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Summary of the Generalised Theory of Electrical Machines as Applied to Synchronous Generators and Induction Motors

20.1 INTRODUCTION

A summarised description of the 'generalised theory' of electrical machines is given, with an emphasis on synchronous generators and induction motors. Many texts are available that provide detailed mathematical treatments of the subject, for example References 1 to 6. Some texts develop the theory from a more practical perspective such as References 7 to 12.

The mathematical treatments are very similar, but there are some subtle differences in the matrix transformations that are needed. Examples of these differences are, constants in the matrix inversions, directions of rotation of stator applied voltages, directions of rotation of the rotor shaft, invariance of power in transformation, base quantities for per-unit systems. In most cases the derived quantities e.g. synchronous reactances, transient reactances, sub-transient reactances, time contents are either the same or very nearly the same, after the inherent simplifications have been made. Usually the data used in power system studies are subject to reasonably large tolerances e.g. $\pm15\%$, $\pm25\%$. For some machinery and transformers the maximum ranges of these tolerances are given in the international standards e.g. IEC60034 Part 1, BS4999 Part 1. The results of the studies will therefore be subject to similar tolerances and so the benefit of applying a highly detailed set of equations in the study is questionable. Hence, a set of equations that has been simplified by reasonable assumptions will provide adequate results in most cases e.g. Reference 5, Chapters 12 and 13.

The theory described herein is primarily applicable to balanced three-phase circuits and to balanced disturbances such as the three-phase short circuit, changes of loading, switching lines or cables in or out of circuit. The theory as presented is not directly suitable for unbalanced conditions such as line-to-ground faults, line-to-line faults, single-phase loading and unbalanced loading. For unbalanced analysis the references given in this chapter should be studied in depth. When a power system is being designed for an oil industry plant the most important studies are those for balanced faults and disturbances. Usually the unbalanced situations are less severe and are of lesser importance. For unbalanced situations it is necessary to be sure that a proprietary computer

Handbook of Electrical Engineering: For Practitioners in the Oil, Gas and Petrochemical Industry. Alan L. Sheldrake
 ISBN: 0-471-49631-6

program does contain the appropriate mathematical equations, and that they are based on the properly applied theory.

20.2 SYNCHRONOUS GENERATOR

The theory described will assume that the synchronous generator (and motor) can be adequately presented by three balanced stator windings for connection to the supply, one field winding on the rotor and two damper windings on the rotor. The d-axis has the field winding (f) and one of the damper windings (kd) ascribed to it, whilst the q-axis has only one damper winding (kq). The theory presented starts from the well-established definitions of the most frequently encountered resistances, inductances and reactances that are used in proprietary computer programs and for which numerical data can usually be obtained from manufacturers. Numerical data can often be the design data before the machine is built, unless the machine is the standard product of the manufacturer in which case the data may have been derived from actual factory tests. Otherwise some of these data are verified during the testing of the machine before it is delivered to the customer. Testing is usually limited to obtaining the resistances and reactances in the d-axis and the stator windings. Special tests are required for obtaining the q-axis data (IEEE112, IEC60034), but these tests are not normally required by the customer. This means that the q-axis data are subject to a wider tolerance than the d-axis data by the time the machine is delivered to the customer.

The established definitions are:-

a) Resistances

R_a	Resistance of a stator or armature winding.
R_{kd}	Resistance of the d-axis rotor damping winding.
R_{kq}	Resistance of the q-axis rotor damping winding.
R_{fd}	Resistance of the d-axis rotor field winding.
R_{ext}	Resistance of a component connected in series with the stator winding, one in each phase.

Note: A lower case R is often used in the literature.

b) Inductances

M_{md}	Mutual inductance between windings in the d-axis.
M_{mq}	Mutual inductance between windings in the q-axis.
L_{la}	Leakage inductance of a stator winding.
L_{lkd}	Leakage inductance of the d-axis rotor damping winding.
L_{lkq}	Leakage inductance of the q-axis rotor damping winding.
L_{lfd}	Leakage inductance of the d-axis rotor field winding.
L_{ext}	Inductance of a component connected in series with the stator winding, one in each phase.

c) Reactances at the nominal system frequency ω_n

X_{md}	Mutual reactance between windings in the d-axis.
X_{mq}	Mutual reactance between windings in the q-axis.
X_{la}	Leakage reactance of a stator winding.

X_{lkd} Leakage reactance of the d-axis rotor damping winding.
X_{lkq} Leakage reactance of the q-axis rotor damping winding.
X_{lfd} Leakage reactance of the d-axis rotor field winding.
X_{ext} Reactance of a component connected in series with the stator winding, one in each phase.

d) Frequencies

f_n nominal cyclic frequency of the power system in cycles per second or hertz.
ω_n nominal angular frequency of the power system in radians per second $= 2\pi f_n$.
f any cyclic frequency of the power system within its normal operating range in cycles per second.
ω any angular frequency of the power system within its normal operating range in radius per second $= 2\pi f$.

Note: ω and f could be the frequency of the system when speed governor action is present, or the fundamental frequency of a variable frequency e.g. as used in the speed control of synchronous or induction motors.

e) Leakage inductances are due to flux which only links with its own winding and is caused by its own current.

f) Mutual inductances are due to flux which links two windings that share the same magnetic circuit. The same flux is created by either of the currents in the two windings. Mutual inductances are defined between two windings, not three or more, even though there may be several windings sharing the same magnetic circuit e.g. the d-axis of a synchronous machine, a three-winding three-phase transformer. The mutual inductance between the three pairs of windings in a magnetic circuit of three windings can often be assumed to be equal e.g. $M_{12} = M_{13} = M_{23} = M$.

g) Self or total inductances are the addition of the leakage inductance of a particular winding and the mutual inductance between it and another winding. See Figure 20.1.

h) Reactances at frequencies other than the nominal frequency.

Each of the reactances X_{md} through X_{ext} and others yet to be defined or that exist in the power system could be modified as the frequency of the system changes, e.g. during a long disturbance such as starting a large motor with a high inertia load. The necessary modification is simply to apply the ratio of the disturbance frequency (ω) to the nominal frequency (ω_n) as a multiplying factor e.g., X_{md} changes to $X_{md}\omega/\omega_n$. This modification applies especially to machines supplied from variable frequency power sources. In systems where the frequency deviations are small during a disturbance the modification is usually ignored. The difference in computed results will be small compared with the tolerances on the data used in the program.

i) Flux linkages

A coil or winding carrying a current I will produce a proportional amount of flux Ø, provided the permeability of the magnetic circuit remains constant for all values of the current. The winding will usually consist of N closed loops of conductor connected in series, with each loop being one turn. Hence the winding has N turns. The total amount of flux linking the N turns of the winding

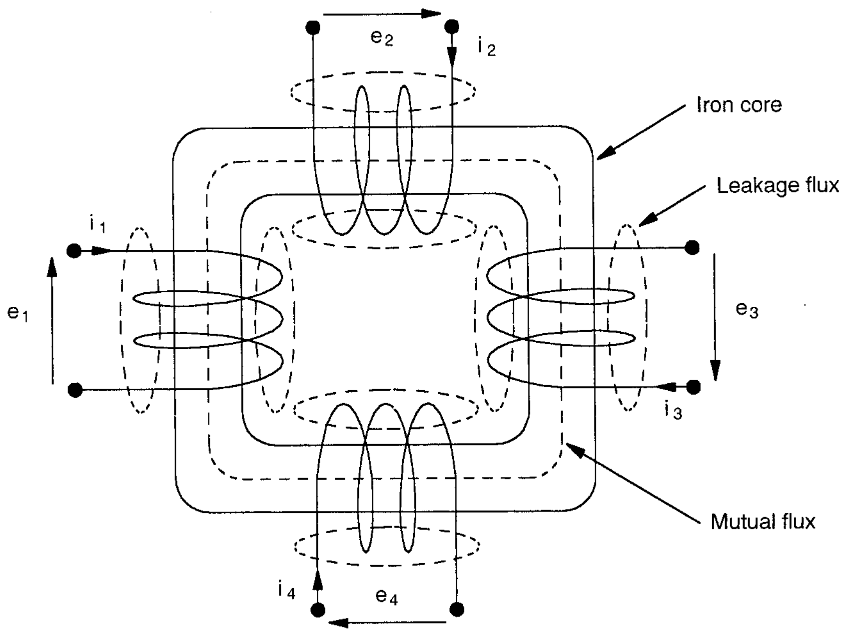

Figure 20.1 Mutually coupling and leakage fluxes in coils that share a common magnetic iron core.

is called the flux linkage ψ. An emf is induced in the winding when the current is changed and therefore when the flux linkage is changed in sympathy with the current. The emf e induced is:-

$$\begin{aligned} e &= \frac{d\psi}{dt} \\ &= \frac{d\psi}{dI} \cdot \frac{dI}{dt} = L\frac{dI}{dt} \end{aligned} \tag{20.1}$$

which opposes the applied voltage at the terminals of the winding.

Where L is the inductance of the winding in henrys (or flux linkages per ampere). If I varies sinusoidally then the emf is induced in a co-sinusoidal manner,

$$\begin{aligned} e &= L\frac{d(\hat{I}\sin\omega t)}{dt} = \omega L\hat{I}\cos\omega t \quad \text{volts} \\ &= X\hat{I}\cos\omega t \end{aligned}$$

where $X = \omega L$ is the inductive reactance at the frequency ω.

When the emf is induced in one winding by the current changing in a second winding the process is called 'induction by transformer action' or 'transformer induced emf'.

The rate of change of flux linkages can be brought about by rotating one winding with respect to a second winding, as is the fundamental situation in a motor or generator. If the current that produces the flux linkages is kept constant but its winding is rotated at an angular velocity ω_r then the emf induced is,

$$e = \omega_r \psi \quad \text{volts}$$

This process is called 'induction by rotating action' or 'rotationally induced emf'.

These two processes are fundamental to the induction of emfs in all the windings of a motor or generator.

20.2.1 Basic Mathematical Transformations

The generalised theory when applied in a suitable manner has the very convenient effect of removing the sinusoidal variations that are at the frequency of the power system. The frequency variations are those which are associated with the instantaneous currents, voltages and emfs. Their removal occurs, when these variables are transformed to the d and q axes. In effect the d and q-axes stator currents and voltages become envelope values of their corresponding stator three-phase sinusoidal quantities. This is very advantageous when digital computers are used to solve single machine and especially multi-machine transient problems. This is similar to using rms quantities in circuit analysis instead of instantaneous quantities. The labour and calculation times are greatly reduced. Two commonly used matrix transformations for currents, voltages and emfs are:-

a) Transform a, b, c variables to d, q, o variables

$$\begin{pmatrix} v_d \\ v_q \\ v_o \end{pmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \; k \begin{pmatrix} \cos\theta & \cos(\theta - 120^\circ) & \cos(\theta + 120^\circ) \\ \sin\theta & \sin(\theta - 120^\circ) & \sin(\theta + 120^\circ) \\ 0.5 & 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \\ v_c \end{pmatrix} \tag{20.2}$$

b) Transform d, q, o variables to a, b, c variables

$$\begin{pmatrix} v_a \\ v_b \\ v_c \end{pmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \; k_i \begin{pmatrix} \cos\theta & \sin\theta & 1.0 \\ \cos(\theta - 120^\circ) & \sin(\theta - 120^\circ) & 1.0 \\ \cos(\theta + 120^\circ) & \sin(\theta + 120^\circ) & 1.0 \end{pmatrix} \begin{pmatrix} v_d \\ v_q \\ v_c \end{pmatrix} \tag{20.3}$$

Where (20.3) is the inverse transformation of (20.2) and the lower-case letter 'v' represent the instantaneous variation of the corresponding peak value of voltage 'v'. The same transformations apply to the instantaneous currents i_a through i_o. The suffices 'o' are attached to the zero sequence instantaneous quantities, which are essentially added to the matrices to make them invertable. Under balanced circuit conditions and balanced disturbances the zero sequence components have no effect on the computed results. Their use in the 'generalised theory' to study line-to-ground faults and single-phase unbalanced loading should be approached with some caution. The combining of the symmetrical component theory with the 'general theory' should be undertaken with care, the additional mathematics becomes formidable, see Reference 5, Chapters 9 and 10, Reference 13, and Reference 3, Chapter X.

The two constants k and k_i have different values in the literature and occur as interrelated pairs e.g. where $k = 2/3$, $k_i = 1.0$ see References 5, 7, 8 and 13, when $k = \sqrt{2/3}$, $k_i = \sqrt{2/3}$ and the 0.5 and 1.0 constant become $\sqrt{1/2}$ see References 10, 13 and 14. The most commonly used constants are $k = 2/3$ and $k_i = 1.0$. Harris *et al*, Reference 13, Chapter 3, discuss this subject at length, in relation to power invariance and the choice of base parameters for per-unit systems. Bimbhra, Reference 10, also discusses transformations in considerable detail.

From (20.1) the emf induced in a winding is,

$$e = \frac{d\psi}{dt}$$

The voltage (v) applied to the winding must always balance this emf (e) and the resistive volt-drop (IR) of the winding conductor carrying the current, hence:-

$$v = RI + \frac{d\psi}{dt}$$

Where $d\psi/dt$ will in some windings be a combination of transformer induced and rotationally induced emfs. The flux linkages ψ will be the sum of its own linkages due to its own currents and all the linkages from windings sharing the same magnetic circuit. For the synchronous generator which has three stator windings and three rotor windings, as described in sub-section 20.2 a) to g), the set of voltage equations are:-

$$\begin{pmatrix} v_a \\ v_b \\ v_c \\ v_f \\ v_{kd} \\ v_{kq} \end{pmatrix} = \begin{pmatrix} R_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_a & 0 & 0 & 0 & 0 \\ 0 & 0 & R_a & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{fd} & 0 & 0 \\ 0 & 0 & 0 & 0 & R_{kd} & 0 \\ 0 & 0 & 0 & 0 & 0 & R_{kq} \end{pmatrix} \begin{pmatrix} i_a \\ i_b \\ i_c \\ i_f \\ i_{kd} \\ i_{kq} \end{pmatrix}$$

$$+ p \begin{pmatrix} L_{aa} & M_{ab} & M_{ac} & M_{af} & M_{akd} & M_{akq} \\ M_{ba} & L_{bb} & M_{bc} & M_{bf} & M_{bkd} & M_{bkq} \\ M_{ca} & M_{cb} & L_{cc} & M_{cf} & M_{ckd} & M_{ckq} \\ M_{fa} & M_{fb} & M_{fc} & L_{fdfd} & M_{fkd} & M_{fkq} \\ M_{kda} & M_{kdb} & M_{kdc} & M_{kdf} & L_{kdkd} & M_{kdkq} \\ M_{kqa} & M_{kqb} & M_{kqc} & M_{kqf} & M_{kqkd} & L_{kqkq} \end{pmatrix} \begin{pmatrix} i_a \\ i_b \\ i_c \\ i_f \\ i_{kd} \\ i_{kq} \end{pmatrix} \quad (20.4)$$

Where, p is the differential operator $\dfrac{d}{dt}$.

Equation (20.4) has the matrix form, $[v] = [R][i] + p[L]\ [i]$.

The mutual inductances M_{ij} in the triangle above the leading diagonal are equal to those M_{ji} in the lower triangle and represent the mutual inductance between winding i and winding j. Where i and j take the suffices a, b, c through to k_q. For a salient pole synchronous generator or motor some of the mutual and self-inductances are sinusoidal functions of the rotor position θ.

For a squirrel cage induction motor none of the mutual and self-inductances are functions of the rotor position.

Equation (20.2) can be applied to v_a, v_b and v_c and again to i_a, i_b and i_c. The zero sequence terms can be neglected.

The substitution exercise is very tedious, but eventually yields the following expression:-

$$\begin{pmatrix} v_d \\ v_q \\ v_f \\ v_{kd} \\ v_{kq} \end{pmatrix} = \begin{pmatrix} R \end{pmatrix} \begin{pmatrix} i_d \\ i_q \\ i_f \\ i_{kd} \\ i_{kq} \end{pmatrix} + \begin{pmatrix} p & +\omega & 0 & 0 & 0 \\ -\omega & p & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & p \end{pmatrix} \begin{pmatrix} \psi_d \\ \psi_q \\ \psi_f \\ \psi_{kd} \\ \psi_{kq} \end{pmatrix} \tag{20.5}$$

Where, $[R] = [R_a, R_a, R_f, R_{kd}, R_{kq}]^T$ and superscript T means transpose.

Note: Since the damper circuits have no external connections and are short circuited by end rings, the terminal voltages v_{kd} and v_{kq} are zero, as shown in Figure 20.2.

c) Mutual inductances

Most authors identity the various mutual inductances in each axis of (20.4) e.g. M_{ab}, M_{af}, M_{akd}, and then assume them to be equal as, M_d for the d-axis and M_q for the q-axis. Some analyses have been published in which these mutual inductances have been assumed to be unequal, particularly when two or more damper windings have been included in each axis, see References 6, 15 and 16.

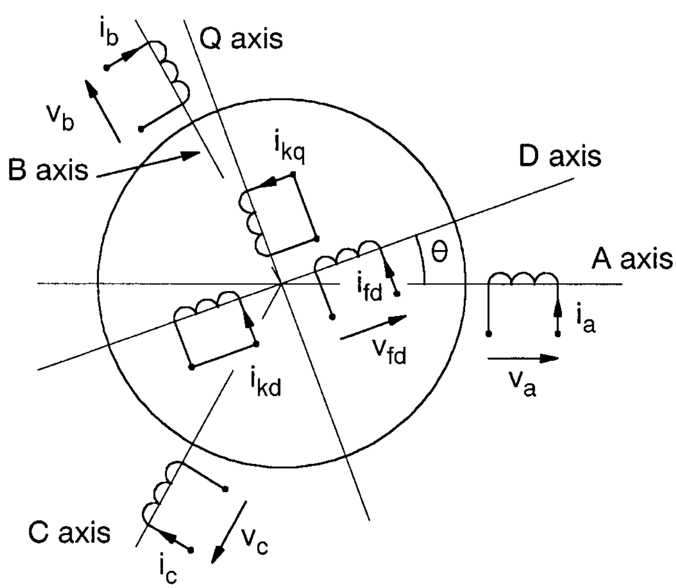

Figure 20.2 Mutually coupled circuits in the A-B-C phase and d-q axis reference frames.

d) Flux linkage equations

The flux linkage variables in (20.5) can now be established in terms of equal mutual inductances.

$$\begin{pmatrix} \psi_d \\ \psi_q \\ \psi_f \\ \psi_{kd} \\ \psi_{kq} \end{pmatrix} = \begin{pmatrix} (M_d + L_{la}) & 0 & M_d & M_d & 0 \\ 0 & (M_q + L_{la}) & 0 & 0 & M_q \\ M_d & 0 & (M_d + L_{lfd}) & M_d & 0 \\ M_d & 0 & M_d & (M_d + L_{lkd}) & 0 \\ 0 & M_q & 0 & 0 & (M_q + L_{lkq}) \end{pmatrix}$$

$$\times \begin{pmatrix} i_d \\ i_q \\ i_f \\ i_{kd} \\ i_{kq} \end{pmatrix} \qquad \begin{matrix} (20.6) \\ (20.7) \\ (20.8) \\ (20.9) \\ (20.10) \end{matrix}$$

A set of first-order differential equations can be obtained by rearranging the leading diagonal terms in the square matrix on the right-hand side of (20.5). Hence:-

$$\begin{pmatrix} p\psi_d \\ p\psi_q \\ p\psi_f \\ p\psi_{kd} \\ p\psi_{kq} \end{pmatrix} = \begin{pmatrix} v_d \\ v_q \\ v_f \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \\ \\ R \\ \\ \\ \end{pmatrix} \begin{pmatrix} i_d \\ i_q \\ i_f \\ i_{kd} \\ i_{kq} \end{pmatrix} - \begin{pmatrix} 0 & +\omega & 0 & 0 & 0 \\ -\omega & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_d \\ \psi_q \\ \psi_f \\ \psi_{kd} \\ \psi_{kq} \end{pmatrix} \qquad (20.11)$$

Equation (20.11) in conjunction with equations (20.6) to (20.10), the external stator network and field excitation equations can be used to compute the flux linkages. These equations represent the machine in its full form. Later some simplifications will be made, which make very little loss of accuracy in the solution and will substantially speed up the digital integration of the differential equations.

e) Shaft torque and shaft power

The per-unit torque T_e developed in the shaft is given by:-

$$T_e = \psi_d i_q - \psi_q i_d$$

The power P_e developed can be calculated from the mechanical expression, power = torque $\times$ speed. Hence the per-unit power developed in the machine is:-

$$P_e = \frac{\omega}{\omega_n} T_e$$

f) Operational impedances and derived reactances

In order to derive the familiar reactances e.g. X_d'' the sub-transient reactance, it is first necessary to obtain the 'operational impedances'. (In control theory terminology these would be called 'transfer functions'.)

Since the inductances in (20.6) to (20.10) are constant it is a simple exercise to differentiate both sides of the equation. Equations (20.6) to (20.10) and its differentiated form can now be substituted into (20.11) to obtain voltage equations that are functions of the currents, and thereby eliminate the flux linkages. The resulting equations are,

$$\begin{pmatrix} v_d \\ v_q \\ v_f \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} R_a + L_{ad}p & \omega L_q & M_d p & M_d p & \omega M_q \\ -\omega L_d & R_a + L_{aq}p & -\omega M_d & -\omega M_d & M_q p \\ M_d p & 0 & R_f + L_{ff}p & M_d p & 0 \\ M_d p & 0 & M_d p & R_{kd} + L_{kd}p & 0 \\ 0 & M_q p & 0 & 0 & R_{kq} + L_{kq}p \end{pmatrix}$$

$$\times \begin{pmatrix} i_d \\ i_q \\ i_f \\ i_{kd} \\ i_{kq} \end{pmatrix} \qquad \begin{matrix} (20.12) \\ (20.13) \\ (20.14) \\ (20.15) \\ (20.16) \end{matrix}$$

In the steady state the transformation of the three-phase currents and voltages into their d and q axis equivalents, when the rotor is rotating at the synchronous speed, causes them to become constant values. The magnitude of these constant values is equal to the peak value of their corresponding rms values in the phase windings. This is because the transformations have been made with a synchronous reference frame.

In addition the differential terms in (20.12) to (20.16) become zero and so do the currents in the damper windings. Hence by using suffix 'ss' the steady state version of (20.12) to (20.16) become:

$$\begin{pmatrix} v_{dss} \\ v_{qss} \\ v_{fss} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} R_a & \omega L_q & 0 & 0 & \omega M_q \\ -\omega L_d & R_a & -\omega M & -\omega M & 0 \\ 0 & 0 & R_f & 0 & 0 \\ 0 & 0 & 0 & R_{kd} & 0 \\ 0 & 0 & 0 & 0 & R_{kq} \end{pmatrix} \begin{pmatrix} i_{dss} \\ i_{qss} \\ i_{fss} \\ 0 \\ 0 \end{pmatrix} \tag{20.17}$$

The steady state flux linkages become from (20.6) to (20.10),

$$\begin{pmatrix} \psi_{dss} \\ \psi_{qss} \\ \psi_{fss} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (M_d + L_{la}) & 0 & M_d & M_d & 0 \\ 0 & (M_q + L_{la}) & 0 & 0 & M_q \\ M_d & 0 & (M_d + L_{lfd}) & M_d & 0 \\ M_d & 0 & M_d & (M_d + L_{lkd}) & 0 \\ 0 & M_q & 0 & 0 & (M_q + L_{lkq}) \end{pmatrix} \begin{pmatrix} i_{dss} \\ i_{qss} \\ i_{fss} \\ 0 \\ 0 \end{pmatrix} \tag{20.18}$$

These equations can be used to determine the initial conditions of the synchronous machine in a computer program.

If i_f and i_{kd} are eliminated in (20.12) and (20.14) and (20.6) and after much manipulation the following reactances and time constants can be determined. References 3, 5 and 17 describe the elimination process and the necessary assumptions required to obtain the time constants.

By referring to Chapter VI of Reference 3 sub-section 25, in particular, the algebraic substitutions and a sequence of approximations can be studied, from which the following results are most frequently used. In sub-section 20.2c herein the symbols for the leakage reactances are usually quoted slightly differently, X_{la}, X_{lkd}, X_{klq} and X_{lfd} become X_a, X_{kd}, X_{kq} and X_f respectively. It should be remembered that these are leakage reactances, wherein the suffix '1' emphasises the fact.

g) Derived reactances

D-axis synchronous reactance $X_d = X_a + X_{md}$

D-axis transient reactance $X'_d = X_a + \dfrac{X_{md} X_f}{X_{md} + X_f}$

D-axis sub-transient reactance $X''_d = X_a + \dfrac{X_{md} X_f X_{kd}}{X_{md} X_f + X_{md} X_{kd} + X_f X_{kd}}$

Q-axis synchronous reactance $X_q = X_a + X_{mq}$

Q-axis sub-transient reactance $X''_q = X_a + \dfrac{X_{mq} X_{kq}}{X_{mq} + X_{kq}}$

Q-axis transient reactance X'_q does not exist when only one winding is present in the rotor. If a second winding is placed on the q-axis, such as used to represent the deep-bar effect in an induction motor then X'_q does exist. In most synchronous generator and synchronous motor studies the use of X'_q does not arise, but in some situations it is given a value equal to X_q, for example a computer program may be written to accept a value of X'_q to suit the form in which the equations have been presented in the program. If a value of zero or 'infinity' were to be inserted into the program than a strange result may be given.

h) Time constants

D-axis transient open-circuit time constant $T'_{do} = \dfrac{1}{\omega R_f}(X_f + X_{md})$

D-axis transient short-circuit time constant $T'_d = \dfrac{1}{\omega R_{kd}}\left(X_f + \dfrac{X_{md} X_a}{X_{md} + X_a}\right)$

D-axis sub-transient open-circuit time constant $T''_{do} = \dfrac{1}{\omega R_{kd}}\left(X_{kd} + \dfrac{X_{md} X_f}{X_{md} + X_f}\right)$

D-axis sub-transient short-circuit time constant

$$T''_d = \frac{1}{\omega R_{kd}}\left(X_{kd} + \frac{X_{md} X_a X_f}{X_{md} X_a + X_{md} X_f + X_a X_f}\right)$$

D-axis damper leakage time constant $T_{kd} = \dfrac{1}{\omega R_{kd}} X_{kd}$

Armature time constant $T_a \simeq \dfrac{X_2}{\omega R_a}$

Q-axis sub-transient open-circuit time constant $T''_{qo} = \dfrac{1}{\omega R_{kq}}(X_{kq} + X_{mq})$

Q-axis sub-transient short-circuit time constant $T''_q = \dfrac{1}{\omega R_{kq}}\left(X_{kq} + \dfrac{X_{mq}X_a}{X_{mq} + X_a}\right)$

Q-axis damper leakage time constant $T_{kq} = \dfrac{1}{\omega R_{kq}} X_{kq}$

Negative phase sequence reactance

$$X_2 = \sqrt{X''_d \cdot X''_q} \quad \text{or} \quad \frac{X''_d + X''_q}{2} \quad \text{or} \quad \frac{2X''_d X''_q}{X''_d + X''_q}$$

Zero phase sequence reactance X_o has a value lower than X''_d and is a complex function of the slot pitching of the stator windings and the leakage reactance present in their end windings, see Reference 7, Chapter XII.

i) Operational impedances in the d-axis.

The equation for the operational impedance that relates the d-axis flux linkages to the stator current i_d and the rotor excitation v_f is,

$$\Psi_d = \frac{X_d(p)}{\omega} i_d + \frac{G(p)}{\omega} v_f \tag{20.19}$$

where,
$$X_d(p) = \frac{(1 + T''_d p)(1 + T''_d p)}{(1 + T''_{do} p)(1 + T''_{do} p)} X_d$$

and,
$$G(p) = \frac{(1 + T_{kd} p)}{(1 + T''_{do} p)(1 + T''_{do} p)} \frac{X_{md}}{R_f}$$

j) Operational impedance in the q-axis.

The equation that relates the q-axis flux linkages to the stator current i_q is,

$$\Psi_q = \frac{X_q(p)}{\omega} i_q \tag{20.20}$$

Where,
$$X_q(p) = \frac{(1 + T''_q p)}{(1 + T''_{qo} p)} X_q$$

The process of obtaining expressions for the derived reactances, operational impedances and time constants was based on the notion that only one damper winding exists on each axis. Krause in Reference 5 applied the process to a synchronous machine that has two damper windings on the q-axis. This would be advantageous when studies are being performed with large solid pole machines such as steam power plant generators, which are nowadays rated between 100 and 660 MW. Very similar functions are formed for the q-axis as are formed for the d-axis. To represent three windings on the d-axis would require a formidable amount of algebraic manipulation, from which the benefits may only be small and there will then be the problem of obtaining the extra parameters from either design data or factory tests.

Hence the machine with one damper winding on each axis is adequate for most practical situations, certainly for those in the oil industry.

20.3 SOME NOTES ON INDUCTION MOTORS

At this stage it can be noted that equations (20.5) to (20.11) can be applied to induction motors, but with the following modifications:-

a) Omit the line and row pertaining to the field winding.
b) There is no saliency and so corresponding d-axis and q-axis parameters are equal. The mutual inductances are all equal, which can be denoted as M_{dq} or M.
c) The damper windings *kd* and *kq* have identical structures and parameters.
d) The d, q notation for the rotor axes will be retained for comparison purposes. Some authors, e.g. Reference 11, use the notation r, s to denote the stator and the rotor circuits where as many others use a combination of both notations i.e. *dr*, *qr*, *ds*, *qs*: *rd*, *rq*, *sd*, *sq*, e.g. References 5, 15, 18, 19, 20 and 21, Also used is the notation *ld*, *lq*, 2*d*, and 2*q* e.g. in Reference 12, where l and 2 are used in equivalent circuits of induction motors to represent the stator (primary −1) and rotor (secondary −2) windings.
e) Additional three phase to two axis transformations are required for the following reasons:-
   i) The rotor has a uniform construction. The conductors consist of solid copper bars fixed in slots axially along and near the surface of the rotor. Usually one conductor fills a slot. The ends of the conductors at the drive end of the shaft are short circuited with a copper ring. The ends at the non-drive end are also short circuited by a similar ring. The conductors form what is called a 'single cage' or 'squirrel cage' design. There are no external connections by way of slip rings or commutators.
   ii) A cage design has no wound or physical poles, as with a synchronous machine. The cage creates its own poles as it rotates. A three-phase winding with the same number of poles as the stator is automatically formed by the induction of rotor currents.
   iii) The three-phase rotor windings need to be replaced by equivalent two-axis windings fixed to the rotor. A second transformation is required to convert these windings to a set that rotates at the frequency of the phase voltages applied to the stator. Although the induction machine is simpler in construction and operation than the synchronous machine, the transformation mathematics are more complicated. A basic explanation of the above is given by Cotton in Reference 12 and a more sophisticated mathematical treatment is given by Krause in Reference 5 for machines with a greater number of windings, i.e. additional rotor windings. Cotton in Chapter 31 presents equations of stator-applied voltages in terms of the stator resistive volt-drops and the $d-q$ axis flux linkages. He shows that these are of identical form to those of the synchronous machine. (It can be implied from this conclusion that a computer program could be written using the same form of equations for both type of machines. This observation has been commented upon in the literature e.g. References 22 and 23. Reference 23 considers double-cage induction motors in which the 'deep bar' effect is included, and results were obtain for motors having ratings in the range of 2500 hp to 22,000 hp.) The form of these stator equations are:-

$$v_d = R_a i_d + p\psi_d - \omega\psi_q$$
$$v_q = R_a i_q + p\psi_q + \omega\psi_d$$

f) The rotor equations involving the flux linkages are:-

For the synchronous machine rotor

$$
\begin{aligned}
v_f &= R_f i_f + p\psi_f \\
0 &= R_{kd} \cdot i_{kd} + p\psi_{kd} \\
0 &= R_{kq} \cdot i_{kq} + p\psi_{kq} \\
v_f &= R_f i_f + pL_{fd} i_f + pM_d i_d + pM_d i_{kd} \\
0 &= R_{kd} \cdot i_{kd} + pL_{kd} i_{kd} + pM_d i_d + pM_d i_f \\
0 &= R_{kq} \cdot i_{kq} + pL_{kq} \cdot i_{kq} + pM_q i_q
\end{aligned}
$$

a) Induction machine rotor

$$
\begin{aligned}
0 &= R_{kd} \cdot i_{kd} + p\psi_{kd} \\
0 &= R_{kq} \cdot i_{kq} + p\psi_{kq} \\
0 &= R_{kd} \cdot i_{kd} + pL_{kd} \cdot i_{kd} + pM_d i_d \\
0 &= R_{kq} \cdot i_{kq} + pL_{kq} \cdot i_{kq} + pM_q i_q
\end{aligned}
$$

At this stage operational impedances and time constants have been derived for synchronous machines, and for induction machines, if appropriate substitutions are made as shown in Reference 23.

20.3.1 Derived Reactances

The derived reactances are those most frequently used to specify synchronous generators and motors. They are the synchronous, transient and sub-transient reactances in the d and q-axes. The most convenient method of deriving these is from the application of a three-phase short circuit at the terminals of the unloaded machine, whether it be a generator or a motor. For a motor the testing procedure is more complicated as described in sub-section 5 of Reference 23. The d-axis reactances are easily obtained from normal factory tests. The q-axis are usually taken as their design values because the necessary factory tests are more difficult to perform. The tests are described in for example IEEE standard 112 and BS4296.

20.3.2 Application of Three-phase Short Circuit

The following derivations are made for a synchronous generator, after which the derivations applicable to induction motors are given by a heuristic comparison.

Generators and motors are often connected to their associated switchboards or networks by an impedance. This impedance can be a cable, an overhead line, a unit transformer or a combination of these components. The intermediate circuit introduced in the stator circuit will contain resistance and inductive reactance, the effect of which is to modify the time constants in the generator and motor equations, and the performance of these machines under most transiently disturbed conditions. This aspect has been mentioned in the literature e.g. References 24, 25 and 26 but is easily overlooked when developing computer programs.

In a multi-machine network the generators and motors should be considered in relation to the 'source impedance' to which they are connected. This impedance will also be dependent upon the location and type of disturbance e.g. near to a generator, remote from a generator, three-phase fault, line-to-ground fault, change in the state of the load such as starting a large motor direct-on-line.

The following discussion applies to a synchronous machine that has one field and two damper windings.

There are various methods of solving the equations for a three-phase short circuit on the basic that the set of equations are linear and where the use of Laplace transforms, or the Heaviside calculus, is appropriate. See References 3, 5, 6 and 8 for examples. These methods are complicated and appropriate assumptions concerning the relative magnitudes of resistances, inductances and time constants need to be made in order to obtain practical solution. The relative magnitudes of the parameters are derived from typical machinery data. Adkins in Reference 3 gives a solution of the following form,

$$\begin{aligned} i_a &= \sqrt{2}\ V_{o/c}\left[\frac{1}{X_d} + \left(\frac{1}{X''_d} - \frac{1}{X_d}\right)e^{\frac{-t}{T''d}} + \left(\frac{1}{X''_d} - \frac{1}{X'_d}\right)e^{\frac{-t}{T''d}}\right]\cos(\omega t + \theta) \\ &\quad - \left(\frac{X''_d + X''_q}{2X''_d X''_q}e^{\frac{-t}{Ta}}\right)\cos\theta - \left(\frac{(X''_q - X''_d)}{2X''_d X''_q}e^{\frac{-t}{Ta}}\right)\cos(2\omega t + \theta) \\ &= \sqrt{2}\ V_{o/c}\ (A + B + C + D + E) \end{aligned}$$

Where A, B and C are the fundamental frequency synchronous, transient and sub-transient AC components,

E is due to the sub-transient saliency and contributes a small double frequency component, usually small enough to be neglected.

D is the DC offset caused by the switching angle θ and the values of the sub-transient reactances.

θ is the angle of the open-circuit sinusoidal terminal voltage when the short circuit is applied.

All the reactances and time constants are the same as those defined in sub-section 20.2.1g) and h)

In a situation where the disturbance is remote from the machine the short circuit time constants and the derived reactances X_d, X'_d, X''_d, X_q, (X'_q), X''_q and X_2 are all functions of the external reactance X_e since it should be added to X_a. Likewise R_e should be added to R_a. R_a does not appear in the time constants except for T_a.

An example of the decrement in the short-circuit current for a synchronous generator is given in sub-section 7.2.10 where its relevance to switchgear is described.

The worst-case situation for calculating the fault current in phase A is when the switching angle θ is zero, the DC offset is then at its maximum value.

The above expression is adequate for data that are typically available for the industry. The armature resistance R_a is only present in the time constant T_a. (Krause offers a more complete solution in which the omission of R_a is minimised. The effect is then to modify the time constant T_a

in the terms for the DC offset and the sub-transient saliency.) The inclusion of an external impedance such as a unit transformer that has both reactance and resistance will only have the modifying effect as mentioned above because the external reactance will be much greater than the external resistance. The ratios of reactance to resistance in high voltage circuits is usually at least 10:1. The external reactance added to X_d, X_d', X_q'' will also reduce the magnitude of the instantaneous short-circuit current for all values of time.

The time constant T_a is important because it influences the lower envelope of the short-circuit current wave form to such an extent that the current can fail to cross the time axis until several cycles have been completed. This is demonstrated in 7.2.10 and Figure 7.1 shows the result. The behaviour of the instantaneous current imparts a heavy duty on the stator circuit breakers. Should this be anticipated in practice, from preliminary design studies, then the equipment involved should be specified accordingly.

20.3.3 Derived Reactances and Time Constants for an Induction Motor

The absence of the field winding can be used to convert the mathematical model of the synchronous machine into one for an induction machine. In addition the mutual inductance in the q-axis is made equal to mutual inductance in the d-axis, i.e. the machine becomes symmetrical in both axes. The matrix equations (20.6) to (20.16) are modified as shown below. In these equations the mutual inductances M_d and M_q become M, L_{lkd} and L_{lkq} become L_{lk}, R_{kd} and R_{kq} become R_k. All the derived reactances and time constants for an induction machine are equivalent to those applicable to the q-axis of the synchronous machine.

Some of the literature use ‘transient’ notation, e.g. References 3, 22 and 28. Others use ‘sub-transient’ notation particularly in relation to fault current contribution in power systems, e.g. Reference 24.

Most literature use transient notation, Adkins, Ramsden IEE68 Fitzgerald and Kingsley. Others use sub-transient notation particularly in relation to fault current contribution in power systems.

Equations (20.6) to (20.10) become:-

$$\begin{pmatrix} \psi_d \\ \psi_q \\ \psi_{kd} \\ \psi_{kq} \end{pmatrix} = \begin{pmatrix} M + L_{la} & 0 & M & 0 \\ 0 & M + L_{la} & 0 & M \\ M & 0 & M + L_{kd} & 0 \\ 0 & M & 0 & M + L_{kq} \end{pmatrix} \begin{pmatrix} i_d \\ i_q \\ i_{kd} \\ i_{kq} \end{pmatrix} \tag{20.21}$$

It is reasonable to regard the rotor windings as damper windings and use the notation of sub-transient reactances. Hence the following derived reactances and time constants are appropriate to induction machines.

Equation (20.11) becomes:-

$$\begin{pmatrix} p\psi_d \\ p\psi_q \\ p\psi_{kd} \\ p\psi_{kq} \end{pmatrix} = \begin{pmatrix} v_d \\ v_q \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} R \end{pmatrix} \begin{pmatrix} i_d \\ i_q \\ i_{kd} \\ i_{kq} \end{pmatrix} - \begin{pmatrix} 0 & +\omega & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_d \\ \psi_q \\ \psi_{kd} \\ \psi_{kq} \end{pmatrix} \tag{20.22}$$

Equations (20.12) to (20.16) become:-

$$\begin{bmatrix} v_d \\ v_q \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_a + L_a p & \omega L_{dq} & Mp & \omega M \\ -\omega L_{dq} & R_a + L_a p & -\omega M & Mp \\ Mp & 0 & R_k + L_k p & 0 \\ 0 & Mp & 0 & R_k + L_k p \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_{kd} \\ i_{kq} \end{bmatrix} \tag{20.23}$$

where ω is the rotor speed,

$$L_{dq} = M + L_{la}$$

and
$$L_k = M + L_{kd} = M + L_{kq}$$

The operational impedances become:-

$$X_d(p) = X_q(p) = \frac{(1 + T_d'' p)}{1 + T_{do}'' p} X_d \qquad \text{where} \quad T_q'' = T_d'' \quad \text{and} \quad T_{qo}'' = T_{do}''$$

And $G(p)$ does not exist.

$$T_{do}'' = T_{qo}'' = \frac{1}{\omega R_k}(X_k + X_m)$$

$$T_d'' = T_q'' = \frac{1}{\omega R_k}\left[X_k + \frac{X_m X_a}{X_m + X_a}\right]$$

$$T_k = \frac{X_k}{\omega R_k}$$

T_{do}' and T_d' do not exist.

The flux linkage equations can be rewritten using the symmetrical parameters and the rotor speed as ω_r:-

$$v_d = R_a i_d + p\psi_d - \omega_r \psi_q$$

$$v_q = R_a i_q + p\psi_q + \omega_r \psi_d$$

$$0 = R_k i_{kd} + p\psi_{kd}$$

$$0 = R_k i_{kq} + p\psi_{kq}$$

Application of a three-phase short circuit to the terminals of an unloaded induction motor is not a practical factory test, especially for a large high-voltage motor, because the motor can only be excited at its stator windings from the power supply. A three-phase short circuit at or near the stator terminals can occur in practice e.g. damaged supply cable, damage in the cable terminal box. The parameters of the stator and rotor windings can be obtained from other factory tests. However, the derived reactance can be defined in the same manner as those for the synchronous machine, but with the assumptions regarding symmetry and the deletion of the field winding taken into account. The derived reactances become:-

$$X_d = X_q = X_a + X_m$$

$$X'_d = X'_q = X''_d = X''_q = X_a + \frac{X_m X_k}{X_m + X_k}$$

$$X_2 = X''_d \qquad \text{(negative sequence reactance)}$$

$$T_a = \frac{X'_d}{\omega R_a}$$

20.3.4 Derivation of an Equivalent Circuit

Equation (20.23) can be rewritten with the rotationally induced emfs correctly represented by the rotor speed ω_r instead of ω as in the case of the synchronous machine:-

$$\begin{bmatrix} v_d \\ v_q \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_a + L_a p & \omega_r L_{dq} & Mp & \omega_r M \\ -\omega_r L_{dq} & R_a + L_a p & -\omega_r M & Mp \\ Mp & 0 & R_k + L_k p & 0 \\ 0 & Mp & 0 & R_k + L_k p \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_{kd} \\ i_{kq} \end{bmatrix} \tag{20.24}$$

Where $\omega_r = (1 - s)\omega$, ω is the frequency of the power supply, and s is the slip of the rotor speed.

The familiar equivalent circuit for the induction motor will be developed from (20.24). The oil industry occasionally uses variable frequency power supplies to start and run variable speed pumps and compressors. The nominal frequency applied to the motor is ω_n. The inductances in (20.24) can be changed to their nominal reactances by using the nominal frequency ω_n. The steady state variables replace the instantaneous variables and the differential operator p is replaced by the steady state frequency in conjunction with the j operator.

$$\begin{bmatrix} V_d \\ V_q \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_a + \frac{X_d}{\omega_n} j\omega & (1-s)\frac{\omega}{\omega_n} X_{dq} & \frac{X_{md}}{\omega_n} j\omega & (1-s)\frac{\omega}{\omega_n} X_{md} \\ -(1-s)\frac{\omega}{\omega_n} X_{dq} & R_a + \frac{X_d}{\omega_n} j\omega & -(1-s)\frac{\omega}{\omega_n} X_{md} & \frac{X_{md}}{\omega_n} j\omega \\ \frac{X_{md}}{\omega_n} j\omega & 0 & R_k + \frac{X_k}{\omega_n} j\omega & 0 \\ 0 & \frac{X_{md}}{\omega_n} j\omega & 0 & R_k + \frac{X_k}{\omega_n} j\omega \end{bmatrix} \begin{bmatrix} I_d \\ I_q \\ I_{kd} \\ I_{kq} \end{bmatrix} \tag{20.25}$$

Where V_d, V_q, I_d, I_q, I_{kd} and I_{kq} are the phasor equivalents of their instantaneous variables.

For the balanced three-phase operation of the motor the following discussion applies. In the above equation the magnitude of the q-axis variables are equal to their corresponding d-axis variables. The operator j is required in the q-axis variables to identify its 90° phase advance from the d-axis.

It can now be seen that the two q-axis equations are identical in form to the d-axis equations. Hence the solution of one pair gives the same form of solution for the second pair. Consider the d-axis pair:-

$$\begin{bmatrix} V_d \\ 0 \end{bmatrix} = \begin{bmatrix} R_a + \dfrac{X_d}{\omega_n} j\omega & (1-s)\dfrac{\omega}{\omega_n} X_{dq} & \dfrac{X_{md}}{\omega_n} j\omega & (1-s)\dfrac{\omega}{\omega_n} X_{md} \\ \dfrac{X_{md}}{\omega_n} p & 0 & R_k + \dfrac{X_k}{\omega_n} j\omega & 0 \end{bmatrix} \begin{bmatrix} I_d \\ I_{kd} \end{bmatrix}$$

Equation (20.25) has the same form as (20.12) to (20.16) and (20.21) the same for as (20.6) to (20.10). When the machine runs at a speed that is different from the synchronous speed, and is only changing slowly, the d and q axis variables are sinusoids not constants. The frequency of the d and q variables is the slip frequency, see Reference 3, Chapter VII, and Reference 27 Art 6-6.

Consider a synchronous generator supplying a load consisting of a static element and an induction motor. Let the motor be small in rating compared with the generator. The motor is to be started direct-on-line to drive a pump. Before closing the circuit breaker to the motor the d and q axis currents and voltages in the generator and static load will be constant values. After the circuit breaker is closed the motor will carry its starting currents, which will be sinusoidal. These sinusoidal currents and their associated voltages will be superimposed on the currents and voltages of the static load. The generator will have similar superimposed variables in its stator circuits. The actual phase variables can be found by a suitable inverse transformation in the synchronous reference frame. The root-mean-square values of the phase variables can then be found on a cycle-by-cycle basis.

20.3.5 'Re-iteration or Recapitulation'

The two-axis 'generalised theory' has been applied to the synchronous and induction machines in a similar manner thus far. It has been assumed that an idealised 'mathematical' machine adequately represents them under most practical operating conditions. The idealised machine has a few subtle differences from the practical machines normally encountered. The differences are carefully made to simplify the mathematical analysis. The practical machines have their primary windings fixed in the stator. These are fed from the three-phase supply at the synchronous frequency. The secondary windings are fixed in the rotor. The d and q-axes are fixed on the rotor. In the generalised machine theory the relative motion is obtained by transposing the windings. The field and damper windings are placed in the reference (d–q axes) frame. The reference frame can be taken to be stationary, to rotate at the synchronous velocity or to rotate at the rotor velocity. Adkins in Reference 3 calls these pseudo-stationary windings or coils. Krause in Reference 5 explains in detail the various choices that appear in the literature, and which choice is appropriate to a particular analysis. Some of the graphical results given in Reference 5, sub-section 4.11 for example, may appear strange at first sight but are peculiar to the particular frame of reference used.

The synchronous generator has been considered as a set of coupled windings in which the primary windings are in the rotor and the secondary windings are in the stator. A practical motor has the primary-secondary notation reversed. The primary windings are in the stator and the secondary windings in the rotor, i.e. similar to a static transformer. The main difference in the winding configuration is that of the primary in the machines. The synchronous machine has a two-phase winding and the induction machine a three-phase winding. Therefore the three-phase winding needs to be converted into an equivalent two-phase primary winding. The three-phase currents and voltages in the primary are transformed to their equivalent two-phase variables. These transformations are detailed in References 3, 5 and 6 for example. The result is a transposition of the rows in the voltage-current equation (20.24) and the insertion of suffices 1 and 2, 1 for the primary and 2 for the secondary (as with static transformers). Equation (20.24) becomes:-

$$\begin{bmatrix} v_{d1} \\ v_{q1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Mp & 0 & R_2 + L_2 p & 0 \\ 0 & Mp & 0 & R_2 + L_2 p \\ R_1 + L_1 p & \omega_r L_{dq} & Mp & \omega_r M \\ -\omega_r L_{dq} & R_1 + L_1 p & -\omega_r M & Mp \end{bmatrix} \begin{bmatrix} i_{d1} \\ i_{q1} \\ i_{d2} \\ i_{q2} \end{bmatrix} \tag{20.26}$$

Where: $R_1 = R_a$, $R_2 = R_k$, $L_1 = L_a$, $L_2 = L_k$ and $L_{dq} = M + L_{la}$.

Replace suffix 'a' with '1', and suffices 'kd' and 'kq' with '2'.

The corresponding flux linkage equation, derived from (20.21), becomes:-

$$\begin{bmatrix} \psi_{d1} \\ \psi_{q1} \\ \psi_{d2} \\ \psi_{q2} \end{bmatrix} = \begin{bmatrix} M + L_{l1} & 0 & M & 0 \\ 0 & M + L_{l1} & 0 & M \\ M & 0 & M + L_{l2} & 0 \\ 0 & M & 0 & M + L_{l2} \end{bmatrix} \begin{bmatrix} i_{d1} \\ i_{q1} \\ i_{d2} \\ i_{q2} \end{bmatrix} \tag{20.27}$$

And similarly from (20.21) and differentiating:-

$$\begin{bmatrix} p\psi_{d1} \\ p\psi_{q1} \\ p\psi_{d2} \\ p\psi_{q2} \end{bmatrix} = \begin{bmatrix} (M + L_{l1})p & 0 & Mp & 0 \\ 0 & (M + L_{l1})p & 0 & Mp \\ Mp & 0 & (M + L_{l2})p & 0 \\ 0 & Mp & 0 & (M + L_{l2})p \end{bmatrix} \begin{bmatrix} i_{d1} \\ i_{q1} \\ i_{d2} \\ i_{q2} \end{bmatrix} \tag{20.28}$$

And the voltage equation (20.5) becomes:-

$$\begin{bmatrix} v_{d1} \\ v_{q1} \\ v_{d2} \\ v_{q2} \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} i_{d1} \\ i_{q1} \\ i_{d2} \\ i_{q2} \end{bmatrix} + \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & \omega \\ 0 & 0 & -\omega & p \end{bmatrix} \begin{bmatrix} \psi_{d1} \\ \psi_{q1} \\ \psi_{d2} \\ \psi_{q2} \end{bmatrix} \tag{20.29}$$

Substituting (20.28) into (20.29) are rearranging the terms gives,

$$\begin{bmatrix} v_{d1} \\ v_{q1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + (M + L_{l1})p & 0 & Mp & 0 \\ 0 & R_1 + (M + L_{l1})p & 0 & Mp \\ Mp & \omega M & R_2 + (M + L_{l2})p & \omega(M + L_{l2}) \\ -\omega M & Mp & -\omega(M + L_{l2}) & R_2 + (M + L_{l2})p \end{bmatrix} \begin{bmatrix} i_{d1} \\ i_{q1} \\ i_{d2} \\ i_{q2} \end{bmatrix} \tag{20.30}$$

The two upper rows represent the stator and the two lower rows the rotor.

Similarly (20.25) becomes:

$$\begin{bmatrix} V_{d1} \\ V_{q1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + j\dfrac{\omega}{\omega_n}X_{L1} & 0 & j\dfrac{\omega}{\omega_n}X_m & 0 \\ 0 & R_1 + j\dfrac{\omega}{\omega_n}X_{L1} & 0 & j\dfrac{\omega}{\omega_n}X_m \\ j\dfrac{\omega}{\omega_n}X_m & \dfrac{\omega_r}{\omega_n}X_m & R_2 + j\dfrac{\omega}{\omega_n}X_{L2} & \dfrac{\omega_r}{\omega_n}X_m \\ -\dfrac{\omega_r}{\omega_n}X_m & j\dfrac{\omega}{\omega_n}X_m & -\dfrac{\omega_r}{\omega_n}X_m & R_2 + j\dfrac{\omega}{\omega_n}X_{L2} \end{bmatrix} \begin{bmatrix} I_{d1} \\ I_{q1} \\ I_{d2} \\ I_{q2} \end{bmatrix} \tag{20.31}$$

Where V_{d1}, V_{q1}, I_{d1}, I_{q1}, I_{d2} and I_{q2} are the phasor equivalents of their instantaneous values, X_{L1} is the total reactance of the primary and X_{L2} that of the secondary. For the balanced three-phase operation of the induction motor the following discussion applies.

$$\begin{bmatrix} V_{d1} \\ -jV_{d1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + j\dfrac{\omega}{\omega_n}X_{L1} & 0 & j\dfrac{\omega}{\omega_n}X_m & 0 \\ 0 & R_1 + j\dfrac{\omega}{\omega_n}X_{L1} & 0 & j\dfrac{\omega}{\omega_n}X_m \\ j\dfrac{\omega}{\omega_n}X_m & \dfrac{\omega_r}{\omega_n}X_m & R_2 + j\dfrac{\omega}{\omega_n}X_{L2} & \dfrac{\omega_r}{\omega_n}X_m \\ -\dfrac{\omega_r}{\omega_n}X_m & j\dfrac{\omega}{\omega_n}X_m & -\dfrac{\omega_r}{\omega_n}X_m & R_2 + j\dfrac{\omega}{\omega_n}X_{L2} \end{bmatrix} \begin{bmatrix} I_{d1} \\ -jI_{d1} \\ I_{d2} \\ -jI_{d2} \end{bmatrix} \tag{20.32}$$

Consider the first and third rows in (20.32), and define the rotor slip speed as $s\omega = \omega - \omega_r$. The comments following (20.25) regarding pairs of equations also apply here.

These two rows become:-

$$V_{d1} = \left[R_1 + j\frac{\omega}{\omega_n}X_{L1}\right]I_{d1} + j\frac{\omega}{\omega_n}X_m I_{d2} \tag{20.33}$$

$$0 = \frac{s\omega}{\omega_n}X_m I_{d1} + \left[R_2 + j\frac{s\omega}{\omega_n}X_{L2}\right]I_{d2} \tag{20.34}$$

Divide the secondary equation by the slip s:-

$$0 = j\frac{\omega}{\omega_n}X_m I_{d1} + \left[\frac{R_2}{s} + j\frac{\omega}{\omega_n}X_{L2}\right]I_{d2} \tag{20.35}$$

The equations (20.33) and (20.35) represent the familiar stationary coupled circuit shown in Figure 20.3.

The magnitude of the axes variables is equal due to the symmetry of the winding construction, as shown in (20.32). Hence $|V_{q1}| = |V_{d1}|$, $|I_{q1}| = |I_{d1}|$ and $|I_{q2}| = |I_{d2}|$. The relationship between

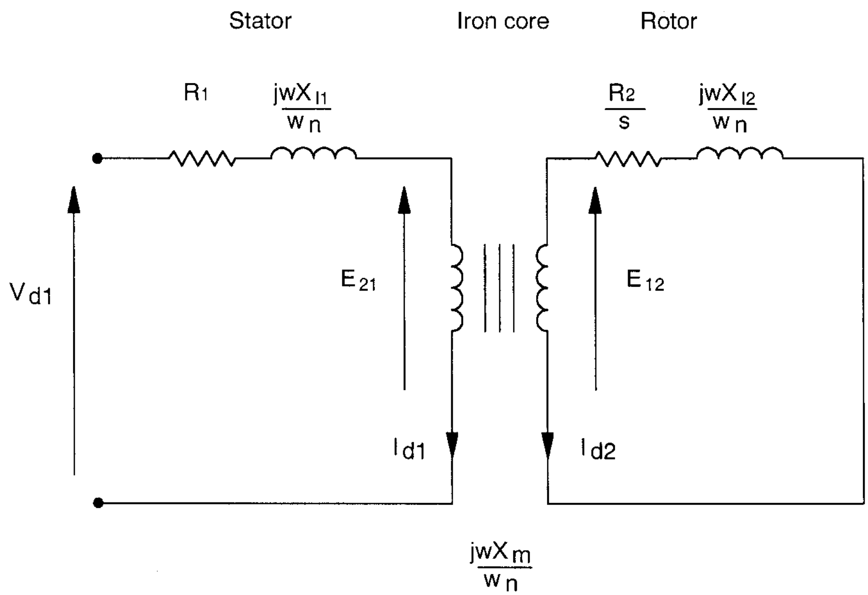

Figure 20.3 Equivalent circuit of an induction motor using mutual coupling.

these variables and the magnitude of the phase-to-neutral stator supply variables is simply:-

$$V_1 = V_{d1} + jV_{q1} = \frac{1}{\sqrt{2}}(v_{d1} - jv_{q1})$$

$$|V_1| = \sqrt{V_{d1}{}^2 + V_{q1}{}^2} = \frac{1}{\sqrt{2}}\sqrt{v_{d1}{}^2 + v_{q1}{}^2}$$

$$\therefore \quad |V_1| = \sqrt{V_{d1}{}^2 + V_{d1}{}^2} = \sqrt{2}V_{d1}$$

An equivalent Tee circuit can be formed by assuming the inductive shunt impedance carries the sum of the stator and rotor currents.

In (20.33) delete the suffix 'd', subtract $j\dfrac{\omega}{\omega_n}X_m I_1$ from the first term on the right-hand side and add it to the second term:-

$$V_d = [R_1 + j\omega(X_{L1} - X_m)]I_1 + j\frac{\omega}{\omega_n}X_m(I_2 + I_1)$$

In (20.35) subtract $j\dfrac{\omega}{\omega_n}X_m I_2$ from the second term on the right-hand side and add it to the first term:-

$$0 = j\frac{\omega}{\omega_n}X_m(I_1 + I_2) + \left[\frac{R_2}{s} + j\frac{\omega}{\omega_n}(X_{L2} - X_m)\right]I_2$$

The following resistances and reactances can be defined when referred to the stator at the nominal power system frequency:-

$X_1 = X_{L1} - X_m$ is the stator leakage reactance.
$X_2 = X_{L2} - X_m$ is the rotor leakage reactance, not to be confused with the negative sequence reactance.
X_m is the magnetising reactance.
R_1 is the stator winding resistance.
R_2 is the rotor winding resistance.
s is the rotor slip with respect to frequency ω.

The resistance R_2/s represents the total power dissipated in one phase winding of the rotor. This consists of a resistive loss in the winding itself and the shaft power transmitted to the mechanical load. The resistance can be divided into two parts:-

$$\frac{R_2}{s} = R_2 + \frac{(1-s)}{s} R_2$$

Where R_2 is the rotor winding resistance and $(1-s)R_2/s$ is the equivalent rotor resistance of the mechanical load.

A practical motor has two additional losses that are significant. One is a resistive loss in the iron core due to eddy currents. The other is a mechanical loss due to the cooling fans and bearings on the shaft, plus the frictional losses due to the presence of the air in the air gap and which surrounds the moving parts of the rotor. The total amount of these losses can be presented approximately by a constant shunt resistance R_c placed in parallel with the magnetising shunt reactance $\omega \mathrm{X}_m/\omega_n$. The equivalent circuit for a motor fed from a fixed frequency supply is shown in Figure 5.1 and from a variable frequency supply in Figure 15.11.

Note: The direction of I_2 can be reversed and its sign changed in the shunt circuit R_c and X_m.

The sub-transient reactance of the motor at the nominal frequency can be defined in terms of the familiar reactances in the equivalent circuit.

$$X''_d = X''_q = X_1 + \frac{X_m X_2}{X_m + X_2}$$

and

$$X_d = X_q = X_1 + X_m$$

Many practical motors are designed to give efficient performance near to their rated speed, high starting torque and a reasonably low starting current. These are somewhat conflicting requirements for the design of a single squirrel-cage motor that has fixed resistances and reactances. The motor designer is able to make the rotor resistance and rotor reactance functions of the slip. The resistance is caused to increase with an increase in slip and the reactance to decrease with slip. The change in the performance of the motor at different values of slip becomes comparable with a double squirrel-cage motor. The variable characteristic is obtained by placing part of the rotor winding bars in the bottom of a deep and narrow slot. Some slots have specially shaped cross-sectional areas to obtain a pronounced effect in the bottom of the slot. The variable characteristic is also described as the 'deep-bar effect' or 'skin effect', see Reference 8, Chapter XIII and Reference 27, Chapter 10. The effect of variable resistance is more significant than the change in reactance. The designer is able to achieve a typical resistance change ratio of 4:1 and an accompanying reactance ratio of 0.5:1, when the slip changes from unity at standstill to approximately 0.01 at full-load. See Figures 5.2 and 5.3. Approximate formulae for these changes are:-

$$\text{Rotor resistance } R_2 = (R_{21} - R_{20})s + R_{20}$$

$$\text{Rotor reactance } X_2 = (X_{21} - X_{20})s + X_{20}$$

Where suffix '0' represents the full-load value and suffix '1' represents the standstill value.

Some motor designers apply the reactive change to the sum of the stator and rotor leakage reactances:-

$$X_{12} = X_1 + X_2 = (X_{121} - X_{120})s + X_{120}$$

Most designers consider the stator resistance as a constant value.

The equivalent circuit may be used for transient performance studies such as determining the starting, or run-up, time of the motor when coupled to its load. Its currents and voltages are usually their rms values at the supply frequency. It is known that large and rapid oscillations in electrical torque occur when an induction motor is started direct-on-line.

These oscillatory torques are approximately symmetrical about the torque calculated form the simple equivalent circuit and decay to zero as the rotor accelerates.

The equivalent circuits such as those in Figures 5.1 and 15.11 cannot be used for this type of study, and the more precise d–q axes equations involving the stator flux linkages must be used, see Reference 5. These equations would be more useful to the motor designer than the power system designer, where he is concerned with the stresses, strains and materials used in the construction of the motor windings, shafts, couplings and their keys.

When the equivalent circuit is suitable it can be treated as a passive circuit in that no differential equations need to be solved for the currents or voltages in the circuit. The only differential equation associated with the circuit is the torque necessary to accelerate the rotor and its coupled load. For this purpose the standard form of equations for the electrical torque are appropriate, in which the air-gap voltage V_m should be used.

20.3.6 Contribution of Three-phase Short-circuit Current from Induction Motor

20.3.6.1 Fault at the motor

When a running induction motor has a short-circuit applied to its terminals the air-gap flux creates an emf that drives a current into the fault. The motor is then driven by the inertia of its load. The speed may be assumed to be unchanged for the duration of the fault current, which in practice for small motors is only a few cycles at the supply frequency i.e. less than 60 milliseconds. For large motors the duration may as long as 250 milliseconds, see Reference 23. This is due to the higher X-to-R ratio in the short circuit than is the case with small motors. The impedance to the fault current consists of the transient reactance (equal to the sub-transient reactance) and the stator resistance. This will be shown below.

The authors of References 6 and 27 give analyses of the short-circuit current of an induction motor that has only one winding in each axis of the rotor. These analyses result in a simple equation of the form,

$$I_1'' = \left(\frac{E_1''}{X''} e^{\frac{-t}{T''}}\right)$$

where E'' is the air-gap phase-to neutral voltage before the fault was applied,

$$T'' = \frac{X''}{R_1}$$

and
$$X'' = X_1 + X_m - \frac{X^2{}_m}{X_2 + X_m}$$

Which approaches X_1 when X_m is large compared with X_1 and X_2.

The DC off-set has been ignored in the above equation, which is a reasonable assumption for small motors.

A more comprehensive treatment of the subject is given in Reference 23 in which comparisons were made with actual test results taken from large motors. The treatment also takes account of the DC off-set and the 'deep-bar' effect of the rotor conductors and slots. These are important factors to consider, especially with large high-voltage motors. The problem of delayed zero crossing is discussed in sub-section 7.2.11 in relation to the breaking current duty of circuit breakers. The problem arose with generators from the possibility that a poor combination of the armature time constant T_a and the sub-transient reactance X_d'' could occur. A very similar effect can occur with large motors. Kalsi *et al* in Reference 23 showed that the peak value of the current in the first half-cycle could be as high as 12 times the rated peak current, largely due to the full DC off-set that can occur, see Figure 20.4.

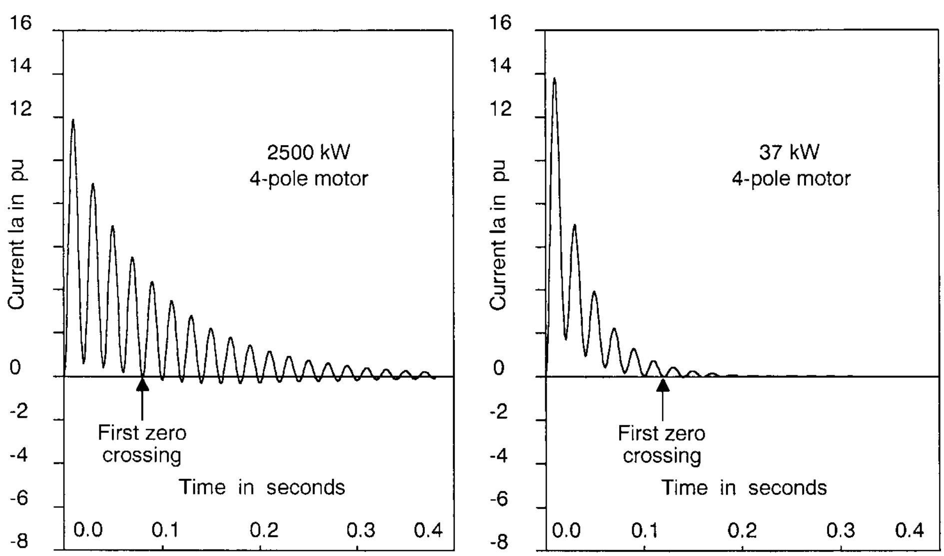

Figure 20.4 Short-circuit current decrement for a 2500 kW and a 37 kW induction motor. These motors have a relatively high armature time constant Ta that causes the initial offset of the waveform. The 'deep bar' effect in the rotor has been taken into account.

Note that Figure 20.4 shows the short circuit current for a fault inside the terminal box of the motor when its internal emf is acting alone, i.e. the stator is isolated from its supply. In practice there will be the transient and the steady state in-feeds of fault current from the upstream switchgear, which will act in addition to that created within the motor. When a motor feeds current back into its faulted upstream system, e.g. short circuit at the busbars of the motor control center, then the motor feeder cables will attenuate the motor current to some extent. A low voltage motor feeder cable usually has a low X-to-R ratio and, for long route lengths, reasonably significant impedance when it is compared with the one per-unit impedance of the motor. Hence the attenuation effect will be more pronounced than with high voltage motors. In addition the reduction in the X-to-R ratio in the stator circuit will usually cause the initial decay of the motor contribution to be faster than for a high voltage motor. The absence of current zero-crossings in the initial period may also be much reduced or even eliminated altogether.

Oil industry power systems often have generators and large motors connected to the same high-voltage switchboards. Hence there is the possibility of, more than may be expected, contributions of sub-transient current from the generators and motors. This will unduly stress the switchgear.

It can be noted that equation 8 in Reference 23 has a very similar form to the equation for the phase current i_a of a generator in sub-section 7.2.7. With appropriate assumptions and approximations the phase current i_a of an induction motor can be presented in the same manner.

The motor parameters normally given by a manufacturer are those given in sub-section 5.2.1, i.e. R_1, X_1, R_{20}, R_{21}, X_{20}, X_{21}, X_m and R_c. The parameters take account of the 'deep-bar' effect in the rotor. The following reactances and time constants can be defined in the same manner as for a generator.

Synchronous reactance

$$X = X_1 + X_m$$

Transient reactance

$$X' = X_1 + \frac{X_m X_{20}}{X_m + X_{20}}$$

Sub-transient reactance

$$X'' = X_1 + \frac{X_m X_{20} X_{21}}{X_m X_{20} + X_{20} X_{21} + X_m X_{21}}$$

Armature time constant

$$T_a = \frac{X''}{\omega R_1}$$

Transient short-circuit time constant

$$T' = \frac{X_{20} + \dfrac{X_m X_1}{X_m + X_1}}{\omega R_{20}}$$

Sub-transient short-circuit time constant

$$T'' = \frac{X_{21} + \dfrac{X_m X_1 X_{20}}{X_m X_1 + X_1 X_{20} + X_m X_{20}}}{\omega R_{21}}$$

These reactance and time constants can now be used to replace their corresponding ones in the equation for the short-circuit current in phase A of the motor, i_a, as previously used for a generator.

As with the generator short circuit, the worst-case condition of the equation for the motor is when the switching angle ϕ_o is zero. The equation becomes:-

$$I_a = V_{pk}\left[\left[\frac{1}{X''}-\frac{1}{X'}\right]\exp\frac{-t}{T''}+\frac{1}{X'}\exp\frac{-t}{T''}\right]\cos(\omega t)+V_{pk}\frac{1}{X''}\exp\frac{-t}{T_a} \tag{20.36}$$

Figure 20.4 was drawn from equation (20.36).

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