CHAPTER 10

Stresses, Effective Stress, Water Stress, Air Stress, and Strains

10.1 GENERAL

A soil mass is subjected to internal and boundary forces due to loading by a building, bridge, dam, retaining wall, or even rain and evaporation near the ground surface. Under these forces, displacements take place. The objective of the design process is to ensure that the displacements are tolerable and safe for the structure. It is difficult to use forces and displacements as parameters in the design process because they are not normalized quantities and therefore cannot be compared between, for example, the full-scale behavior in the field and a small-scale test in the laboratory. The concept of stress and strain is used to normalize forces and displacements to the point where such comparisons can be made. Note that although using stresses and strains makes some problems easier to deal with, it may create some difficulties at the same time. For example, you cannot add stresses as you would add forces. Any time one wishes to compose stresses, it is much preferable to use the stresses to calculate the forces, then add the forces by conventional means to find the resultant, and then calculate the resultant stress.

10.2 STRESS VECTOR, NORMAL STRESS, SHEAR STRESS, AND STRESS TENSOR

A *stress* is a force divided by the area over which it applies. The force is not necessarily perpendicular or tangent to the area. Because the force is a vector, so is the stress. For a given point in a soil mass and for a given plane at that point, there is one stress vector:

$$\vec{t} = \lim_{A \to 0} \frac{\vec{F}}{A} \tag{10.1}$$

where t is the stress vector, F is the resultant force at the point considered, and A is the area of the plane on which F is acting. The stress vector, like the force, can always be decomposed into a normal stress σ and a shear stress τ. If the force is perpendicular to the area, the stress is a normal stress. If the force is tangential to the area, the stress is a shear stress:

$$\sigma = \frac{N}{A}, \quad \tau = \frac{T}{A}, \tag{10.2}$$

where σ is the normal stress, N is the force normal to the surface of area A, τ is the shear stress, and T is the force tangent to the surface of area A. For a given point in a soil mass, there is one resultant force but there is an infinity of stress vectors because, though there is only one force, one can choose an infinity of planes with different orientations through that point. By swiveling the plane around that point, one will find three planes where the shear stresses are zero. These planes are perpendicular to each other and are called the *principal planes*; the normal stresses on the principal planes are called *principal stresses* and are denoted σ_1, σ_2, and σ_3. The largest of the three is the *major principal stress* σ_1, the smallest is the *minor principal stress* σ_3, and σ_2 is called the *intermediate principal stress*.

The stress state at one point is usually represented by drawing a cube with axes in the x, y, and z directions. The stress vector on each face of the cube is decomposed into a normal stress (e.g., direction of x) and two shear stresses (e.g., directions of y and z). The definitions refer to the following labeling system:

- σ_{xx} is the stress on the plane perpendicular to x and in the direction of x; it is a normal stress.
- τ_{xy} is the stress on the plane perpendicular to x and in the direction of y; it is a shear stress.
- τ_{xy} is the stress on the plane perpendicular to x and in the direction of z; it is a shear stress.

Those three stresses are the decomposition of the stress vector t along the three orthogonal directions associated with the plane perpendicular to x (Figure 10.1).

For reasons of moment equilibrium, the shear stresses on two perpendicular planes must be equal ($\tau_{xy} = \tau_{yx}$). For reasons of symmetry and equilibrium, and because the cube is at the infinitesimal scale, stresses on opposite faces are equal and opposite. Therefore, while there are a total of 18 stresses

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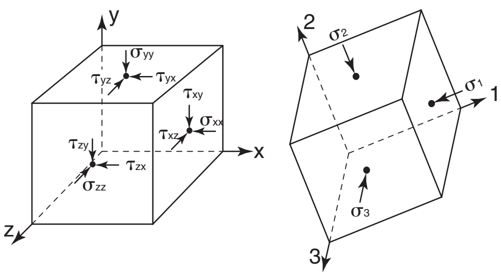


Figure 10.1 Stresses on an elementary cube.

(6 faces times 3 stresses), there are only 6 independent stresses (3 normal stresses and 3 shear stresses). These stresses are organized and presented in a stress tensor, which is a 3 × 3 matrix. That matrix has 9 elements, but is symmetric because the shear stresses on perpendicular planes are equal. Once the stress tensor is known at one point, all stresses are known at that point by simple geometric rules:

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{10.3}$$

The stress tensor Σ can be decomposed into the spherical tensor S and the deviatoric tensor *D*:

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = S + D$$

$$= \begin{bmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{bmatrix} + \begin{bmatrix} \sigma_{xx} - \sigma_M & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - \sigma_M & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \sigma_M \end{bmatrix} \tag{10.4}$$

where

$$\sigma_M = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \tag{10.5}$$

The spherical tensor represents a confinement effect at the point considered in the soil; it creates consolidation of the soil element with no shear. The deviatoric tensor represents the effect of various shear stresses on the soil element; it creates distortion with no mean normal stress.

10.3 SIGN CONVENTION FOR STRESSES AND STRAINS

Sign conventions are necessary in engineering because equations can differ for different conventions. Here, we will use compression stresses as positive because compression stresses are the most common case in soil mechanics. Note that in structures it is the contrary: there tension stresses are chosen to be positive normal stresses. Shear stresses are more complicated, so two cases must be considered.

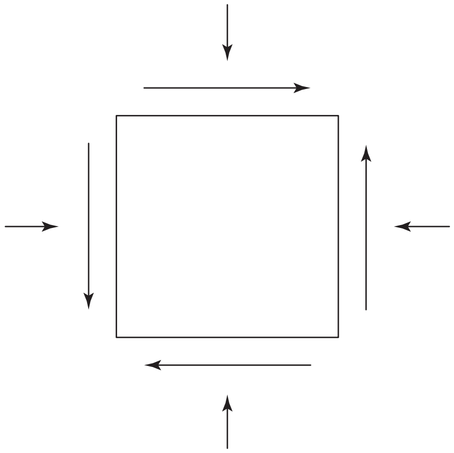


Figure 10.2 Positive sign convention for stress relationship equations.

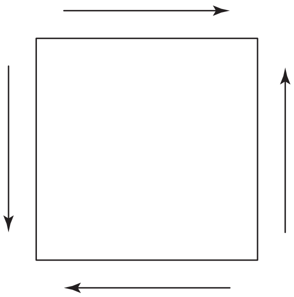


Figure 10.3 Positive sign convention for Mohr circle.

When dealing with the equations that relate stresses on two perpendicular planes to the stresses on an inclined plane, the positive convention for shear stress is as shown in Figure 10.2. However, when dealing with the Mohr circle representation of shear stresses, then the positive convention for shear stresses is as shown in Figure 10.3. Note that Figure 10.3 does not represent a feasible state of stress in a material, but simply the sign convention for the Mohr circle.

For normal strains, compressive strains will be considered positive. For shear strains, positive strains will be those that decrease an initially right angle.

10.4 CALCULATING STRESSES ON ANY PLANE: EQUILIBRIUM EQUATIONS FOR TWO-DIMENSIONAL ANALYSIS

At a specific point in the soil mass, and given the stresses on two perpendicular planes, the normal and shear stress on any other plane forming a wedge with the two perpendicular planes (Figure 10.4) can be related to the stresses on the two perpendicular planes as follows.

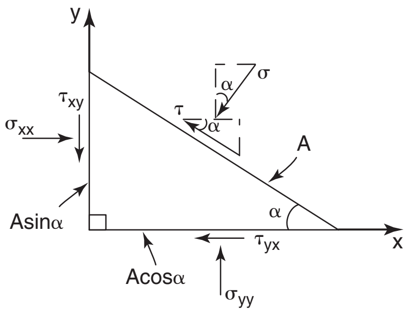


Figure 10.4 Wedge subjected to normal and shear stresses in equilibrium.

Referring to Figure 10.4, horizontal and vertical equilibrium of forces lead to equations 10.6 and 10.7:

$$\sigma_{\mathrm{y}} \mathrm{A}\cos\alpha - \tau_{xy}\mathrm{A}\sin\alpha + \tau\,\mathrm{A}\sin\alpha - \sigma\,\mathrm{A}\cos\alpha = 0 \qquad (10.6)$$

$$\sigma_{\mathrm{x}} \mathrm{A}\sin\alpha - \tau_{yx}\mathrm{A}\cos\alpha - \tau\,\mathrm{A}\cos\alpha - \sigma\,\mathrm{A}\sin\alpha = 0 \qquad (10.7)$$

where σ_{y} and σ_{x} are the normal stresses on the planes perpendicular to the y and x directions respectively, σ is the normal stress on the oblique surface, τ_{xy} and τ_{yx} are the shear stresses on the planes perpendicular to the x and y directions respectively, τ is the shear stress on the oblique surface, A is the area of the oblique surface, and α is the angle of the oblique surface as shown on Figure 10.4. From Eqs. 10.6 and 10.7 we get:

$$\sigma = \frac{\sigma_{\mathrm{y}} + \sigma_{\mathrm{x}}}{2} + \frac{\sigma_{\mathrm{y}} - \sigma_{\mathrm{x}}}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \qquad (10.8)$$

$$\tau = -\frac{\sigma_{\mathrm{y}} - \sigma_{\mathrm{x}}}{2}\sin 2\alpha - \tau_{xy}\cos 2\alpha \qquad (10.9)$$

If the planes perpendicular to the x and y directions are principal planes (zero shear), then equations 10.8 and 10.9 become:

$$\sigma = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\cos 2\alpha \qquad (10.10)$$

$$\tau = -\frac{\sigma_1 - \sigma_3}{2}\sin 2\alpha \qquad (10.11)$$

where σ_1 and σ_3 are the major and minor principal stresses respectively, σ is the normal stress on the oblique surface, τ is the shear stress on the oblique surface, A is the area of the oblique surface, and α is the angle of the oblique surface as shown in Figure 10.4.

10.5 CALCULATING STRESSES ON ANY PLANE: MOHR CIRCLE FOR TWO-DIMENSIONAL ANALYSIS

In a set of coordinates where the shear stress on a plane is plotted on the vertical axis and the normal stress on the same

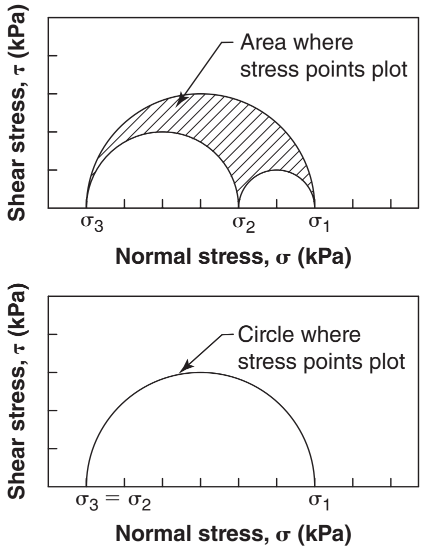


Figure 10.5 Shear stress vs. normal stress space and Mohr circle.

plane is plotted on the horizontal axis, three circles bound the zone where the stress points are located (Figure 10.5). Indeed, all the stress points with τ, σ coordinates obtained for all the planes at that point fall in an area bounded by three circles centered on the horizontal axis. The reason why the center of the circles is on the horizontal axis goes back to the fact that shear stresses on perpendicular planes are equal. The circles intersect the normal stress axis at the principal stress values σ_1, σ_2, and σ_3; therefore, the circles have common points at the end of the diameter on the normal stress axis (Figure 10.5).

If the intermediate principal stress σ_2 is equal to the minor or the major principal stress, then there are only two principal stresses and the three circles collapse into one (Figure 10.5). This circle is called the *Mohr circle*. Otto Mohr was a German civil engineer who demonstrated in 1882 how this single circle could be used to find stresses on any plane at a point.

The case in which the intermediate principal stress is equal to the minor or the major principal stress occurs in a number of common situations (unconfined compression test, column loading, triaxial test, tension test). In this case, the zone representing all the stress points becomes the circle itself, and simple geometric constructions can be used to find the normal stress and the shear stress given a plane at that point in the soil mass (e.g., the Pole method). The Mohr circle can be defined as the graphical representation of the stresses at one point in a mass for the case where the intermediate principal stress is equal to the minor or the major principal stress. If one considers a different point, then the Mohr circle will be different. However, in the general case, there are three circles at one point and most stress points are not on the circles.

In the simpler case where the three principal stresses reduce to two, the following construction can be used to find the stresses on a randomly chosen plane (Figure 10.6). Although the problem can be posed in many different ways, the key, once the Mohr circle is known, is the relationship between:

1. a stress point on the Mohr circle for which we know the plane on which these stresses act
2. the direction of another plane in the two-dimensional (2D) space
3. the stresses on that other plane

If you known 1 and 2, you can find 3. If you know 1 and 3, you can find 2. The relationship is that if α is the angle between the two planes in space, the angle between the two stress points on the Mohr circle is 2α. This is due to equations 10.8 and 10.9, which have 2α in them. The angle 2α on the Mohr circle could be taken clockwise or counterclockwise from the known stress point, and that would lead to two different answers. The correct direction is such that if you go from the known plane to the plane where you seek the stresses by an angle α in space, you have to go from the known stress point to the unknown stress point through 2α in the same direction on the Mohr circle. Figure 10.6 illustrates the case for the triaxial test; Figure 10.7 illustrates the case for the direct shear test.

The Pole method is another popular method for solving the same problem. The *Pole* is a point on the Mohr circle such that a line on the Mohr circle passing through the stress point and parallel to the plane on which the stresses act will intersect the Mohr circle at two points: the stress point and the Pole. The Pole method always has three components:

1. the Pole on the Mohr circle
2. the stress point on the Mohr circle
3. the plane on which the stresses act in space

You always have to know 2 of these 3 components to solve a problem. Typically, the first step is to find where the Pole is. For this you need to know the Mohr circle and a plane on which you know the stresses and therefore the stress point on the Mohr circle. The steps are as follows:

1. Draw a line from the known stress point on the Mohr circle parallel to the plane in the two-dimensional space on which the stresses act.
2. That line intersects the Mohr circle at 2 points: the stress point and the Pole. This gives the location of the Pole.
3. From the Pole on the Mohr circle, draw a line parallel to the plane on which the stresses are to be found.
4. That line intersects the Mohr circle at 2 points: the Pole and the stress point. The coordinates of this point are the stresses on the chosen plane, and the direction of these stresses are given by the sign convention discussed in section 10.3.

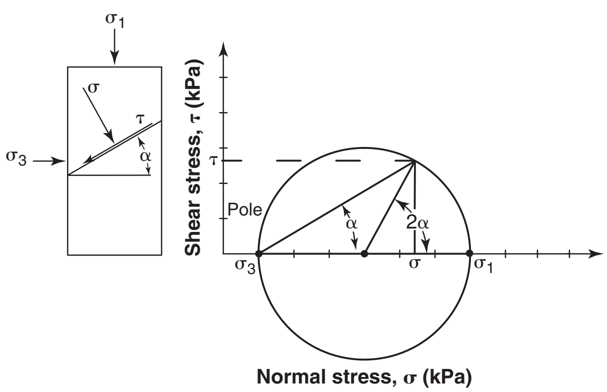


Figure 10.6 Relationship between physical space and Mohr circle (triaxial test).

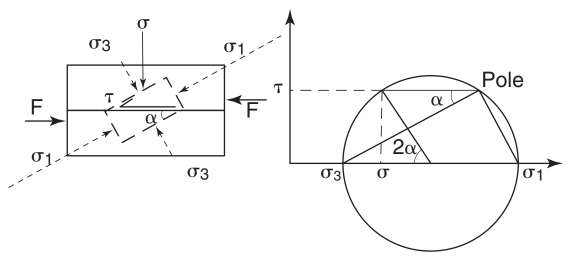


Figure 10.7 Relationship between physical space and Mohr circle (direct shear test).

10.6 MOHR CIRCLE IN THREE DIMENSIONS

Section 10.5 dealt with the special case in which the intermediate principal stress σ_2 is equal to the minor principal stress σ_3 or to the major principal stress σ_1. In this case, there is only one Mohr circle and the stress points are on the circle. In the general case, the intermediate principal stress σ_2 is not equal to σ_1 or σ_3. As a result, there are three Mohr circles (Figure 10.5). In this general case, the stress point is located within the area bound by the three circles. The construction to find the stress point is more complicated than in the 2D case, as might be expected. It requires knowledge of the location of the point and plane considered in spherical coordinates, and the graphical solution defines the stress point at the intersection of three circles centered at the centers of the Mohr circles. Most advanced mechanics books describe this solution.

10.7 STRESS INVARIANTS

Stress invariants are combinations of stresses. There are three stress invariants:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \quad (10.12)$$

$$I_2 = \frac{1}{6}((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2) \quad (10.13)$$

$$I_3 = \sigma_1\sigma_2\sigma_3 \quad (10.14)$$

where I_1, I_2, I_3 are the first, second, and third stress invariants, and $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses. These stress invariants are quite useful in describing yield criteria for soils. For example, the Drucker-Prager yield criterion (see Chapter 12) is:

$$\sqrt{I_2} = A + BI_1 \tag{10.15}$$

where A and B are constants for a given material.

10.8 DISPLACEMENTS

Displacements take place as a result of many possible factors: loading change, temperature change, and water content changes are common. They can occur in the three directions x, y, and z. Any point A in a soil mass can experience displacements in the three directions x, y and z. These displacements can also be a function of time t. We will call the displacements u, v, and w, corresponding to the directions x, y, and z respectively. Figure 10.8 illustrates the displacements in a two-dimensional space. For a point B different from A but very close to A, the displacements will be slightly different, so the displacements are a function of the location of the point considered: $u(x,y)$, $v(x,y)$, $w(x,y)$ for the two-dimensional space of Figure 10.8.

Point A moves to A′ such that the displacements are $u(x,y)$ in the direction of x and $v(x,y)$ in the direction of y. Point B (Figure 10.8) is at a distance dx from A in the direction of x. Point B moves to B′ such that the displacements are $u(x + dx, y)$ and $v(x + dx, y)$. The displacement $u(x + dx, y)$ can be written as u(x,y) plus or minus a little bit. This little bit is expressed mathematically as $\frac{\partial u}{\partial x}dx$, which is the product of the partial derivative of u with respect to x times the distance dx. So:

$$u(x + dx, y) = u(x, y) + \frac{\partial u}{\partial x}dx \tag{10.16}$$

This equation can be understood by looking at the diagram of Figure 10.9. In the same way, $v(x + dx, y)$ can be written as:

$$v(x + dx, y) = v(x, y) + \frac{\partial v}{\partial x}dx \tag{10.17}$$

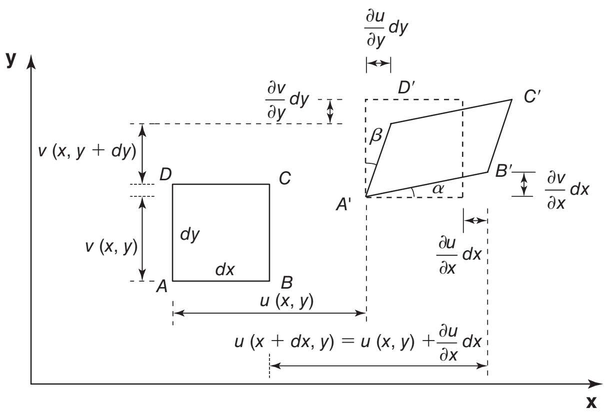


Figure 10.8 Illustration of displacements in a two-dimensional space.

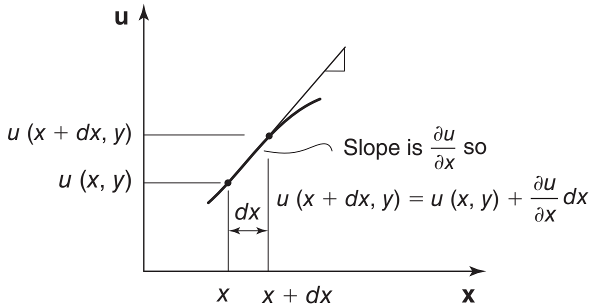


Figure 10.9 Visual illustration of equation 10.16.

Now consider point D on Figure 10.8. Point D is at a distance dy from A in the direction of y. Point D moves to D′ such that the displacements are $u(x, y + dy)$ and $v(x, y + dy)$. These displacements satisfy equations similar to 10.16 and 10.17, as follows:

$$u(x, y + dy) = u(x, y) + \frac{\partial u}{\partial y}dy \tag{10.18}$$

$$v(x, y + dy) = v(x, y) + \frac{\partial v}{\partial y}dy \tag{10.19}$$

10.9 NORMAL STRAIN, SHEAR STRAIN, AND STRAIN TENSOR

Strains are used to quantify the deformation of a material as a result of a loading process, a temperature change, a water content change, or some other change. Six strains are defined at one point: three normal strains and three shear strains. These six strains are defined from the knowledge of the three displacements (u, v, w) at a given point. Therefore, the six strains are not independent variables, and three strain relationships can be written linking the six strains to one another. *Normal strains* are used to quantify the change in length between two points. *Shear strains* are used to quantify the distortion of an angle.

Considering a point in a mass and an infinitesimal length in the x direction, the normal strain ε_{xx} at that point in the x direction is defined as the change in length of that infinitesimal length divided by the original length. The same definition applies for the normal strains in the y and z direction. More precisely, and referring to Figure 10.8, the normal strain ε_{xx} is defined as:

$$\varepsilon_{xx} = \frac{length\ \ A'B' - length\ \ AB}{length\ \ AB}$$
$$= \frac{dx + u(x + dx, y) - (dx + u(x, y))}{dx} = \frac{\partial u}{\partial x} \tag{10.20}$$

This equation assumes that the displacements are small and that the error in taking the length $A'B'$ equal to its projection on the x axis is very small. This is called the *small strain theory*. By the same reasoning in this theory, the two other normal strains are defined as:

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \tag{10.21}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{10.22}$$

Now consider the same point in the mass and two initially perpendicular directions *x* and *y* (DAB on Figure 10.8). In the deformed state, the right angle is deformed and becomes the angle formed by D′A′B′. The shear strain at point A is defined as one-half of the change in angle between DAB and D′A′B′ expressed in radians:

$$\varepsilon_{xy} = \frac{1}{2}(DAB - D'A'B') = \frac{1}{2}(\alpha + \beta)$$
$$= \frac{1}{2}(\tan\alpha + \tan\beta) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{10.23}$$

This equation assumes that the displacements are small because the angles α and β in radians are taken to be equal to $\tan\alpha$ and $\tan\beta$ respectively, and that the projection of A′B′ on the x axis and the projection of A′D′ on the y axis are equal to dx and dy respectively. The other shear strains are then:

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{10.24}$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{10.25}$$

These six strains (Eqs. 10.20–10.25) form the strain tensor, which is a 3 × 3 matrix where the shear strains are repeated on either side of the diagonal. Mathematically, these six strains are defined from the knowledge of the three independent displacements; therefore, the six strains represent only three independent variables.

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix} \tag{10.26}$$

Note that there is a factor ½ in front of the shear strain. This is because that shear strain is an average of the shear strains in both directions. In engineering practice, the factor ½ is not used and the engineering shear strains are defined as follows:

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{10.27}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{10.28}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{10.29}$$

Note also that the same Mohr circle concepts apply to strains as apply to stresses. One can draw Mohr circles for strains on the shear strain vs. normal strains set of axes (Figure 10.10). Note too that the Mohr circle for strains applies to the ε values and not the γ values of shear strains.

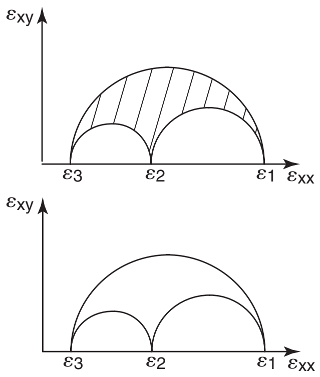


Figure 10.10 Mohr circle for strain.

10.10 CYLINDRICAL COORDINATES AND SPHERICAL COORDINATES

Sometimes the geometry of a problem makes it convenient to use cylindrical coordinates or even spherical coordinates to solve the problem. In cylindrical coordinates (Figure 10.11),

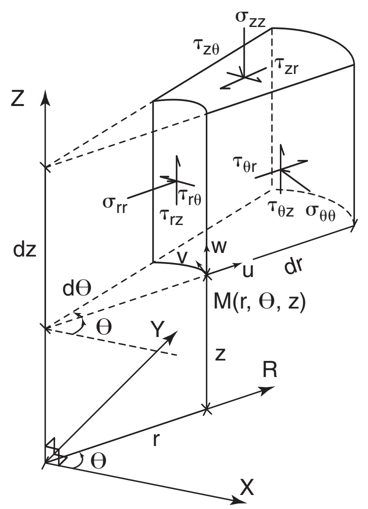


Figure 11 Stresses in cylindrical coordinates.

point M has coordinates r, θ, and z and the displacements of point M are u, v, and w in the directions of r, θ, and z respectively. The stresses are shown in Figure 10.11 and the stress tensor is:

$$\Sigma = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_{zz} \end{bmatrix} \tag{10.30}$$

The strains definitions are:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \tag{10.31}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{10.32}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{10.33}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} \tag{10.34}$$

$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} \tag{10.35}$$

$$\gamma_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{10.36}$$

In spherical coordinates (Figure 10.12), point M has coordinates r, θ, and φ and the displacements are u, v, and w, in the directions of r, θ, and φ respectively. The stresses are shown in Figure 10.12 and the stress tensor is:

$$\Sigma = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{r\varphi} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta\varphi} \\ \tau_{\varphi r} & \tau_{\varphi\theta} & \sigma_{\varphi\varphi} \end{bmatrix} \tag{10.37}$$

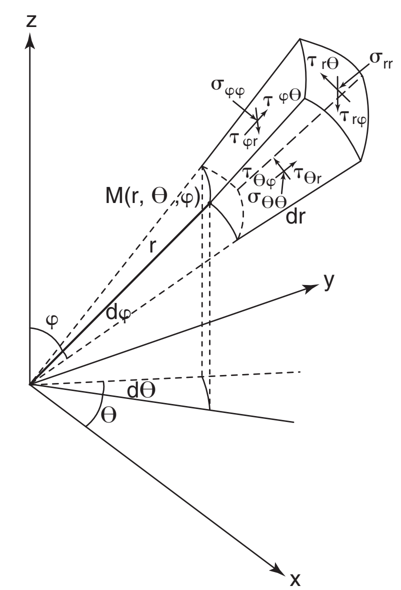


Figure 10.12 Stresses in spherical coordinates.

The strains definitions are:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \tag{10.38}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{10.39}$$

$$\varepsilon_{\varphi\varphi} = \frac{1}{r\sin\theta}\left(\frac{\partial w}{\partial \varphi} + u\sin\theta + v\cos\theta\right) \tag{10.40}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} \tag{10.41}$$

$$\gamma_{\theta\varphi} = \frac{1}{r}\left(\frac{\partial w}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial v}{\partial \varphi} - w\cot\theta\right) \tag{10.42}$$

$$\gamma_{\varphi r} = \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi} + \frac{\partial w}{\partial r} - \frac{w}{r} \tag{10.43}$$

10.11 STRESS-STRAIN CURVES

Stress-strain curves are often obtained when one tests a material in the laboratory or in the field. They usually relate one of the six stresses applied to an element of the material or to the mass to one of the six strains measured as a result of the stress applied. These stress-strain curves are very useful because they give fundamental soil properties that enter into the design process. As a result of this stress-strain curve relationship, one might be tempted to conclude that stresses and strains are intimately linked. However, there are exceptions to that intuitive statement. Take the example of the rails of high-speed trains. These rails have very few joints so that the very fast ride will be smooth. The rails change temperature during the daily temperature cycle; this temperature change would induce a change in length if such a length change were possible—but the anchors of the track do not permit such change and a stress develops because the strain is being suppressed. There is stress but no strain. Alternatively, consider a wire between two power-line poles. When the temperature increases, the wire gets longer but there is no change in stress. In this case there is strain but no stress. Nevertheless, in most cases stresses and strains are in fact intimately related.

10.12 STRESSES IN THE THREE SOIL PHASES

Concrete and steel are considered to be mono-phase materials (only one material). Soils, however, are three-phase materials, and stresses exist in each of the phases. The water can experience compression (also called *positive pore pressure*), or tension (also called *suction* or *negative pore pressure*). The air can also experience compression or tension. The shear stresses in the water and the air are neglected because they are very small compared to the shear stresses existing between the grains. The normal stress between the grains is very important because it has a significant influence on the shear strength and the compressibility of the soil. Note that failure in shear is the most common failure mechanism in soils.

10.13 EFFECTIVE STRESS (UNSATURATED SOILS)

Effective stress is a normal stress, and one of the most important parameters to know when dealing with soils. The effective stress equation gives the relationship between the various normal stresses that exist in the three phases. The derivation of this equation proceeds as follows. Consider a half space of soil in equilibrium and then within that half space consider an imaginary vertical cylinder. The top of the cylinder is the ground surface and the bottom of the cylinder is a generally horizontal plane that goes through the grain contacts and cuts through the voids (Figure 10.13). The external forces acting on that soil cylinder in the vertical direction are the total weight, including the grains, the water, and the air (acting downward); the vertical components of the contact forces between the grains on the bottom plane (acting upward); the vertical forces on the bottom plane corresponding to the water stress times the area of the water; and the vertical forces on the bottom plane corresponding to the air stress times the area of the air. The water area plus the air area plus the area of the contacts is equal to the total area. There are no vertical forces on the sides of the cylinder (shear forces) because there is no relative movement at that boundary.

Writing vertical equilibrium leads to the following equation:

$$F = \sum f_{ci} + \sum f_{wi} + \sum f_{ai} \tag{10.44}$$

where F is the weight of the soil mass plus any surcharge, and f_{ci}, f_{wi}, and f_{ai} are the vertical components of the forces between the grains, transmitted through the water, and transmitted through the air along the lower boundary of the free-body respectively. The forces f_{wi}, and f_{ai} are equal to:

$$f_{wi} = u_{wi} a_{wi} \tag{10.45}$$

$$f_{ai} = u_{ai} a_{ai} \tag{10.46}$$

where $\mathrm{u_{wi}}$ and $\mathrm{u_{ai}}$ are the water stress and air stress respectively, and a_{wi} and a_{ai} are the horizontal projections of the areas of water and air respectively on the bottom surface

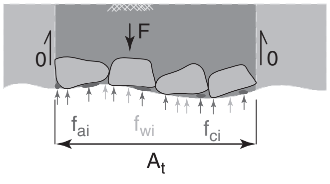


Figure 10.13 Free-body diagram for derivation of effective stress equation.

of the free body. It is further assumed that u_{wi} and u_{ai} are constant along the bottom surface and equal to u_w and u_a respectively. Therefore, equation 10.44 becomes:

$$F = \sum f_{ci} + u_w \sum a_{wi} + u_a \sum a_{ai} \tag{10.47}$$

Now divide both sides of the equation by the total horizontal projected area $\mathrm{A_t}$ of the bottom of the cylinder:

$$\frac{F}{A_t} = \frac{\sum f_{ci}}{A_t} + u_w \frac{\sum a_{wi}}{A_t} + u_a \frac{\sum a_{ai}}{A_t} \tag{10.48}$$

On the left-hand side, we get a quantity that is the total weight divided by the total area; this is called the *total (normal) stress* σ_t. On the right-hand side, the first term is the sum of the vertical components of the contact forces divided by the total area; this is the *effective (normal) stress* σ'. The second term is the water stress times the water area divided by the total area. This ratio of areas is lower than or equal to 1 and is called α. The third term is the air stress times the air area divided by the total area. This ratio of areas is lower than or equal to 1 and is called β. The total area can be written as:

$$A_t = \sum a_{ci} + \sum a_{wi} + \sum a_{ai} \tag{10.49}$$

Then

$$1 = \frac{\sum a_{ci}}{A_t} + \frac{\sum a_{wi}}{A_t} + \frac{\sum a_{ai}}{A_t} \tag{10.50}$$

And

$$1 = \frac{\sum a_{ci}}{A_t} + \alpha + \beta \tag{10.51}$$

where a_{ci} is the contact areas between particles. If it is assumed that Σa_{ci} is negligible compared to Σa_{wi} and Σa_{ai}, then:

$$\alpha + \beta = 1 \tag{10.52}$$

So, in summary:

$$\sigma = \sigma' + \alpha \mathrm{u}_w + \beta \mathrm{u_a} \tag{10.53}$$

Or

$$\sigma' = \sigma - \alpha \mathrm{u}_w - \beta \mathrm{u_a} \tag{10.54}$$

where $\sigma' = \frac{\sum f_{ci}}{A_t}$ is the effective stress, $\sigma = \frac{F}{A_t}$ is the total stress, α and β are the water and air area ratios ($\alpha + \beta = 1$), and u_w and u_a are the water stress and the air stress respectively.

Note that σ' is not the contact stress σ_c, which is the sum of the vertical components of the contact forces divided by the contact areas. This real stress σ_c is not used in geotechnical engineering because it is very difficult to know the area of the contacts. The contact stress σ_c is much higher than the effective stress σ'.

Note also that the effective shear stress is equal to the total shear stress, because the shear stress in the water τ_w and the shear stress in the air τ_a are neglected. The stresses τ_w and τ_a are not zero, however, and are responsible in part for the process of erosion (τ_w) and the drag force on airplanes (τ_a). Nevertheless, their order of magnitude is in N/m^2 rather than kN/m^2 as in the shear strength of soils.

$$\tau = \tau' \tag{10.55}$$

Again, the shear stress calculated is the shear force at the particle contacts divided by the total area rather than the contact area; therefore, it does not represent the shear stress at the contacts, but instead a much lower, well-defined value.

10.14 EFFECTIVE STRESS (SATURATED SOILS)

If the soil is saturated, Eq. 10.53 is simpler, as there is no air. The left-hand side is unchanged and equal to the total (normal) stress. The first term on the right-hand side is unchanged and equal to the effective (normal) stress. The second term reduces to u_w because the α value becomes equal to one, and the third term vanishes because there is no air in the soil:

$$\sigma' = \sigma - u_w \tag{10.56}$$

In unsaturated soils, the water stress can be significantly negative (high water tension); in this case the water stress can contribute significantly to increasing the effective stress between particles. For saturated soils with water in compression, that water stress detracts from the effective stress between particles. As in the case of unsaturated soils, however, the effective shear stress is the same as the total shear stress and Eq. 10.55 is equally valid for saturated soils and for unsaturated soils.

10.15 AREA RATIO FACTORS α AND β

In nature, the degree of saturation is either high enough that the air is occluded (air bubbles surrounded by water) or low enough that there is a continuous air path to the surface. The transition from occluded air to continuous path occurs at a degree of saturation approximately equal to 85%. If the air is occluded, the air stress u_a can be taken as being equal to the water stress u_w, as there is equilibrium at the bubble boundary between the air and the water. In this case, Eq. 10.54 reduces to Eq. 10.56 and the soil behaves as if it were saturated, except that the water phase is much more compressible due to the air bubbles. This increase in water compressibility can have a beneficial effect, as in reducing the potential for liquefaction. If the air phase is continuous, then the air path to the atmospheric pressure ensures that the air stress is zero and the term involving the air stress drops out. In this case, the effective stress equation expresses that the total stress is equal to the sum of the effective stress plus the product of the water stress (negative) by the ratio of the water area divided by the total area:

$$\sigma' = \sigma - \alpha u_w \tag{10.57}$$

Therefore, for most common cases, the general effective (normal) stress equation is Eq. 10.57. Figure 10.14 summarizes these situations.

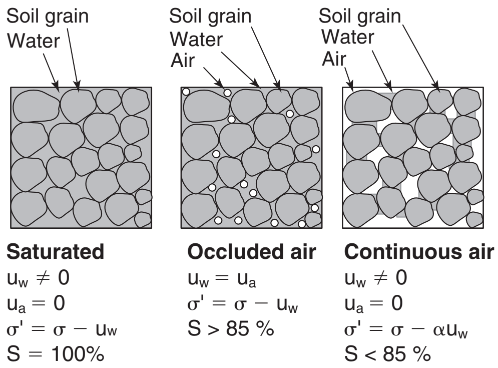


Figure 10.14 Effective stress equation for various common situations.

Note that in the case of occluded air, there can be a difference between u_w and u_a because of the contractile skin. Indeed, that membrane allows for a difference in pressure that can be obtained by writing equilibrium of the free-body diagram of half the bubble (Figure 10.15):

$$u_a \pi \frac{D^2}{4} = u_w \pi \frac{D^2}{4} + \pi D T \tag{10.58}$$

$$u_a - u_w = \frac{4T}{D} \tag{10.59}$$

Therefore, the expression of the effective stress for the case of the occluded air is an approximation. This approximation is reasonable, as the value of β is much smaller than the value of α in this case.

Because the effective stress has such a fundamental impact on the behavior of soils, it is very important to be able to evaluate the coefficient α in Eq. 10.57. This coefficient was first proposed by Bishop in the 1960s as the factor χ. This factor has been correlated with the degree of saturation S.

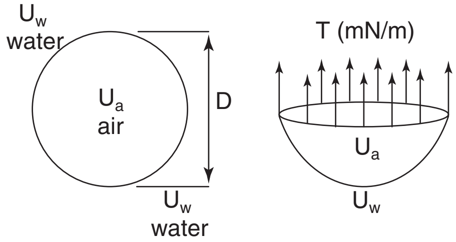


Figure 10.15 Pressure difference across an air-bubble boundary.

This makes some sense, because when the soil has no water (S = 0), α should also be zero, and when the soil is saturated (S = 1), α should also be equal to 1. Furthermore, if it is assumed that the area of the contacts A_c is negligible compared to the area of the voids A_v, then the definition of α becomes:

$$\alpha = \frac{A_w}{A_t} = \frac{A_w}{A_v} \tag{10.60}$$

Recall that the degree of saturation is defined as:

$$S = \frac{V_w}{V_v} \tag{10.61}$$

The analogy is tempting, but it must be said that the ratio of areas A_w/A_v is not likely equal to the ratio of volumes V_w/V_v, because the plane that cuts through the contacts in Figure 10.13 does not represent the general situation in the soil volume. As a result, there is quite a bit of scatter in the correlation between α and S (Figure 10.16).

Khalili and Khabbaz (1998) proposed a better relationship to predict α (Figure 10.17):

$$\alpha = \left(\frac{(u_a - u_w)}{(u_a - u_w)_{ae}}\right)^{-0.55} \tag{10.62}$$

which can be simplified without much loss of accuracy when u_a is zero as:

$$\alpha = \sqrt{\frac{u_{wae}}{u_w}} \tag{10.63}$$

where u_a is the air stress, u_w is the water stress, and $(u_a - u_w)_{ae}$ refers to the difference between u_a and u_w at the air entry value. At the beginning of the drying process of a saturated sample of soil, the water tension u_w increases (becomes more negative) as the water is evaporating out of the soil and into the surrounding air, but the soil remains saturated. As the drying continues, u_w continues to become more negative and gets to a point where air first enters the pores. This value of the water tension is called the *air entry value* u_{wae}. As the drying continues, the water tension continues to become more negative. The area ratio for air is β, and because $\alpha + \beta$ is equal to 1 ($A_c \sim 0$), once α is known so is β.

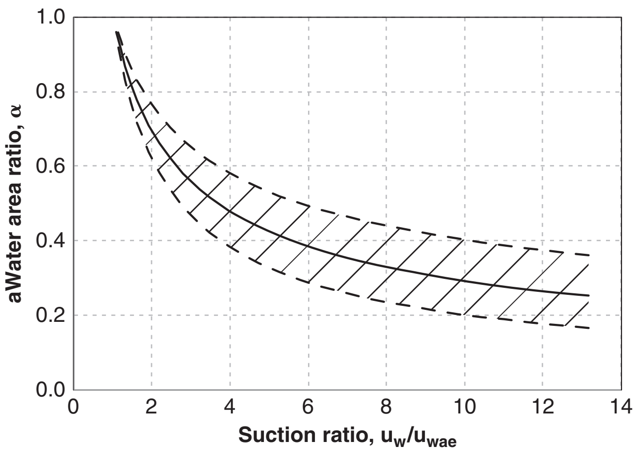


Figure 10.17 Water area ratio α vs. suction ratio. (After Khalili and Khabbaz 1998. Courtesy of Nasser Khalili)

10.16 WATER STRESS PROFILES

The water normal stress can be positive (pore pressure, compression) or negative (suction, tension). In the field, the groundwater level (GWL) is found at some depth below the ground surface (Figure 10.18).

In some cases that depth is very large (deserts); in others it is very shallow (regions close to oceans, lakes, or rivers). At the GWL, the water stress is zero. Below the GWL, the water is in compression (pore pressure) and, in the most common case, the water stress profile shows a linear increase with

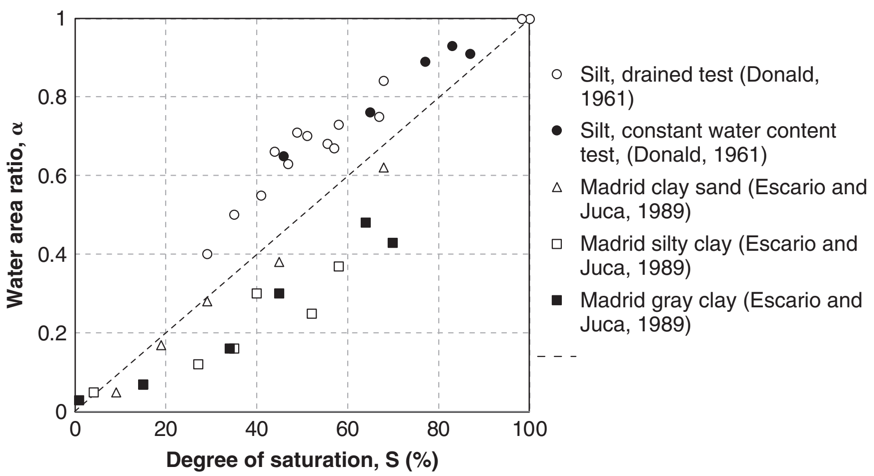


Figure 10.16 Water area ratio α vs. degree of saturation S. (After Lu and Likos 2004)

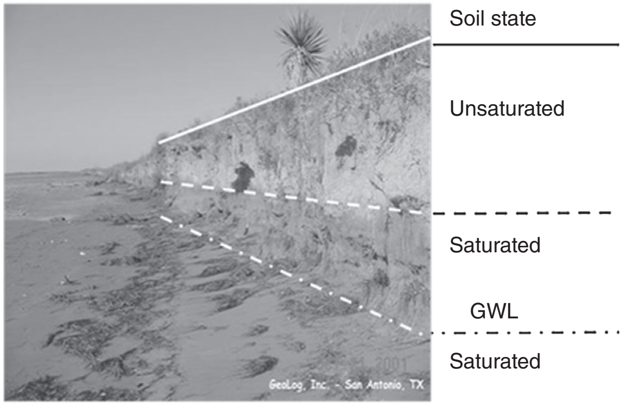


Figure 10.18 Groundwater level (GWL) and zones above the GWL. (Adapted from a photo of Art Koenig, reproduced with permission)

depth (hydrostatic pressure) and can be calculated as $\gamma_w z$ where γ_w is the unit weight of water and z is the depth below the GWL. Sometimes the water stress profile below the GWL is complicated by the presence of perched aquifers (water bodies sandwiched between dry soil) or artesian conditions (water body connected to a pressure higher than the local hydrostatic pressure). Figure 10.19 shows examples of such conditions.

Above the GWL, the water is in tension (suction). In the zone above the GWL and deep enough to be unaffected by the weather at the ground surface, the water stress is linear and given by $(-\gamma_w z)$ where z is the absolute value of the vertical distance above the water table. In the zone above the GWL and close enough to the ground surface that the weather can influence the water stress profile by evapotranspiration and rainfall (generally a few meters), the water stress profile becomes curved to reach an equilibrium between the weather and the soil (Figure 10.20). This part of the water stress profile is very difficult to calculate and varies daily with the weather.

10.17 WATER TENSION AND SUCTION

Water tension is the tension in the water expressed in kN/m^2. *Suction* is the potential that the water has to achieve a certain water tension; it is also expressed in kN/m^2. This suction potential is not always realized. If the suction potential is fully realized, the suction is equal to the water tension. If the suction is not fully realized, the suction is higher than the water tension. It is a bit like standing on top of a building but not jumping: you have potential energy, but you are not transforming it into velocity because you are not jumping. Later we will discuss cases in which the suction is not transformed into water tension. Although the suction is important, the water stress is the one that enters into most calculations. Note that suction is often defined as the difference between the air stress and the water stress $(u_a - u_w)$. Because the air stress is often zero in the field

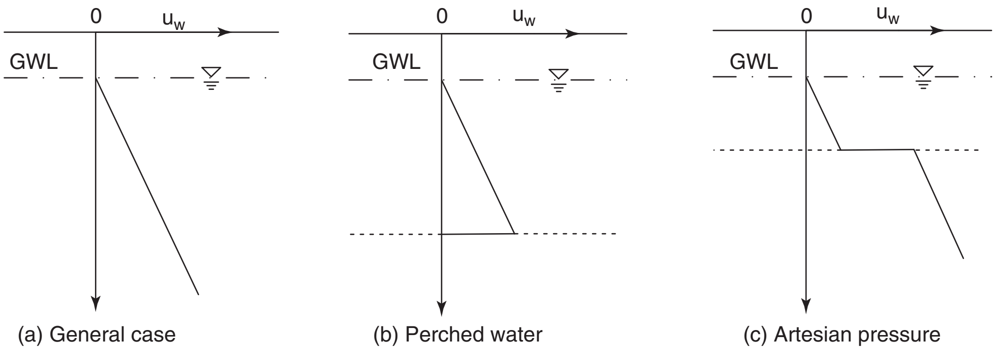


Figure 10.19 Examples of water stress profiles below the groundwater level.

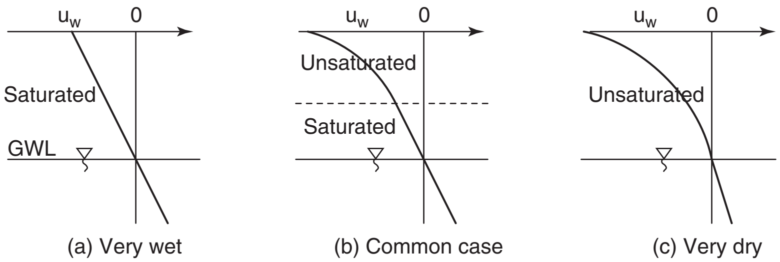


Figure 10.20 Examples of water stress profiles above the groundwater level.

(continuous air voids), suction is defined here as u_w. Note further that with $u_a - u_w$, suction is positive, whereas with u_w, suction is negative. Suction and water tension will always be negative in the rest of this book, as compression has been chosen as the positive sign convention for stresses.

The water tension and suction come from two different sources: attraction of water to the minerals in the soil particles and attraction of distilled water to salty water. The first one is called *matric suction*; the second one *osmotic suction*.

10.17.1 Matric Suction

Matric suction is due to the attraction between water molecules and the minerals in soil particles. If the mineral is silica, the phenomenon is called *capillary action*. The attraction between water and silica generates a force of 73 mN/m. Other minerals, such as smectite ($Al_2Si_4O_{10}(OH)_2$), can generate much higher attraction forces and therefore much higher water tension. Let's discuss capillary attraction first. The force of 73 mN/m is given per unit of length because it exists along the contact line of the meniscus interface between the water, the air, and the silica. Recall that one Newton is about the weight of a small apple, so 73 mN is a very small force, yet it is responsible for some major phenomena when dealing with very small scales. For example, when a very-small-diameter glass (silica) tube open at both ends is placed in water, that force lifts the water in the tube like one would pull up a sock. If the glass tube is small enough, the water can rise more than 10 m in the tube. Note that if the tube were made of a different mineral, the water would not rise to the same level. Also, if the tube were made of glass, but instead of water you had mercury in the container, the mercury would actually go down in the small tube rather than up, because there is a basic repulsion between mercury and silica.

The water rising in the silica tube does so up to a height where the volume of water lifted in the tube has a weight equal to the vertical component of the attraction force at the top of the column times the contact length of the meniscus. Equating the weight of the column of water to the vertical component of the attraction force leads to the height of the water column or capillary rise (Figure 10.21):

$$h_c \frac{\pi d^2}{4} \gamma_w = \pi d T \cos\alpha \tag{10.64}$$

Therefore,

$$h_c = \frac{4T \cos\alpha}{\gamma_w d} \tag{10.65}$$

It is clear that the capillary rise depends on the diameter of the tube; the capillary rise will be high in small-diameter tubes and small in larger-diameter tubes. If the tube has a diameter equal to the size of clay particles—say, 0.001 mm—Eq. 10.65 gives a height of capillary rise equal to 29.2 m (height of a 10-story building). The continuous voids in a soil play the role of the tiny glass tube because, like glass, many soil particles are made of silica. Continuous clay voids are similar to tiny tubes and the water can saturate the clay high above the groundwater level (15 m or more). In sands, the height to which the water can rise is more limited.

Let's study the water stress profile in the capillary tube (Figure 10.21). Below the water level, in the big container, the water is in compression and the water stress is positive. Above that level, in the tiny glass tube, the water is in tension because the water is pulled up into the tube by the force πdTcosα. The water tension increases (becomes more negative) linearly with the height in the tube, as shown in Figure 10.21. At the top of the column, the water tension is maximum and equal to $-h_c\gamma_w$. Yet in the air immediately above the water level in the small tube, the pressure is atmospheric or zero gage pressure. It is not possible for such a discontinuity to exist between two fluids unless there is a membrane separating the water from the air: this membrane is the *contractile skin*. It is similar to a car tire: the pressure in the tire is much higher than outside the tire, and this is made possible by the membrane represented by the tire. We will discuss the contractile skin a bit later.

Consider now two soil particles in the form of spheres (Figure 10.22). The soil is allowed to dry and the water

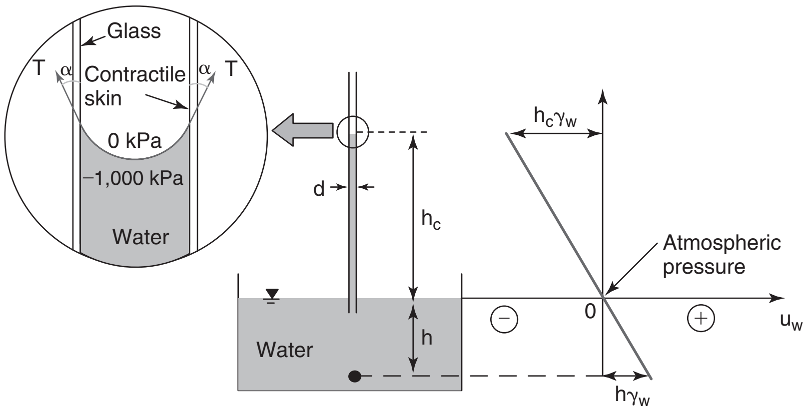
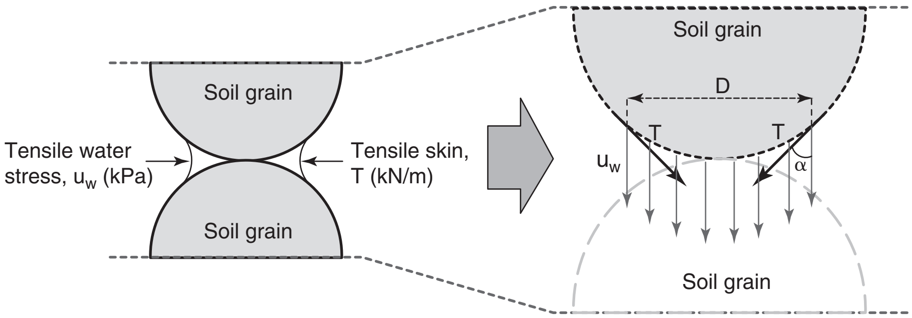


Figure 10.21 Capillary tube experiment.



If $u_w = -1000$ kPa, T = 73 mN/m, D = 0.001 mm, $\alpha = 45°$
$F = u_w \pi D^2/4 + (T\cos\alpha)\,\pi D = 10^{-9}$ kN
$\sigma = F/A = 10^3$ kN/m^2

Figure 10.22 Water tension at the contact between two spherical particles.

between the particles evaporates. When the water is almost gone, the water is only found around the contact between the two particles (Figure 10.22). The water is in tension and the air is at atmospheric pressure. The contractile skin allows large stress difference to exist between the two fluids. Now let's calculate the force at the contact. We draw a free-body diagram of the upper particle and show the forces imparted by the water and the contractile skin on the particle. The water is under a tension stress u_w, so the water pulls on the particles above and below the contact area A with a force u_wA. The contractile skin is also in tension and pulls on the particle at an angle α. The calculations are shown in Figure 10.22. The force is a compression force equal to 10^{-6} N. Remember that 1 N is about the weight of a small apple, so the force is extremely small—yet the stress is very large (1000 kPa). These stresses develop when the soil dries and are the reasons why dry soils are a lot harder than saturated soils.

The preceding discussion focused on the case of water attraction to silica and the water tension that can be generated due to this phenomenon. Some clay minerals, such as smectite ($Al_2Si_4O_{10}(OH)_2$), can generate much higher attraction forces and therefore water tension which can reach 100,000 kPa or even 1,000,000 kPa (Figure 10.23). These water tension values correspond to soils that are very dry yet have a little bit of water between particles. It is not clear in these cases whether the water is still in liquid form, or in viscous form, or possibly approaching solid form.

10.17.2 Contractile Skin

The membrane called the *contractile skin* exists at the interface between the water and the air. The existence of this membrane is rooted in the Van der Waals forces, which are elementary attractive forces between molecules. In the water, these forces act in all three directions and give water its tensile strength. This tensile strength can be measured by placing water in a cylinder and pulling on the piston until the water breaks in tension, at about 20 MPa. This is remarkably large, approaching the strength of concrete in compression.

At the interface between the water and the air, the molecules of water attract those that are below the interface but are unable to attract water molecules above the surface, as there are none available. Instead, the water molecules enhance their attraction in the horizontal direction, thereby creating a membrane. Figure 10.24 shows a water strider resting on that contractile skin. (So it is possible to walk on water, at least for the water strider.)

This water membrane is able to generate 73 mN of force for every meter of linear contact with silica. This represents

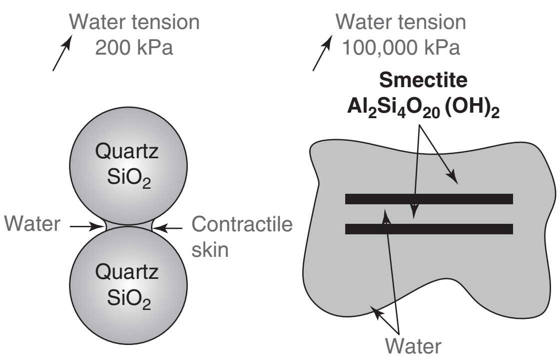


Figure 10.23 Water tension between soil particles.

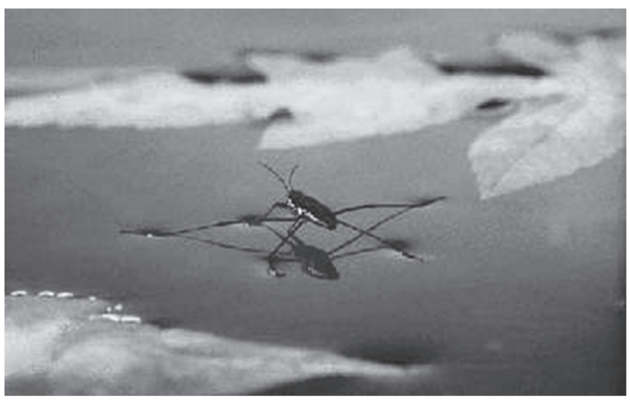


Figure 10.24 Water strider resting on contractile skin.

a very small force, but the membrane is extremely thin. Its thickness is estimated at 20 to 30 nanometers; therefore, the stress in the contractile skin under 73 mN/m is larger than 20 MPa.

10.17.3 Osmotic Suction

There is a second reason why water can go into tension in a soil: osmotic suction. Osmotic suction is due to the basic attraction that exists between water and salt. The phenomenon can be explained as follows. Imagine a container with two sides (Figure 10.25). On one side is distilled water, and on the other side is water with salt in it. Imagine that there is an imaginary screen separating the two sides that allows water molecules to travel across it but not salt molecules. This imaginary screen therefore prevents the two water bodies from mixing. In this experiment, the distilled water will be attracted to the salt water and therefore a difference in elevation will be generated, as shown in Figure 10.25. This difference is a suction potential called the osmotic suction. To help you remember that the distilled water goes towards the salt water, just remember that when you eat salty food, you get thirsty!

Osmotic suction depends on the salt concentration in the water on the right side and on the type of salt in that water. Osmotic suction exists in a soil if the soil contains dissolved salts. This suction exists as a potential and is realized into a water tension if there is a change in salt concentration between two locations. This can happen when a sprinkler system is installed in the backyard of a home. In the majority of real situations, the osmotic suction is much smaller than the matric suction. The sum of the matric suction plus the osmotic suction is the total suction. Figure 10.26 shows values of total suction or water tension for a range of conditions.

If the salt concentration is high, as would be the case in a prepared solution, the osmotic suction can be very high. Table 10.1 shows the values of osmotic suction associated with various concentrations and various salt types. Note that osmotic suction exists in saturated soils as well as in unsaturated soils, as it is related only to the chemistry of the pore fluid.

10.17.4 Relationship between Total Suction and Relative Humidity

If you place water at the bottom of a container with air above it and then you close the container, the humidity of the air in the container will increase or decrease until it comes to an equilibrium. This equilibrium depends on the pressure and temperature in the container. At atmospheric pressure and at a temperature of 25°C, dry air consists of nitrogen (~78% by volume), oxygen (~21% by volume), and a few other

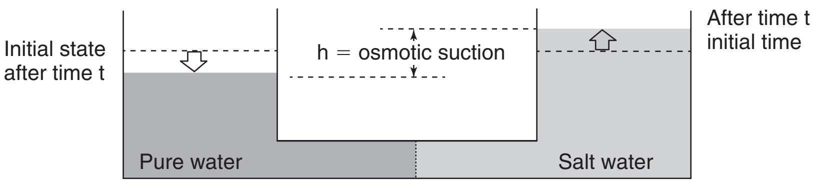


Figure 10.25 Osmotic suction experiment.

Water state	Examples	Water stress pF	Water stress cm	Water stress kPa	Degree of saturation	Water content	Swell	Shrink
Tension	Oven dry	7	-10^7	-10^6	0	0	Yes	No
	Air dry	6	-10^6	-10^5				
	Shrinkage limit	4	-10^4	-10^3	Near 100 %	8 to 15 %		
	Swell limit	2	-10^2	-10^1		25 to 50 %		Yes
		0	0	0	100 %		No	
Compression	Large river		10^3	10^2				
	Deepest offshore platforms		10^5	10^4				
	Bottom of deepest ocean		10^9	10^8				

Figure 10.26 Range of water tension and water compression for various conditions.

Table 10.1 Osmotic Suction in kPa of Some Salt Solutions at 25°C

	Osmotic Suction in kPa at 25°C						
Molality (mol/kg)	N_aCl	KCl	NH_4Cl	Na_2SO_4	$CaCl_2$	$Na_2S_2O_3$	$MgCl_2$
0.001	5	5	5	7	7	7	7
0.002	10	10	10	14	14	14	14
0.005	24	24	24	34	34	34	35
0.010	48	48	48	67	67	67	68
0.020	95	95	95	129	132	130	133
0.050	234	233	233	306	320	310	324
0.100	463	460	460	585	633	597	643
0.200	916	905	905	1115	1274	1148	1303
0.300	1370	1348	1348	1620	1946	1682	2000
0.400	1824	1789	1789	2108	2652	2206	2739
0.500	2283	2231	2231	2582	3396	2722	3523
0.600	2746	2674	2671	3045	4181	3234	4357
0.700	3214	3116	3113	3498	5008	3744	5244
0.800	3685	3562	3558	3944	5880	4254	6186
0.900	4159	4007	4002	4384	6799	4767	7187
1.000	4641	4452	4447	4820	7767	5285	8249
1.200	5616	5354	5343	N/A	N/A	N/A	N/A
1.400	6615	6261	6247	N/A	N/A	N/A	N/A
1.500	N/A	N/A	N/A	6998	13391	7994	14554
1.600	7631	7179	7155	N/A	N/A	N/A	N/A
1.800	8683	8104	8076	N/A	N/A	N/A	N/A
2.000	9757	9043	9003	9306	20457	11021	22682
2.500	12556	11440	11366	11901	29115	14489	32776

*All suction values are in kPa.
(After Bulut et al. 2001)

gasses. If such a dry air is in the container, there is plenty of room for water molecules to become part of the air, thereby increasing the relative humidity of the air. Part of the liquid water at the bottom of the container will become vaporized, and join the air phase by fitting vaporized water molecules between the molecules of nitrogen and oxygen. This process will continue until an equilibrium is reached.

Each gas component in the air has a partial pressure, and the partial pressures add up to the total pressure, according to the ideal gas law:

$$p_{air} = p_{nitrogen} + p_{oxygen} + p_{water} + \cdots \tag{10.66}$$

At a certain relative humidity, the air has a corresponding partial water vapor pressure p_{water}. At 100% relative humidity, the partial water vapor pressure p_{water} equals the saturated water vapor pressure $p_{water,sat}$. This pressure is 3.17 kPa for conditions of atmospheric pressure (101.3 kPa) and a temperature of 25°C. The general equation for the saturated partial vapor pressure of water in air $p_{water,sat}$ at atmospheric pressure for different temperatures is (Tetens 1930):

$$p_{water,sat}(\text{kPa}) = 0.611\, e^{\left(17.27\, \frac{T(^oC)}{T(^oC)+237.2}\right)} \tag{10.67}$$

where T is the temperature in degree Celsius. The relative humidity of the air is defined as the ratio:

$$R_H = \frac{p_{water}}{p_{water,sat}} \tag{10.68}$$

The relationship between the relative humidity R_H of the air in the void of an unsaturated soil and the suction potential ψ is given by Kelvin's equation (Fredlund and Rahardjo 1993; Lu and Likos 2004):

$$\Psi = \frac{\rho_w RT}{M} Ln\, R_H \tag{10.69}$$

where ψ is the suction potential in Pa, ρ_w is the mass density of the water (1000 kg/m^3), M is the molecular weight of water

(0.01802 kg/mol), T is the absolute temperature in Kelvin, R is the universal gas constant (8.314 N m/mol K), and R_H is the relative humidity expressed as a ratio rather than a percent. This suction potential in the void of the unsaturated soil can develop into a water tension, in which case:

$$u_w = \frac{\rho_w RT}{M} Ln\, R_H \tag{10.70}$$

At 20°C and given the same constants used earlier, the relationship is:

$$u_w(\text{kPa}) = 135000\, Ln R_H \tag{10.71}$$

where R_H is taken as a fraction. This equation is shown in Figure 10.27. It indicates, among other interesting observations, that a humidity room at 95% relative humidity has a water tension potential of almost 7000 kPa and therefore is a drying room.

10.17.5 Trees

Water is drawn up to the top of trees through suction. Osmotic suction in the tree is due to the difference in mineral concentration of the water in the tree and of the water in the soil. Capillary suction is due to the very small size of the tiny tubes (*xylem conduits*) that exist through the stem or tree trunk; water is attracted to the walls of the xylem conduits much like water is attracted to the glass (silica) wall of a capillary tube. In trees, suction or water tension can reach 2000 kPa. Evaporation takes place from the leaf surfaces and a continuous flow of water is generated in this fashion. This flow can reach 1 m^3 per day.

The tree absorbs carbon dioxide (CO_2) from the air and pumps water (H_2O) from the ground. It then uses the energy from the sun (photosynthesis) to combine the carbon dioxide with the water to make sugar ($C_6H_{12}O_6$) and release oxygen (O_2). Sugar is the essential basis for all plant growth. Trees and plants in general are extremely important to humankind because they absorb what we exhale (CO_2) and produce what we inhale (O_2).

10.18 PRECISION ON WATER CONTENT AND WATER TENSION

Water tension is more complicated to measure than water content. Water content also typically varies much less than water tension. A typical range of water content variation is 5 to 50%, whereas the typical range for water tension is −10 to −1,000,000 kPa. In an experiment conducted by Garner (2002, unpublished), three samples were sent to eight laboratories in Texas requesting that the water content and the suction be measured. Most laboratories used the filter paper method for the suction determination. The results were collected and an error band was created for each sample. The results are shown in Figure 10.28. They confirm that the arithmetic value of the suction varies a lot more than the water content. They also indicate that the error band for identically prepared samples is much larger for the determination of suction than for water content. If the log of the suction is used instead of the arithmetic value, then the error band of log(suction) approaches the error band of water content (Figure 10.28).

10.19 STRESS PROFILE AT REST IN UNSATURATED SOILS

The total vertical stress at rest σ_{ov} at any depth z in a uniform soil is equal to the total unit weight of the soil γ_t times the depth z:

$$\sigma_{ov} = \gamma_t z \tag{10.72}$$

If the soil above the depth z is made of n layers, the total vertical stress at rest σ_{ov} at depth z is:

$$\sigma_{ov} = \sum_{i=1}^{n} \gamma_{ti} h_i \tag{10.73}$$

where γ_{ti} is the total unit weight of layer i and h_i is the thickness of layer i. Note that if there is water above the

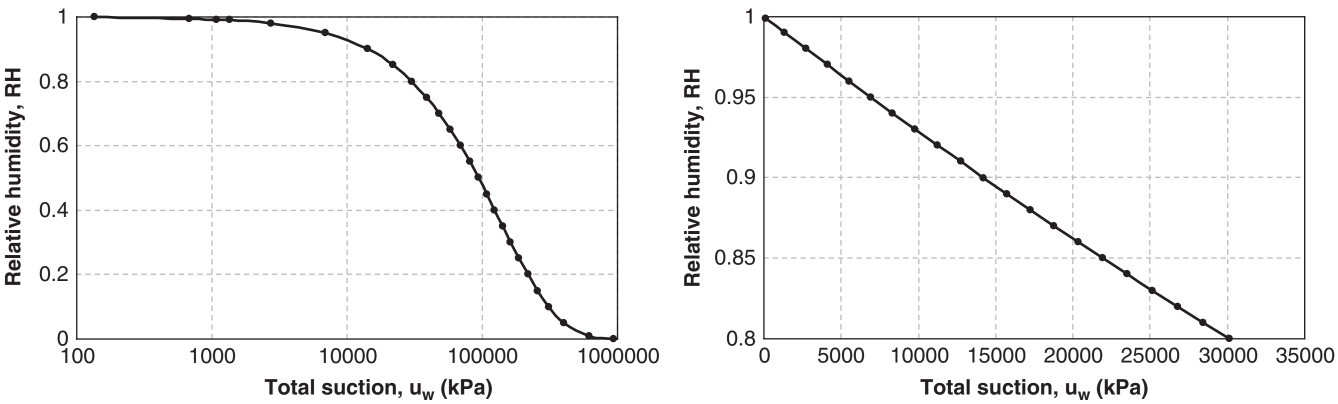


Figure 10.27 Water tension vs. relative humidity: *(a)* Relative humidity 0–100%. *(b)* Relative humidity 80-100%.

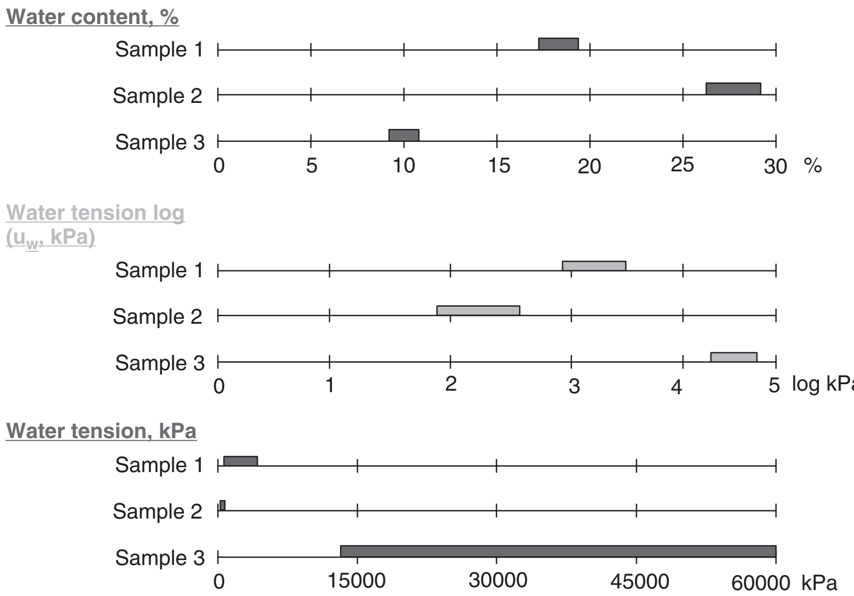


Figure 10.28 Error bands for suction and water content determination by eight different laboratories in Texas for three identically prepared samples. (After Garner 2002)

ground surface, the water must be included as a layer to calculate the total vertical stress. This is the case with a river, a lake, or an ocean. At the bottom of the deep oceans, the total vertical stress is very large and compresses any object tremendously. For example a Styrofoam coffee cup going to 3000 m of water depth comes back the size of a thimble.

Below the groundwater level, the water stress at rest u_{wo} is calculated under normal circumstances as:

$$u_{wo} = \gamma_w z_w \quad (10.74)$$

where γ_w is the unit weight of water and z_w is the depth below the GWL. Note that this water stress acts equally in all directions (hydrostatic), as it is assumed that water has no shear strength. This stress is a compressive stress. If an artesian condition exists, then information about the water stress in the artesian layer must be known or inferred from the global aquifer analysis. If a perched GWL condition exists, then information must be gathered where the groundwater layer ends.

Above the GWL, in the zone saturated by capillary action, the water stress u_w is calculated as:

$$u_{wo} = -\gamma_w z_w \quad (10.75)$$

where z_w is positive and represents the vertical distance above the GWL.

The water stress in this case is a tensile stress. Close to the surface, the water tension no longer exhibits a linear profile (Figure10.29). In that zone there is a power struggle between the soil particle minerals, which tend to attract the water, and the low relative humidity in the soil pores caused by the sun, which tends to draw the water away from the particles. The water is pulled hard in both directions, so high tensile stresses develop. Quantifying the variation of u_w with depth within that region requires advanced computations and depends on many factors, including rainfall, wind speed, solar radiation, temperature, soil hydraulic conductivity, extent of the cracks in the soil, and so on. Such computations are beyond the scope of this book.

Once the total vertical stress at rest σ_{ov} is known, and once the water stress at rest u_{wo} is known, the vertical effective stress at rest σ'_{ov} is calculated as:

$$\sigma'_{ov} = \sigma_{ov} - \alpha \, u_{wo} \quad (10.76)$$

where α is the water area ratio estimated as the degree of saturation or obtained from Eq. 10.63.

One of the important initial steps in solving a geotechnical problem is to prepare the profile of vertical stresses at rest for the site. This is done in the following steps:

1. Identify the layers and their thicknesses for the deposit considered.
2. Determine the total unit weight of each layer.
3. Determine the location of the GWL and any irregularity associated with the water regime (artesian pressure, perched water table).
4. Identify the points of discontinuity versus depth. These points include boundaries between two layers and depth to the GWL.
5. Calculate the total vertical stress at rest, σ_{ov}, at each discontinuity using Eq. 10.73.
6. Calculate the water stress at rest, u_w, at each discontinuity using Eq. 10.74 below the GWL and Eq. 10.75 above the GWL).

7. Calculate the water area ratio α at each discontinuity, estimated as the degree of saturation or by using Eq. 10.63 (α will be 1 under the GWL and in the zone saturated by capillary action).
8. Calculate the effective vertical stress at rest, σ'_{ov}, using Eq. 10.76.
9. Plot the values of σ'_{ov} on a graph at the depths corresponding to the discontinuities and join these points by straight lines.

Figure 10.29 is an illustration of this step-by-step procedure under the following conditions:

1. The soil is uniform.
2. The total unit weight of the soil is equal to 20 kN/m^3, and the unit weight of water is taken as 10 kN/m^3.
3. The GWL is at a depth of 5 m.
4. The points of discontinuity are the bottom of the profile (z = 7 m), the GWL (z = 5 m), the top of the capillary zone (z = 3 m), and the ground surface (z = 0 m).
5. The total vertical stress at rest at the bottom of the profile is equal to $\sigma_{ov} = 20 \times 7 = 140$ kN/m^2. At the top of the profile, it is $\sigma_{ov} = 0$. Because there is no discontinuity in total unit weight between these two discontinuities, the profile is a straight line between the two values.
6. The water stress at rest at the bottom of the profile is equal to $u_{wo} = 2 \times 10 = 20$ kN/m^2. At the GWL, the water stress u_{wo} is zero. At the top of the capillary zone, the water stress is $u_{wo} = -2 \times 10 = -20$ kN/m^2. The profile of water stress in the unsaturated zone above the top of the capillary zone is estimated as shown in Figure 10.29.
7. The water area ratio is equal to 1 in the zone where the soil is saturated. In the unsaturated zone, a linear decrease of α from 1 at the top of the capillary zone to 0 at the ground surface is assumed in this case.
8. The effective stress is calculated according to Eq. 10.76. For the point at the bottom of the profile, $\sigma'_{ov} = 140 - 20 = 120$ kN/m^2. For the point at the GWL, it is $\sigma'_{ov} = 100 - 0 = 100$ kN/m^2. For the point at the top of the saturated capillary zone, it is $\sigma'_{ov} = 60 - 1 \times (-20) = 80$ kN/m^2. In the unsaturated zone above the capillary zone, the profile is obtained by using Eq. 10.76.
9. The values of effective stress are plotted in Figure 10.29. Note that the effective stress decreases linearly as the depth decreases when the soil is saturated, but increases as the depth continues to decrease in the unsaturated zone.

We can then calculate the shear strength of the soil on horizontal planes by multiplying the vertical effective stress by the tangent of the friction angle, assuming that the soil has no effective stress cohesion: s = $\sigma'_{ov}\tan\varphi$. Therefore, the shear strength profile has the same shape as the effective stress profile. The increase in effective stress, and therefore strength close to the surface due to higher water tension, often leads to a crust that can be a few meters thick.

10.20 SOIL WATER RETENTION CURVE

The soil water retention curve (SWRC), also known as the soil water characteristic curve, is a property of the soil much like the shear strength parameters (Figure 10.30). It is a plot of the water content of the soil as a function of the water tension stress (suction) in the soil pores.

Figure 10.30 is a SWRC on a semilog plot; the water content is on a natural scale while the water tension is on a log scale. From point A to point B on Figure 10.30, the soil remains nearly saturated while the water tension increases. At the air entry value (point B), the water content decreases while the water tension increases. Up to point C on Figure 10.30, the water content is usually well represented

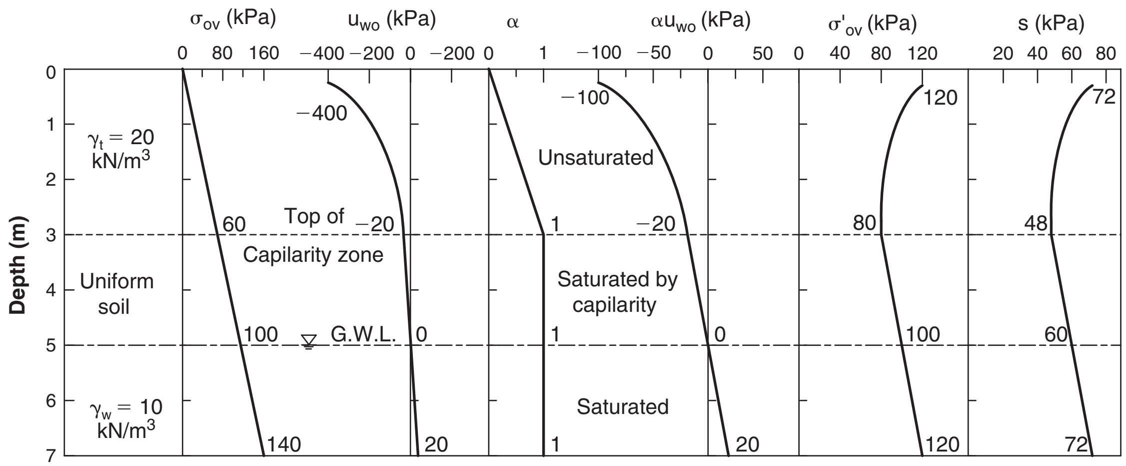


Figure 10.29 Stress profiles in a soil deposit.

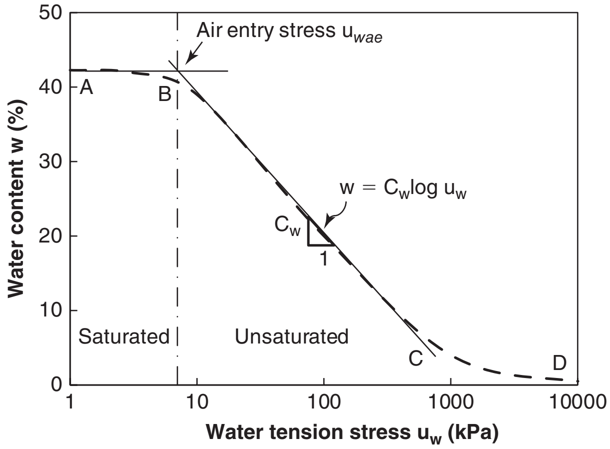


Figure 10.30 Soil water retention curve.

by a straight line and the slope of that line is the coefficient C_w:

$$\Delta w = C_w \log \frac{u_w}{u_{wae}} \tag{10.77}$$

where Δw is the change in water content, C_w is the slope of the SWRC, u_w is the water tension, and u_{wae} is the air entry value of the water tension. From C to D, the water content continues to decrease while the water tension continues to increase, but at a much higher rate.

If a saturated soil sample is placed on a table top and is strong enough to stand by itself, it is likely held together by water tension unless it has some cementation (effective stress cohesion). As the soil dries, it initially shrinks while remaining saturated. The water tension increases, and at a given water tension stress (suction), air enters the pores. This water tension is called the *air entry value* (u_{wae}). From this point on during the drying process, the soil is unsaturated. By definition, the water content at the air entry value is the undisturbed shrinkage limit because, during the shrinkage process, it is the last water content where the soil is saturated.

The gravimetric water content is the water content definition most commonly used in geotechnical engineering, but for the SWRC, the volumetric water content is often used. They are defined as follows:

$$\text{Gravimetric water content: } w = W_w/W_s \tag{10.78}$$

$$\text{Volumetric water content: } \theta_w = V_w/V \tag{10.79}$$

When the term *water content* is used in this book, it means gravimetric water content. Example SWRCs are presented in Figure 10.31. Different soils have different SWRCs; for instance, a sand will not retain water the same way a clay would. Imagine that you insert a straw into a sand. It would not take much sucking to get the water out of the sand. Now imagine that your straw is inserted into a clay. In this case it would take a lot of sucking to get a little bit of water out. The suction or water tension that you would exert through the straw would be much higher for the clay than for the sand. This phenomenon is what the SWRC characterizes.

Soils under the groundwater level are generally saturated and the water is in compression. Soils above the GWL can be saturated or unsaturated, but in both cases the water is in tension (suction). The SWRC is a property of a soil where the water is in tension. Thus, the SWRC for a saturated soil refers to the case where the soil is saturated above the GWL by capillary action and other electrochemically

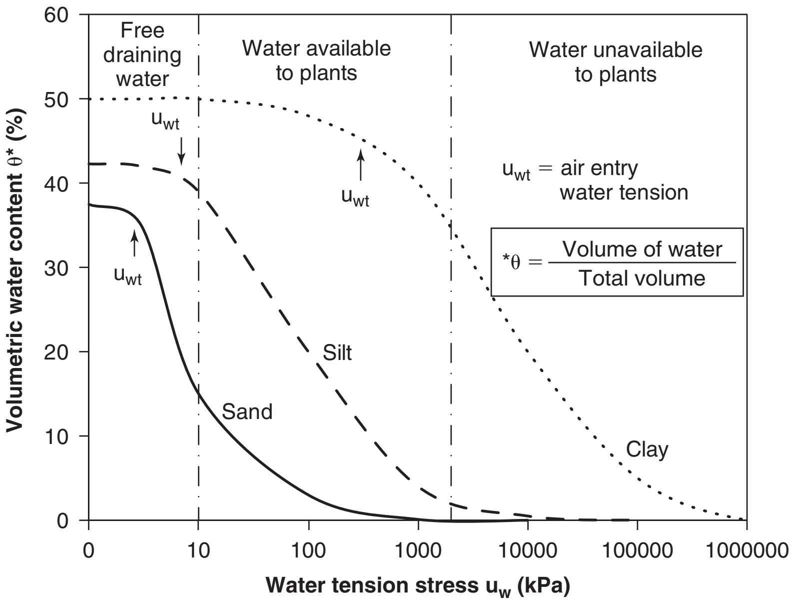


Figure 10.31 Example of soil water retention curve.

based phenomena such as the affinity between water and clay minerals (point A to B on Figure 10.30). Beyond point B the soil is unsaturated.

10.21 INDEPENDENT STRESS STATE VARIABLES

Effective stress σ', as defined in Eq. 10.54, is:

$$\sigma' = \sigma - \alpha \mathrm{u}_w - \beta \mathrm{u}_\mathrm{a} \tag{10.80}$$

Effective stress is defined on the basis of three stresses (σ, u_w, $\mathrm{u_a}$) and two soil properties (α, β). Therefore, it depends not only on the state of stress in a soil, but also on the soil properties. Hence, it cannot be considered an independent stress variable, even if it is a very useful stress in solving many soil problems. Equation 10.54 can be rewritten as follows:

$$\begin{aligned}\sigma' &= \sigma - \alpha\, u_w - \beta\, u_a - u_a + u_a \\ &= (\sigma - u_a) - \alpha\, u_w + (1-\beta) u_a \\ &= (\sigma - u_a) + \alpha\,(u_a - u_w)\end{aligned} \tag{10.81}$$

In this form, it becomes clear that two independent stress state variables are necessary to describe the effective stress: the net normal total stress in excess of air stress ($\sigma - u_a$) and the net water tension with respect to the air stress ($u_a - u_w$).

In terms of total stresses, the stress tensor at a point is defined in Eq. 10.3. This stress tensor does not include information on the water stress or the air stress. Keeping in mind that shear stresses are unaffected by water or air stress, the stress state in a soil can be fully described by the following two stress tensors, which include all the stress information necessary to solve an unsaturated soil problem.

$$\Sigma_1 = \begin{bmatrix} \sigma_{xx} - u_a & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - u_a & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - u_a \end{bmatrix} \tag{10.82}$$

$$\Sigma_2 = \begin{bmatrix} u_a - u_w & 0 & 0 \\ 0 & u_a - u_w & 0 \\ 0 & 0 & u_a - u_w \end{bmatrix} \tag{10.83}$$

In the case of a saturated soil, only one tensor is necessary:

$$\Sigma_1 = \begin{bmatrix} \sigma_{xx} - u_w & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - u_w & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - u_w \end{bmatrix} \tag{10.84}$$

PROBLEMS

10.1 A wedge has applied stress vectors on two faces as shown in Figure 10.1s*a* and Figure 10.1s*b*. Calculate the stress on the third face in both cases. Hint: You can compose forces, but you cannot compose stresses unless they act on the same area.

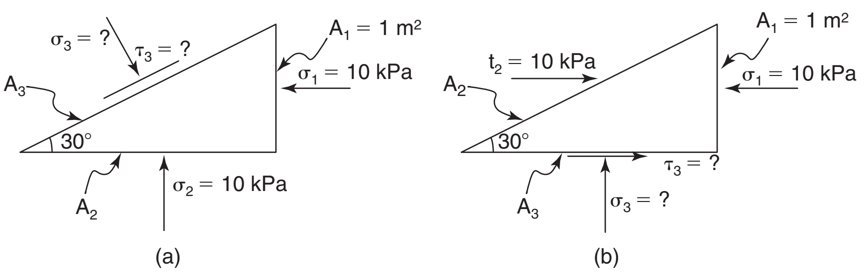


Figure 10.1s Stress vectors on wedge faces.

10.2 In a triaxial test, the confining stress (minor principal stress) σ_3 is 50 kPa, and the vertical stress (major principal stress) σ_1 is 150 kPa.

a. Form the total stress tensor shown in Eq. 10.3. Decompose the tensor into the deviatoric and spherical tensor forms shown in Eq. 10.4.
b. The soil is saturated, and under the given stresses, the water stress is 20 kPa. Form the stress tensor in terms of effective stress.
c. The soil is unsaturated, and under the given stresses, the air stress is 30 kPa and the water tension is −1000 kPa. Form the two tensors describing the state of stress in the sample in terms of independent stress state variables.

10.3 A simple shear test is performed in a plane strain condition. The vertical normal stress on the plane of failure is 80 kPa, the horizontal normal stress is 40 kPa, and the shear stress is 30 kPa on the horizontal plane. The Poisson's ratio for the soil is 0.35. Form the total stress tensor (Eq. 10.3). Decompose this tensor into the deviatoric and spherical tensor forms shown in Eq. 10.4.

10.4 A sample of cohesionless silt is tested in a direct shear test. At failure, the vertical normal stress is 100 kPa and the shear stress on the horizontal plane where failure occurs is equal to 40 kPa. The water stress is 20 kPa.
   a. Calculate the effective principal stresses by using the equilibrium equations approach in two dimensions.
   b. Calculate the effective principal stresses by using the Mohr circle approach in two dimensions.

10.5 For problem 10.4, use the Pole method to locate the planes where the principal stresses act.

10.6 For the sample in Figure 10.6s:
   a. Find the stresses on the plane shown.
   b. On what plane does the maximum shear stress exist?

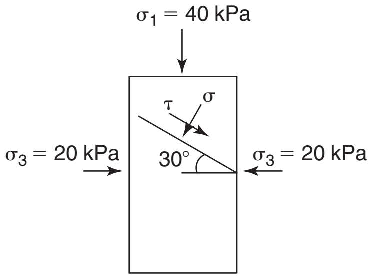


Figure 10.6s Stress state.

10.7 What happens to the Pole method when the diagram of a stress element in space is rotated by an angle θ? Does the Mohr circle change? Do the stresses on any plane change? Does the Pole location change?

10.8 In a simple shear test, the horizontal displacement at the top of the sample is 1 mm and the vertical displacement is a reduction in height of 0.5 mm. The original height of the sample is 25 mm.
   a. Calculate the shear strain and the vertical normal strain.
   b. Is the sample dilating or contracting?

10.9 In a triaxial test, the sample has an initial height of 150 mm and an initial diameter of 75 mm. During the loading in the vertical direction, the vertical displacement is 3 mm and the increase in diameter is 2 mm.
   a. Calculate the normal strains ε_{zz} and ε_{rr}.
   b. Form the strain tensor.
   c. Calculate the shear strain on a 45-degree plane.

10.10 Consider the sphere-shaped soil particles shown in Figure 10.10s. The degree of saturation S is 1, the porosity n is 0.4, and the ratio between the sum of the contact areas and the total area (A_c/A_t) is 0.01. Calculate the following quantities and show the relationship between the total stress and the effective stress if the water stress is +40 kPa.
   a. The average effective normal stress
   b. The average normal stress at the contacts
   c. The average normal total stress

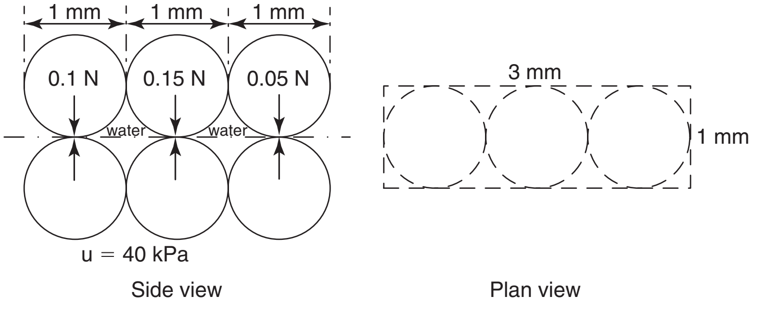


Figure 10.10s Sphere-shaped soil particles.

10.11 The surface tension of water is T = 73 mN/m, the diameter of a glass tube plunged into water is 0.002 mm, and the contact angle between the wall of the clean glass and the water is $\alpha = 10$ degrees. Find the height to which the water will rise in the small tube.

10.12 Consider the sphere-shaped soil particles shown in Figure 10.11s. The porosity n is 0.4, the ratio between the sum of the contact areas and the total area (A_c/A_t) is 0.01, and the ratio between the sum of the areas of water and the total area (A_w/A_t) is 0.1. Calculate the following quantities and show the relationship between the total stress and the effective stress if the water stress is −6000 kPa.

a. The average normal effective stress
b. The average normal stress at the contacts
c. The average normal total stress

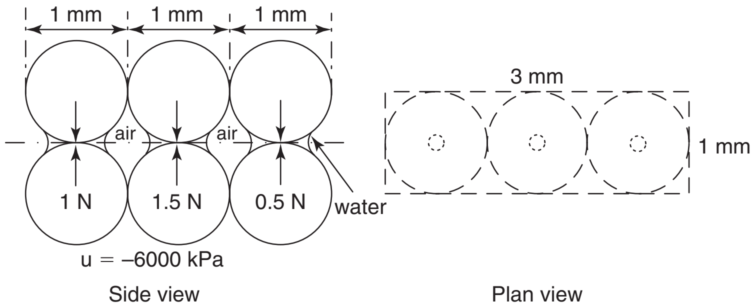


Figure 10.11s Sphere-shaped soil particles.

10.13 A soil has a degree of saturation of 92%. The air is occluded and the bubbles are 1 mm in diameter. Knowing that the water tension can reach 73 mN/m, what is the maximum difference in pressure that can exist between the water stress and the air stress?

10.14 A soil has a degree of saturation of 35%, an air entry value of −150 kPa, and a water tension stress of −1500 kPa at a depth of 2 m. Estimate the vertical effective stress at rest at a depth of 2 m below the ground surface, assuming that the unit weight of the soil is 19 kN/m^3.

10.15 Draw the three profiles (σ_{ov}, u_o, σ'_{ov}) for the layered system shown in Figure 10.12s.

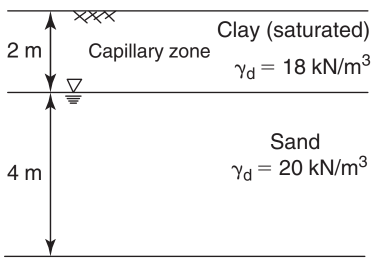


Figure 10.12s Soil profile

10.16 Draw the effective stress profiles (σ'_{ov}) for the layered system shown in Figure 10.14s.

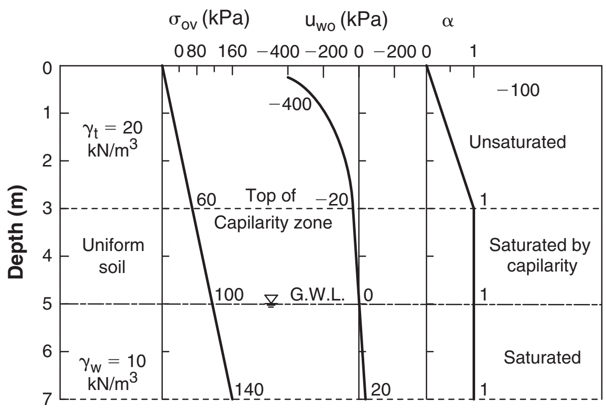


Figure 10.14s Soil and stress profile.

10.17 Draw the three profiles (σ_{ov}, u_w, σ'_{ov}) at the center of the river for the layered system shown in Figure 10.16s.

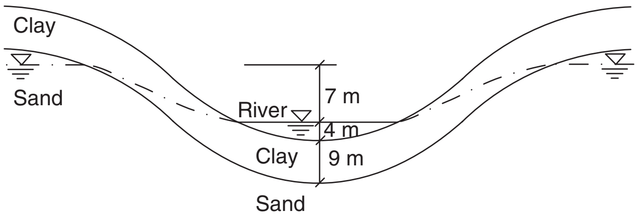


Figure 10.16s River profile.

10.18 An insect has 4 legs and is able to walk on water. The depression created under each foot is a sphere, as shown in Figure 10.18s. What is the maximum possible weight of the insect?

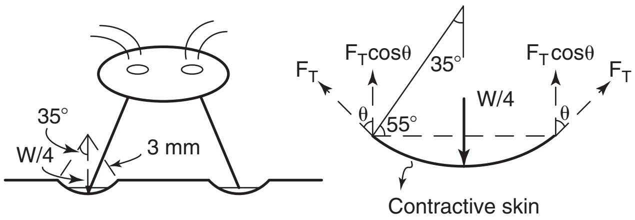


Figure 10.18s Free-body diagram of insect.

10.19 A soil has a water content of 42% and an air entry value of −8 kPa. If the slope of the soil water retention curve is 0.2 per log cycle of water tension in kPa, calculate the water tension for a water content of 10%.

10.20 A tree's root system occupies a volume equal to 1000 m^3. How much water is available to that tree if it is rooted in the three soils described by the retention curves of Figure 10.31?

Problems and Solutions

Problem 10.1

A wedge has applied stress vectors on two faces as shown in Figure 10.1s*a* and Figure 10.1s*b*. Calculate the stress on the third face in both cases. Hint: You can compose forces, but you cannot compose stresses unless they act on the same area.

Solution 10.1

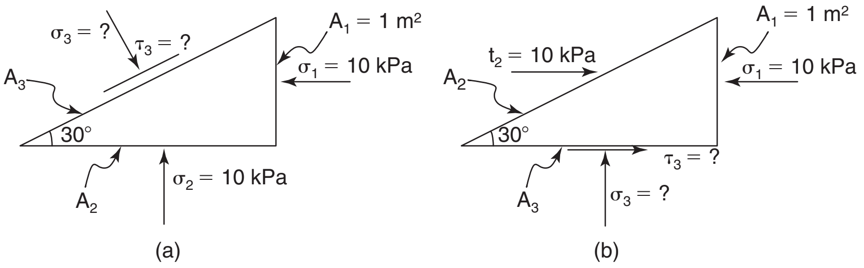


Figure 10.1s Stress vectors on wedge faces.

Part a:

$$F_1 = \sigma_1.A_1 = 10 \text{ (kN)}$$

$$A_2 = \frac{A_1}{\tan\theta} = \frac{1}{\tan 30^\circ} = 1.732 \text{ (m}^2\text{)}$$

$$F_2 = \sigma_2.A_2 = 10 * 1.732 = 17.32 \text{ (kN)}$$

$$A_3 = \frac{A_1}{\sin\theta} = \frac{1}{\sin 30^\circ} = 2\ (\text{m}^2)$$

$$\sum_{i=1}^{3} F_{xi} = 0 \rightarrow F_{x3(shear)}.\cos 30 + F_{x3(normal)}.\sin 30 - 10 = 0 \rightarrow (I)$$

$$\sum_{i=1}^{3} F_{yi} = 0 \rightarrow F_{y3(shear)}.\sin 30 - F_{y3(normal)}.\cos 30 + 17.32 = 0 \rightarrow (II)$$

$$(I)\&(II) \rightarrow \begin{cases} F_{S3} = 0\ (\text{kN}) \\ F_{N3} = 20\ (\text{kN}) \end{cases} \rightarrow \begin{cases} \tau_3 = 0\ (\text{kPa}) \\ \sigma_3 = 10\ (\text{kPa}) \end{cases}$$

Part b:

$$F_1 = \sigma_1.A_1 = 10\ (\text{kN})$$

$$A_2 = \frac{A_1}{\sin\theta} = \frac{1}{\sin 30^\circ} = 2\ \text{m}^2$$

$$A_3 = \frac{A_1}{\tan\theta} = \frac{1}{\tan 30^\circ} = 1.732\ (\text{m}^2)$$

$$\sum_{i=1}^{3} F_{xi} = 0 \rightarrow -F_1 + t_2 A_2 + \tau_3 A_3 = 0 \rightarrow \tau_3 = -5.77\ \text{kPa}$$

$$\sum_{i=1}^{3} F_y = 0 \rightarrow \sigma_3 = 0$$

Problem 10.2

In a triaxial test, the confining stress (minor principal stress) σ_3 is 50 kPa, and the vertical stress (major principal stress) σ_1 is 150 kPa.

a. Form the total stress tensor shown in Eq. 10.3. Decompose the tensor into the deviatoric and spherical tensor forms shown in Eq. 10.4.
b. The soil is saturated, and under the given stresses, the water stress is 20 kPa. Form the stress tensor in terms of effective stress.
c. The soil is unsaturated, and under the given stresses, the air stress is 30 kPa and the water tension is −1000 kPa. Form the two tensors describing the state of stress in the sample in terms of independent stress state variables.

Solution 10.2

a.

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 150 \end{bmatrix} (\text{kPa})$$

$$\sigma_M = \frac{(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})}{3} = \frac{(50 + 50 + 150)}{3} = 83.33\ (\text{kPa})$$

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{bmatrix} + \begin{bmatrix} \sigma_{xx} - \sigma_M & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - \sigma_M & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \sigma_M \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 83.33 & 0 & 0 \\ 0 & 83.33 & 0 \\ 0 & 0 & 83.33 \end{bmatrix} + \begin{bmatrix} 50 - 83.33 & 0 & 0 \\ 0 & 50 - 83.33 & 0 \\ 0 & 0 & 150 - 83.33 \end{bmatrix} (\text{kPa})$$

$$\Sigma = \begin{bmatrix} 83.33 & 0 & 0 \\ 0 & 83.33 & 0 \\ 0 & 0 & 83.33 \end{bmatrix} + \begin{bmatrix} -33.33 & 0 & 0 \\ 0 & -33.33 & 0 \\ 0 & 0 & 66.67 \end{bmatrix} (\text{kPa})$$

b.

$$\sigma' = \sigma - u$$

$$\Sigma = \begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{yx} & \sigma'_{yy} & \tau'_{yz} \\ \tau'_{zx} & \tau'_{zy} & \sigma'_{zz} \end{bmatrix} = \begin{bmatrix} 50-20 & 0 & 0 \\ 0 & 50-20 & 0 \\ 0 & 0 & 150-20 \end{bmatrix} = \begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 130 \end{bmatrix} \text{(kPa)}$$

c.

$$\sigma' = (\sigma - u_a) + \alpha(u_a - u_w)$$

$$\Sigma_1 = \begin{bmatrix} \sigma_{xx} - u_a & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - u_a & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - u_a \end{bmatrix} = \begin{bmatrix} 50-30 & 0 & 0 \\ 0 & 50-30 & 0 \\ 0 & 0 & 150-30 \end{bmatrix} \text{(kPa)}$$

$$\Sigma_1 = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 120 \end{bmatrix} \text{(kPa)}$$

$$\Sigma_2 = \begin{bmatrix} u_a - u_w & 0 & 0 \\ 0 & u_a - u_w & 0 \\ 0 & 0 & u_a - u_w \end{bmatrix} = \begin{bmatrix} 30-(-1000) & 0 & 0 \\ 0 & 30-(-1000) & 0 \\ 0 & 0 & 30-(-1000) \end{bmatrix} \text{(kPa)}$$

$$\Sigma_2 = \begin{bmatrix} 1030 & 0 & 0 \\ 0 & 1030 & 0 \\ 0 & 0 & 1030 \end{bmatrix} \text{(kPa)}$$

Problem 10.3

A simple shear test is performed in a plane strain condition. The vertical normal stress on the plane of failure is 80 kPa, the horizontal normal stress is 40 kPa, and the shear stress is 30 kPa on the horizontal plane. The Poisson's ratio for the soil is 0.35. Form the total stress tensor (Eq. 10.3). Decompose this tensor into the deviatoric and spherical tensor forms shown in Eq. 10.4.

Solution 10.3

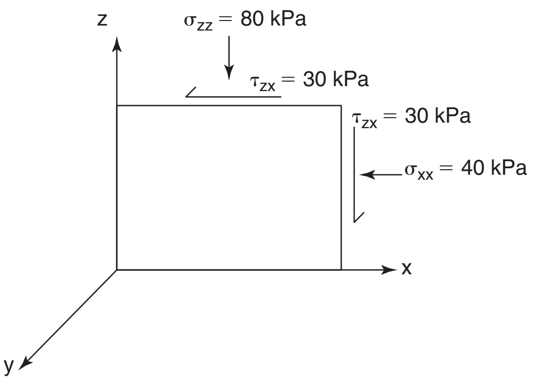


Figure 10.2s Stresses during the simple shear test.

Eq. 10.4:

$$\sum = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} = S + D = \begin{pmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{pmatrix} + \begin{pmatrix} \sigma_{xx} - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \sigma_m \end{pmatrix} \text{kPa}$$

where $\sigma_m = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$.

From the problem statement: $\sigma_{xx} = 40$ kPa, $\sigma_{zz} = 80$ kPa, $\tau_{xz} = \tau_{zx} = 30$ kPa, and $\tau_{xy} = \tau_{yz} = 0$ due to the plane strain condition. The value of σ_{yy} is found using the plain strain condition:

$$\varepsilon_{yy} = 0 = \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}))$$

so that

$$\sigma_{yy} = \nu(\sigma_{xx} + \sigma_{zz}) = 0.35(40 + 80) = 42 \text{ kPa}$$

Therefore,

$$\sigma_m = \frac{1}{3}(40 + 42 + 80) = 54 \text{ kPa.}$$

The deviatoric and spherical tensor forms are:

$$\sum = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} = S + D = \begin{pmatrix} 54 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 54 \end{pmatrix} + \begin{pmatrix} -14 & 0 & 30 \\ 0 & -12 & 0 \\ 30 & 0 & 26 \end{pmatrix} \text{kPa}$$

Problem 10.4

A sample of cohesionless silt is tested in a direct shear test. At failure, the vertical normal stress is 100 kPa and the shear stress on the horizontal plane where failure occurs is equal to 40 kPa. The water stress is 20 kPa.

a. Calculate the effective principal stresses by using the equilibrium equations approach in two dimensions.
b. Calculate the effective principal stresses by using the Mohr circle approach in two dimensions.

Solution 10.4

a. The effective principal stresses are related to the shear and normal stress on the failure plane through the equilibrium equations (Eq. 10.10 and Eq. 10.11):

$$\sigma' = \frac{\sigma'_1 + \sigma'_3}{2} + \frac{\sigma'_1 - \sigma'_3}{2}\cos 2\alpha$$

$$\tau = -\frac{\sigma'_1 - \sigma'_3}{2}\sin 2\alpha$$

Because the sample is at failure and the silt is cohesionless, the shear strength equation can be written as:

$$\tau = \sigma' \tan \varphi'$$

This also means that:

$$\sin \varphi' = \frac{\sigma'_1 - \sigma'_3}{\sigma'_1 + \sigma'_3}$$

The last four equations, together with the given values of $\sigma' = 80$ kPa and $\tau = 40$ $\tau = 40$ kPa, give the values of the four unknowns: φ', σ'_1, σ'_3, and α. The solution is $\varphi' = 26.6^{\text{o}}$, $\sigma'_1 = 144.4$ kPa, $\sigma'_3 = 55.4$ kPa, and $\alpha = 59^{\text{o}}$.

b. The effective principal stresses can be found using the Mohr circle as shown in Figure 10.3s.

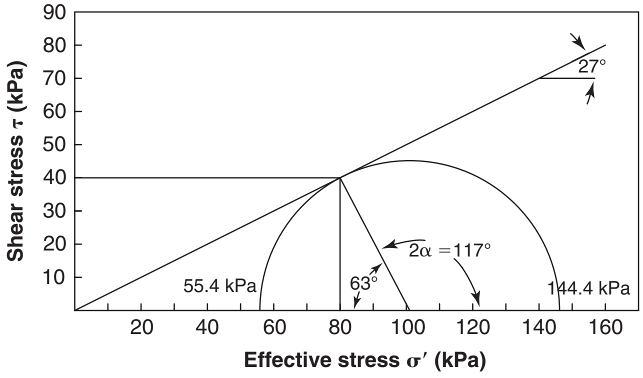


Figure 10.3s Mohr circle for direct shear test.

Problem 10.5

For problem 10.4, use the Pole method to locate the planes where the principal stresses act.

Solution 10.5

First we draw the failure stress point on the shear stress vs. effective normal stress set of axes ($\tau = 40$ kPa, $\sigma = 80$ kPa). This point is on the failure envelope, and because the soil has no cohesion intercept, the failure envelope can be drawn through the origin and the failure point. The Mohr circle is found tangent to the failure envelope at the failure stress point. According to the Pole method, the line parallel to the plane on which the stresses act (horizontal plane) intersects the Mohr circle at two points: the stress point and the Pole. This allows us to find the Pole (Figure 10.4s). Knowing the Pole, we draw the lines that join the Pole to the two principal stress points σ_1' and σ_3'. These lines define the directions of the planes on which the principal stresses are acting (Figure 10.5s).

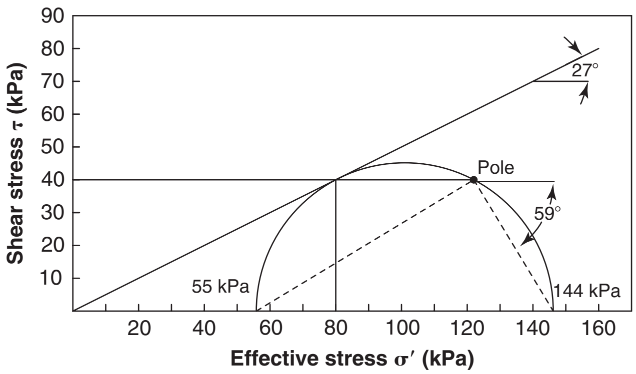


Figure 10.4s Pole method.

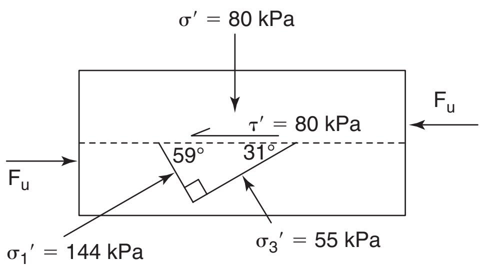


Figure 10.5s Principal planes in direct shear test.

Problem 10.6

For the sample in Figure 10.6s,

a. Find the stresses on the plane shown.
b. On what plane does the maximum shear stress exist?

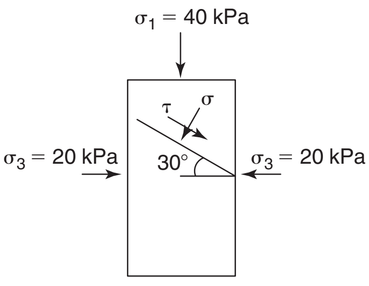


Figure 10.6s Stress state.

Solution 10.6

a. We can solve this problem with the equilibrium equations or with the Mohr circle. Recall Eqs 10.10 and 10.11 from the text:

$$\sigma = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\cos 2\alpha$$
$$\tau = -\frac{\sigma_1 - \sigma_3}{2}\sin 2\alpha$$

By using these equations, we obtain:

$$\sigma = \frac{40 + 20}{2} + \frac{40 - 20}{2}\cos(2 \times 30^\circ) = 35 \text{ kPa}$$
$$\tau = -\frac{40 - 20}{2}\sin(2 \times 30^\circ) = -8.67 \text{ kPa}$$

We then confirm the solutions by use of the Mohr circle (Figure 10.7s).

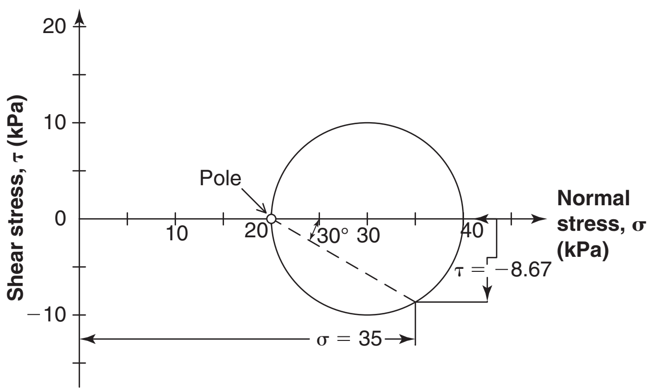


Figure 10.7s Mohr circle.

b. To find the plane where the maximum shear stress acts, we use the Pole method. We first find the Pole by drawing a line parallel to the plane where σ_1 acts (horizontal). That line intersects the Mohr circle at two points: the σ_1 stress point and the Pole. Then we join the Pole to the largest shear stress point of the Mohr circle. That line is a 45-degree line and gives the plane on which the highest shear stress acts (Figure 10.8s).

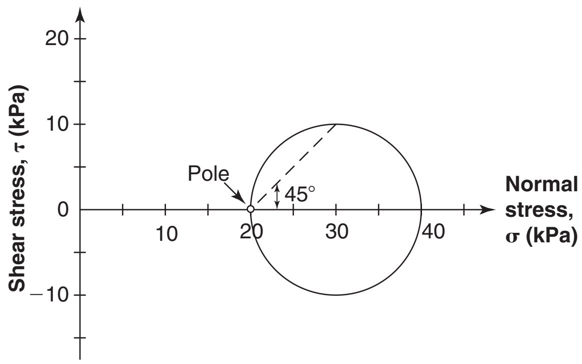


Figure 10.8s Pole method.

Problem 10.7

What happens to the Pole method when the diagram of a stress element in space is rotated by an angle θ? Does the Mohr circle change? Do the stresses on any plane change? Does the Pole location change?

Solution 10.7

When the diagram of the stress element in space is rotated by an angle θ, the Mohr circle does not change because the principal stresses do not change; accordingly, the stresses on any plane do not change either. However, the location of the Pole on the Mohr circle rotates with the diagram to maintain the rule of parallelism.

Problem 10.8

In a simple shear test, the horizontal displacement at the top of the sample is 1 mm and the vertical displacement is a reduction in height of 0.5 mm. The original height of the sample is 25 mm.

a. Calculate the shear strain and the vertical normal strain.
b. Is the sample dilating or contracting?

Solution 10.8

a. The shear strain and the vertical normal strain are (Figure 10.9s):

$$\varepsilon_{shear} = \tan^{-1}\frac{1}{25} = 0.04 \quad \text{or } 4\% \text{ shear strain}$$

$$\varepsilon_{normal} = \frac{0.5}{25} = 0.02 \quad \text{or } 2\% \text{ compression normal strain}$$

b. The sample is contracting.

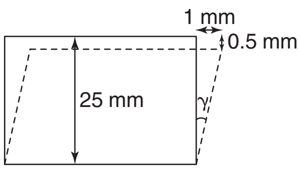


Figure 10.9s Normal and shear strain.

Problem 10.9

In a triaxial test, the sample has an initial height of 150 mm and an initial diameter of 75 mm. During the loading in the vertical direction, the vertical displacement is 3 mm and the increase in diameter is 2 mm.

a. Calculate the normal strains ε_{zz} and ε_{rr}.
b. Form the strain tensor.
c. Calculate the shear strain on a 45-degree plane.

Solution 10.9

a. The normal strains ε_{zz} and ε_{rr} are:

$$\varepsilon_{zz} = \frac{3}{150} = 0.02, \varepsilon_{rr} = \frac{2}{75} = -0.027$$

b. The strain tensor is:

$$\begin{bmatrix} \varepsilon_{rr} & \frac{1}{2}\gamma_{rz} \\ \frac{1}{2}\gamma_{zr} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} -0.027 & 0 \\ 0 & 0.02 \end{bmatrix}$$

c. The shear strain on a 45-degree plane is:

$$\gamma_{45} = (\varepsilon_{zz} - \varepsilon_{rr}) \sin 2\alpha = (0.02 - (-0.027)) * \sin(90) = 0.047$$

Problem 10.10

Consider the sphere-shaped soil particles shown in Figure 10.10s. The degree of saturation S is 1, the porosity n is 0.4, and the ratio between the sum of the contact areas and the total area (A_c/A_t) is 0.01. Calculate the following quantities and show the relationship between the total stress and the effective stress if the water stress is +40 kPa.

a. The average effective normal stress
b. The average normal stress at the contacts
c. The average normal total stress

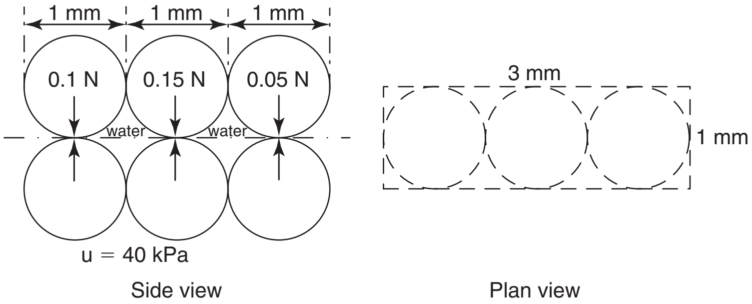


Figure 10.10s Sphere-shaped soil particles.

Solution 10.10

$$A = 3 \times 1 \times 10^{-6} = 3 \times 10^{-6}\ \mathrm{m}^2$$

The average effective normal stress is:

$$\sigma'_{aver} = \frac{(0.1 + 0.15 + 0.05) \times 10^{-3}}{3 \times 10^{-6}} = 100\ \mathrm{kPa}$$

The average normal stress at the contacts is:

$$\sigma_{c-aver} = \frac{(0.1 + 0.15 + 0.05) \times 10^{-3}}{3 \times 10^{-6} \times 0.01} = 10000\ \mathrm{kPa}$$

The average normal total stress is:

$$\sigma_{aver} = \frac{(0.1 + 0.15 + 0.05) \times 10^{-3} + 40 \times (1 - 0.01) \times 3 \times 10^{-6}}{3 \times 10^{-6}} = 139.6\ \mathrm{kPa}$$

By definition, the relation between the total stress and the effective stress is:

$$\sigma = \sigma' + u = 100 + 40 = 140 \approx 139.6\ \mathrm{kPa}$$

Problem 10.11

The surface tension of water is T = 73 mN/m, the diameter of a glass tube plunged into water is 0.002 mm, and the contact angle between the wall of the clean glass and the water is $\alpha = 10$ degrees. Find the height to which the water will rise in the small tube.

Solution 10.11

$$d = 0.002\ \mathrm{mm}$$

$$T = 73\ \mathrm{mN/m}$$

$$\alpha = 10\ \deg$$

$$h_c = \frac{4T\cos\alpha}{d\gamma_w} \rightarrow h_c = \frac{4 \times 73 \times 10^{-6} \times \cos(10)}{0.002 \times 10^{-3} \times 10} = 14.37\ \mathrm{m}$$

Problem 10.12

Consider the sphere-shaped soil particles shown in Figure 10.11s. The porosity n is 0.4, the ratio between the sum of the contact areas and the total area (A_c/A_t) is 0.01, and the ratio between the sum of the areas of water and the total area (A_w/A_t) is 0.1. Calculate the following quantities and show the relationship between the total stress and the effective stress if the water stress is −6000 kPa.

a. The average normal effective stress
b. The average normal stress at the contacts
c. The average normal total stress

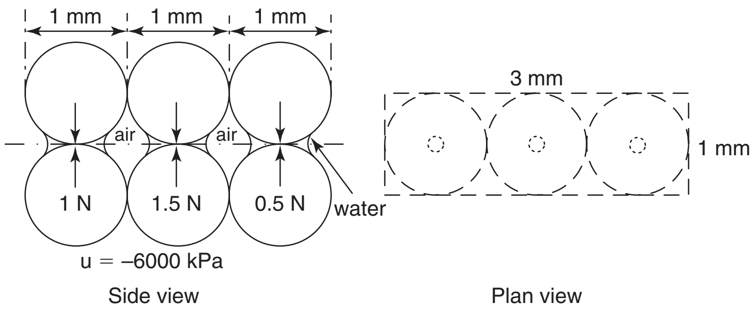


Figure 10.11s Sphere-shaped soil particles.

Solution 10.12

$$A = 3 \times 1 \times 10^{-6} = 3 \times 10^{-6}\text{m}^2$$

a. The average normal effective stress is:

$$\sigma'_{aver} = \frac{(1 + 1.5 + 0.5) \times 10^{-3}}{3 \times 10^{-6}} = 1000 \text{ kPa}$$

b. The average normal stress at the contacts is:

$$\sigma_{c-aver} = \frac{(1 + 1.5 + 0.5) \times 10^{-3}}{3 \times 10^{-6} \times 0.01} = 100000 \text{ kPa}$$

c. The average normal total stress is:

$$\sigma_{aver} = \frac{(1 + 1.5 + 0.5) \times 10^{-3} - 6000 \times 0.1 \times 3 \times 10^{-6}}{3 \times 10^{-6}} = 400 \text{ kPa}$$

The relation between the total stress and the effective stress is:

$$\sigma' = \sigma - \alpha u = 400 - 0.1 \times (-6000) = 1000 \text{ kPa}$$

Problem 10.13

A soil has a degree of saturation of 92%. The air is occluded and the bubbles are 1 mm in diameter. Knowing that the water tension can reach 73 mN/m, what is the maximum difference in pressure that can exist between the water stress and the air stress?

Solution 10.13

It seems reasonable that the air is occluded, as the degree of saturation of the soil is 92%, which is larger than 85%. Based on the equilibrium of the free-body diagram of half the bubble, and knowing that the water tension is 73 mN/m, we have (Eq. 10.59):

$$u_a - u_w = \frac{4T}{D} = \frac{4 \times 73 \times 10^{-6} \text{ kN/m}}{1 \times 10^{-3} \text{ m}} = 0.29 \text{ kPa}$$

Problem 10.14

A soil has a degree of saturation of 35%, an air entry value of −150 kPa, and a water tension stress of −1500 kPa at a depth of 2 m. Estimate the vertical effective stress at rest at a depth of 2 m below the ground surface, assuming that the unit weight of the soil is 19 kN/m^3.

Solution 10.14

The soil has a degree of saturation of 35%, which means the soil is unsaturated and the relationship between the effective stress and total stress is:

$$\sigma' = \sigma - \alpha\, u_w$$

Here, α can be obtained based on Khalili and Khabbaz (1998):

$$\alpha = \sqrt{\frac{u_{wae}}{u_w}} = \sqrt{\frac{-150}{-1500}} = 0.316$$

Therefore, at the given depth of 2 m below the ground surface, the vertical effective stress at rest is:

$$\sigma' = \sigma - \alpha\ \ u_w = \gamma\ \ Z - \alpha\ \ u_w = 19 \times 2 - 0.316 \times (-1500) = 512 \text{ kPa}$$

Problem 10.15

Draw the three profiles (σ_{ov}, u_o, σ'_{ov}) for the layered system shown in Figure 10.12s.

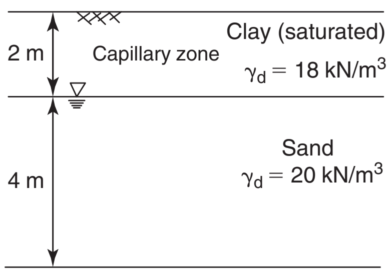


Figure 10.12s Soil profile.

Solution 10.15

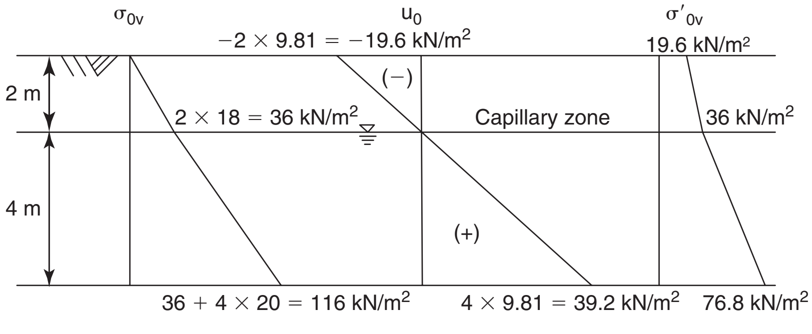


Figure 10.13s Stress profiles.

Problem 10.16

Draw the effective stress profiles (σ'_{ov}) for the layered system shown in Figure 10.14s.

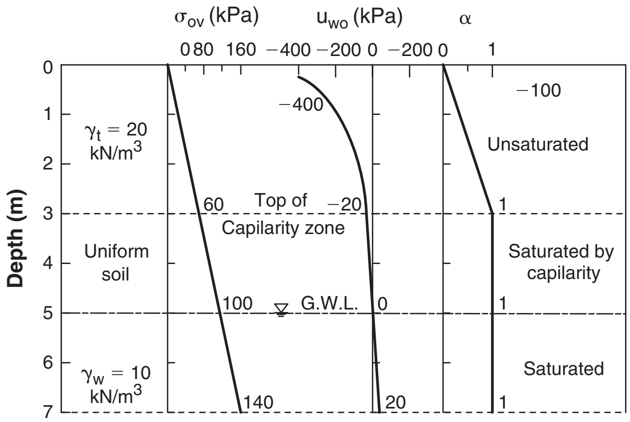


Figure 10.14s Soil and stress profile.

Solution 10.16

The effective vertical stress σ'_{ov} profile is shown in Figure 10.15s.

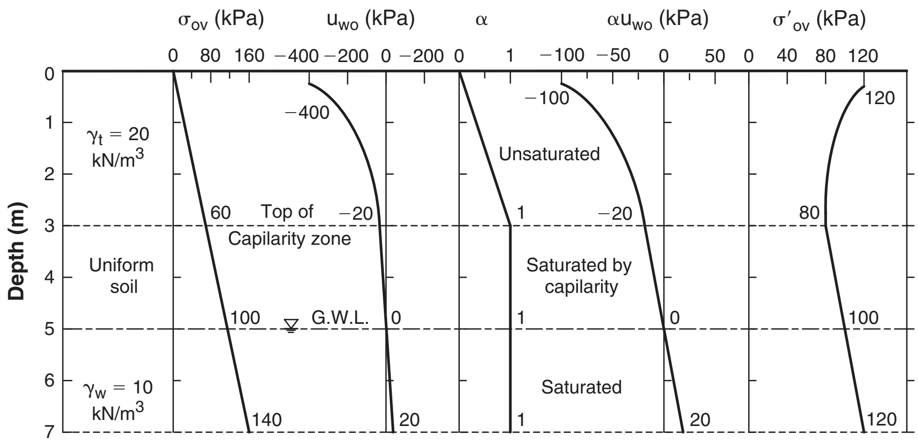


Figure 10.15s Stress profiles.

Problem 10.17

Draw the three profiles (σ_{ov}, u_w, σ'_{ov}) at the center of the river for the layered system shown in Figure 10.16s.

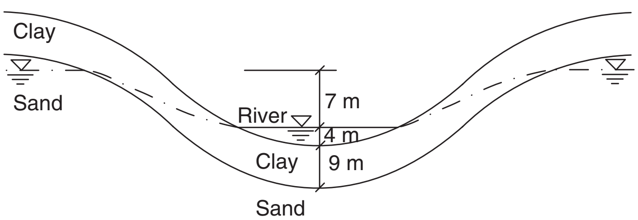


Figure 10.16s River profile.

Solution 10.17

The stress profile for σ_{ov}, u_w, σ'_{ov} is shown in Figure 10.17s:

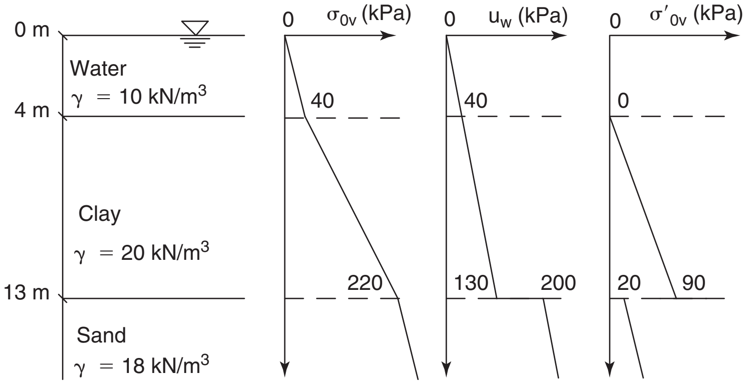


Figure 10.17s Stress profile in the river.

Problem 10.18

An insect has 4 legs and is able to walk on water. The depression created under each foot is a sphere, as shown in Figure 10.18s. What is the maximum possible weight of the insect?

Solution 10.18

The contact radius of the insect foot with the contractile skin is:

$$r = 3 \text{ mm} \times \sin 35^\circ = 1.72 \text{ mm}$$

For a water temperature of 20°C, the surface tension (σ_T) is 73 mN/m.

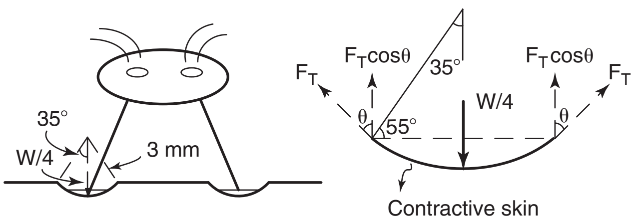


Figure 10.18s Free-body diagram of insect.

$$\sum F_v = 0$$

$$F_T \times 2\pi r \cos\theta - W/4 = 0$$

$$W = 8\pi r F_T \cos\theta$$

$$W = 8\pi(0.00172\text{m})(73 \text{ mN/m}) \cos 55^\circ = 1.79 \text{ mN}$$

Problem 10.19

A soil has a water content of 42% and an air entry value of −8 kPa. If the slope of the soil water retention curve is 0.2 per log cycle of water tension in kPa, calculate the water tension for a water content of 10%.

Solution 10.19

Given values:

$$C_w = -0.2$$
$$u_{wae} = -8 \text{ kPa}$$
$$w_1 = 42\%$$
$$w_2 = 10\%$$
$$\Delta w = 0.42 - 0.10 = -0.2 \log\left(\frac{-8}{u_w}\right)$$
$$\log\left(\frac{-8}{u_w}\right) = \frac{0.32}{-0.2} = -1.6$$
$$10^{-1.6} = \frac{-8}{u_w}$$
$$u_w = \frac{-8}{10^{-1.6}}$$
$$u_w = -318.5 \text{ kPa}$$

Problem 10.20

A tree's root system occupies a volume equal to 1000 m^3. How much water is available to that tree if it is rooted in the three soils described by the retention curves of Figure 10.31?

Solution 10.20

From the portion in the graph referring to "Water available to plants":

Change in volumetric water content (Θ):

1. Clay: $\Delta\Theta = 0.50 - 0.34 = 0.16$
2. Silt: $\Delta\Theta = 0.40 - 0.02 = 0.38$
3. Sand: $\Delta\Theta = 0.15 - 0 = 0.15$

$\Delta\Theta \times$ Volume of soil in root system = Water available to the tree

1. Clay: $0.16 \times 1000 \text{ m}^3 = 160 \text{ m}^3$
2. Silt: $0.38 \times 1000 \text{ m}^3 = 380 \text{ m}^3$
3. Sand: $0.15 \times 1000 \text{ m}^3 = 150 \text{ m}^3$