9

CONSERVATION LAW FOR MOMENTUM

Momentum transfer is introduced by reviewing the units and dimensions of momentum, time rate of change of momentum, and force. The phenomenological law governing the transfer of momentum by molecular diffusion—Newton's second law—was briefly discussed in Chapter 5. In addition to molecular diffusion, momentum (and energy) may also be transferred by bulk motion. Since bulk motion involves transfer of mass from one point in a system to another, the equation of continuity (conservation law for mass) was also discussed earlier. These serve as an excellent warm-up for the equation of motion (equation of momentum transfer or conservation law for momentum that receives treatment in Section 9.2 of this chapter).

9.1 MOMENTUM BALANCES

A momentum balance (also termed the *impulse-momentum principle*) is important in flow problems where forces need to be determined. This analysis is inherently more complicated than those previously presented (i.e., forces possess both magnitude and direction), because the force, F, and momentum, M, are vectors. In order to describe force and momentum vectors, both direction and magnitude must be specified; for mass and energy, only the magnitude is required.

Newton's law is applied in order to derive the linear momentum balance equation. Newton's law states that the sum of all forces equals the rate of change of linear

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momentum

$$\sum F = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv}{g_c}\right) = \frac{\mathrm{d}M}{\mathrm{d}t} = \dot{M} \tag{9.1}$$

Here $\dot{M}$ is the rate (with respect to time) of linear momentum, and m and v represent the mass and velocity, respectively. Newton's law must be applied in a specified direction (e.g., horizontal or vertical). The product $(m)(v)$ is called the *linear momentum*. When this is applied to a fluid entering or leaving a control volume, the following terms may be defined:

$\dot{M}_{\text{out}}$ = momentum rate of the fluid leaving the control volume

$\dot{M}_{\text{in}}$ = momentum rate of the fluid entering the control volume

Equation (9.1) may be rewritten in finite form

$$\sum F = \dot{M}_{\text{out}} - \dot{M}_{\text{in}} \tag{9.2}$$

This balance essentially means that for steady-state flow, the force on the fluid equals the net rate of outflow of momentum across the control surface. Equation (9.2) also may be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv}{g_c}\right)_{\text{in}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv}{g_c}\right)_{\text{out}} - \sum F \tag{9.3}$$

or

$$\dot{M}_{\text{in}} = \dot{M}_{\text{out}} - \sum F \tag{9.4}$$

This may be compared with the generalized steady-state balance equation for momentum:

$$\{\text{rate of momentum in}\} = \{\text{rate of momentum out}\} + \{\text{generation rate of momentum}\} \tag{9.5}$$

Thus, the generation rate of momentum may be viewed as the negative of the net force acting on the fluid mass.(1) When a momentum balance is used to calculate the forces in different (but perpendicular) directions (e.g., F_x and F_y), the net (or resultant) force is obtained

$$F_{\text{res}} = \sqrt{F_x^2 + F_y^2} \tag{9.6}$$

Application of the above principles is provided in the following two Illustrative Examples.

Illustrative Example 9.1 A horizontal water jet impinges on a vertical plate. The jet splits into several jets traveling in the vertical direction. The water flow rate, q, is 0.5 ft^3/s, the water's horizontal velocity, v, is 100 ft/s, and the water density, ρ, is 62.4 $\mathrm{lb}/\mathrm{ft}^3$. Determine the force required to hold the plate stationary.

Solution The momentum balance equation in the horizontal direction is

$$F = \dot{M}_{\mathrm{out}} - \dot{M}_{\mathrm{in}}$$

The momentum rate of the inlet water in the horizontal direction is given by

$$\dot{M}_{\mathrm{in}} = \frac{\rho q v}{g_c}$$

The horizontal momentum rate of the exit water is $\dot{M}_{\mathrm{out}} = 0$. The net force in the horizontal direction, F, is therefore

$$F = 0 - \frac{\rho q v}{g_c} = -\frac{(62.4)(0.5)(100)}{32.2} = -97\,\mathrm{lb_f}$$

The net horizontal force can be recalculated if the jet had an angle of 10° to the horizontal. For this case

$$\dot{M}_{\mathrm{in}} = \frac{\rho q v}{g_c}\cos(10^\circ) = 97(0.985) = 95.5\,\mathrm{lb_f}$$

$$F = -95.5\,\mathrm{lb_f}$$

The negative answer above indicates that to hold the plate in place, a force must be exerted in a direction opposite to that of the water flow.

Illustrative Example 9.2 A 10 cm diameter horizontal line carries saturated steam at a velocity of 420 m/s. Water is entrained (carried along) by the steam at the rate 0.15 kg/s. The line has a 90° bend. Calculate the force required to hold the bend in place due to the entrained water (see Fig. 9.1).

Solution Select the control volume as the fluid in the bend and apply a mass balance.

$$\dot{m}_1 = \dot{m}_2$$

In addition,

$$v_1 = v_2$$

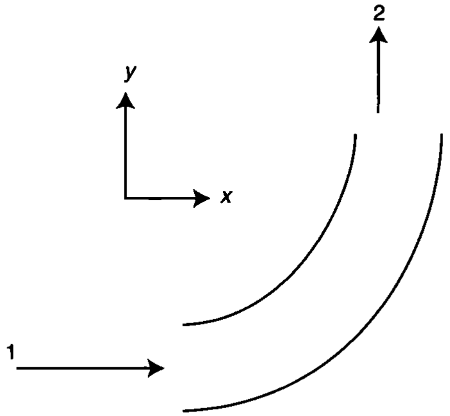

Figure 9.1 90° turn.

Apply a linear momentum balance in the horizontal (x) direction, neglecting the momentum of the steam

$$F_x = \frac{\mathrm{d}}{\mathrm{d}t}(mv)_{\mathrm{out},x} - \frac{\mathrm{d}}{\mathrm{d}t}(mv)_{\mathrm{in},x} = 0 - \dot{m}v_{\mathrm{in},x} = -0.15(420) = -63\,\mathrm{N}$$

The x-direction force acting on the 90° elbow is therefore $F_x = +63$ N.

Apply a linear momentum balance in the vertical (y) direction

$$F_y = \dot{M}_{\mathrm{out},y} - \dot{M}_{\mathrm{in},y} = \dot{m}v_{\mathrm{out},y} - 0 = 0.15(420) = 63\,\mathrm{N}$$

The y-direction force acting on the 90° elbow is therefore $F_y = -63$ N.

The resultant force may now be calculated from Equation (9.6)

$$F_{\mathrm{res}} = \sqrt{F_x^2 + F_y^2} = \sqrt{(-63)^2 + 63^2} = 89.1\,\mathrm{N}$$

The resultant force is the force required to hold the elbow in place.

Illustrative Example 9.3 Water (density = 62.4 lb/ft^3) flows in a 2 inch diameter pipe. The pipe has a 90° bend. The bend support can withstand a maximum force in the x-direction of 5 lb$_\mathrm{f}$. Determine the maximum water flow rate in the pipe bend.

Solution Select the control volume to be the fluid in the bend and apply a mass balance.

$$\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho S v$$

For steady incompressible flow,

$$q_1 = q_2 = q = \rho v$$

Therefore,

$$v_1 = v_2 = v$$

Apply a linear momentum balance on a rate basis in the horizontal x-direction [see Equation (9.4)]

$$\dot{M}_{\text{in},x} = \dot{M}_{\text{out},x} + F_x = \dot{m}\frac{v}{g_c} = 0 - (-5)$$

$$\frac{\rho S v^2}{g_c} = 5$$

$$v = \sqrt{\frac{5g_c}{\rho S}}$$

The use of g_c is necessary to obtain the proper units on both sides of the equation. Substitute numerical values to generate the flow velocity.

$$v = \sqrt{\frac{5g_c}{\rho(\pi D^2/4)}} = \sqrt{\frac{5(32.174)}{62.4(\pi)(0.167^2/4)}} = 10.8\,\text{ft/s}$$

Finally, the volumetric and mass flow rates can be calculated

$$q = Sv = (0.0219)(10.8) = 0.238\,\text{ft}^3/\text{s}$$
$$\dot{m} = \rho q = 62.4(0.238) = 14.8\,\text{lb/s}$$

This represents the maximum water flow rate that the elbow can handle. However, the practicing engineer employs a safety factor so that the possibility of a failure or problem arising is decreased.

Illustrative Example 9.4 Water (density = 1000 kg/m^3, viscosity = 0.001 kg/(m · s)) is discharged through a horizontal fire hose (see Fig. 9.2) at a rate of

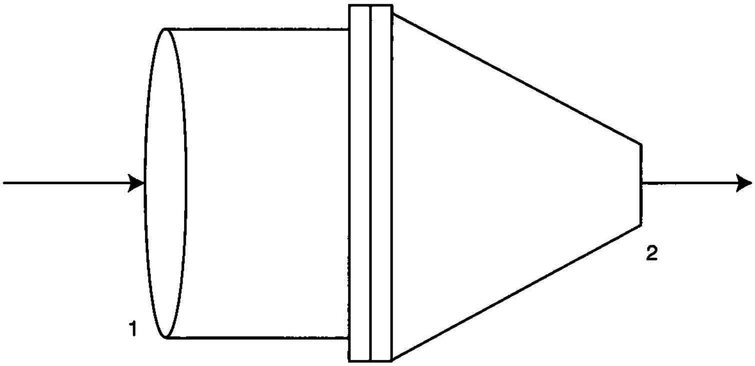

Figure 9.2 Fire hose.

1.5 m^3/min. The fire hose is 10 cm in diameter. The nozzle's diameter reduces from 10 cm to 3 cm. The nozzle discharges the water into the atmosphere. Calculate water velocities and the pressures in the fire hose and at the nozzle tip, the x-direction momentum at both ends of the nozzle, the force required to hold the hose, and the type of flow in the fire hose.

Solution Apply a mass balance on the CV.

$$q = q_1 = v_1 S_1 = 0.025\,\mathrm{m^3/s}$$
$$\dot{m} = \rho q = 1000(0.025) = 25\,\mathrm{kg/s}$$

Calculate the velocities v_1 and v_2.

$$v_1 = \frac{q}{S_1} = \frac{0.025}{\pi(0.1)^2/4} = 3.2\,\mathrm{m/s}$$
$$v_2 = \frac{q}{S_2} = \frac{0.025}{\pi(0.03)^2/4} = 35.4\,\mathrm{m/s}$$

Determine the pressure, P_1, by applying Bernoulli's equation between points 1 and 2 (see Fig. 9.2).

$$z_1 = z_2$$
$$P_2 = 0\ \text{Pag (Pascal gauge)}$$
$$P_1 = \frac{\rho}{2}\frac{(v_2{}^2 - v_1{}^2)}{g_c} = \frac{1000}{2}[(35.4)^2 - (3.2)^2] = 620{,}000\,\mathrm{Pag}$$

Calculate the x-direction momentum rates.

$$\dot{M}_{1,x} = (\dot{m}_1 v_1)_x = (25)(3.2) = 80\,\mathrm{N}$$
$$\dot{M}_{2,x} = (\dot{m}_2 v_2)_x = (25)(35.4) = 885\,\mathrm{N}$$

Obtain the force from the momentum balance in the x-direction.

$$F_x = \dot{M}_{2,x} - \dot{M}_{1,x} - P_1 S_1 = 885 - 80 - (620{,}000)\left(\frac{\pi}{4}(0.1)^2\right) = -4067\,\mathrm{N}$$
$$= -915\,\mathrm{lb_f}$$

The magnitude of the force (915 $\mathrm{lb_f}$) explains why it often takes several firefighters to hold a fire hose steady at full discharge.

9.2 MICROSCOPIC APPROACH: EQUATION OF MOMENTUM TRANSFER

The equation of momentum transfer—more commonly called the equation of motion—describes the velocity distribution and pressure drop in a moving fluid. It is derived from momentum considerations by applying a momentum balance on a rate basis in conjunction with Newton's law to a volume element in a moving field. Once again, this microscopic derivation is available in the literature.(2,3)

If the fluid is Newtonian, the components of the shear-stress may be replaced by the shear-stress components given by Newton's law (see Table 5.1). In addition, the density and the viscosity of the fluid are often constant, and the only significant external force concerned is that due to gravity. The resulting equation has been referred to as the Navier–Stokes equation. This equation is also expanded into rectangular, cylindrical and spherical coordinates; the results are presented in Tables 9.1, 9.2, and 9.3.

Illustrative Example 9.5 Derive Equation (5.16), as presented in Chapter 5. A fluid is flowing through a long vertical cylindrical duct of radius R under steady-state laminar flow conditions (see Fig. 9.3). Calculate the velocity profile as a function of the pressure drop per unit length in the direction of motion. Also, calculate the volumetric flow rate, the average velocity, the maximum velocity, and the ratio of the average to the maximum velocity.

Solution This problem is solved using cylindrical coordinates. The describing equations are now "extracted" from Table 9.2. Since the flow is one-dimensional

$$v_r = 0$$
$$v_\phi = 0$$
$$v_z \neq 0$$

Table 9.1 The equation of motion: expansion in rectangular coordinates

x-component:

$$\frac{\rho}{g_c}\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \frac{\mu}{g_c}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho\frac{g_x}{g_c}$$

y-component:

$$\frac{\rho}{g_c}\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \frac{\mu}{g_c}\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho\frac{g_y}{g_c}$$

z-component:

$$\frac{\rho}{g_c}\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \frac{\mu}{g_c}\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho\frac{g_z}{g_c}$$

Table 9.2 The equation of motion: expansion in cylindrical coordinates

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r-component:

$$\frac{\rho}{g_c}\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+v_\phi\frac{\partial v_r}{\partial \phi}-\frac{v_\phi^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial P}{\partial r}$$

$$+\frac{\mu}{g_c}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\{rv_r\}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \phi^2}-\frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi}+\frac{\partial^2 v_r}{\partial z^2}\right]+\rho\frac{g_r}{g_c}$$

ϕ-component:

$$\frac{\rho}{g_c}\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r v_\phi}{r}+v_z\frac{\partial v_\phi}{\partial z}\right)=-\frac{1}{r}\frac{\partial P}{\partial \phi}$$

$$+\frac{\mu}{g_c}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\{rv_\phi\}\right)+\frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \phi}+\frac{\partial^2 v_\phi}{\partial z^2}\right]+\rho\frac{g_\phi}{g_c}$$

z-component:

$$\frac{\rho}{g_c}\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\phi}{r}\frac{\partial v_z}{\partial \phi}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}$$

$$+\frac{\mu}{g_c}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \phi^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho\frac{g_z}{g_c}$$

Table 9.3 The equation of motion: expansion in spherical coordinates

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r-component:

$$\frac{\rho}{g_c}\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)=-\frac{-\partial P}{\partial r}$$

$$+\frac{\mu}{g_c}\left(\nabla^2 v_r-\frac{2}{r^2}v_r-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2}{r^2}v_\theta\cot\theta-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho\frac{g_r}{g_c}$$

θ-component:

$$\frac{\rho}{g_c}\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}$$

$$+\frac{\mu}{g_c}\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho\frac{g_\theta}{g_c}$$

ϕ-component:

$$\frac{\rho}{g_c}\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta\right)=-\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}$$

$$+\frac{\mu}{g_c}\left(\nabla^2 v_\phi-\frac{v_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right)+\rho\frac{g_\phi}{g_c}$$

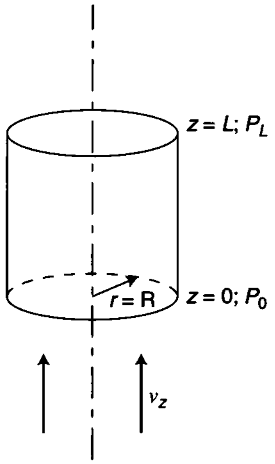

Figure 9.3 Tubular flow.

The terms v_r, v_ϕ, and all their derivatives must be zero. From Table 7.1,

$$\frac{\partial v_z}{\partial z} = 0$$

Based on physical grounds

$$\frac{\partial v_z}{\partial \phi} = 0$$

Based on the problem statement

$$\partial v_z / \partial t = 0$$

It is reasonable to conclude that v_z might vary with r, i.e.,

$$v_z = v_z(r)$$

This means

$$\frac{\partial v_z}{\partial r} \neq 0$$

or perhaps

$$\frac{\partial^2 v_z}{\partial r^2} \neq 0$$

Examining the equation of motion in cylindrical coordinates in Table 9.2, one notes that

$$\frac{\partial P}{\partial r} = 0$$

$$\frac{\partial P}{\partial \phi} = 0$$

$$\frac{\partial P}{\partial z} = \frac{\mu}{g_c}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$$

The last equation may be rewritten

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{\mu}{g_c}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v_z}{\mathrm{d}r}\right)\right] \tag{9.7}$$

The left-hand side is a constant or a function of z. The right-hand side is either a constant or a function of r. One can then conclude that both must equal a constant. Since $\mathrm{d}P/\mathrm{d}z$ is a constant, it is written in the finite form

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}z} &= +\frac{\Delta P}{\Delta z}\\ &= -\frac{\Delta P}{L}\end{aligned}$$

The negative sign appears because P decreases as z increases. Equation (9.7) now becomes

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v_z}{\mathrm{d}r}\right) = -\frac{g_c\Delta P}{\mu L}$$

It would be wise to multiply both sides of the equation by $r\,\mathrm{d}r$; otherwise, some difficulty would be encountered on integrating the equation.

$$\mathrm{d}\left(r\frac{\mathrm{d}v_z}{\mathrm{d}r}\right) = -\frac{g_c\Delta P}{\mu L}r\,\mathrm{d}r$$

Integrating once

$$r\frac{\mathrm{d}v_z}{\mathrm{d}r} = -\frac{g_c\Delta P}{2\mu L}r^2 + A \tag{9.8}$$

Multiplying both sides by $\mathrm{d}r/r$

$$\mathrm{d}v_z = -\frac{g_c \Delta P}{2\mu L} r\,\mathrm{d}r + \frac{A}{r}\mathrm{d}r$$

and integrating

$$v_z = -\frac{g_c \Delta P}{4\mu L} r^2 + A \ln r + B \tag{9.9}$$

What about the BCs? Note that the procedure for the evaluation of integration constants A and B is also available in Chapter 5.

BC(1)

$$v_z = 0 \quad \text{at } r = R$$

BC(2)

$$\left.\begin{array}{ll} v_z = \text{finite} & \text{at } r = 0 \\ \text{or the equivalent} & \\ \dfrac{\mathrm{d}v_z}{\mathrm{d}r} = 0 & \text{at } r = 0 \end{array}\right\} \text{based on physical grounds}$$

Substituting BC(2) into Equation (9.8) or (9.9) yields

$$A = 0$$

BC(1) gives

$$0 = -\frac{g_c \Delta P}{4\mu L} R^2 + B$$

$$B = \frac{g_c \Delta P}{4\mu L} R^2$$

Substitution of A and B leads to Equation (5.16), as given in Chapter 5 and shown again below

$$v_z = \frac{g_c \Delta P}{4\mu L}(R^2 - r^2)$$

Illustrative Example 9.6 With reference to Illustrative Example 9.5, comment on the nature of the velocity profile.

Solution An examination of Equation (5.16) indicates that the velocity profile is parabolic. Parabolic velocity profiles are the norm for laminar flow in pipes. The reader is left the exercise of plotting v_z as a function of r in order to verify the above statement.

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