10

LAW OF HYDROSTATICS[1]

10.1 INTRODUCTION

When a fluid is at rest, there is no shear stress and the pressure at any point in the fluid is the same in all directions. The pressure is also the same across any longitudinal section parallel with the Earth's surface; it varies only in the vertical direction, that is, from height to height. This phenomenon gives rise to hydrostatics, the subject title for this chapter. Following this introduction, this chapter addresses (once again) pressure principles, buoyancy effects (including Archimedes' Law), and manometry principles.

10.2 PRESSURE PRINCIPLES

Consider a differential element of fluid of height, dz, and uniform cross-section area, S. The pressure, P, is assumed to increase with height, z. The pressure at the bottom surface of the differential fluid element is P; at the top surface, it is $P + dP$. Thus, the net pressure difference, dP, on the element is acting downward. A force balance on this element in the vertical direction yields:

$$\text{downward pressure force} - \text{upward pressure force} + \text{gravity force} = 0 \qquad (10.1)$$

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so that

$$(P + \mathrm{d}P)S - PS + \rho S \frac{g}{g_c} \mathrm{d}z = 0$$

$$-(\mathrm{d}P)S - \rho \frac{g}{g_c} S(\mathrm{d}z) = 0 \tag{10.2}$$

As described in Chapter 2, one has a choice as to whether to include g_c in the describing equation(s). As noted, the term g_c is a conversion constant with a given magnitude and units, e.g., 32.2 $(\mathrm{lb/lb_f})(\mathrm{ft/s^2})$ or dimensionless with a value of unity, for example, $g_c = 1.0$. In this development, g_c is retained. Rearrangement of Equation (10.2) yields

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\rho \frac{g}{g_c} = -\gamma \tag{10.3}$$

The term γ is the specific weight of fluid with units of $\mathrm{lb_f/ft^3}$ or $\mathrm{N/m^3}$. This equation is the hydrostatic or barometric differential equation. The term $\mathrm{d}P/\mathrm{d}z$ is often referred to as the pressure gradient.

Equation (10.3) is a first-order ordinary differential equation. It may be integrated by separation of variables

$$\int \mathrm{d}P = -\int \rho \frac{g}{g_c} \mathrm{d}z = -\frac{g}{g_c} \int \rho \, \mathrm{d}z \tag{10.4}$$

For most engineering applications involving liquids, and many applications involving gases, the density may be considered constant, i.e., the fluid is incompressible. Taking ρ outside the integration sign and integrating between any two limits in the fluid (Station 1 is where the pressure equals P_1 and the elevation is z_1, and Station 2 has a pressure of P_2 and elevation z_2), the pressure–height relationship is

$$P_2 - P_1 = -\rho \frac{g}{g_c}(z_2 - z_1) \tag{10.5}$$

Equation (10.3) may also be written as:

$$\frac{P_1}{\rho(g/g_c)} + z_1 = \frac{P_2}{\rho(g/g_c)} + z_2 \tag{10.6}$$

The term $P/\rho(g/g_c)$ is defined as the *pressure head* of the fluid, with units of m (or ft) of fluid. Equation (10.6) states that the sum of the pressure head and potential head is constant in hydrostatic "flow". Equation (10.6) is sometimes termed Bernoulli's hydrostatic equation. It is useful in calculating the pressure at any liquid depth.

Illustrative Example 10.1 Given the height of a column of liquid whose top is open to the atmosphere, determine the pressure exerted at the bottom of the column and calculate the pressure difference. The height of the liquid (mercury) column is 2.493 ft, the density of mercury is 848.7 lb/ft^3, and atmospheric pressure is 2116 psf.

Solution Refer to Equation (10.5). If P_1 is assumed to represent atmospheric pressure, $P_2 - P_1$ reduces to the gauge pressure at the bottom of the column. The equation describing the gauge pressure in terms of the column height and liquid density is:

$$P_g = \rho \frac{g}{g_c} h; \; h = \Delta z$$

Calculate the gauge pressure, a term that also represents the pressure difference in psf.

$$P_g = \rho \frac{g}{g_c} h = (848.7)(1)(2.493) = 2116 \text{ psf}$$

Determine the pressure in psfa (psf absolute)

$$P = P_g + P_a = 2116 + 2116 = 4232 \text{ psfa} = 29.4 \text{ psia}$$
$$= 14.7 \text{ psfg} = 2 \text{ atm absolute (atma)}$$

Illustrative Example 10.2 Suppose that one is interested in determining the depth in the Atlantic Ocean at which the pressure is equal to 10 atm absolute (atma). Assume that sea water is incompressible with a density of 1000 kg/m^3, and the pressure at the ocean surface is 1.0 atm absolute (atma) or 0.0 atm gauge (atmg).

Solution Take Station 1 to be at the ocean surface ($z_1 = 0$) with Station 2 at a depth equal to z_2 where the pressure, P_2, equals 10 atma = (10)(101,325) = 1.013 MPa abs. Apply Equation (10.6) between points 1 and 2

$$\frac{P_1}{\rho(g/g_c)} + z_1 = \frac{P_2}{\rho(g/g_c)} + z_2$$

$$z_2 = z_1 - \frac{(P_2 - P_2)}{\rho g} = 0 - \frac{(10 - 1)(101{,}325)}{(1000)(9.807)} = -93 \text{ m} = -305 \text{ ft}$$

The pressure is 10 atma, or 9 atmg at a depth of 93 m or 305 ft.

Calculations of fluid pressure force on submerged surfaces is important in selecting the proper material and thickness. If the pressure on the submerged surface is not uniform, then the pressure force is calculated by integration. Also note that a nonmoving, simple fluid exerts only pressure forces; moving fluids exert both pressure and shear forces.

Illustrative Example 10.3 A cylindrical tank is 20 ft (6.1 m) in diameter and 45 ft high. It contains water ($\rho = 1000$ kg/m^3) to a depth of 9 ft (2.74 m) and 36 ft

(10.98 m) of an immiscible oil (SG = 0.89) above the water. The tank is open to the atmosphere (see Fig. 10.1). Calculate the density of the oil, the gauge pressure at the oil–water interface, the gauge pressure and pressure force at the bottom of the tank and the resultant pressure force on the bottom 9 ft of the side of the tank. Assume the elevation, z, to be zero at the oil–air interface.

Solution Calculate the density of oil in kg/m^3

$$\rho_{\text{oil}} = (\text{SG})\rho_w = (0.89)(1000) = 890\ \text{kg/m}^3$$

Apply Bernoulli's equation between points 1 and 2 to calculate the gauge pressure at the water–oil interface. Note that since this is a static fluid application, the velocity is zero and Equation (10.5) applies

$$P_1 + \rho_{\text{oil}}\frac{g}{g_c}z_1 = P_2 + \rho_{\text{oil}}\frac{g}{g_c}z_2$$

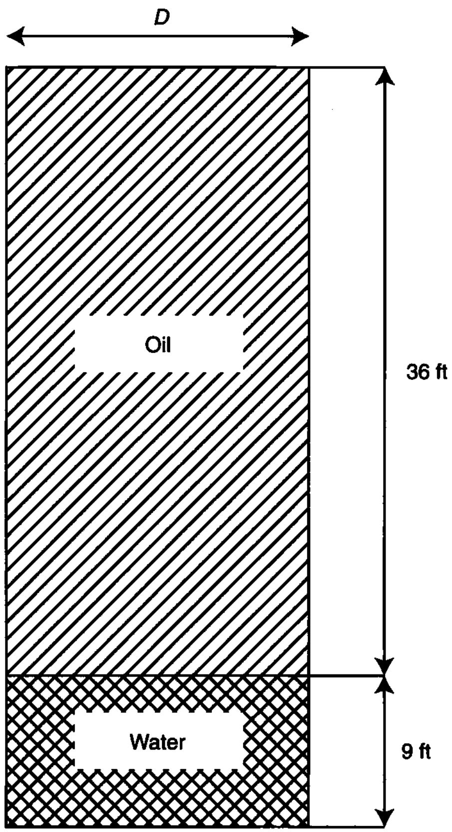

Figure 10.1 Oil–water open tank.

Note that $z_1 = 0$, $P_1 = 1$ atm, and $z_2 = -36\,\text{ft} = -10.98\,\text{m}$. Substitution yields

$$P_{2,g} = \rho_{\text{oil}}\frac{g}{g_c}(z_1 - z_2) = (890)(9.807)(10.98)$$
$$= 95{,}771 \text{ Pag} = 0.945 \text{ atmg}$$

The gauge pressure at the bottom (z_3) of the tank is

$$P_2 + \rho_w\frac{g}{g_c}z_2 = P_3 + \rho_w\frac{g}{g_c}z_3$$
$$P_3 = P_2 + \rho_w\frac{g}{g_c}(z_2 - z_3) = 95{,}771 + (1000)(9.807)(2.74)$$
$$= 122{,}673 \text{ Pag} = 1.2 \text{ atmg}$$

The pressure force at the bottom of the tank is

$$F = P_3S = (122{,}673 + 101{,}325)\frac{\pi(6.1)^2}{4}$$
$$= 6{,}537{,}684 \text{ N} = 1{,}469{,}671 \text{ lb}_f$$

Obtain an equation describing the variation of pressure with height in the water layer.

$$P = P_2 + \rho_w\frac{g}{g_c}(z_2 - z) = (890)(9.807)(10.98)$$
$$= 95{,}771 + (1000)(9.807)(-10.98 - z)$$
$$P = 95{,}771 + 9807(-10.98 - z) = -11{,}910 - 9807z$$

Calculate the force on the side of the tank, within the water layer.

$$\mathrm{d}F = P\,\mathrm{d}S$$

$$F = \int \mathrm{d}F = \int P(\pi D)\,\mathrm{d}z = (\pi D)\int_{z_3}^{z_2} P\,\mathrm{d}z = (\pi)(6.1)\int_{z_3}^{z_2}(-11{,}910 - 9807z)\,\mathrm{d}z$$

$$F = 6.1\pi[-11{,}910(z_2 - z_3) - 4903.5(z_2^2 - z_3^2)]$$

Setting $z_3 = -13.72$ m and $z_2 = -10.98$ m, results in

$$F = 6.1\pi[-11{,}910(-10.98 + 13.72) - 4903.5(-10.98^2 - (-13.72)^2)]$$
$$= 5.73 \times 10^6 \text{ N}$$

10.2.1 Buoyancy Effects; Archimedes' Law

Buoyancy force is the force exerted by a fluid on an immersed or floating body. Archimedes' Law states that for a body floating in a fluid, the volume of fluid displaced equals the volume of the immersed portion of the body, and the weight (force) of fluid displaced equals the weight (force) of the body. The buoyancy force on the body, F_B, is

$$\begin{aligned} F_B &= (\text{displaced volume of fluid})(\text{fluid density})(g/g_c) \\ &= (\text{displaced volume of fluid})(\text{fluid specific weight})(1/g_c) \end{aligned}$$

Thus,

$$F_B = V_{disp}\rho_{fl}\frac{g}{g_c} = V_{disp}\gamma_{fl}\frac{1}{g_c} \tag{10.7}$$

where γ_{fl} is the specific weight of the fluid (see Equation 10.3).

In the case where the density of the fluid and the body are equal, the body remains at its point or location in the fluid where it is placed. This is termed neutral buoyancy. In the case where the body density, ρ_{body}, is greater than the fluid density, ρ_{fluid}, the body will sink in the fluid.

Illustrative Example 10.4 A block of some material weighs 200 lb_f in air. When placed in water (specific weight $\gamma_{H_2O} = 62.4\ lb_f/ft^3$), it weighs 120 lb_f. Determine the density of the material. Assume that the material density of the block is greater than the water density so that the block sinks in water, and that the volume of water displaced equals the volume of the block. Also calculate the buoyancy force and the block volume, V.

Solution The buoyant force is

$$F_B = 200 - 120 = 80\ lb_f$$

$$F_B = V_{disp}(\rho g)_{H_2O}\frac{1}{g_c} = V_{disp}\gamma_{H_2O}\frac{1}{g_c} = V(62.4)/(1)$$

$$V = 80/62.4 = 1.282\ ft^3$$

Next, use the block weight in air and its volume to calculate the density of the block, since the density is the mass (in air) per unit volume of the material

$$\text{Weight in air} = 200\ lb_f = F_B = V\rho_{block}\frac{g}{g_c} = 1.282\rho_{block}(1)$$

$$\rho_{block} = \frac{200}{1.282} = 150\ lb/ft^3 = 2500\ kg/m^3$$

The assumption of $\rho_{block} > \rho_{water}$ (1000 kg/m^3) is justified.

Illustrative Example 10.5 A hydrometer is a liquid specific gravity indicator, with the value being indicated by the level at which the free surface of the liquid intersects the stem when floating in a liquid (see Fig. 10.2). Three hydrometer scales are commonly used. The API scale is used for oils, and the two Baumé scales are used for liquids—one for liquids heavier than water and the other for liquids lighter than water. The relationship between the hydrometer API scales and the specific gravity, SG, is

$$SG = \frac{141.15}{131.5 + \text{deg API}}$$

The relationship between the hydrometer degree Baumé scale and the specific gravity, SG, of liquids lighter than water is

$$SG = \frac{140}{130 + \text{deg Baumé}}$$

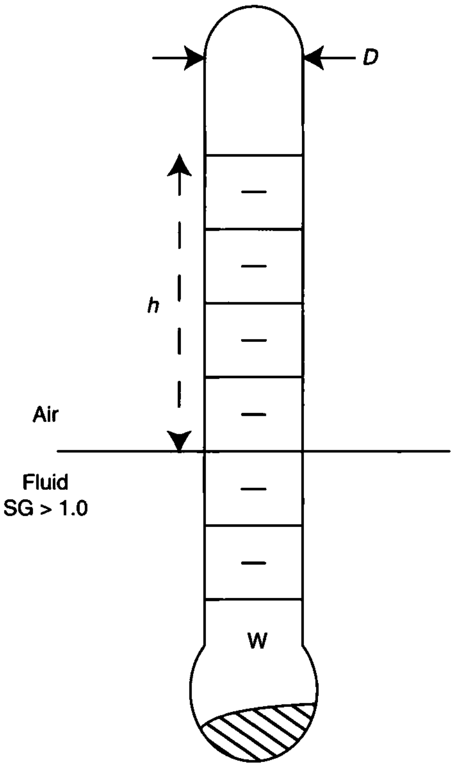

Figure 10.2 Hydrometer.

For liquids heavier than water the relation is

$$\text{SG} = \frac{145}{145 + \text{deg Baumé}}$$

When placed in a liquid, the hydrometer floats at a level which is a measure of the specific gravity of the liquid. The 1.0 mark is the level when the liquid is distilled water ($\rho = 1000\,\text{kg/m}^3$). The stem is of constant diameter and has a weight in the bottom of the bulb, W.

If the total hydrometer weight, W, is 0.13 N and the stem diameter is 8 mm, calculate the height, h, where it will float. The liquid is heavier than water and has a specific gravity of 33.5 deg. Baumé.

Solution Apply Achimedes' Law

$$F = V_{\text{disp}}(\text{SG})\rho_{\text{water}}\frac{g}{g_c}$$

For water, (SG = 1.0),

$$V_{\text{disp}} = \frac{\pi D^2}{4}z_1$$

where z_1 is the hydrometer reading. In the case of liquid of SG > 1.0,

$$V_{\text{disp}} = \frac{\pi D^2}{4}z_2$$

Therefore, two equations may be written

$$F = \frac{\pi D^2}{4}z_1\rho_{\text{water}}\frac{g}{g_c}$$

$$F = \frac{\pi D^2}{4}z_2(\text{SG})\rho_{\text{water}}\frac{g}{g_c}$$

Divide the above two equations to obtain

$$\frac{z_1}{z_2} = \text{SG}$$

This equation may be rearranged to give

$$\frac{z_1 - z_2}{z_1} = \frac{h}{z_1} = 1 - \frac{1}{\text{SG}}$$

or

$$h = z_1\left(1 - \frac{1}{\text{SG}}\right)$$

Substitute for z_1 (see above) to obtain

$$h = \frac{4F}{\pi D^2 \left(\rho_{\text{water}} \dfrac{g}{g_c}\right)} \left(1 - \frac{1}{\text{SG}}\right)$$

This is the hydrometer equation. Substitute the numerical values provided to calculate h.

$$h = \frac{4(0.13)}{\pi(0.008)^2(9807)} \left(1 - \frac{1}{1.3}\right) = 61 \text{ mm}$$

The hydrometer is a simple device to estimate liquid densities. It is used widely in various industries. By varying the stabilizing weight or the stem diameter it is possible to design the hydrometer to be sensitive to different ranges of specific gravities.

10.3 MANOMETRY PRINCIPLES

As noted earlier, the fundamental equation of fluid statics indicates that the rate of change of the pressure P is directly proportional to the rate of change of the depth z, or

$$\frac{dP}{dz} = -\rho \frac{g}{g_c} \tag{10.3}$$

where z = vertical displacement (upward is considered positive)
ρ = fluid density
g = acceleration due to gravity
g_c = unit conversion factor

For constant density, the above equation may be integrated to give the hydrostatic equation

$$P_2 = P_1 + \frac{\rho g h}{g_c}; \quad h = z_1 - z_2 \tag{10.5}$$

Here point 2 is located at a distance h below point 1.

Manometers are often used to measure pressure differences. This is accomplished by a direct application of the above equation. Pressure differences in manometers may be computed by systematically applying the above equation to each leg of the manometer.

Illustrative Example 10.6 Consider the system pictured in Fig. 10.3.

Solution Since the density of air is effectively zero, the contribution of the air to the 3-ft manometer reading can be neglected. The contribution to the pressure

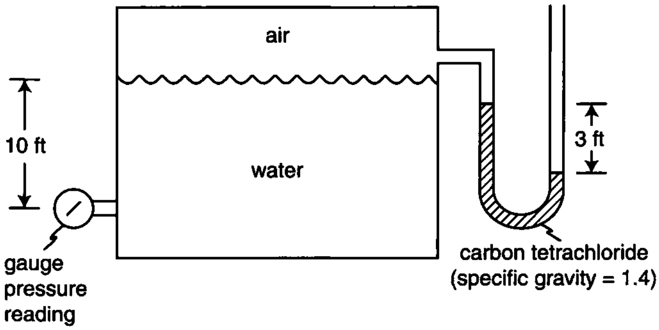

Figure 10.3 Diagram for Illustrative Example 10.6.

due to the carbon tetrachloride in the manometer is found by using the hydrostatic equation.

$$\begin{aligned}\Delta P &= \rho g h / g_c \\ &= (62.4)(1.4)(3) = 262.1 \text{ psf} \\ &= 1.82 \text{ psi}\end{aligned}$$

Since the right leg of the manometer is open to the atmosphere, the pressure at that point is atmospheric:

$$P = 14.7 \text{ psia}$$

Note that this should be carried as a positive term.

The contribution to the pressure due to the height of water above the pressure gauge is similarly calculated using the hydrostatic equation.

$$\Delta P = (62.4)(3) = 187.2 \text{ psf} = 1.3 \text{ psi}$$

The pressure at the gauge is obtained by summing the results of the steps above, but exercising care with respect to the sign(s):

$$\begin{aligned}P &= 14.7 - 1.82 + 1.3 = 14.18 \text{ psia} \\ &= 14.18 - 14.7 = -0.52 \text{ psig}\end{aligned}$$

The pressure may now be converted to psfa and psfg:

$$\begin{aligned}P &= (14.18)(144) = 2042 \text{ psfa} \\ &= (-0.52)(144) = -75 \text{ psfg}\end{aligned}$$

Care should be exercised when providing pressure values in gauge and absolute pressure. The key equation is once again

$$P(\text{gauge}) = P(\text{absolute}) - P(\text{ambient}); \quad \text{consistent units}$$

The subject of manometry will be revisited in Chapter 19.

REFERENCE

Much of the material in this chapter was adopted from:

1. J. Reynolds, J. Jeris, and L. Theodore. "Handbook of Chemical and Environmental Engineering Calculations," John Wiley & Sons, Hoboken, NJ, 2004.

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