, every disk $D \in \mathcal{P}$ with $A_c(D) \le K\ell_2$ satisfies*

$$C \cdot A_c(D) \le |\partial D| \,.$$

Then every disk in $\mathcal{P}$ satisfies

$$C' \cdot A_c(D) \le |\partial D|$$

where $C' = 10^{-15}C(\ell_1/\ell_2)$.

PROOF. We have $\ell_1 \le \frac{A_{\mathrm{E}}(D)}{A_{\mathrm{c}}(D)} \le \ell_2$ for any disk D in X, since this holds face-by-face. Reducing K if necessary we may assume $K \approx 10^{24}(\ell_2/\ell_1)C^{-2}$, and set $k^2 = K\ell_1\ell_2/10^6$. Then very disk $D \in \mathcal{P}$ with $10^{-3}k^2 \le A_{\mathrm{E}}(D) \le 10^6 k^2$ has $A_{\mathrm{c}}(D) \le K\ell_2$ and hence also $A_{\mathrm{c}}(D) \le C^{-1}|\partial D|$. We now calculate:

$$|\partial D|^2 \ge C^2 A_{\mathrm{c}}(D)^2 \ge C^2 \ell_2^{-2} A_{\mathrm{E}}(D)^2 \ge 10^{-3} C^2 \ell_2^{-2} k^2 A_{\mathrm{E}}(D) \ge 10^{-9} C^2 K(\ell_1/\ell_2) A_{\mathrm{E}}(D) \,.$$

Since $10^{-9}C^2K(\ell_1/\ell_2) \ge 2 \cdot 10^{14}$ and $k \ge 10^{10}\ell_2$ (note that $C \le 2$) we may apply the Corollary, to see that for any disk $D \in \mathcal{P}$,

$$|\partial D| \ge \frac{1}{10^4 k} A_{\mathrm{E}}(D) \ge \frac{10^3 \ell_1}{10^4\sqrt{K\ell_1\ell_2}} A_{\mathrm{c}}(D) \approx 10^{-13}(\ell_1/\ell_2) \cdot C \,.$$

□

1.19.2. Boostrapping the isoperimetric constant a-la [9]. We fix a finite presentation $\langle S|R\rangle$ where every relator has length between ℓ_1 and ℓ_2. Let $\mathcal{P}$ be a hereditary class of van-Kampen diagrams for this presentation, and assume that every $D \in \mathcal{P}$ satisfies

$$|\partial D| \ge C' \cdot A_{\mathrm{c}}(D) \,,$$

where we may assume $C' < 1$. We then set $\alpha = -\frac{1}{\log(1-C')} \le \frac{1}{C'}$.

We assume that small diagrams satsify $|\partial D| \ge CA_{\mathrm{c}}(D)$ and would like to extend this to all diagrams, perhaps with a small loss in the constant. We thus fix $\frac{1}{4} > \varepsilon > 0$.

LEMMA 124. [7, Lem. 9-10] *Let $D \in \mathcal{P}$. Then*

(1) D can be written as a disjoint union $D_1 \cup D_2$ where $D_1$is connected and all of its faces are within $\alpha \log(A_c(D)/\ell_2)$ of the boundary, and D_2 has area at most ℓ_2.
(2) D can be paritioned into two diagrams D', D'' by a path of length at most $\ell_2 + 2\alpha\ell_2\log(A_c(D)/\ell_2)$ connecting two boundary points, such that each of the two diagrams contains at least one quarter of the bounary of D.

PROOF. For D (or any disjoint union of simply connected subdiagrams) we have $|\partial D| \ge C'A_{\mathrm{c}}(D)$.

(1) The faces at distance 1 from the boundary have area at least $C'A_{\mathrm{c}}(D)$, the faces at distance at least 2 are at most $(1-C')A_{\mathrm{c}}(D)$. Removing the boundary faces and continuing by induction, the faces at distance at least k from the boundary have area at most $(1-C')^k A_{\mathrm{c}}(D)$. Now take $k = 1 + \alpha\log(A_{\mathrm{c}}(D)/\ell_2)$ (rounded to the nearest integer).

(2) Let $L = |\partial D|$. Assume first that D_2 is empty, and mark x,y,z,w on ∂D at distance $L/4$ from each other. There then exists a path of length at most $2\alpha\log(A_{\mathrm{c}}(D)/\ell_2)$ connecting a point of xy to a point of zw or xz and yw. If D_2 is non-empty, retracting each of its components to a point take a path as above. Now the total diameter of all components of D_2 is at most ℓ_2.

□

PROPOSITION 125. *(Induction step) Let $A \geq 50/(\varepsilon C')^2$ and suppose that every $D \in \mathcal{P}$ with boundary length at most $A\ell_2$ satisfies*

$$|\partial D| \geq C \cdot A_c(D)\,.$$

Then every diagram with boundary length at most $\frac{7}{6}A\ell_2$ satisfies

$$|\partial D| \geq (C-\varepsilon)A_c(D)\,.$$

PROOF. Since $\alpha \leq 1/C'$, we have $2+4\alpha\log(7A/6C') \leq \varepsilon A \leq A/4$.

Let $D \in \mathcal{P}$ be a diagram with $A\ell_2 \leq |\partial D| \leq \frac{7}{6}A\ell_2$. Partition D into D', D'' as in the Lemma, in which case:

$$\left|\partial D'\right|, \left|\partial D''\right| \leq \frac{3}{4}|\partial D| + \ell_2\left(1+2\alpha\log\frac{7A}{6C'}\right) \leq \ell_2\left(\frac{7A}{8}+\frac{A}{8}\right) = A\ell_2\,.$$

We thus have:

$$\begin{aligned}
|\partial D| &= \left|\partial D'\right| + \left|\partial D''\right| - 2\left|\partial D' \cap \partial D''\right| \\
&\geq \left|\partial D'\right| + \left|\partial D''\right| - 2\ell_2\left(1+2\alpha\log\frac{7A}{6C'}\right) \\
&\geq C\left(A_{\mathrm{c}}(D') + A_{\mathrm{c}}(D'')\right) - 2\varepsilon A\ell_2 \\
&\geq (C-\varepsilon)A_{\mathrm{c}}(D)\,,
\end{aligned}$$

since $A\ell_2 \leq |\partial D| \leq A_{\mathrm{c}}(D)$. □

REMARK 126. Note that the assumption on A is independent of C.

THEOREM 127. *Let $\varepsilon_0 \in (0,1/4)$, let $B \geq 50/(\varepsilon_0^2 C'^3)$ and assume that any diagram $D \in \mathcal{P}$ with area at most $B\ell_2$ satisfies*

$$|\partial D| \geq C_0 A_c(D)\,.$$

Then any diagram in $\mathcal{P}$ satisfies

$$|\partial D| \geq (C_0 - 14\varepsilon_0)A_c(D)\,.$$

PROOF. Let $A_0 = C'B$ and, recursively, $A_{n+1} = \frac{7}{6}A_n$, $\varepsilon_{n+1} = \sqrt{\frac{6}{7}}\varepsilon_n$ and $C_{n+1} = C_n - \varepsilon_n$. Note that then $A_n \geq 50/(\varepsilon_n C')^2$ for all n.

Let $D \in \mathcal{P}$ have $|\partial D| \leq A_0\ell_2$. Then $A_{\mathrm{c}}(D) \leq (C')^{-1}|\partial D| \leq B\ell_2$.

Assume now, by induction, that every digram with boundary size at most $A_n\ell_2$ satisfies $|\partial D| \geq C_n \cdot A_{\mathrm{c}}(D)$ (the case $n=0$ is the assumption of the Theorem). By the Lemma it follows that every diagram with boundary size at most $A_{n+1}\ell_2$ satisfies

$$|\partial D| \geq C_{n+1}A_{\mathrm{c}}(D)\,.$$

Finally, note that $C_n \geq C_0 - \varepsilon_0\sum_{n=0}^{\infty}\left(\frac{7}{6}\right)^{n/2} \geq C_0 - 14\varepsilon_0$. □

COROLLARY 128. *Let $\langle S|R\rangle$ be a finite presentation with every relation having length between ℓ_1 and ℓ_2. Let $\mathcal{P}$ be a hereditary class of van-Kampen diagrams for this presentation. Let $C > 0$, $\varepsilon \in (0,1/4)$ and suppose that for some $K \geq 10^{50}(\ell_2/\ell_1)^3\varepsilon^{-2}C^{-3}$, any diagram $D \in \mathcal{P}$ of area at most $K\ell_2$ satisfies*

$$|\partial D| \geq CA_c(D).$$

Then every diagram $D \in \mathcal{P}$ satisfies

$$|\partial D| \geq (C-\varepsilon)A_c(D).$$

PROOF. Let $C' = 10^{-15}(\ell_1/\ell_2)C$, $\varepsilon_0 = \varepsilon/14$, $B = 50/(\varepsilon_0^2 C'^3)$. By Proposition 123, any $D \in \mathcal{P}$satisfies $|\partial D| \geq C'A_{\mathrm{c}}(D)$. Since $B < K$ we can now apply the Theorem. □

1.20. Random Reduced Relators

Let $d \in (0,1/2)$, and let R_l be $\sim (2k-1)^{dl}$ reduced words of length l chosen uniformly at random.

THEOREM 129. *(Ollivier; Gromov) For every $\varepsilon > 0$ and $K \in \mathbb{N}$, a.a.s. every reduced van-Kampen diagram with at most K faces w.r.t. $\langle S|R_l\rangle$ satisfies*

$$|\partial D| \geq (1-2d-\varepsilon)lA_f(D).$$

PROOF. If D is an abstract decorated diagram involving m_i relators r_i with $m_1 \geq m_2 \geq \cdots$,
Let δ_i defined as before, then

$$|\partial D| \geq (1-2d)\ell\,|D|$$

□

Part 5

Fixed Point Properties

1.21. Introduction: Lipschitz Involutions and averaging

Let Y be a Hilbert space, and let $\sigma\colon Y \to Y$ be a Lipschitz involution. In other words, $\sigma^2 = \mathrm{id}$ and there exists $C \geq 1$ such that for all $x, y \in Y$ we have $\|\sigma x - \sigma y\| \leq C\|x - y\|$.

PROBLEM 130. Does σ have a fixed point?

If $C = 1$ this is clearly the case: σ is then an isometry, hence an affine map, and it follows that $\frac{y+\sigma y}{2}$ is fixed by σ for all y. When C is close to 1, we can still use the map $Ty = \frac{1}{2}y + \frac{1}{2}\sigma y$ to find a fixed point.

LEMMA 131. *Let Y be a complete convex metric space. If $C < 2$ then σ fixes a point of Y.*

PROOF. For $y \in Y$ set $\delta(y) = d(y, \sigma y)$, the *displacement length*. Our goal is to find y such that $\delta(y) = 0$. Consider $\delta(Ty)$. We have:

$$\begin{aligned}
\delta(Ty) = d(\sigma Ty, Ty) &\leq \frac{1}{2}d(\sigma Ty, y) + \frac{1}{2}d(\sigma Ty, \sigma y) \\
&\leq \frac{C}{2}d(Ty, \sigma y) + \frac{C}{2}d(Ty, y) \\
&= \frac{C}{2}d(y, \sigma y) = \frac{C}{2}\delta(y).
\end{aligned}$$

Fix any $y_0 \in Y$ and let $y_{n+1} = Ty_n$. Then $\delta(y_n) \leq \left(\frac{C}{2}\right)^n \delta(y_0)$. Since $d(Ty, y) = \frac{1}{2}\delta(y)$, it follows that $d(y_{n+1}, y_n) \leq \frac{1}{2}\left(\frac{C}{2}\right)^n \delta(y_0)$. When $\frac{C}{2} < 1$ this implies $d(y_n, y_{n+k}) \leq \frac{\delta(y_0)}{2-C}\left(\frac{C}{2}\right)^n$, in other words that $\{y_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Its limit y_∞ must satisfy $\delta(y_\infty) = 0$ by the continuity of δ. □

In a CAT(0) space we can prove a stronger result.

LEMMA 132. *Let Y be a complete CAT(0) space. If $C < \sqrt{5} \approx 2.23$ then σ fixes a point of Y.*

PROOF. Let $E(y) = d^2(y, \sigma y)$. Applying the CAT(0) inequality to the same triangle as in the previous Lemma gives:

$$\begin{aligned}
E(Ty) = d^2(\sigma Ty, Ty) &\leq \frac{1}{2}d^2(\sigma Ty, y) + \frac{1}{2}d^2(\sigma Ty, \sigma y) - \frac{1}{4}d^2(y, \sigma y) \\
&\leq \frac{C^2}{2}d^2(Ty, \sigma y) + \frac{C^2}{2}d^2(Ty, y) - \frac{1}{4}d^2(y, \sigma y) \\
&= \frac{C^2 - 1}{4}E(y).
\end{aligned}$$

Again take any y_0 and set $y_{n+1} = Ty_n$, for which we have $\delta(y_n) \leq \frac{\sqrt{C^2-1}}{2}\delta(y_0)$. If $C < \sqrt{5}$ the displacements decrease exponentially and the proof proceeds as before. □

PROPOSITION 133. *Let Y be a Hilbert space. If $C <$?? then σ fixes a point of Y.*

PROOF. Fix $\varepsilon > 0$. Given $y \in Y$ and $0 \leq t \leq 1$ set $T_t y = (1-t)y + t\sigma y$, and consider the vector $V_y(t) = \sigma T_t y - T_t y$. We have $V_y(0) = \sigma y - y$ and $V_y(1) = y - \sigma y = -V_y(0)$. Assume first that for all y there exists $t = t(y)$ such that $\delta(T_t y) = \left\|V_y(t)\right\| \leq (1-\varepsilon)\delta(y)$, and set $Ty = T_{t(y)}y$. Since $\|y - T_t y\| \leq \delta(y)$, it follows as before that $T^n y$ converge to a fixed point. Otherwise, there exists y such that the curve $t \mapsto V_y(t)$ connects $V_y(0)$ to $-V_y(0)$ while remaining outside the disc of radius

$(1-\varepsilon)R$ where $R = \|V_y(0)\|$. It is clear that the length of such a curve is at least $(1-O(\varepsilon))\pi R$. On the other hand,

$$\begin{aligned} \|V_y(t)-V_y(s)\| &\leq \|\sigma T_t y - \sigma T_s y\| + \|T_t y - T_s y\| \\ &\leq (C+1)\,|t-s|\,R\,. \end{aligned}$$

It follows that the length of the curve is at most $(C+1)R$, and hence that

$$C \geq (1-O(\varepsilon))\,\pi - 1\,.$$

Now for $C < \pi - 1$ we can choose ε small enough to derive a contradiction. □

LEMMA 134. *Let Y be a complete metric space, $\delta\colon Y \to \mathbb{R}_{>0}$ a continuous function. Fix $a > 2$. Then for each $y \in Y$ there exists $y' \in B(y, a\delta(y))$ such that for all $z \in B(y', \frac{a}{2}\delta(y'))$, $\delta(z) \geq \frac{1}{2}\delta(y')$.*

PROOF. Assume that, for each $y' \in B(y, a\delta(y))$ there exists $z = z(y') \in B(y', \frac{a}{2}\delta(y'))$ such that $\delta(z) < \frac{1}{2}\delta(y')$. Let $y_0 = y$, and by induction assume that $d(y_n, y) \leq a\delta(y)\sum_{k=1}^{n} 2^{-k} < a\delta(y)$ and that $\delta(y_n) < 2^{-n}\delta(y)$. There exists then y_{n+1} with $d(y_{n+1}, y_n) \leq \frac{a}{2}\delta(y_n) \leq a2^{-(n+1)}\delta(y)$ and such that $\delta(y_{n+1}) < \frac{1}{2}\delta(y_n) < 2^{-(n+1)}\delta(y)$. The bound on $d(y_{n+1}, y)$ follows.

As before, the sequence $\{y_n\}_{n=0}^{\infty}$ is Cauchy and therefore converges. It follows that δ vanishes somewhere, a contradiction. □

PROPOSITION 135. *Let $\Gamma = \langle S \rangle$ be a finitely generated group, Y a complete metric space, $\rho\colon \Gamma \to Lip(Y)$. For each $y \in Y$ set $\delta(y) = \max\{d_Y(y, sy) \mid s \in S\}$. Assume that $\inf_{y\in Y}\delta(y) = 0$ but that Γ does not fix a point on Y. Then there exists an asymptotic cone $Y_\omega = \lim_\omega \left(Y, y'_n, \frac{2}{\delta(y'_n)} d_Y\right)$ and an action $\rho_\omega\colon \Gamma \to Lip(Y_\omega)$ such that $\|\rho_\omega(\gamma)\|_{\mathrm{Lip}} \leq \|\rho(\gamma)\|_{\mathrm{Lip}}$ for each $\gamma \in \Gamma$, and $\delta(\underline{z}) \geq 1$ for each $\underline{z} \in Y_\omega$. The action and the displacement bound extend to the completion $\bar{Y}$ of Y_ω.*

PROOF. Choose $y_n \in Y$ such that $\delta(y_n) \to 0$. Let $a_n = 2 + \frac{1}{\sqrt{\delta(y_n)}}$. By the Lemma, for each n there exists $y'_n \in B(y_n, a_n\delta(y_n))$ such that for all $z_n \in B(y'_n, \frac{a_n}{2}\delta(y'_n))$, $\delta(z_n) \geq \frac{1}{2}\delta(y'_n)$. Let ω be a non-principal ultrafilter, and let Y_ω be as in the statement of the Lemma. First, for each $\gamma \in \Gamma$ and $y \in Y$, $d_Y(y, \gamma y) \leq |\gamma|_S\,\delta(y)$. This implies that $\frac{2}{\delta(y'_n)} d(y'_n, \rho(\gamma)y'_n) \leq 2\,|\gamma|_S$ is bounded independently of n. Since rescaling the metric does not change Lipschitz constants, it follows that the Γ action passes to a limiting action ρ_ω and clearly the Lipschitz constant at the limit cannot grow. Finally, let $\underline{z} = (z_n)$ be a representative for a point in Y_ω. Since $\frac{2}{\delta(y'_n)} d(y'_n, z_n)$ is bounded while $a_n \to \infty$, from some point onward we have $z_n \in B(y'_n, \frac{a_n}{2}\delta(y'_n))$. It follows that $\delta_\rho(z_n) \geq \frac{1}{2}\delta(y'_n)$. Rescaling the metric and passing to the limit we conclude $\delta_{\rho_\omega}(\underline{z}) \geq 1$. The last claim is obvious. □

COROLLARY 136. *Let K be the set of Lipschitz constants $C \geq 1$ such that every involution on a Hilbert space with Lipschitz constant at most C has a fixed point. Then $K = [1, L)$ for some $1 < L \leq \infty$.*

PROOF. Let σ_n be an involution of the Hilbert space Y_n with Lipschitz constant $L + \varepsilon_n$ without fixed points, where ε_n are positive and tend to zero. We would like to show that $L \notin K$. By the Proposition we may assume that, for every $z_n \in Y_n$, we have $d(\sigma_n z_n, z_n) \geq 1$, and rescaling the metrics we may assume that $\inf\{d(\sigma_n z_n, z_n) \mid z_n \in Y_n\} = 1$. We now choose $y_n \in Y_n$ such that $d(\sigma_n y_n, y_n) \leq 2$, fix a non-principal ultrafilter ω on the integers and let $Y_\omega = \lim_\omega \left(Y_n, y_n, \|\cdot\|_{Y_n}\right)$. It is clear that the σ_n induce a limiting action σ_ω on Y_ω (which is a pre-Hilbert space), an involution of Lipschitz constant at most L. It is also clear that this action displaces each $\underline{z} \in Y_\omega$ by at least 1 (this is the case at each co-ordinate). Taking the completion shows that $L \notin K$. □

Summary.

- Replace points $y \in Y$ with *orbits* $\{y, \sigma y\}$, that is equivariant functions $f\colon \Gamma \to Y$.
- Measure the "energy" of an orbit; $E(y) = d_Y^2(y, \sigma y)$ was used here.
- Construct an averaging operator on the orbit; we mostly used $Ty = \frac{1}{2}y + \frac{1}{2}\sigma y$.
- Show that averaging redues energy exponentially, and that the distance between y and Ty can be bounded using the energy of y.
- Conclude that iterated averaging converges to a fixed point.

1.22. Expander graphs

Let $G = (V, E)$ be a (possibly infinite) locally finite graph. We allow self-loops and multiple edges. For $x \in V$ the *neighbourhood of* x is the multiset $N_x = \{y \in V | (x, y) \in E\}$. Let $E(A, B) = |E \cap A \times B|$, $e(A, B) = |E(A, B)|$, $e(A) = e(A, V)$ for $A, B \subseteq V$. $A \mapsto e(A)$ is a measure on V, with density $d_x = \#N_x$ w.r.t. counting measure. Note that $e(V)$ is *twice* the (usual) number of edges in the graph, and let $\nu_G(A) = \frac{1}{2\#E} e(A)$ be the associated probability measure. Let $\mu_G(u \to v)$ be the standard random walk on G:

$$\mu_G(x \to y) = \frac{e(\{x\}, \{y\})}{d_x}.$$

This is a *reversible* Markov chain: we have $d\nu_G(x) d\mu_G(x \to y) = d\nu_G(y) d\mu_G(y \to x)$ as measures on $V \times V$.

DEFINITION 137. The "local average" operator $A_G : L^2(V) \to L^2(V)$ of G is:

$$(A_G f)(x) = \int d\mu_G(x \to y) f(v) = \frac{1}{d_x} \sum_{y \in N_x} f(y).$$

The reversibility of the Markov chain is equivalent to the self-adjointness of A_G as an operator on $L^2(\nu)$. Furthermore,

$$|\langle Af, g \rangle| \le \int d\nu_G(x) d\mu_G(x \to y) |f(x)| |g(y)| = \frac{1}{2\#E} \sum_{x \in V} |f(x)| \sum_{y \in N_x} |g(y)|.$$

Two applications of Cauchy-Schwarz give:

$$\begin{aligned} |\langle Af, g \rangle| &\le \tfrac{1}{2\#E} \sum_{x \in V} |f(x)| \left(\sum_{y \in N_x} 1 \right)^{1/2} \left(\sum_{y \in N_x} |g(y)|^2 \right)^{1/2} \\ &\le \left(\sum_{x \in V} |f(x)|^2 \frac{d_x}{2\#E} \right)^{1/2} \left(\frac{1}{2\#E} \sum_{x \in V} \sum_{y \in N_x} |g(y)|^2 \right)^{1/2} \\ &= \|f\|_{L^2(\nu)} \|g\|_{L^2(\nu)}. \end{aligned}$$

In other words, $\|A\|_{L^2(\nu)} \le 1$.

From now on we assume that G has finite connected components. Then by the maximum principle, $Af = f$ iff f is constant on connected components of G and $Af = -f$ iff f takes opposing values on the two sides of each bipartite component.

DEFINITION 138. The *discrete Laplacian* on V is the opeartor $\Delta_G = I - A_G$.

By the previous discussion it is self-adjoint, positive definite and of norm at most 2. The kernel of Δ is spanned by the characteristic functions of the components (e.g. if G is connected then zero is a non-degenerate eigenvalue). Its orthogonal complement $L^2_0(V)$ is the space of *balanced* functions (i.e. the ones who average to zero on each component of G). The infimum of the positive eigenvalues of Δ will be an important parameter, the *spectral gap* $\lambda_1(G)$. If $\lambda_1(G) \geq \lambda$ we call G a λ*-expander*. If, furthermore, G is connected, d-regular, and $\#V = n$ we say (Alon) that G is an (n,d,λ)-graph.

DEFINITION 139. Let $A \subset V$. The *edge boundary of* A is $\partial A = E(A, \neg A)$. The *Cheeger constant* of the graph G is:

$$h(G) = \min\left\{ \frac{e(A,\neg A)}{e(A,V)} \middle| A \subseteq V, e(A\cap X) \leq \frac{1}{2} e(X) \text{ for every component } X \subseteq V \right\}.$$

PROPOSITION 140. *(Buser inequality)* $h(G) \geq \frac{\lambda_1(G)}{2}$.

PROOF. We may assume that G is connected and take $A \subset X$ be such that $\nu_G(A) \leq \frac{1}{2}$. Let $B = V \setminus A$, and choose α, β so that $f(x) = \alpha 1_A(x) + \beta 1_B(x)$ is balanced. Then we have: $\lambda_1(G) \leq \frac{\langle \Delta f, f\rangle}{\langle f, f\rangle}$. Now,

$$\Delta f(x) = \begin{cases} \alpha - \frac{|N_x\cap A|}{|N_x|}\alpha - \frac{|N_x\cap B|}{|N_x|}\beta & x\in A \\ \beta - \frac{|N_x\cap A|}{|N_x|}\alpha - \frac{|N_x\cap B|}{|N_x|}\beta & x\in B \end{cases} = \begin{cases} \frac{|N_x\cap B|}{|N_x|}(\alpha-\beta) & x\in A \\ \frac{|N_x\cap A|}{|N_x|}(\beta-\alpha) & x\in B \end{cases},$$

so that $\langle \Delta f, f\rangle = (\alpha-\beta)\alpha|\partial A| + \beta(\beta-\alpha)|\partial B| = (\alpha-\beta)^2|\partial A|$ and thus

$$\lambda_1(G) \leq \frac{(1-\frac{\beta}{\alpha})^2}{e(A)+e(B)(\beta/\alpha)^2}|\partial A|.$$

$\langle f, \mathbb{1}\rangle = e(A)\alpha + e(B)\beta$, so that the choice $\beta/\alpha = -e(A)/e(B)$ makes f balanced. This means:

$$\lambda_1(G) \leq |\partial A| \frac{(e(B)+e(A))^2}{e(A)e(B)^2 + e(B)e(A)^2} = 2\frac{|\partial A|}{e(A)}\frac{e(B)+e(A)}{2e(B)}.$$

But $2e(B) \geq e(X) = e(A)+e(B)$ and we are done. □

Conversely,

PROPOSITION 141. *(Cheeger inequality)* $h(G) \leq \sqrt{2\lambda_1(G)}$.

PROOF. Let f be an eigenfunction of Δ of e.v. $\lambda \leq \lambda_1 + \varepsilon$, w.l.g. supported on a component X and everywhere real-valued. Let $A = \{x \in V | f(x) > 0\}$, $B = X \setminus A$. We can assume $e(A) \leq \frac{1}{2}e(X)$ by taking $-f$ instead of f if necessary. Let $g(x) = \mathbb{1}_A(x)f(x)$. Then for $x \in A$,

$$\Delta f(x) = f(x) - \frac{1}{d_x}\sum_{y\in N_x} f(x) = g(x) - \frac{1}{d_x}\sum_{y\in N_x\cap A} f(x) - \frac{1}{d_x}\sum_{y\in N_x\cap B} f(x)$$

$$= \Delta g(x) + \frac{1}{d_x}\sum_{y\in N_x\cap B}(-f(x)) \geq \Delta g(x).$$

Since also $(\Delta f)(x) = \lambda f(x)$ for all x, we have:

$$\lambda \sum_{x\in A} d_x g(x)^2 = \sum_{x\in A} d_x \Delta f(x)\cdot g(x) \geq \sum_{x\in A} d_x \Delta g(x)\cdot g(x),$$

or ($g\restriction_B = 0$):

$$\lambda_1 + \varepsilon \geq \lambda \geq \frac{\langle \Delta g, g\rangle}{\langle g, g\rangle}.$$

we now estimate $\langle \Delta g, g\rangle$ in a different fashion. Motivated by the continuous fact: $\nabla g^2 = 2g\nabla g$, we evaluate

$$I = \sum_{x\in V} d_x \frac{1}{d_x} \sum_{y\in N_x} |g(x)^2 - g(y)^2|$$

in two different ways. On the one hand,

$$I = \sum_{(x,y)\in E} |g(x)+g(y)|\cdot|g(x)-g(y)| \leq \left(\sum_{(x,y)\in E} (g(x)+g(y))^2\right)^{1/2} \left(\sum_{(x,y)\in E} (g(x)-g(y))^2\right)^{1/2},$$

and we note that

$$\begin{aligned}\sum_{(x,y)\in E} (g(x)-g(y))^2 &= \sum_{x\in V} d_x g(x) \frac{1}{d_x} \sum_{y\in N_x} (g(x)-g(y)) - \sum_{y\in V} d_y g(y) \frac{1}{d_x} \sum_{x\in N_y} (g(x)-g(y)) \\ &= 2\langle \Delta g, g\rangle\end{aligned}$$

and

$$\sum_{(x,y)\in E} (g(x)+g(y))^2 \leq 2 \sum_{(x,y)\in E} (g(x)^2+g(y)^2) = 4\langle g, g\rangle,$$

so:

$$I^2 \leq 8\langle \Delta g, g\rangle \cdot \langle g, g\rangle \leq 8\lambda_1 \langle g, g\rangle^2. \tag{1.22.1}$$

On the other hand, let $g(x)$ take the values $\{\beta_i\}_{i=0}^r$ where $0 = \beta_0 < \beta_1 < \cdots < \beta_r$, and let $L_i = \{x \in V | g(x) \geq \beta_i\}$ (e.g. $L_0 = V$). Then write:

$$I = 2 \sum_{(x,y)\in E} \sum_{a(x,y)<i\leq b(x,y)} (\beta_i^2 - \beta_{i-1}^2)$$

where $\{\beta_{a(x,y)}, \beta_{b(x,y)}\} = \{g(x), g(y)\}$ (i.e. replace $\beta_b^2 - \beta_a^2$ with $(\beta_b^2 - \beta_{b-1}^2) + \cdots + (\beta_{a+1}^2 - \beta_a^2)$). Then the difference $\beta_i^2 - \beta_{i-1}^2$ appears for every pair $(x,y) \in E$ such that $a(x,y) < i \leq b(x,y)$ or such that $\max\{g(x), g(y)\} \geq \beta_i^2$ while $\min\{g(x), g(y)\} < \beta_i^2$. This exactly means than $(x,y) \in \partial L_i$ and

$$I = 2\sum_{i=1}^r \left(\beta_i^2 - \beta_{i-1}^2\right) |\partial L_i|.$$

By definition of h, $L_i \subseteq A$ and $e(A) \leq E$ imply $|\partial L_i| \geq h\cdot e(L_i)$ so:

$$I \geq 2h\sum_{i=1}^r \left(\beta_i^2 - \beta_{i-1}^2\right) e(L_i) = 2h\sum_{i=1}^{r-1} \beta_i^2 (e(L_i) - e(L_{i+1})) + 2h\cdot e(L_r)\beta_r^2.$$

Also, $e(L_i) - e(L_{i+1}) = e(L_i \setminus L_{i+1})$ so:

$$I \geq 2h\sum_{i=1}^{r-1} \sum_{g(x)=\beta_i} \beta_i^2 d_x + 2h\cdot \sum_{g(x)=\beta_r} \beta_r^2 d_x = 2h\sum_{x\in V} d_x g(x)^2 = 2h\cdot\langle g, g\rangle. \tag{1.22.2}$$

We now combine Equations 1.22.1 and 1.22.2 to get:

$$2h\langle g, g\rangle \leq I \leq 2\sqrt{2(\lambda_1 + \varepsilon)}\langle g, g\rangle$$

for all $\varepsilon > 0$, or

$$h(G) \leq \sqrt{2\lambda_1(G)}.$$

□

Let us restate the previous two propositions in:

$$\frac{1}{2}\lambda_1(G) \leq h(G) \leq \sqrt{2\lambda_1(G)}.$$

1.22.1. References, examples and applications. The above propositions can be found in [**2**], with slightly different conventions. We also modify their definitions to read:

DEFINITION 142. Say that G is an *h_0-expander* if $h(G) \geq h_0$. Say that G is a *λ-expander* if $\lambda_1(G) \geq \lambda$.

The previous section showed that both these notions are in some sense equivalent. Being well-connected, sparse (in particular regular) expanders are very useful. See the survey [**3**].

The existence of expanders can be easily demonstrated by probabilistic arguments (see [**3**]). Infinite families of expanders are not difficult to find, e.g. the incidence graphs of $\mathbb{P}^1(\mathbb{F}_q)$ have $\lambda = 1 - \frac{\sqrt{q}}{q+1}$ (as computed in [**6**] and later in [**1**]). However families of *regular* expanders are more difficult. The next section discusses the generalization by Alon and Milman in [**1**] of a construction due to Margulis [**11**]. For a different explicit family of regular expanders which enjoys additional useful properties see [**10**].

We remark here that there exists a bound for the asymptotic expansion constant of a family of expanders:

THEOREM 143. *(Alon-Boppana) For every $d \geq 3$ and $\varepsilon > 0$ there exists $C = C(d, \varepsilon) > 0$ such that if G is a connected d-regular graph on n vertices, the number of eigenvalues of A in the interval*

$$[(2-\varepsilon)\frac{\sqrt{d-1}}{d}, 1]$$

is at least $C \cdot n$.

COROLLARY 144. *Let $\{G_m\}_{m=1}^{\infty}$ be a family of connected k-regular graphs such that $|V_m| \to \infty$. Then*

$$\limsup_{m\to\infty} \lambda_1(G_m) \leq 1 - \frac{2\sqrt{d-1}}{d}.$$

This leads to the following definition (the terminlogy is justified by [**10**]):

DEFINITION 145. A d-regular graph G such that $|\lambda| \leq 2\frac{\sqrt{d-1}}{d}$ for every eigenvalue $\lambda \neq \pm 1$ of A_G is called a *Ramanujan graph*.

THEOREM 146. *(??) Let $\{G_m\}_{m=1}^{\infty}$ be a family of connected d-regular graphs such that, for each k, the number of k-cycles in G_m is $o(|G_m|)$. Then the spectral measures of G_m converge to that of the tree.*

Problem Set

1. Concentration of measure on expanders application of expanders.
2. Spectrum of the regular tree.
3. Spectral gap for random graphs.

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