CHAPTER 13

Flow of Fluid and Gas Through Soils

13.1 GENERAL

The fluid and gas phase can either flow through the voids between the soil particles or stay static in the voids. The flow is affected by the resistance offered by friction between the soil particles, by the size of the voids, by the blockage posed by the gas phase to the fluid flow, by the blockage posed by the fluid phase to the gas flow, and by the energy gradient in the fluid or gas. Most of the time the fluid is water and the gas is air, so from this point on we will talk about water and air. If other fluids or gasses are involved, the viscosity of the fluid or gas will change from that of water and air; additionally, any chemical reaction that may occur between the fluid and the particles can increase or decrease the size of the voids.

Some of the output quantities of interest in a flow problem are the water stress, the air stress, the water velocity and its direction, the air velocity and its direction, and the quantity of water flowing per unit time. In geoenvironmental studies, the future location of a moving body of a contaminant is of interest in predicting the extent of contamination. The soil can be saturated or unsaturated. The flow of water in a saturated soil is the simplest case, so we will start with that.

13.2 FLOW OF WATER IN A SATURATED SOIL

13.2.1 Discharge Velocity, Seepage Velocity, and Conservation of Mass

One of the two main equations used to solve flow problems in soils is the conservation of mass equation, which in this case states that the flow Q in m^3/s is equal to the cross-sectional area A times the water velocity v:

$$Q = vA \tag{13.1}$$

One distinguishes between the discharge velocity and the seepage velocity (Figure 13.1). The *seepage velocity* v_s is the actual velocity of the water. In other words, if you were riding on the water molecule, what you would read on the speedometer would be the seepage velocity. Also, if you put a dye in the water and you could see through the soil, you would see the colored water propagate at the seepage velocity. The cross section associated with that velocity is the actual cross section of the voids A_v.

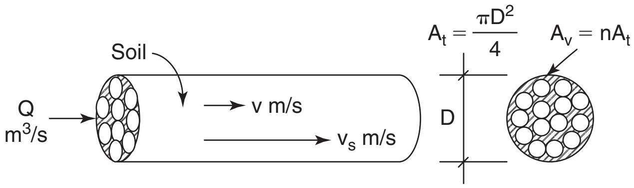

Figure 13.1 Steady flow of water through soil.

$$Q = v_s A_v \tag{13.2}$$

Because A_v is difficult to estimate, the discharge velocity v is used instead in almost all calculations. The *discharge velocity* is the ratio of the flow Q divided by the total cross-sectional area A_t of the soil being traversed by the water (voids plus grains). The discharge velocity is not the actual water velocity, but it is a convenient value to use for calculation purposes:

$$Q = vA_t \tag{13.3}$$

Using Eqs. 13.2 and 13.3 gives the relationship between the seepage and discharge velocity as

$$vA_t = v_s A_v \quad \text{or} \quad v = nv_s \tag{13.4}$$

where n is the porosity of the soil. This shows that the seepage velocity is higher than the discharge velocity (2 to 3 times higher). Although v_s is the actual water velocity, in most geotechnical problems we will use the discharge velocity v. One exception concerns the propagation of contaminated plumes, where it is important to know where the contaminated water is going as a function of time; in this case the seepage velocity must be used. In any case, switching from one to the other can be achieved simply by using Eq. 13.4.

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13.2.2 Heads

The energy level in the water is measured in height of water or *head*. The total head h_t represents the total energy available to the water to drive through the soil voids. The pressure head h_p represents the energy stored as pressure in the water. The elevation head h_e represents the potential energy, and the velocity head h_v represents the kinetic energy. The total head is the sum of the pressure head plus the elevation head plus the velocity head:

$$h_t = h_p + h_e + h_v \tag{13.5}$$

Because water flows very slowly through soils (mm per second at most), the velocity head is assumed to be zero for all practical purposes. The pressure head times the unit weight of water γ_w gives the water stress u_w or pore pressure:

$$u_w = h_p \gamma_w \tag{13.6}$$

The elevation head is measured with reference to an arbitrarily chosen datum. This arbitrary choice does not affect the results because all calculations involve changes in quantities, not absolute quantities. The total head of a water molecule at the surface of a lake is the same as the total head of a water molecule at the bottom of that lake (Figure 13.2). Indeed, at the surface, the pressure head is zero but the elevation head is the water depth (if the lake bottom is chosen as the datum), whereas at the bottom of the lake the elevation head is zero but the pressure head is equal to the water depth (water pressure divided by unit weight).

As the water drives through the voids of the soil, it burns energy or total head. The loss of energy is due to the friction that exists between the water molecules and the soil particles. This friction force is called the *seepage force* S. At any point M in the soil, the elevation head can be obtained as the vertical distance between M and the arbitrarily chosen datum (Figure 13.3). At any point M in the soil, the pressure head can be measured by placing a standpipe connected to M at one end and to the atmosphere at the other and measuring the vertical distance between M and the water level in the pipe. The standpipe does not have to be vertical or even straight as long as there is a clear water path from M to the atmospheric pressure. Figure 13.4 shows an example of head diagrams from a constant head permeameter and a falling head permeameter.

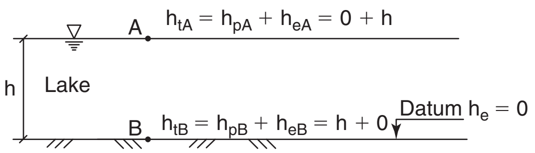

Figure 13.2 Heads in a lake with no flow.

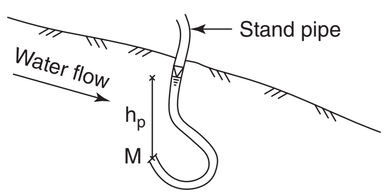

Figure 13.3 Pressure head at M.

13.2.3 Hydraulic Gradient

The *hydraulic gradient* is defined between two points A and B along the path of water travel, called the *flow path*. The hydraulic gradient is the ratio of the loss of total head between A and B over the actual distance traveled by the water to go from A to B (not always the straight line joining the two points):

$$i_{AB} = \frac{h_{tA} - h_{tB}}{l_{AB}} = \frac{\Delta h_{tAB}}{l_{AB}} \tag{13.7}$$

where i_{AB} is the hydraulic gradient between A and B, Δh_{tAB} is the loss of total head between A and B, and l_{AB} is the length of the flow path from A to B. The hydraulic gradient is unitless and varies from about 0.1 to 2 in the field. The hydraulic gradient represents a rate of energy consumption. It is similar in concept to gas consumption for a car. Gas in the tank of a car is energy that is burned when traveling from one point to another; the amount of gas burnt per actual distance traveled on the highway is the gas consumption. If you wish to figure your car consumption from one point to another, you use the actual distance travelled, not the straight-line distance between the two cities. The same applies for the hydraulic gradient. The hydraulic gradient between D and C in Figure 13.4*a* is:

$$i_{DC} = \frac{h_{tD} - h_{tC}}{l_{DC}} = \frac{1.12 - 0.96}{0.2} = 0.8 \tag{13.8}$$

The hydraulic gradient between C and D on Figure 13.4*b* is:

$$i_{CD} = \frac{h_{tC} - h_{tD}}{l_{CD}} = \frac{0.8 - 0.4}{0.2} = 2 \tag{13.9}$$

13.2.4 Darcy's Law: The Constitutive Law

This law is named after Henry Darcy, a French engineer, who discovered it in 1855 as he was working on a problem with the public fountains in Dijon, France (yes, that's the mustard city). The experiment that Darcy set up is essentially the same as the constant head permeameter shown in Figure 13.4*a*. Darcy varied the water level and the length of the sample while measuring the flow coming out of the sample. He found that there was a linear relationship between the water velocity and the hydraulic gradient (Figure 13.5). The slope of that line is called the *hydraulic conductivity* k and Darcy's law states that the discharge velocity v is equal to the hydraulic conductivity k times the hydraulic gradient i:

$$v = ki \tag{13.10}$$

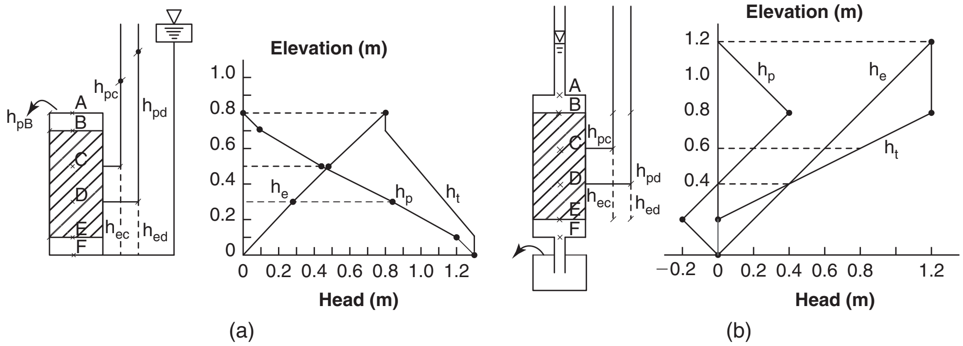

Figure 13.4 Head diagram examples: (a) Constant heat permeameter. (b) Falling head permeameter.

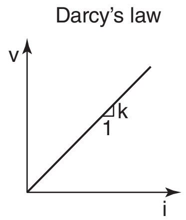

Figure 13.5 Darcy's law.

Going back to the analogy with the car and its gas consumption, it would mean that the gas consumption of the car is linearly proportional to the speed of the car. Darcy's law is the most important constitutive law related to the flow of fluids through soils.

In Figure 13.4*a*, if the hydraulic conductivity of the sand is 10^{-5} m/s, and considering the hydraulic gradient of Eq. 13.8, the discharge velocity is:

$$v = 10^{-5} \times 0.8 \text{ m/s} = 0.008 \text{ mm/s} = 691 \text{ mm/day} \tag{13.11}$$

In Figure 13.4*b*, if the hydraulic conductivity of the clay is 10^{-10} m/s, the discharge is:

$$v = 10^{-10} \times 2 \text{ m/s} = 0.0000002 \text{ mm/s} = 6.3 \text{ mm/year} \tag{13.12}$$

Darcy's law applies to fluid flow and is parallel to Fourier's law for heat flux, to Ohm's law for electrical flux, and to Fick's law for diffusive flux. All these laws express that the propagation of a phenomenon is linearly related to a gradient of potential through a conductivity constant specific to the material through which the propagation is taking place. The analogy can be taken further, as shown in Table 13.1.

13.2.5 Hydraulic Conductivity

The hydraulic conductivity of saturated soils varies widely, from about 10^{-2} m/s for some gravels to about 10^{-11} m/s for some clays. It is measured in the laboratory with a constant head permeameter for sands and gravels, or with a falling head permeameter for silts and clays (see Chapter 9). In the field, it is measured by using a borehole and performing either a pumping test or an infiltration test (see Chapter 7). It can also be obtained by using the piezocone penetrometer test through pore-pressure decay as a function of time (see Chapter 7). The hydraulic conductivity obtained by laboratory tests can be 10 times to 100 times lower than the field value, because the lab test may be testing the intact soil between fissures while the field test may include a network of fissures.

The hydraulic conductivity of saturated soils depends on many factors, including the void ratio, the shape and roughness of the particles, the structure of the soil skeleton, and

Table 13.1 Equivalency between Hydraulic, Heat, and Electricity

Parameter	Hydraulic	Heat	Electricity
Law	Darcy	Fourier	Ohm
Soil property	Hydraulic conductivity	Thermal conductivity	Electrical conductivity
Quantity	Volume V(m^3)	Heat Q(J)	Charge q (C)
Potential	Head (m)	Temperature (K)	Potential (V)
Gradient	Hydraulic gradient (m/m)	Temperature gradient (K/m)	Potential gradient (V/m)
Flux	Flow rate (m^3/s)	Heat transfer rate (J/s)	Current I(A)
Flux density	Velocity v(m/s)	Heat flux (W/m^2)	Current density j(A/m^2)

the fluid properties (viscosity and unit weight). To separate the influence of the fluid from that of the soil skeleton on the hydraulic conductivity k, the intrinsic hydraulic conductivity or simply *permeability*, K is used:

$$K = k\frac{\mu_f}{\gamma_f} \tag{13.13}$$

where μ_f is the dynamic viscosity of the fluid, and γ_f is the unit weight of the fluid. The value of μ_f for water is 10^{-6} kPa.s at 20°C, and γ_f for water is 9.79 kN/m^3 at 20°C. Note that K is in units of m^2.

Some empirical relationships have been proposed over the years for estimating the hydraulic conductivity of coarse-grained soils. Hazen (1892), working on sand filters, proposed a formula relating the hydraulic conductivity k to the D_{10} particle size corresponding to 10% finer on the particle size distribution curve. The Hazen formula seems to work best for sands with D_{10} values between 0.1 and 1 mm.

$$k(\text{m/s}) = C(D_{10}(\text{mm}))^2 \tag{13.14}$$

where C is a constant usually taken as 0.01 but with reported values from 0.1 to 0.001. For D_{10} values above 1 mm, the power of 2 for D_{10} decreases. Kozeny (1927) and Carman (1938) proposed a semi-empirical formula also for sands:

$$k(\text{m/s}) = \frac{\gamma e^3}{\mu C S_o^2(1+e)} \tag{13.15}$$

where γ is the unit weight of the permeating fluid (kN/m^3), μ is its dynamic viscosity (kN.s/m^2), e is the void ratio of the soil, C is a constant usually taken equal to 5×10^6, and S_o is the specific surface of the particles (1/m). The *specific surface* is the ratio of the particle surface area over the volume of the particle. For a sphere it would be:

$$S_o = \frac{\pi D^2}{\pi D^3/6} = \frac{6}{D} \tag{13.16}$$

The factor 6 in Eq. 13.16 goes up to 8.5 for very angular coarse-grain particles. Using a value of 5×10^6 for C, a value of 10 kN/m^3 for the unit weight of water, and a value of 10^{-6} kN.s/m for the dynamic viscosity of water at 20°C, the formula becomes:

$$k(\text{m/s}) = \frac{2e^3}{S_o^2(\text{mm}^{-2})(1+e)} \tag{13.17}$$

The specific surface can be measured or estimated from the particle size distribution curve (Carrier 2003). Some equations have been proposed to estimate the hydraulic conductivity of remolded clays from index properties (Carrier and Beckman 1984):

$$k(\text{m/s}) = \frac{0.0174}{1+e}\left(e - \frac{0.027\left(w_L - 0.242PI\right)}{PI}\right)^{4.29} \tag{13.18}$$

Table 13.2 Approximate Range of Hydraulic Conductivity of Soils

Soil type	Hydraulic conductivity (m/s) for water flow in saturated soils
Gravel	10^{-4} to 10^{-2}
Sand	10^{-6} to 10^{-4}
Silt	10^{-8} to 10^{-6}
Clay	10^{-11} to 10^{-8}

where e is the void ratio, w_L is the liquid limit, and PI is the plasticity index. Table 13.2 gives a range of possible hydraulic conductivity values for saturated water flow through soils.

13.2.6 Field vs. Lab Values of Hydraulic Conductivity

One of the difficult issues in soil hydraulic conductivities is the different values obtained in the laboratory at small scale, k_{lab}, and in the field at large scale, k_{field}, particularly for fine-grained soils. The difference can be several orders of magnitude, as shown conceptually in Figure 13.6. Among the reasons for this large difference is the lack of representativeness of the small samples. Indeed, often the small samples do not reflect the influence of the large-scale features of a soil deposit on the hydraulic conductivity k. These features include cracks and fissures formed through successive drying and wetting or simply bending of the soil mass over geologic time. Hence, the lower values given in Table 13.2 may represent the lab values, while the higher values may represent the large-scale field values.

13.2.7 Seepage Force

Seepage force is a drag force that develops at the interface between flowing water and soil particles. It is due to the viscous friction between the two elements. If the water and

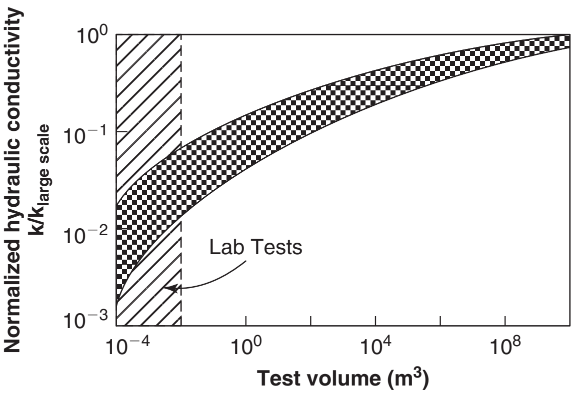

Figure 13.6 Conceptual difference between small-scale and large-scale hydraulic conductivity.

the soil particles are considered together in a free-body diagram, the seepage force is an internal force and does not enter into the equilibrium of the free body. Indeed, only the external forces influence the equilibrium of the free body. However, if the soil skeleton made only of the particles is considered as the free body and the water is external, then the seepage force must be considered in the equilibrium of the free body, as it is now an external force. Similarly, if the water only is considered as the free body, then again the seepage force is an external force.

Let's see how we can calculate the magnitude of this seepage force. For this, we consider the sketch of Figure 13.7 in which water flows from A to B in a cylinder of diameter D filled with soil. As explained earlier, we must consider either the free body of the soil skeleton alone or the free body of the water alone to make the seepage force appear in the equilibrium of the free body. It is easier in this case to consider the free body of the water in the cylinder between point A and point B. So, imagine the body of water with all the soil particles removed; it looks like Swiss cheese. The external forces are the seepage force, which is the summation of all the small friction or drag forces the soil particles exert on the water; the oblique upward force at point A due to the water pressure at that point; the oblique downward force at point B due to the water pressure; and the weight of the water. The weight of the particles is another external force, but it is carried by the container.

The equilibrium of the free body of water in the flow direction is written as:

$$h_{pA}\gamma_w A - h_{pB}\gamma_w A - S - W\sin\alpha = 0 \qquad (13.19)$$

where h_{pA} and h_{pB} are the pressure head at A and B respectively, γ_w is the unit weight of water, A is the total cross-sectional area $(\pi D^2/4)$, S, S is the seepage force, W is the weight of the water body, and α is the angle between the flow direction and the horizontal. Note that A is not the correct area to use, as there are holes in this "Swiss cheese" water body. We consider this area, even though it is wrong, for the same reason that we consider the cross-sectional area, even though it is wrong, for the determination of

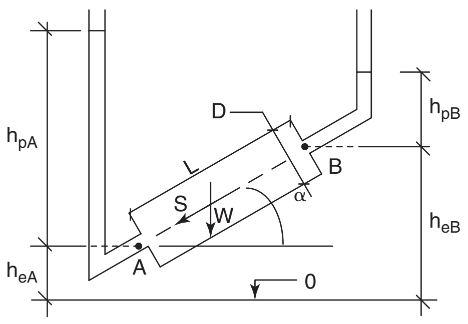

Figure 13.7 Seepage force.

discharge velocity (Section 13.2.1). The term $W\sin\alpha$ can be expressed as:

$$W\sin\alpha = AL\gamma_w \frac{h_{eA} - h_{eB}}{L} \qquad (13.20)$$

where h_{eA} and h_{eB} are the elevation head at A and B respectively and A is the total cross-sectional area. Again, this is not the correct area to use, as there are holes in the "Swiss cheese" water body, but it is consistent with the area chosen for the equilibrium equation (Eq. 13.19). Then, using Eqs. 13.19 and 13.20, the seepage force per unit of soil volume (S/AL) can be written as:

$$\frac{S}{AL} = \frac{h_{pA}\gamma_w - h_{pB}\gamma_w}{L} + \gamma_w \frac{h_{eA} - h_{eB}}{L} = \frac{h_{tA} - h_{tB}}{L}\gamma_w$$
$$= i_{AB}\gamma_w \qquad (13.21)$$

where i_{AB} is the hydraulic gradient between A and B. Therefore, the seepage force s per unit volume of soil is given by $i\gamma_w$ and exists in the direction of the hydraulic gradient. This is where the choice of the "wrong" area—total cross-sectional area—becomes useful; if we had used the correct area, we would have obtained the seepage force per unit of water volume, which would be more difficult to calculate than the simpler volume of soil:

$$\vec{s} = \gamma_w \vec{i} \qquad (13.22)$$

For example, if the hydraulic gradient is 1 (a rather high but not unusual hydraulic gradient for common flow problems), the seepage force for one cubic meter of volume is 10 kN, or about one ton. This is a significant force. Note that the seepage force is zero when there is no flow ($i = 0$) and therefore does not include the buoyancy force, which is always in the vertical upward direction.

13.2.8 Quick Sand Condition and Critical Hydraulic Gradient

If the flow is upward and in the vertical direction, the seepage force adds to the buoyancy force to lighten the soil particles, and can become high enough to make the soil particles weightless (Figure 13.8).

This is called a *quick sand condition* and the corresponding hydraulic gradient is called the *critical hydraulic gradient* i_c. Referring to Figure 13.9, the buoyancy force F_b on the soil volume of cross-sectional area A and length L is equal to:

$$F_b = AL\gamma_w \quad or \quad f_b = \frac{F_b}{AL} = \gamma_w \qquad (13.23)$$

where f_b is the buoyancy force per unit volume. In the case where the flow is vertical upward, the equilibrium of forces per unit volume when the quick condition is reached is:

$$s + f_b = \gamma_w i_c + \gamma_w = \gamma_{sat} \quad or \quad i_c = \frac{\gamma_{sat}}{\gamma_w} - 1 \qquad (13.24)$$

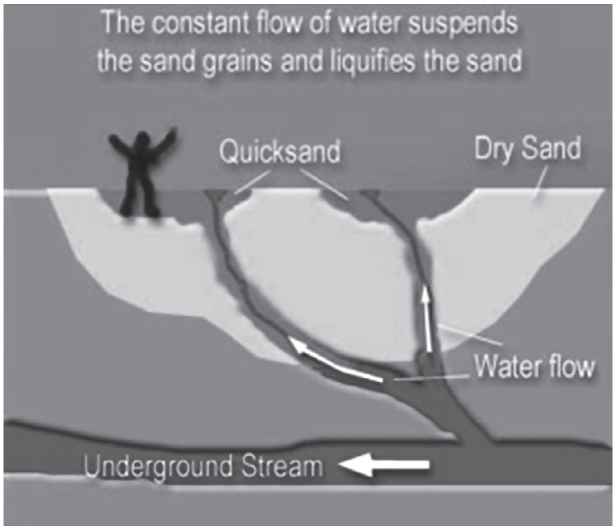

Figure 13.8 Quick sand condition. (Courtesy of Lee Krystek)

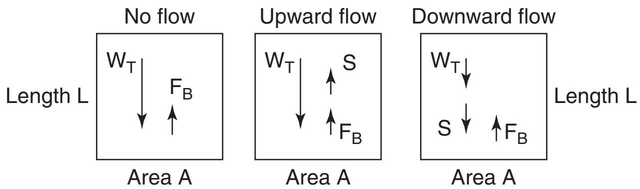

Figure 13.9 Seepage force for upward and downward flow.

Another way to arrive at this result for the critical gradient i_c is to consider the experiment of Figure 13.4 and ask when the total head difference between A and B will be sufficient to generate an effective stress equal to zero at the bottom of the sample:

$$\sigma' = \sigma - u_w = \gamma_{sat} L + \gamma_w h_{pB} - \gamma_w h_{pE} = 0 \qquad (13.25)$$

$$\gamma_{sat} L + \gamma_w (h_{tB} - h_{eB}) - \gamma_w (h_{tE} - h_{eE}) = \gamma_{sat} L - \gamma_w (h_{tE} - h_{tB}) - \gamma_w L = 0 \qquad (13.26)$$

$$i_c = \frac{h_{tE} - h_{tB}}{L} = \frac{\gamma_{sat}}{\gamma_w} - 1 \qquad (13.27)$$

Note that because γ_{sat}/γ_w is about 2, the critical gradient is about 1. If you fall into a quick sand, it is like falling into a very thick liquid. This liquid has the unit weight of the soil ($\sim$20 kN/m^3), which is typically equal to about two times the unit weight of the human body ($\sim$10 kN/m^3). Therefore, theoretically you should sink halfway into the quick sand until the buoyancy force counterbalances your weight. One problem is that if you do not stay still, you will go down, as there is no bearing capacity under your feet, and it will be difficult to go back up as this heavy liquid can develop friction resisting your movement upward. So, if you fall in such a quick sand, stay still and hope that the theory is correct! In contrast, if the flow is downward, the seepage force increases the weight of the particles artificially and the bearing capacity is improved compared to a no-flow condition. Figure 13.9 illustrates these conditions.

13.2.9 Quick Clay

A quick clay is something completely different from a quick sand condition. A *quick clay* is a clay that is a solid in its natural state, but turns into a liquid when disturbed. This disturbance can come from shearing because of loading. Such clays typically were slowly deposited in a seawater, offshore environment and, through geologic aging, are now in an onshore environment. One mechanism is loading by glaciers, which were as thick as 300 m some 10,000 years ago but have now melted, thereby allowing the offshore clays to rebound and become above ground in the process. The slow offshore deposition can lead to a card-castle type of structure for the clay particles (edge-to-face contacts) with a high porosity and abundant salt content in the pore water and at the contacts. Then, in the onshore environment, the clay has been permeated by rainwater (distilled water) or groundwater (usually low-salinity water) and the salt has been washed away from the voids in the clay, leaving only some salt at the contacts between the clay particles and the low-salinity water in the voids. The salt strengthens the particle contacts and therefore the structure because it decreases the repulsion that typically exists between electrically charged clay platelets. The intact clay may have an undrained shear strength of 25 kPa and a water content of 30% in the undisturbed case, for example. The low-salinity water leaches the salt away and weakens the bond between particles. If this fragile structure is disrupted by shearing or vibrations, for example, the structure collapses and the mixture of water from the voids and clay particles becomes a thick liquid (Figure 13.10). If salt is then added to the thick liquid and mixed by stirring, the clay regains some strength (Figure 13.10).

13.2.10 Sand Liquefaction

The phenomenon of *sand liquefaction* should be distinguished from the quick sand condition. Quick sand conditions are due to sufficiently rapid upward flow, whereas sand liquefaction is typically related to earthquake shaking. During such violent, repeated shaking, the water in the saturated sand does not have time to escape the pores (undrained behavior), so the pressure in the water goes up. If the water stress u_w becomes so high as to equal the total stress σ, then the effective stress $\sigma'(\sigma' = \sigma - u_w)$ becomes equal to zero and the sand liquefies. This heavy liquid can flow to the surface and create sand boils, which are often found at the ground surface after a severe earthquake (Figure 13.11).

13.2.11 Two-Dimensional Flow Problem

Some of the structures involving problems associated with steady-state flow through saturated soils include earth dams, cofferdams, spillways, cutoff walls, retaining walls, and

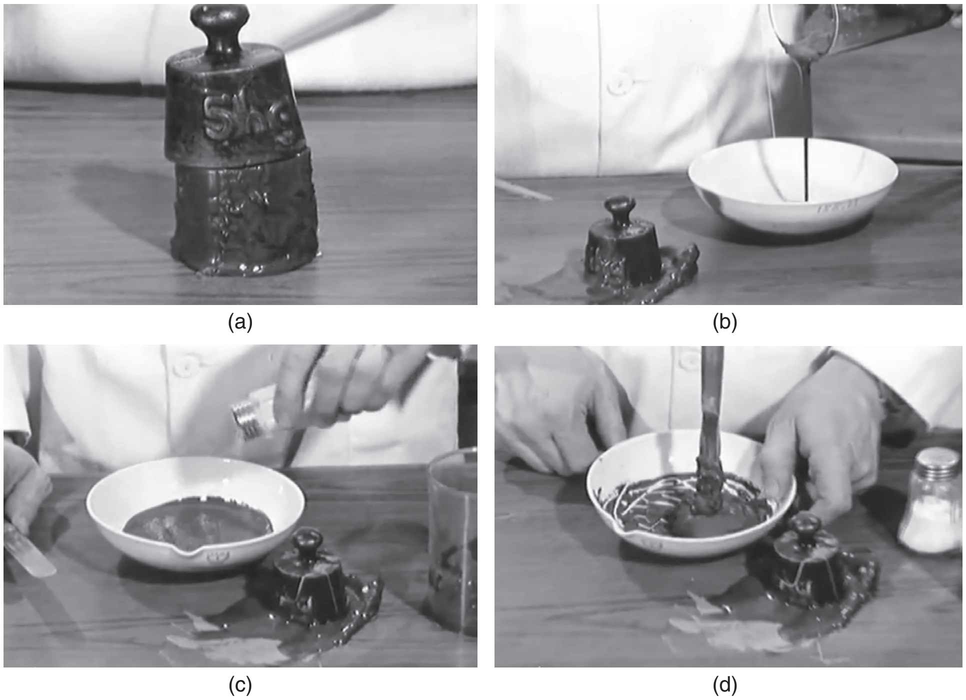

Figure 13.10 Rissa landslide clay, Norway 1978: (a) Intact quick clay. (b) Remolded clay. (c) Adding salt to remolded clay. (d) Remolded clay strengthened by salt. (Pictures/images are from the film *The Quick Clay Landslide in Rissa, Norway*, Made by Norwegian Geotechnical Institute [NGI])

Figure 13.11 Sand boil. (Courtesy of USGS.)

slopes. Some of the questions that must be answered in design are:

1. What is the water stress (pore pressure) at any point in the soil mass?
2. What is the amount of water flowing through the soil?
3. What is the uplift force exerted on a structure buried in the soil?
4. What is the factor of safety against a quick sand condition developing under or near a structure?
5. What happens when the hydraulic conductivity is different in two directions?
6. What happens if the soil is layered rather than being uniform?

To answer these questions, it is necessary to solve the flow problem. The following assumptions are made:

1. The soil is uniform with the same hydraulic conductivity in all directions.
2. The soil is saturated with water and the water is in compression.
3. The water is incompressible.
4. Darcy's law governs the water flow through the soil.
5. The flow is in two directions only (x and z, but no flow in the y direction).
6. The flow is independent of time: steady-state flow.

To solve the problem with these assumptions, we use the problem-solving method outlined in section 11.4.5:

1. Zoom in at the element level. We select an element of soil that has an elementary area dx dz with a dimension of 1 in the y direction (Figure 13.12).
2. Considering the element of Figure 13.12, the water velocity is v_x in the x direction and v_z in the z direction when it enters the element and $v_x + \frac{\partial v_x}{\partial x}dx$ and $v_z + \frac{\partial v_z}{\partial z}dz$ when it exits the element. It is assumed that the water does not flow in the y direction because of the plane strain assumption.

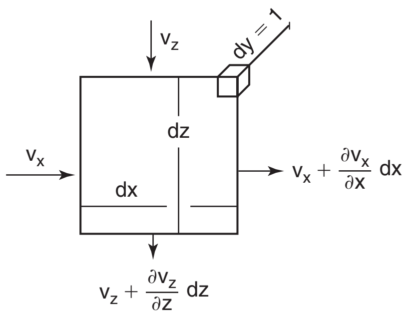

Figure 13.12 Element of soil in the flow mass.

3. The fundamental equation in this case is the conservation of mass equation, expressing that the flow of water in the element is equal to the flow of water out of the element. Use is made of the flow equation (Q = vA):

$$v_x dz \times 1 + v_z dx \times 1 = \left(v_x + \frac{\partial v_x}{\partial x}dx\right) dz \times 1 + \left(v_z + \frac{\partial v_z}{\partial z}dz\right) dx \times 1 \quad (13.28)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \quad (13.29)$$

4. The constitutive equation describes how fast the water flows through the soil (Darcy's law):

$$v_x = ki_x = k\frac{dh_t}{dx} \quad \textit{and} \quad v_z = ki_z = k\frac{dh_t}{dz} \quad (13.30)$$

5. The governing differential equation is obtained by combining Eq. 13.29 and the first derivative of the terms in Eq. 13.30:

$$\frac{d^2h_t}{dx^2} + \frac{d^2h_t}{dz^2} = 0 \quad (13.31)$$

6. This form of differential equation is called the *Laplace equation* and the solutions are called *harmonic functions*.
7. The complexity of the solution is brought about by the complexity of the boundary conditions. These boundary conditions describe the flow conditions at the geometric boundaries of the flow. Equation 13.31 can be solved mathematically or graphically. The most common solution to this problem is a graphical solution called the flow net.

The preceding solution is based on the assumption that the soil mass is uniform, meaning that the hydraulic conductivity k_h is equal to the vertical hydraulic conductivity k_v. If k_h is very different from k_v, then Eq. 13.31 becomes:

$$k_h\frac{d^2h_t}{dx^2} + k_v\frac{d^2h_t}{dz^2} = 0 \quad (13.32)$$

13.2.12 Drawing a Flow Net for Homogeneous Soil

A *flow net* is a graphical solution to the governing differential equation for a steady-state flow of water through a pervious soil. The flow net is made of two sets of lines: the flow lines and the equipotential lines. The flow lines describe the path of the water molecules. The equipotentials are lines of equal potential or total head h_t. A *flow channel* is the soil conduit between two consecutive flow lines (Figure 13.13*d*). A *flow field* is the geometric shape between two consecutive flow lines and two consecutive equipotentials (Figure 13.13*d*). A flow net is a map of the total head h_t giving the value of h_t for any point in the flow net with an x and z coordinate.

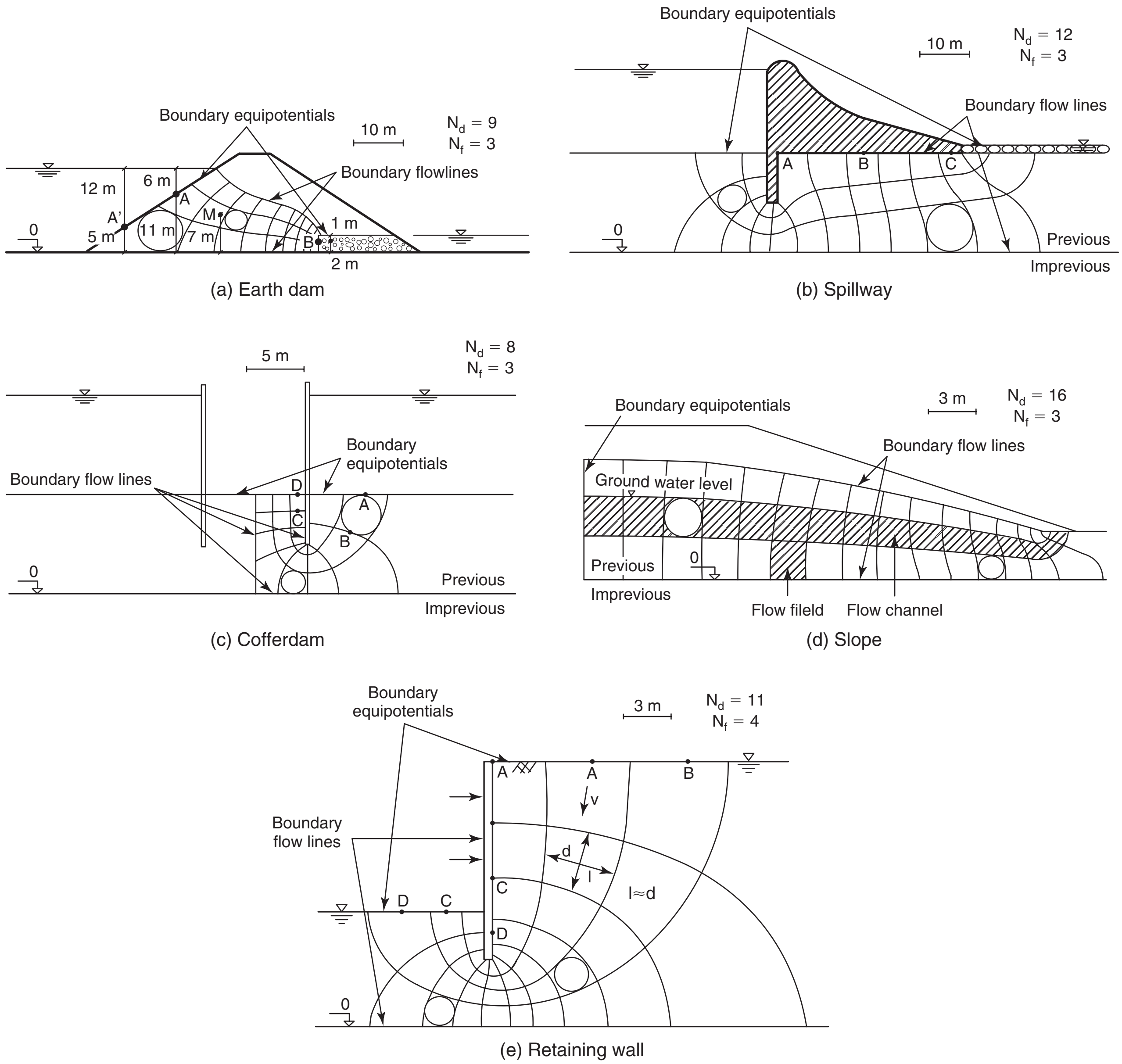

Figure 13.13 Sample flow nets.

Figure 13.13 gives examples of flow nets. To draw a flow net, proceed as follows:

1. Draw the cross section of the flow problem to scale.
2. Draw the boundary flow lines. These are flow lines such that the total flow through the flow net occurs between these flow lines.
3. Draw the boundary equipotential lines (also called boundary equipotentials). These lines define the total head at the beginning of the flow net $h_{t(beg)}$ and the total head at the end of the flow net $h_{t(end)}$.
4. Draw an additional two to three flow lines between the boundary flow lines.
5. Draw the equipotentials such that they cross the flow lines at a right angle; this is the condition expressed by the governing differential equation. Choose the equipotentials in such a way that the flow fields are squares. This is the case if the flow fields are very small, but at the scale at which most flow nets are drawn, this condition should be replaced by: choose the equipotentials in such a way that a circle can be inscribed in each flow field and is tangent to all four sides.
6. Adjust the flow lines and the equipotentials until the conditions of step 5 (perpendicularity and circle inscribed) are satisfied. This step usually takes the longest time and requires some experience.

13.2.13 Properties of a Flow Net for Homogeneous Soil

Two of the most important properties of the flow net are:

1. The potential drop or drop in total head Δh_t from one equipotential to the next is the same across any of the equipotentials.
2. The flow is the same through any of the flow channels.

One of the first things to do when working with a flow net is to choose the datum: the horizontal line where the elevation is equal to zero. The chosen datum is usually located at the location of the bottom impervious layer, so that all elevations will be positive, but theoretically it can be set at any level within the diagram. The total head at the beginning of the flow net $h_{t(beg)}$ and at the end of the flow net $h_{t(end)}$ can be calculated as follows: The elevation heads $h_{e(beg)}$ and $h_{e(end)}$ are simply measured to scale on the diagram (e.g., $h_{e(beg)(\mathrm{A})} = 11$ m and $h_{e(end)(\mathrm{B})} = 2$ m on Figure 13.13*a*). The pressure heads $h_{p(beg)}$ and $h_{p(end)}$ are readily available from the free water body connected to the beginning and the end of the flow net (e.g., $h_{p(beg)(A)} = 6$ m and $h_{p(end)(B)} = 1$ m on Figure 13.13*a*). The value of $h_{t(beg)}$ and $h_{t(end)}$ can then be found (e.g., $h_{t(beg)(\mathrm{A})} = 11 + 6 = 17$ m and $h_{t(end)(B)} = 2 + 1 = 3$ m on Figure 13.13*a*). Note that $h_{t(beg)}$ and $h_{t(end)}$ are constant on the equipotential; indeed, if you try a different point on that equipotential you will find the same value (e.g., $h_{t(beg)(A')} = 5 + 12 = 17$ m on Figure 13.13*a*).

The number of flow channels is N_f and the number of equipotential drops is N_d (Figure 13.13*d*). One of the properties of the flow net is that the drop of total head Δh_t across two consecutive equipotential lines is the same for all flow fields; it is given by:

$$\Delta h_t = \frac{(h_{t(beg)} - h_{t(end)})}{N_d} \tag{13.33}$$

Once $h_{t(beg)}$ and $h_{t(end)}$ are known, the total head $h_{t(M)}$ can be found for any point in the flow net by interpolation:

$$h_{t(M)} = h_{t(beg)} - n_d \frac{(h_{t(beg)} - h_{t(end)})}{N_d} \tag{13.34}$$

where n_d is the number of drops to go from the beginning of the flow net to the point considered. For example, consider point M in the flow net of Figure 13.13*a*. The elevation head at point M is 7 m and the total head is:

$$h_{t(M)} = 17 - 1.7\frac{17-3}{9} = 14.35 \text{ m} \tag{13.35}$$

The hydraulic gradient i in any flow field is:

$$i = \frac{\Delta h_t}{l} \tag{13.36}$$

where Δh_t is the loss of total head in the flow field (a constant for all flow fields) and l is the flow path across the flow field (varies from one flow field to another). Therefore, the hydraulic gradient varies throughout the flow net and is inversely proportional to the length of the flow field. Because the velocity is linearly related to the hydraulic gradient through the hydraulic conductivity of the soil, the water velocity increases when the size of the flow field decreases. To illustrate, imagine that you can ride a water molecule in Figure 13.13*e* and that you have the choice between molecule at point A and molecule at point B; which molecule should you choose if you wish to win the flow net race? Molecule A is your best bet because it is associated with smaller flow fields, higher gradients, and therefore higher velocities. Note that although A will burn the same amount of energy (total head) as B to travel through the flow net, it has a shorter trip to travel and can afford to step on the gas and have a higher energy consumption per meter travelled (hydraulic gradient). Thus, molecule A will get to C before molecule B gets to D. You can check this race by using a colored dye in the water at the upstream face of the flow net in a laboratory experiment.

13.2.14 Calculations Associated with Flow Nets

Quantity of Flow

How much water will go through a flow net per unit time? This is important for a dam, for example. The flow q through one flow channel is:

$$q = kiA = kid \times 1 \tag{13.37}$$

where A is the cross section through which the water flows. A is equal to d × 1 where d is the width of the flow field perpendicular to the flow and 1 represents the unit width of the flow net perpendicular to the flow net (Figure 13.13*e*). Because the flow field is a square, its width d is equal to its length l over which the total head drops by Δh_t. Therefore, Eq. 13.37 can be rewritten as:

$$q = k\frac{\Delta h_t}{l} l \times 1 = k\Delta h_t \tag{13.38}$$

Furthermore, the flow through one flow field is the same as the flow through all flow fields in one flow channel, because no water crosses over into other flow channels. If we go back to the car traffic analogy, in water flow no one changes lanes; everybody stays in their own lane, but the highway is totally congested (saturated soil), so all lanes carry the same traffic flow. The flow q is also the same in all flow channels. Because there are N_f flow channels, the flow per unit width of flow net is $N_f \times q$. If the length perpendicular to the plane of the flow net over which the flow takes place is L, the total flow Q through or under the structure is $N_f \times q \times L$. Using Eq. 13.38 leads to the formula for the total flow:

$$Q = k\frac{N_f}{N_d} L(h_{t(beg)} - h_{t(end)}) \tag{13.39}$$

Water Stress

What is the water stress u_w at any point in the flow net? This is important for calculating the effective stress at any point or even calculating the uplift force on a buried solid structure like a spillway. The procedure is as follows:

1. Calculate the total head at the point M considered by using Eq. 13.34.
2. Subtract the elevation head to obtain the pressure head $h_{p(M)}$.
3. Get the water stress by:

$$u_w = \frac{h_{p(M)}}{\gamma_w} \tag{13.40}$$

where γ_w is the unit weight of water. Note that u_w includes the hydrostatic stress, because the pressure head is the level at which the water would rise in a pipe connected to M. If there were no flow, that height would correspond to the hydrostatic height.

Uplift Force on a Buried Structure

What is the upward force generated by the water pressure under a solid structure buried in the flow net? This force F_{up} is the result of the water pressure acting on the bottom of the structure, as in the case of the spillway shown in Figure 13.13*b*. The procedure for determining upward force is as follows:

1. Select a few points in the flow net along the bottom of the structure. A minimum of four points is recommended (A, B, C, D).
2. Calculate the water stress u_A, u_B, u_C, and u_D at A, B, C, and D.
3. Calculate the average water stress u_{av} under the structure.
4. Calculate the uplift force as:

$$F_{up} = u_{av}BL \tag{13.41}$$

where B is the width of the structure (dam) and L is the length.

Exit Gradient

What is the highest hydraulic gradient on the exit face of the flow net? This is called the *exit gradient*. Because the drop in total head is the same for any two consecutive equipotentials, the highest hydraulic gradient on the exit face (exit gradient) is associated with the smallest flow field on the exit face (Eq. 13.36). Because the exit face is often a horizontal plane, the exit gradient i_{exit} is compared to the critical hydraulic gradient i_{crit} to avoid a critical condition (quick sand). A large factor of safety F is usually used:

$$i_{exit} = \frac{i_{crit}}{F} \quad or \quad \frac{\Delta h_t}{l} = \frac{1}{F}\left(\frac{\gamma_{sat}}{\gamma_w} - 1\right) \tag{13.42}$$

If the required factor of safety is not satisfied, the flow must be modified to satisfy the required factor of safety. This can be done by using cutoff walls, deepening barriers, or preventing the flow altogether.

Heave and Critical Block

A calculation similar to the comparison between the exit gradient and the critical gradient can be performed in the case of retaining structures, as shown in Figure 13.14. In this case a block of soil is identified where the flow is upward and could create a quick sand condition. The free body considered is the soil particles only, with the water as an external body. In this case, the weight of the soil particles is the buoyant weight and the seepage force is an external force acting vertically and upward on the soil particles. The factor of safety against heave is the ratio of the buoyant weight of the particles divided by the seepage force:

$$F = \frac{W'}{S} = \frac{(\gamma_{sat} - \gamma_w)D \times \dfrac{D}{2}}{\left(\dfrac{h_{t(A)} - h_{t(B)}}{D}\right)\gamma_w D \times \dfrac{D}{2}} = \frac{(\gamma_{sat} - \gamma_w)D}{(h_{t(A)} - h_{t(B)})\gamma_w} \tag{13.43}$$

13.2.15 Flow Net for Hydraulically Anisotropic Soil

The procedure described in section 13.2.12 is used for a soil that has the same hydraulic conductivity in the vertical and horizontal directions. If the horizontal hydraulic conductivity k_h is significantly different from the vertical hydraulic conductivity k_v, then the flow net is distorted because Eq. 13.32 applies and the flow lines and equipotential lines no longer intersect at right angles. A change of variable can bring Eq. 13.32 back to Eq. 13.31:

$$x = \alpha x' \quad and \quad z = z' \tag{13.44}$$

$$\frac{k_h}{\alpha^2 k_v}\frac{d^2h_t}{dx'^2} + \frac{d^2h_t}{dz'^2} = 0 \tag{13.45}$$

which shows that if:

$$\alpha = \sqrt{k_h/k_v} \tag{13.46}$$

then the flow net can be drawn for the anisotropic soil with a proper scale transformation in the x direction. The steps to

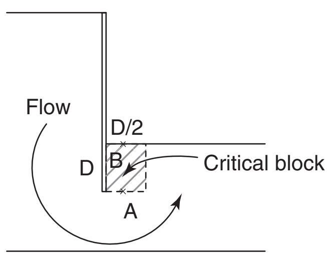

Figure 13.14 Heave of critical block.

draw the flow cross section to scale in the section 13.2.12 procedure are modified as follows:

1. Select a scale for the vertical z direction.
2. Select a scale for the horizontal x direction such that the horizontal scale is equal to $\sqrt{k_h/k_v}$ times the vertical scale.
3. Draw the flow net according to the procedure of section 13.2 12.
4. If needed, use that flow net to go back to the untransformed set of axes and draw the resulting flow net in that space; understand that in that space, the flow lines and equipotential lines will not intersect at right angles and the flow fields will not be squares.

For example, if the hydraulic conductivity k_h was 4×10^{-8} m/s in the horizontal direction and k_v was 10^{-8} m/s in the vertical direction, the transformed cross section of the scaled diagram would be shrunk by a factor of 2 in the horizontal direction while it was kept unchanged in the vertical direction (Figure 13.15). Then the flow net would be drawn as if the soil were uniform. Note that the quantity of flow equation would become:

$$Q = \sqrt{k_h k_v}\frac{N_f}{N_d}L(h_{t(beg)} - h_{t(end)}) \tag{13.47}$$

13.2.16 Flow and Flow Net for Layered Soils

If the flow goes from layer 1 with a hydraulic conductivity k_1 to a layer 2 with a hydraulic conductivity k_2, then the flow lines and the equipotential lines are deflected. If the approach angle of the flow line coming from layer 1 to the interface is θ_1 (Figure 13.16), the angle with which that flow line leaves the interface into layer 2 is θ_2 and is different from θ_1. The angles are linked by the following equations:

$$\frac{k_1}{k_2} = \frac{\tan\theta_1}{\tan\theta_2} \tag{13.48}$$

If the flow is either parallel to the interface or perpendicular to the interface, then the flow lines are not deflected and an equivalent hydraulic conductivity k_e can be found.

In the case where the flow is parallel to the interface of two layers (Figure 13.17), the hydraulic gradient across two equipotentials is the same in both layers:

$$i_1 = i_2 = \frac{h_{tB} - h_{tA}}{L} = i_e \tag{13.49}$$

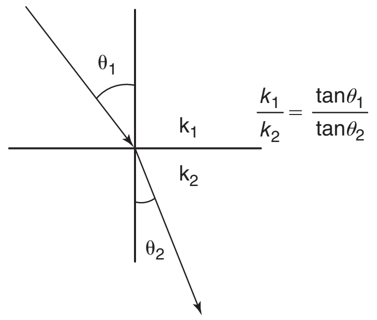

Figure 13.16 Flow line crossing layer interface.

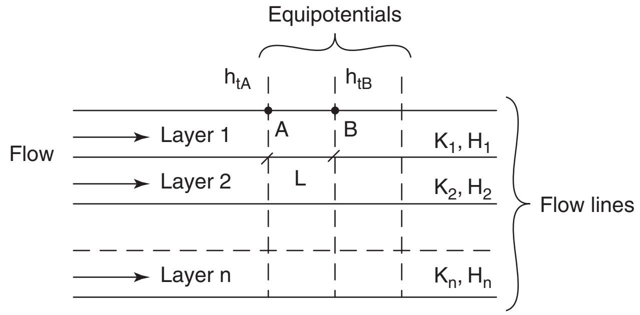

Figure 13.17 Flow parallel to parallel layers.

The flow is additive:

$$q = q_1 + q_2 = v_1 H_1 \times 1 + v_2 H_2 \times 1 \tag{13.50}$$

or

$$k_e i_e (H_1 + H_2) = k_1 i_1 H_1 + k_2 i_2 H_2 \tag{13.51}$$

Therefore:

$$k_e = \frac{k_1 H_1 + k_2 H_2}{H_1 + H_2} \tag{13.52}$$

This result can be generalized for n layers.

$$k_e = \frac{\sum_{i=1}^{n} k_i H_i}{\sum_{i=1}^{n} H_i} \tag{13.53}$$

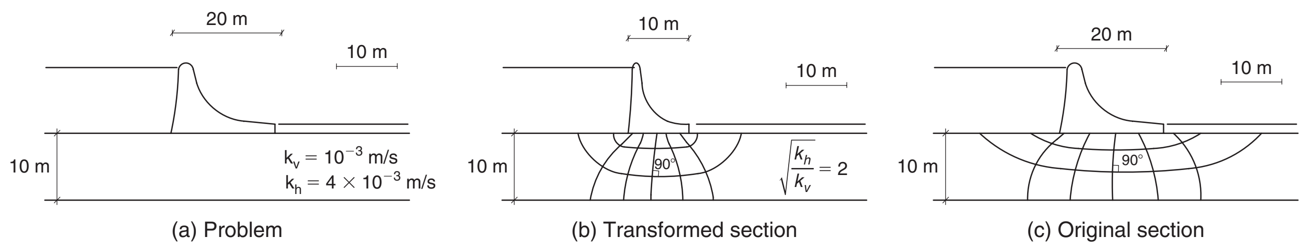

Figure 13.15 Flow net for anisotropic soil.

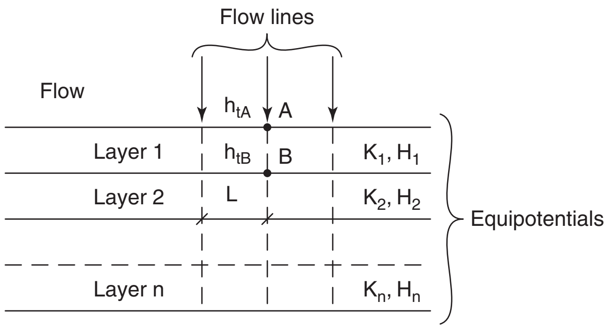

Figure 13.18 Flow perpendicular to parallel layers.

In the case where the flow is perpendicular to two layers (Figure 13.18), the flow across two equipotentials is the same in both layers:

$$q = q_1 = q_2 \quad and$$
$$q = k_e i_e L \times 1 = k_1 i_1 L \times 1 = k_2 i_2 L \times 1 \tag{13.54}$$

The loss of total head, however, is additive:

$$\Delta h_t = \Delta h_{t1} + \Delta h_{t2} \tag{13.55}$$

but

$$i_e = \frac{\Delta h_t}{H_1 + H_2} \quad and \quad i_1 = \frac{\Delta h_{t1}}{H_1} \quad and \quad i_2 = \frac{\Delta h_{t2}}{H_2} \tag{13.56}$$

Therefore,

$$\frac{H_1 + H_2}{k_e L} q = \frac{H_1}{k_1 L} q_1 + \frac{H_2}{k_2 L} q_2 \tag{13.57}$$

and

$$k_e = \frac{H_1 + H_2}{\dfrac{H_1}{k_1} + \dfrac{H_2}{k_2}} \tag{13.58}$$

This result can be generalized for n layers:

$$k_e = \frac{\sum_{i=1}^{n} H_i}{\sum_{i=1}^{n} \dfrac{H_i}{k_i}} \tag{13.59}$$

13.3 FLOW OF WATER AND AIR IN UNSATURATED SOIL

13.3.1 Hydraulic Conductivity for Water and for Air

There is a need to distinguish between the soil hydraulic conductivity for water k_w and the soil hydraulic conductivity for air k_a; k_w expresses how fast water travels through the water phase and k_a expresses how fast air travels through the air phase. One of the fundamental observations regarding water flow in unsaturated soils is that the hydraulic conductivity of water decreases compared to saturated soils. You might think that as the soil becomes drier, there is more room for the water to go through, but that is not the case, because air occupies the voids and cannot get out of the way unless you chase it out somehow. Instead, the water has to go through what is left of water in the soil. The air may be thought of as blocking the flow like particles do. In this sense, the area blocking the flow has increased from the area associated with the particles (solid phase) in the case of a saturated flow to the area associated with the particles plus the air phase in the case of unsaturated flow. This means that the cross-sectional area decreases and that the drag force increases because the water is bound more tightly to the particles. Thus, the hydraulic conductivity decreases because:

1. Cross-sectional area of water flow decreases
2. Tortuosity increases
3. Drag forces increase

Note also that the difference between the discharge velocity v and the seepage velocity v_s (actual water molecule velocity) is increased. Recall that for saturated flow we had:
Saturated flow:

$$vA_t = v_s A_v \quad or \quad v = nv_s \tag{13.60}$$

where A_t is the total cross-sectional area of the soil where the water flows, A_v is the area of voids where the water can flow in the saturated case, and n is the porosity.

In the case of unsaturated flow, the equation becomes:
Unsaturated flow:

$$vA_t = v_s A_w \quad or \quad v = Snv_s \tag{13.61}$$

This gives an indication that the degree of saturation will have a significant influence on the hydraulic conductivity of the water. When the degree of saturation decreases, so does the water content, and the water tension increases in the soil. Therefore, the higher the water tension, the lower the hydraulic conductivity of water is for an unsaturated soil. The reverse is observed for the hydraulic conductivity of air, which increases as the water content decreases and the water tension increases.

Figure 13.19 illustrates what happens to the water hydraulic conductivity k_w when a coarse-grained soil and a fine-grained soil desaturate, which means that they are subjected to higher and higher water tension. At low water tension (~ saturated), the coarse-grained soil has a much higher k_w (e.g., 10^{-4} m/s) than the fine-grained soil (e.g., 10^{-7} m/s). Indeed, the water travels a lot faster through a saturated coarse-grained soil than through a saturated fine-grained soil. The water tension corresponding to the air entry value for the coarse-grained soil (e.g., $u_{wae} = 10$ kPa) is much lower than for the fine-grained soil (e.g., $u_{wae} = 1000$ kPa) because it is a lot easier

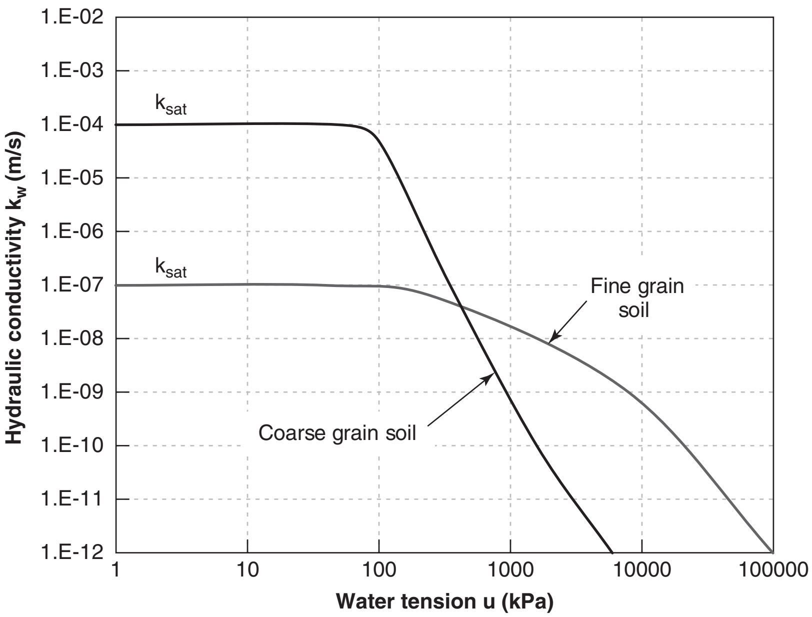

Figure 13.19 Constant head permeameter test results for unsaturated soils.

for the air to enter the large pores than the small pores. The crossover occurs because it does not take much of an increase in water tension to desaturate the pores of a coarse-grained soil compared to what is required to drive the water out of the pores of a fine-grained soil. The fine-grained soil retains more water longer while the water travels through its pores, compared to the coarse-grained soil. As a result, at high water tension (e.g., 5000 kPa), the k_w value of the coarse-grained soil (e.g., 10^{-11} m/s) can be much lower than the k_w value of the fine-grained soil (e.g., 10^{-8} m/s).

When the soil is dry, the value of air hydraulic conductivity k_a is maximum and equal to $k_{a(dry)}$. This trend is contrary to the trend for the hydraulic conductivity of water k_w. Indeed, k_w decreases when the soil gets drier; it is maximum when the soil is saturated and equal to $k_{w(sat)}$. Both hydraulic conductivities are often presented as normalized values as follows:

$$k_w = k_{rw} k_{w(sat)} \tag{13.62}$$

$$k_a = k_{ra} k_{a(dry)} \tag{13.63}$$

Figure 13.20 shows an example of the combined variation of both normalized hydraulic conductivity values k_{rw} and k_{ra} as a function of the degree of saturation S. Note that there is a limiting degree of saturation S_w (0.3 on Figure 13.20) where the water is no longer mobile (bound water) and at the same time a limiting degree of saturation S_a (0.85 on Figure 13.20) where the air is no longer mobile (occluded air). These two stages correspond to the residual stages.

A number of models have been proposed to describe the variation of the hydraulic conductivity as a function of water content, or water tension, or degree of saturation. Among the most popular are:

Averjanov (1950)

$$k_w = k_{ws} S_e^n \tag{13.64}$$

LaLiberte and Correy (1966)

$$k_w = k_{ws} \left(\frac{u_{wae}}{u_w} \right)^n \tag{13.65}$$

Gardner (1958)

$$k_w = \frac{k_{ws}}{1 + a u_w^n} \tag{13.66}$$

where k_w is the hydraulic conductivity to water, k_{ws} is the hydraulic conductivity to water when the soil is saturated, S_e is the effective degree of saturation, u_w is the water tension, u_{wae} is the water tension at the air entry, and a and n are fitting parameters.

The hydraulic conductivity of unsaturated soils depends on many factors, including the degree of saturation, the void ratio, the shape and roughness of the particles, the structure of the soil skeleton, and the fluid properties (viscosity and unit weight). To separate the influence of the fluid from that of the soil skeleton on the hydraulic conductivity k, the intrinsic hydraulic conductivity, or simply permeability K, is used:

$$K = k \frac{\mu_f}{\gamma_f} \tag{13.67}$$

where μ_f is the dynamic viscosity of the fluid, and γ_f is the unit weight of the fluid. At 20°C and one atmosphere,

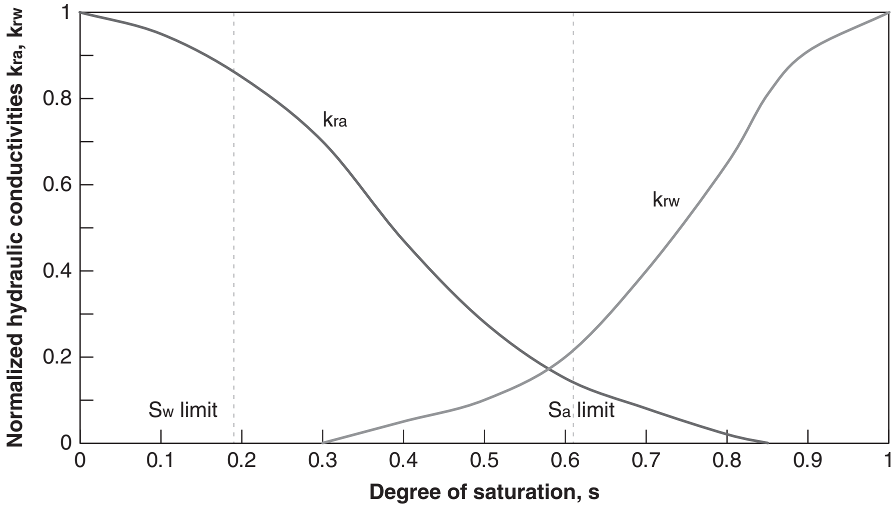

Figure 13.20 Relative hydraulic conductivity of water and air as a function of degree of saturation.

the value of μ_f for water is 10^{-6} kPa.s and γ_f for water is 9.79 kN/m^3 At 20°C and one atmosphere, the value of μ_f for air is 1.82 x 10^{-5} Pa.s and γ_f for air is 11.8 N/m^3. Note that K is in units of m^2.

13.3.2 One-Dimensional Flow

Let's now study the problem of a soil layer sitting in the sun and drying from the inside out or sitting in the rain and getting wet from the outside in. The question here is: What is the change in water stress as a function of time and depth in the soil layer? The assumptions are:

1. The soil is uniform with the same hydraulic conductivity in all directions.
2. The soil is unsaturated, with both water and air present
3. The water is incompressible, but the volume of water in a given soil element can change with time.
4. The soil is not changing volume.
5. Darcy's law governs the water flow through the soil.
6. The change in elevation head is negligible compared to the change in pressure head (water tension).
7. The flow is in one direction only (flow in the z direction, but no flow in the x or y directions).
8. The flow is transient (dependent on time).

The problem is solved (after Aubeny and Lytton 2004) by following a process similar to the case of the saturated soil and as described in the problem-solving method of section 11.4.5.

1. Zoom in at the element level. We select an element of soil that has an elementary area dx dz with a dimension of 1 in the y direction (Figure 13.21).
2. Considering the element of Figure 13.21, the water velocity is v_z in the z direction when it enters the element and $v_z + dv_z$ when it exits the element. It is assumed that the water does not flow in the x and y directions (one-dimensional flow).

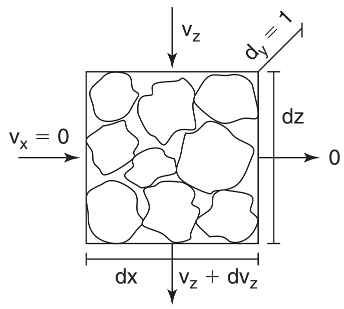

Figure 13.21 Element of unsaturated soil in the flow mass.

3. The fundamental equation in this case is the conservation of mass equation, expressing that the flow of water coming out of the element minus the flow of water entering the element is equal to the time rate of change of the volume of water in the element:

$$q(out) - q(in) = \frac{dV_w}{dt} \tag{13.68}$$

$$\frac{dv_z}{dz}\, dx\, dy\, dz = \frac{dV_w}{dt} \quad or \quad \frac{dv_z}{dz} = \frac{dV_w}{V\, dt} \tag{13.69}$$

4. The first constitutive equation links the water velocity v_z to the hydraulic conductivity k and the hydraulic gradient i_z (Darcy's law):

$$v_z = ki_z = k\frac{dh_t}{dz} \tag{13.70}$$

where h_t is the total head, which is equal to the pressure head h_p plus the elevation head h_e. It is assumed that the change in elevation head is negligible compared to the change in pressure head (water tension):

$$\boxed{dh_e \ll dh_p}$$

Then Eq. 13.70 can be written:

$$v_z = k\frac{dh_p}{dz} \quad (13.71)$$

5. The second constitutive equation describes how the hydraulic conductivity varies with the water stress (suction). Here the Laliberte and Corey model is used, with an exponent n equal to 1, which is not unreasonable but is particularly convenient mathematically:

$$k = \frac{k_0 h_{p0}}{h_p} \quad (13.72)$$

Therefore,

$$v_z = k_0 h_{p0}\frac{dh_p/h_p}{dz} = k_0 h_{p0}\frac{d(\log_e h_p)}{dz} \quad (13.73)$$

6. The third constitutive equation describes how the water content varies with the water stress (suction):

$$dw = c\, d(\log_{10} h_p) = cd(0.434 \log_e h_p) \quad (13.74)$$

where c is the slope of the soil water retention curve (SWRC).

7. Then Eqs. 13.69 to 13.74 are regrouped while making use of a change of variable and phase relationships to obtain the governing differential equation:

Change of variable

$$u = \log_{10} h_p = 0.434 \log_e h_p \quad (13.75)$$

Phase relationship

$$V_w = \frac{W_w}{\gamma_w} = \frac{W_w}{\gamma_w}\frac{W_s}{W_s} = w\frac{\gamma_d}{\gamma_w}V \quad (13.76)$$

$$\frac{dv_z}{dz} = 2.3k_0 h_{p0}\frac{d^2u}{dz^2} = \frac{dw}{dt}\frac{\gamma_d}{\gamma_w} = c\frac{du}{dt}\frac{\gamma_d}{\gamma_w} \quad (13.77)$$

$$\frac{du}{dt} = \frac{2.3k_0 h_{p0}\gamma_w}{c\gamma_d}\,\frac{d^2u}{dz^2} \quad (13.78)$$

With the diffusivity

$$\alpha = \frac{2.3k_0 h_{p0}\gamma_w}{c\gamma_d} \quad (13.79)$$

Then

$$\frac{du}{dt} = \alpha\frac{d^2u}{dz^2} \quad (13.80)$$

Note that this is the same equation as the one-dimensional consolidation equation for a saturated soil, except that u is the $\log_{10}$ of the water stress expressed as a height of water rather than being the excess water stress itself.

8. Now the space and time boundary conditions must be addressed. Let's assume that the entire semi-infinite layer is at an initial water tension stress u_{wi} at a time equal to zero and that the top of the layer is suddenly subjected to a wet condition that permanently imposes a much lower water tension $u_{w(z=0)}$ at that boundary (ground surface).
9. We define the degree of wetting U at any depth z as:

$$U = \frac{u_{w(z,t)} - u_{wi}}{u_{w(z=0,t)} - u_{wi}} \quad (13.81)$$

where $u_{w(z,t)}$ is the water tension at a depth z and a time t, u_{wi} is the initial water tension throughout the layer, and $u_{w(z=0,t)}$ is the wetting value of the water tension permanently applied at the ground surface. We also define the time factor T as:

$$T = \alpha\frac{t}{z^2} \quad (13.82)$$

The solution to the governing differential equation is given in this case by the complementary error function, as follows:

$$U = erfc\left(\frac{1}{2\sqrt{T}}\right) \quad (13.83)$$

Figure 13.22 shows that function.

The average degree of wetting represents the ratio of the area under the water tension (in excess of the wetting value) vs. depth profile at time t over the same area at time t = 0 (Figure 13.22). Another way to present the results is shown in Figure 13.23, where the evolution of the water tension toward the imposed wet value at the surface is shown as a function

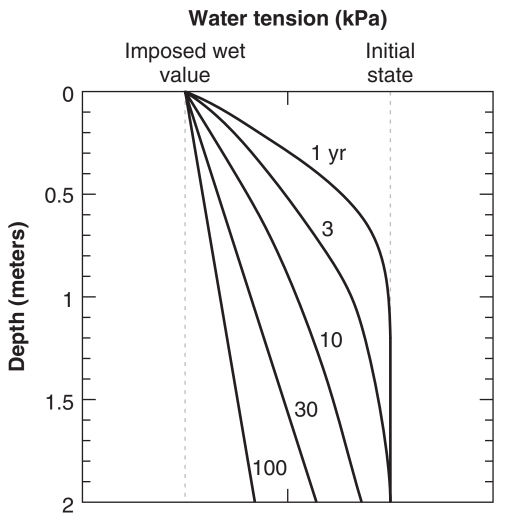

Figure 13.22 Decrease in water tension with depth in an initially high-water-tension soil layer subjected to wetting at the ground surface. (After Aubeny and Lytton 2004)

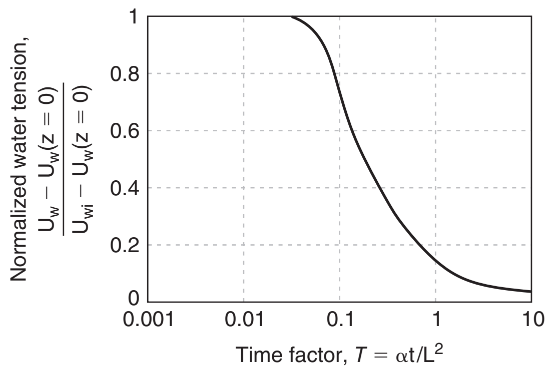

Figure 13.23 Decrease in water tension with time at a chosen depth. (After Aubeny and Lytton 2004)

of time for a given depth z. In that figure the water tension is normalized as:

$$U' = \frac{u_{w(z,t)} - u_{w(z=0,t)}}{u_{wi} - u_{w(z=0,t)}} \tag{13.84}$$

13.3.3 Three-Dimensional Water Flow

The previous example was of a one-dimensional flow of water perpendicular to the surface of an unsaturated soil. Let's look now at what happens in a three-dimensional case. The assumptions are:

1. The flow is in three directions (x, y, z).
2. The flow is transient (dependent on time).
3. The soil is uniform with different hydraulic conductivities in the x, y, and z directions.
4. The soil is unsaturated, with both water and air present.
5. The water is incompressible, but the volume of water in a given soil element can change with time.
6. The soil is not changing volume.
7. Darcy's law governs the water flow through the soil.
8. The hydraulic conductivity k is a function of the water tension u_w.

The problem is solved by following a process similar to the case of the saturated soil and as described in the problem-solving method of section 11.4.5:

1. Zoom in at the element level. We select an element of soil that has an elementary volume V = dx dy dz.
2. The water velocity is v_x in the x direction when it enters the element and $v_x + \frac{\partial v_x}{\partial x}dx$ when it exits the element. The same applies in the y and the z directions (three-dimensional flow).
3. The fundamental equation in this case is the conservation of mass equation, expressing that the flow of water coming out of the element minus the flow of water entering the element is equal to the time rate of change of the volume of water in the element:

$$q(out) - q(in) = \frac{dV_w}{dt} \tag{13.85}$$

Or, using Q = vA on all faces of the element,

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{dV_w}{V\ dt} \tag{13.86}$$

4. The first constitutive equation links the water velocity v_x to the hydraulic conductivity k_{wx} and the hydraulic gradient i_x (Darcy's law):

$$v_x = k_{wx} i_x = k_{wx}\frac{dh_t}{dx} \tag{13.87}$$

where h_t is the total head equal to the pressure head h_p plus the elevation head h_e, which is the coordinate z. Then Eq. 13.87 can be written:

$$v_x = k_{wx}\frac{d(h_p + z)}{dx} = k_{wx}\frac{dh_p}{dx} \quad and \quad v_y = k_{wy}\frac{dh_p}{dy}$$

but

$$v_z = k_{wz}\frac{d(h_p + z)}{dz} = k_{wz}\left(\frac{dh_p}{dz} + 1\right) \tag{13.88}$$

5. The second constitutive equation describes how the hydraulic conductivity varies with the water stress (suction). Here several models could be selected, but in general suffice to say that:

$$k_w = k_w(h_p) \tag{13.89}$$

6. The third constitutive equation describes how the water content varies with the water stress (suction). If a linear semilog model is accepted for this part of the soil water retention curve, then:

$$dw = c\ d(\log_{10} u_w) \tag{13.90}$$

where c is the slope of the SWRC. Using the phase relationship of Eq. 13.76, the following expression is obtained for the term on the right-hand side of Eq. 13.86:

$$\frac{dV_w}{V\ dt} = \frac{\gamma_d}{\gamma_w}\frac{dw}{dt} \tag{13.91}$$

7. Then Eqs. 13.86 to 13.91 are regrouped to obtain the governing differential equation. In this process the pressure head is transformed into the water tension by using:

$$u_w = \gamma_w h_p \tag{13.92}$$

The governing differential equation is then

$$\frac{\partial}{\partial x}\left(k_{wx}\left(u_w\right)\frac{\partial u_w}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{wy}\left(u_w\right)\frac{\partial u_w}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_{wz}\left(u_w\right)\left(\frac{\partial u_w}{\partial z}+1\right)\right)=c\gamma_d\frac{\partial(\log_{10}\mathrm{u}_w)}{\partial t} \tag{13.93}$$

8. Now the space and time boundary conditions must be addressed and the differential equation can be solved. The solution is the function that describes the water tension u_w for any location (x, y, z) and any time t.

13.3.4 Three-Dimensional Air Flow

Section 13.3.3 gave the steps for generating the governing differential equation for the water flow in an unsaturated soil. Now we need to repeat the process for the flow of air in the unsaturated soil. As shown in Figure 13.17, the water hydraulic conductivity decreases when the water tension increases. At the same time, the air hydraulic conductivity increases when the water tension increases, because more of the void space is occupied by air. For the flow of water we made a distinction between the discharge velocity v and the seepage velocity v_s, which were related as follows in the case of unsaturated flow:

$$v_wA_t=v_{ws}A_w \quad or \quad v_w=Snv_{ws} \tag{13.94}$$

For the air flow the relationship becomes:

$$v_aA_t=v_{as}A_a \quad or \quad v_a=(1-S)nv_{as} \tag{13.95}$$

For a degree of saturation of about 85% or more, the air is usually occluded, so we cannot talk about air flow; rather, we address diffusion of the air mass through the water in the soil voids. For a degree of saturation of 20% or less, the air hydraulic conductivity approaches its maximum value.

Blight (1971) and Fredlund and Rahardjo (1993) showed that Darcy's law is applicable to the flow of air in soils, and related the air velocity to the gradient of the total head in the air by:

$$v_{ax}=k_a\frac{\partial h_{ta}}{\partial x} \tag{13.96}$$

The pressure head h_{pa} is related to the air pressure u_a by:

$$u_a=\gamma_a h_{pa} \tag{13.97}$$

Note that the unit weight of air γ_a varies with temperature and pressure (Table 13.3); at 20°C and at atmospheric pressure, γ_a is 11.8 N/m^3.

To develop the solution for the three-dimensional flow of air in soil, we follow the same procedure as in the case of water:

Table 13.3 Unit Weight of Air

Temperature (°C)	Pressure (atm)	Unit weight of air (N/m^3)	Mass density (kg/m^3)
−10	1	13.17	1.341
0	1	12.67	1.316
10	1	12.23	1.247
20	1	11.81	1.204
30	1	11.43	1.164
40	1	11.05	1.127

1. Zoom in at the element level. We select an element of soil that has an elementary volume $V=dx\,dy\,dz$.

2. The air velocity is v_{ax} in the x direction when it enters the element and $v_{ax}+\frac{\partial v_{ax}}{\partial x}dx$ when it exits the element. The same applies in the y and the z directions (three-dimensional air flow).

3. Writing the conservation of mass principle for air poses a problem a bit different from the conservation of mass for water. Because water is considered incompressible at usual pressures, conservation of mass is also conservation of volume, which is what we used for the fundamental equation for water. However, because air is very compressible, the mass of air in a given volume could be very different depending on temperature and pressure. We write that the air mass exiting the element minus the air mass entering the element is equal to the change in air mass corresponding to a decrease in volume of the soil pores of the element over time:

$$\frac{\partial(\rho_a v_{ax})}{\partial x}+\frac{\partial(\rho_a v_{ay})}{\partial y}+\frac{\partial(\rho_a v_{az})}{\partial z}=\frac{1}{V}\frac{\partial(\rho_a V_a)}{\partial t} \tag{13.98}$$

where ρ_a is the mass density of air, V_a is the volume of air in the element, and V is the volume of the soil element.

4. The first constitutive equation is Darcy's law for air flow, which is written for each direction:

$$v_{ax}=\frac{k_{ax}}{\gamma_a}\frac{\partial u_a}{\partial x} \quad \text{and} \quad v_{ay}=\frac{k_{ay}}{\gamma_a}\frac{\partial u_a}{\partial y}$$
$$\text{but} \quad v_{az}=\frac{k_{az}}{\gamma_a}\left(\frac{\partial u_a}{\partial z}+\rho_a g\right) \tag{13.99}$$

where u_a is the air pressure. The second term in the z direction indicates the influence of gravity on the air flow. Note that:

$$\frac{k_{ax}}{\gamma_a}=\frac{K_{ax}}{\mu_a} \tag{13.100}$$

where K is the intrinsic hydraulic conductivity and μ_a is the viscosity of air.

5. The term on the right-hand side of Eq. 13.98 can be rewritten as follows by using phase relationships:

$$\frac{1}{V}\frac{\partial(\rho_a V_a)}{\partial t} = \frac{\partial(\rho_a n(1-S))}{\partial t} \tag{13.101}$$

6. The ideal gas law is an additional constitutive equation describing how the air density ρ_a varies with the air pressure u_a and temperature T:

$$\rho_a = \frac{\omega_a}{RT}u_a \tag{13.102}$$

where ω_a is the molecular weight of air (kg/mol), R is the universal gas constant (J/mol.K), and T is the absolute temperature (K).

7. Then Eqs. 13.98 to 13.102 are regrouped to obtain the governing differential equation for the flow of air through a soil:

$$\frac{\partial\left(\frac{\omega_a}{RT}u_a\frac{K_{ax}}{\mu_a}\frac{\partial u_a}{\partial x}\right)}{\partial x} + \frac{\partial\left(\frac{\omega_a}{RT}u_a\frac{K_{ay}}{\mu_a}\frac{\partial u_a}{\partial y}\right)}{\partial y}$$
$$+\frac{\partial\left(\frac{\omega_a}{RT}u_a\frac{K_{az}}{\mu_a}\left(\frac{\partial u_a}{\partial z}+\frac{\omega_a}{RT}u_a g\right)\right)}{\partial z}$$
$$=\frac{\partial\left(\frac{\omega_a}{RT}u_a n\,(1-S)\right)}{\partial t} \tag{13.103}$$

8. Now the space and time boundary conditions must be addressed and the differential equation can be solved. The solution is the function that describes the air pressure u_a for any location (x, y, z) and any time t.

PROBLEMS

13.1 A soil has a porosity of 40%.
   a. The soil is saturated and water flows through the soil. Calculate the ratio between the discharge velocity v and the seepage velocity v_s.
   b. The soil is unsaturated, with a degree of saturation equal to 35%. Calculate the ratio between the discharge velocity v and the seepage velocity v_s.

13.2 Water is flowing through three soil layers as shown in Figure 13.1s. The cross section is a square with sides of 100 mm. The hydraulic conductivity of each soil layer is given in Table 13.1s.
   a. What is the equivalent hydraulic conductivity of the three layers?
   b. Determine the flow rate exiting the system.
   c. Determine the elevation head diagram, the total head diagram, and the pressure head diagram from point A to point D.

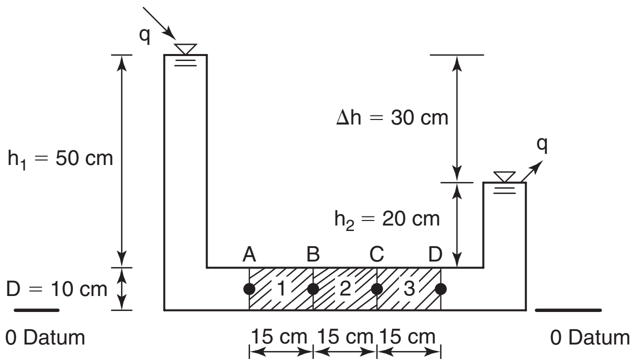

Figure 13.1s Three-layer permeameter.

Table 13.1s Hydraulic Conductivity of Three Soil Layers

Soil	Hydraulic conductivity (m/s)
1	1×10^{-4}
2	5×10^{-6}
3	3×10^{-5}

13.3 Water is flowing through three soil layers as shown in Figure 13.2s. The cross-section is a square with sides of 100 mm. The hydraulic conductivity of each soil layer is given in Table 13.1s.

a. What is the equivalent hydraulic conductivity of the three layers?
b. Determine the flow rate exiting the system.
c. Determine the elevation head diagram, the total head diagram, and the pressure head diagram from point A to point D.

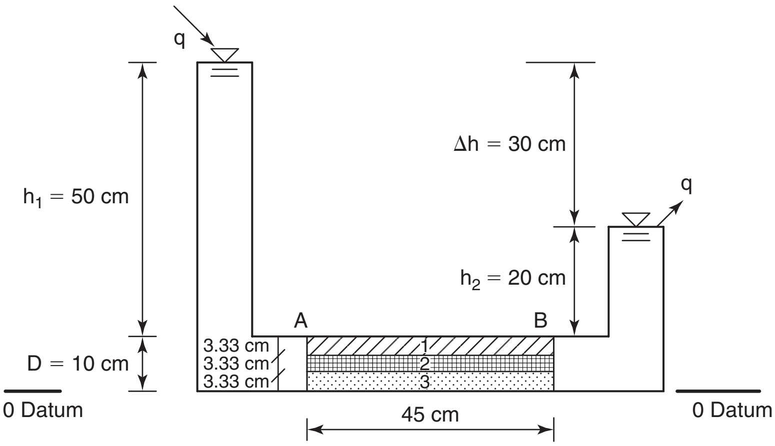

Figure 13.2s Three-layer permeameter.

13.4 Use the uplift force equation (Eq. 13.41) to calculate the uplift force on a ship and demonstrate Archimedes' principle.

13.5 Referring to Figure 13.3s, calculate the following quantities:

a. Elevation head, total head, and pressure head at point M on Figure 13.3s*a*
b. The quantity of water seeping through the dam of Figure 13.3s*a* per day
c. Elevation head, total head, and pressure head at points A, B, and C on Figure 13.3s*b*
d. The uplift force on the bottom of the concrete dam in Figure 13.3s*b*
e. The hydraulic gradient between points A and B and then between points C and D on Figure 1.3s*c*
f. The factor of safety against a quick condition on the exit face of the cofferdam (Figure 13.3s*c*) by the exit gradient method and the critical block method
g. The seepage force applied by the water on a soil grain on the exit face of the slope if the grain has a volume of 1 mm^3 (Figure 13.3s*d*)
h. The water pressure distribution behind the retaining wall of Figure 13.3s*e*

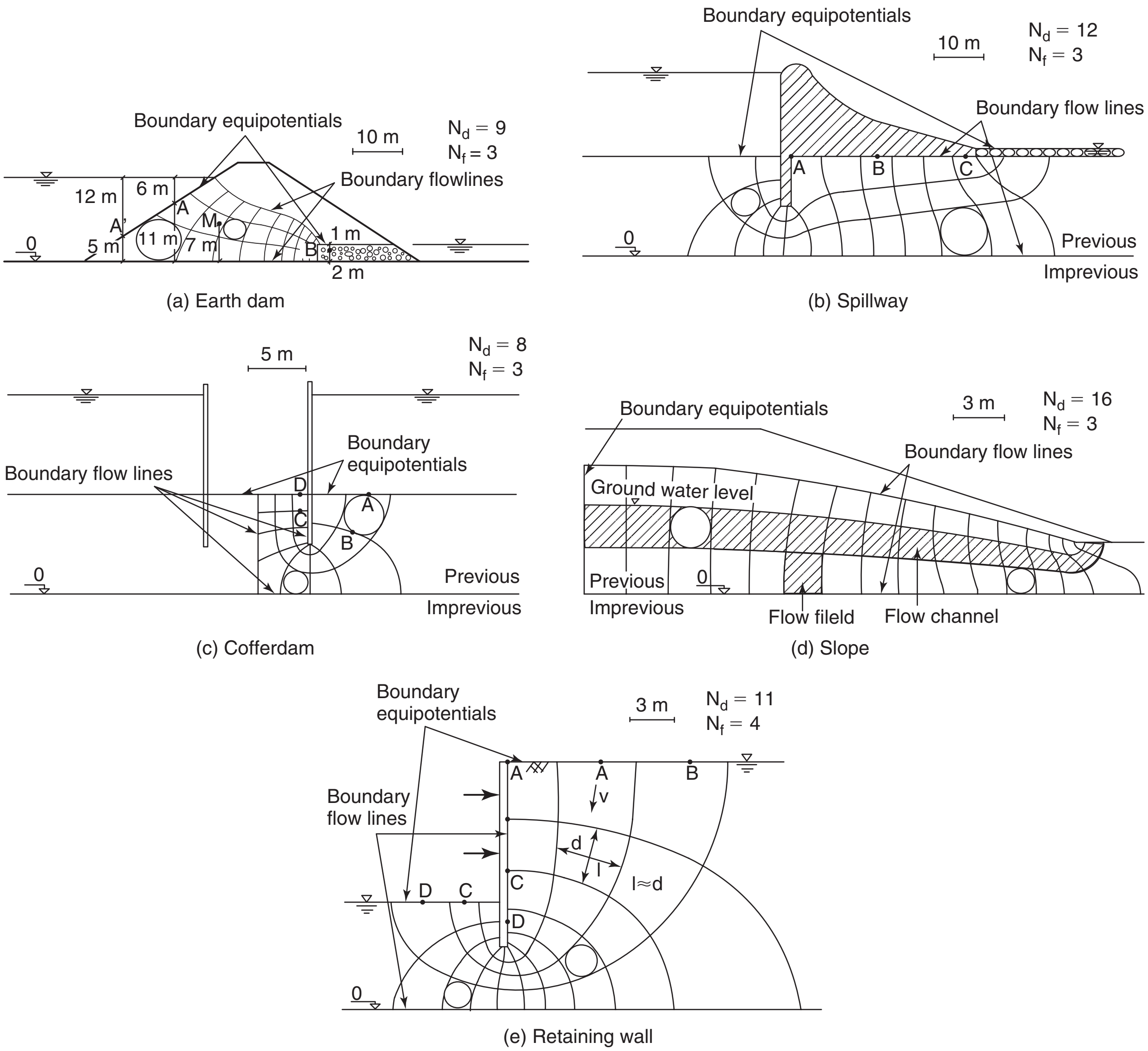

Figure 13.3s Flow nets.

13.6 A tube is filled with a relatively dry soil at a water tension corresponding to a pressure head h_0 and a volumetric water content θ_0. Water is made available at one end of the tube (Figure 13.4s). As a result, a wetting front is created and advances from left to right on the figure. The wetted soil has a water tension corresponding to a pressure head of h_1 and a volumetric water content of θ_1. How fast will the wetting front propagate across the sample?

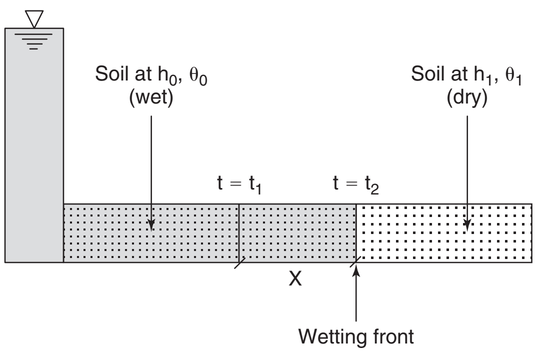

Figure 13.4s Horizontal wetting front propagation.

13.7 A soil sample has a saturated hydraulic conductivity k_{sat} equal to 10^{-8} m/s. Estimate the hydraulic conductivity of the sample if it dries to a degree of saturation equal to 0.9 and then 0.5. Use Figure 13.5s to estimate k_{unsat}.

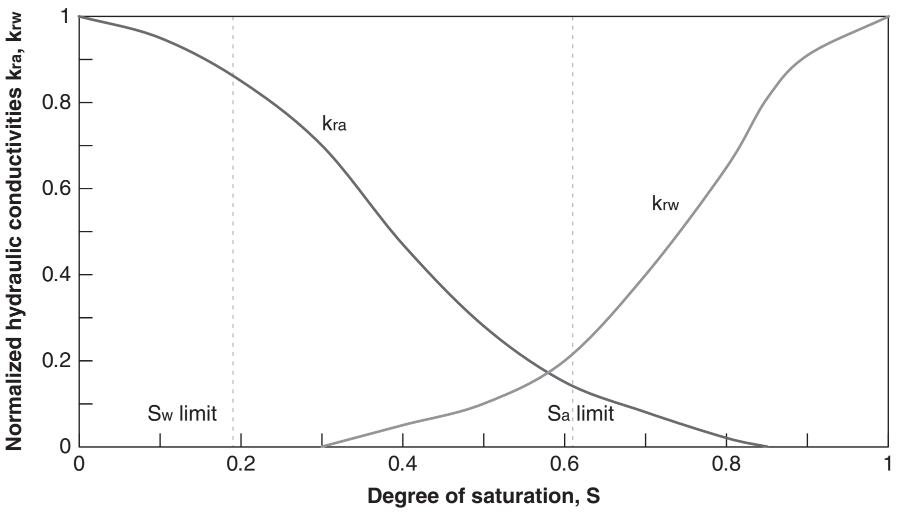

Figure 13.5s Relative hydraulic conductivity of water and air as a function of degree of saturation.

Problems and Solutions

Problem 13.1

A soil has a porosity of 40%.

a. The soil is saturated and water flows through the soil. Calculate the ratio between the discharge velocity v and the seepage velocity v_s.
b. The soil is unsaturated, with a degree of saturation equal to 35%. Calculate the ratio between the discharge velocity v and the seepage velocity v_s.

Solution 13.1

a. For a saturated soil, the relation between seepage and discharge velocity is:

$$v = nv_s$$

Given $n = 40\%$, we have:

$$v = 0.4v_s$$

$$\frac{v}{v_s} = 0.4$$

b. For an unsaturated soil, the relation between seepage and discharge velocity is:

$$v_w = Snv_{ws}$$

Given n = 40% and $S = 35\%$, then:

$$\frac{v_w}{v_{ws}} = Sn = 0.35 \times 0.4 = 0.14$$

Problem 13.2

Water is flowing through 3 soil layers as shown in Figure 13.1s. The cross section is a square with sides of 100 mm. The hydraulic conductivity of each soil layer is given in Table 13.1s.

a. What is the equivalent hydraulic conductivity of the three layers?
b. Determine the flow rate exiting the system.
c. Determine the elevation head diagram, the total head diagram, and the pressure head diagram from point A to point D.

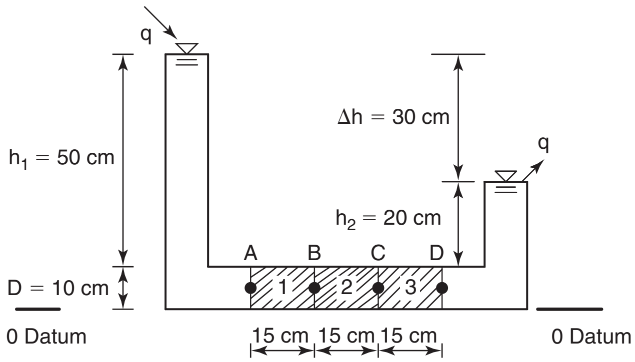

Figure 13.1s Three-layer permeameter.

Table 13.1s Hydraulic Conductivity of Three Soil Layers

Soil	Hydraulic conductivity (m/s)
1	1×10^{-4}
2	5×10^{-6}
3	3×10^{-5}

Solution 13.2

This is a problem about water flowing perpendicularly to the soil layers. The equivalent hydraulic conductivity is calculated as:

$$k_{eq} = \frac{\Sigma H_i}{\Sigma \dfrac{H_i}{k_i}} = \frac{0.45\ \text{m}}{\dfrac{0.15\ \text{m}}{k_1} + \dfrac{0.15\ \text{m}}{k_2} + \dfrac{0.15\ \text{m}}{k_3}} = \frac{0.45\ \text{m}}{\dfrac{0.15\ \text{m}}{1 \times 10^{-4}\ \text{m/s}} + \dfrac{0.15\ \text{m}}{5 \times 10^{-6}\ \text{m/s}} + \dfrac{0.15\ \text{m}}{3 \times 10^{-5}\ \text{m/s}}} = 1.23 \times 10^{-5}\ \text{m/s}$$

The flow rate is calculated as:

$$q = vA = k_{eq} iA = k_{eq} \frac{\Delta h}{L} D^2 = 1.23 \times 10^{-5}\ \text{m/s} \times \frac{0.3\ \text{m}}{0.45\ \text{m}} \times 0.1^2\ \text{m}^2 = 8.2 \times 10^{-8}\ \text{m}^3/\text{s}$$

The zero datum is set at the bottom of the soil layer. The elevation heads at points A, B, C, and D are the same:

$$h_{eA} = h_{eB} = h_{eC} = h_{eD} = 5\ \text{cm}$$

The total head at point A is calculated as:

$$h_{tA} = h_e + h_p = 5\ \text{cm} + 55\ \text{cm} = 60\ \text{cm}$$

The total head at point B is calculated as:

$$q = vA = k_1 iA = k_1 \frac{\Delta h_{AB}}{L_{AB}} D^2 = 1 \times 10^{-4}\,\text{m/s} \times \frac{\Delta h_{AB}}{0.15\,\text{m}} \times 0.1^2\,\text{m}^2 = 8.2 \times 10^{-8}\,\text{m}^3/\text{s}$$

$$\Delta h_{AB} = 0.012\,\text{m} = 1.2\,\text{cm}$$

Because $h_{tA} = 60$ cm from the previous calculation, the total head at point B is:

$$h_{tB} = h_{tA} - \Delta h_{AB} = 60\ \text{cm} - 1.2\ \text{cm} = 58.8\ \text{cm}$$

The total head at point C is calculated as:

$$q = vA = k_2 iA = k_2 \frac{\Delta h_{BC}}{L_{BC}} D^2 = 5 \times 10^{-6}\ \text{m/s} \times \frac{\Delta h_{BC}}{0.15\ \text{m}} \times 0.1^2\ \text{m}^2 = 8.2 \times 10^{-8}\ \text{m}^3/\text{s}$$

$$\Delta h_{BC} = 0.246\ \text{m} = 24.6\ \text{cm}$$

Because $h_{tB} = 58.8$ cm from the previous calculation, the total head at point C is:

$$h_{tC} = h_{tB} - \Delta h_{BC} = 58.8\ \text{cm} - 24.6\ \text{cm} = 34.2\ \text{cm}$$

The total head at point D is calculated as:

$$q = vA = k_3 iA = k_3 \frac{\Delta h_{CD}}{L_{CD}} D^2 = 3 \times 10^{-5}\ \text{m/s} \times \frac{\Delta h_{CD}}{0.15\ \text{m}} \times 0.1^2\ \text{m}^2 = 8.2 \times 10^{-8}\ \text{m}^3/\text{s}$$

$$\Delta h_{CD} = 0.041\ \text{m} = 4.1\ \text{cm}$$

Because $h_{tc} = 34.2$ cm from the previous calculation, the total head at point D is:

$$h_{tD} = h_{tC} - \Delta h_{CD} = 34.2\ \text{cm} - 4.1\ \text{cm} = 30\ \text{cm}$$

The pressure head can be obtained by subtracting the elevation head from the total head at each point:

$$h_{pA} = h_{tA} - h_{eA} = 60\ \text{cm} - 5\ \text{cm} = 55\ \text{cm}$$
$$h_{pB} = h_{tB} - h_{eB} = 58.8\ \text{cm} - 5\ \text{cm} = 53.8\ \text{cm}$$
$$h_{pC} = h_{tC} - h_{eC} = 34.2\ \text{cm} - 5\ \text{cm} = 29.2\ \text{cm}$$
$$h_{pD} = h_{tD} - h_{eD} = 30\ \text{cm} - 5\ \text{cm} = 25\ \text{cm}$$

The elevation head diagram, the total head diagram, and the pressure head diagram from point A to point D are plotted in Figure 13.6s.

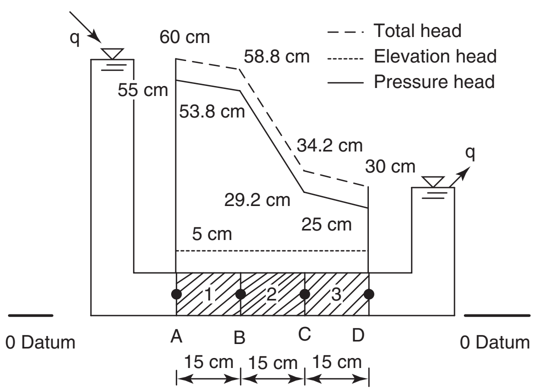

Figure 13.6s Elevation head, total head, and pressure head diagram from point A to point D.

Problem 13.3

Water is flowing through three soil layers as shown in Figure 13.2s. The cross section is a square with sides of 100 mm. The hydraulic conductivity of each soil layer is given in Table 13.1s.

a. Determine the flow rate exiting the system.
b. Determine the elevation head diagram, the total head diagram, and the pressure head diagram from point A to point D.
c. What is the equivalent hydraulic conductivity of the three layers?

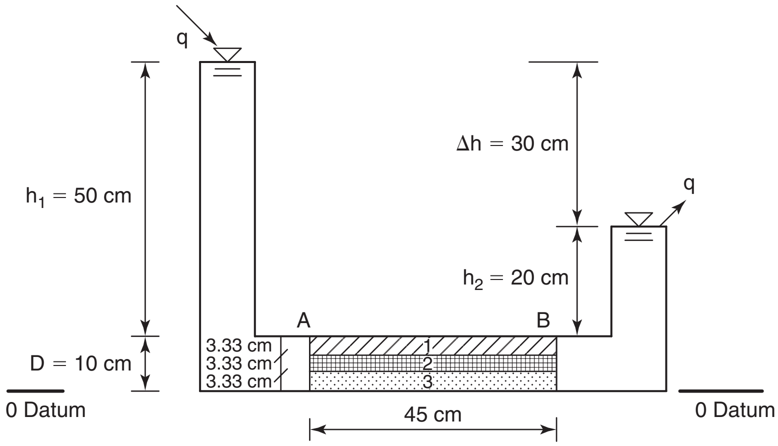

Figure 13.2s Three-layer permeameter.

Solution 13.3

a. This is a problem about water flowing in the direction of the soil layer boundaries. The equivalent hydraulic conductivity is calculated as:

$$k_{eq} = \frac{k_1 + k_2 + k_3}{3} = \frac{(100 + 5 + 30)}{3} \times 10^{-6} = 4.5 \times 10^{-5} \text{ m/s}$$

b. The flow rate is calculated as:

$$q = vA = k_{eq} iA = k_{eq} \frac{\Delta h}{L} D^2 = 4.5 \times 10^{-5} \text{ m/s} \times \frac{0.3 \text{ m}}{0.45 \text{ m}} \times 0.1^2 \text{ m}^2 = 3 \times 10^{-7} \text{ m}^3/\text{s}$$

c. The zero datum is set at the bottom of the soil layer. The total head at point A is calculated as:

$$h_{tA} = h_e + h_p = 5 \text{ cm} + 55 \text{ cm} = 60 \text{ cm}$$

The total head at point B is calculated as:

$$q = vA = k_{eq} iA = k_3 \frac{\Delta h_{AB}}{L_{AB}} D^2 = 4.5 \times 10^{-5} \text{ m/s} \times \frac{\Delta h_{AB}}{0.45 \text{ m}} \times 0.1^2 \text{ m}^2 = 3 \times 10^{-7} \text{ m}^3/\text{s}$$

$$\Delta h_{AB} = 0.3 \text{ m} = 30 \text{ cm}$$

$$h_{tB} = h_{tA} - \Delta h_{AB} = 60 \text{ cm} - 30 \text{ cm} = 30 \text{ cm}$$

The total head at point B can be found by simply using right side condition.

The elevation head diagram, the total head diagram, and the pressure head diagram from point A to point B are plotted in Figure 13.7s.

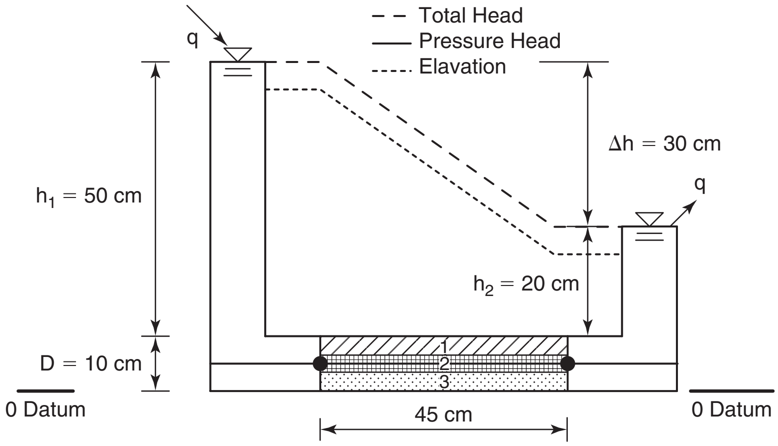

Figure 13.7s Total head, pressure head, and elevation diagram.

Problem 13.4

Use the uplift force equation (Eq. 13.41) to calculate the uplift force on a ship and demonstrate Archimedes' principle.

Solution 13.4

Consider that the height of the ship under water is Z and the total height of the ship is H. The width of the ship is B and the length is L.

Uplift water pressure on the bottom of the ship is:

$$u_{av} = \gamma_w Z$$

Uplift force would be:

$$F_{up} = u_{av} BL \Rightarrow F_{up} = \gamma_w ZBL$$

This is Archimedes' principle, which states that the upward buoyant force exerted on a body immersed in a fluid is equal to the weight of the fluid displaced by the body.

Problem 13.5

Referring to Figure 13.3s, calculate the following quantities:

a. Elevation head, total head, and pressure head at point M on Figure 13.3s*a*
b. The quantity of water seeping through the dam of Figure 13.3s*a* per day
c. Elevation head, total head, and pressure head at points A, B, and C on Figure 13.3s*b*
d. The uplift force on the bottom of the concrete dam in Figure 13.3s*b*
e. The hydraulic gradient between points A and B and then between points C and D on Figure 13.3s*c*
f. The factor of safety against a quick condition on the exit face of the cofferdam (Figure 13.3s*c*) by the exit gradient method and the critical block method
g. The seepage force applied by the water on a soil grain on the exit face of the slope if the grain has a volume of 1 mm^3 (Figure 13.3s*d*)
h. The water pressure distribution behind the retaining wall of Figure 13.3s*e*

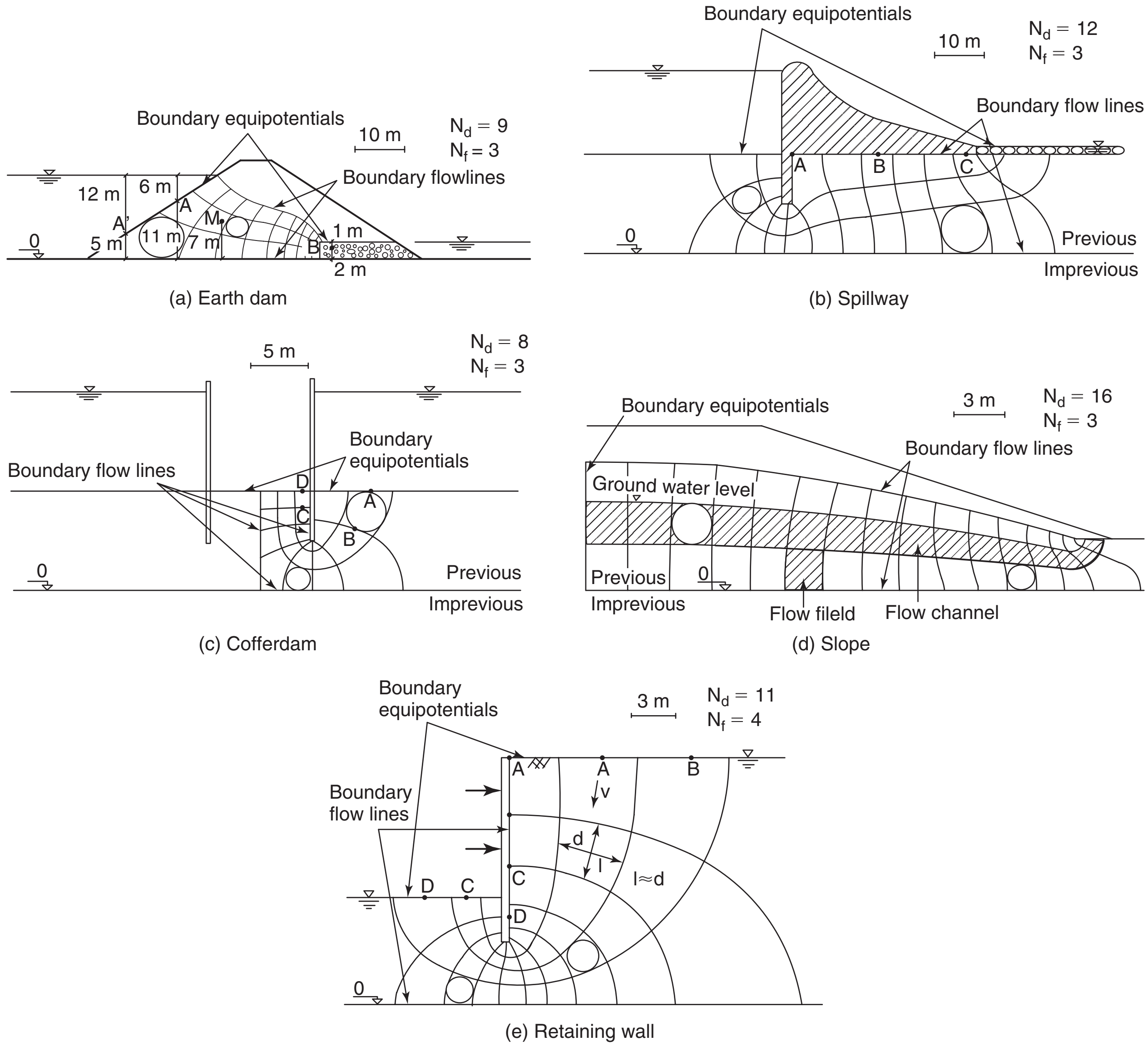

Figure 13.3s Flow nets.

Solution 13.5

a. Elevation head, total head, and pressure head at point M on Figure 13.3s*a*

The head loss between each equipotential line:

$$\Delta h = \frac{\Delta H}{N_d} = \frac{17-3}{9} = 1.56 \text{ m}$$

The total head:

$$h_t = \Delta H_t - \Delta h \times (N_d)_B = 17 - 1.56 \times 2 = 13.9 \text{ m}$$

The elevation head at M:

$$h_e = 7 \text{ m}$$

The pressure head at M:

$$(h_p)_B = h_t - h_e = 13.9 - 7 = 6.9 \text{ m}$$

The pore water pressure at M:

$$u_B = h_p \times \gamma_w = 6.9 \times 9.81 = 67.69 \text{ kN/m}^2 = 67.69 \text{ kPa}$$

b. The quantity of water seeping through the dam of Figure 13.3s*a* per day if the hydraulic conductivity of the soil is 10^{-8} m/s:

$$q = k\Delta H \frac{N_f}{N_d} = 10^{-8} \times (17 - 3) \times \frac{3}{9} = 4.67 \times 10^{-8} \text{ m}^3/\text{ sec /m}$$

$$Q = q \times L = \left(k\Delta H \frac{N_f}{N_d}\right) L = 4.67 \times 10^{-8} \times 1 \text{ m} = 4.67 \times 10^{-8} \text{ m}^3/\text{ sec}$$

$$Q_{1day} = 4.67 \times 10^{-8} \times (24 \times 60 \times 60) = 0.00403 \text{ m}^3$$

c. Elevation head, total head, and pressure head at points A, B, and C on Figure 13.3s*b*
The head loss between each equipotential line:

$$\Delta h = \frac{\Delta H}{N_d} = \frac{16 - 2}{12} = 1.17 \text{ m}$$

The total head at each point:

$$(h_t)_A = \Delta H_t - \Delta h \times (N_d)_B = (20 + 16) - 1.17 \times 5.5 = 29.57 \text{ m}$$
$$(h_t)_B = \Delta H_t - \Delta h \times (N_d)_B = (20 + 16) - 1.17 \times 7.5 = 27.23 \text{ m}$$
$$(h_t)_C = \Delta H_t - \Delta h \times (N_d)_B = (20 + 16) - 1.17 \times 10.5 = 23.72 \text{ m}$$

The elevation head at A, B, and C:

$$h_e = 20 \text{ m}$$

The pressure head at each point:

$$(h_p)_A = h_t - h_e = 29.57 - 20 = 9.57 \text{ m}$$
$$(h_p)_B = h_t - h_e = 27.23 - 20 = 7.23 \text{ m}$$
$$(h_p)_B = h_t - h_e = 23.72 - 20 = 3.72 \text{ m}$$

The pore water pressure at each point:

$$u_A = h_p \times \gamma_w = 9.57 \times 9.81 = 93.88 \text{ kN/m}^2 = 93.88 \text{ kPa}$$
$$u_B = h_p \times \gamma_w = 7.23 \times 9.81 = 70.93 \text{ kN/m}^2 = 70.93 \text{ kPa}$$
$$u_C = h_p \times \gamma_w = 3.72 \times 9.81 = 36.49 \text{ kN/m}^2 = 36.49 \text{ kPa}$$

d. The uplift force on the bottom of the concrete dam in Figure 13.3s*b*
The pressure head at each point:

$$h_p = h_t - h_e = \Delta H - N_d \times \Delta h - h_e$$
$$u = h_p \times 9.81$$
$$u_A = 9.57 \times 9.81 = 93.88 \text{ kPa}$$
$$u_B = 7.23 \times 9.81 = 70.93 \text{ kPa}$$
$$u_C = 3.72 \times 9.81 = 36.49 \text{ kPa}$$
$$h_e \text{ at end of wall} = (20 - 10) \text{ m}$$
$$(h_p)_{end\,of\,wall} = (16 + 20) - 4 \times 1.17 - (20 - 10) = 21.32 \text{ m}$$
$$\therefore u_{end\,of\,wall} = 21.32 \times 9.81 = 209.15 \text{ kPa}$$

The resultant uplift force is:

$$\begin{aligned} F_{up} &= (A_{wall} \times u_{wall}) + A_{AB} \times u_{AB} + A_{BC} \times u_{BC} \\ &= ((2 \times 1) \times 209.15) + (16 \times 1) \times \frac{93.88 + 70.93}{2} + (16 \times 1) \times \frac{70.93 + 36.49}{2} \\ &= 2596.14 \text{ kN/m} \end{aligned}$$

e. The hydraulic gradient between points A and B and then between points C and D on Figure 13.3s*c*
   The head loss between each equipotential line:

$$\Delta h = \frac{\Delta H}{N_d} = \frac{8}{8} = 1 \text{ m}$$

The total head:

$$\begin{aligned} (h_t)_A &= \Delta H_t - \Delta h \times (N_d)_A = (8+8) - 1 \times 0 = 16 \text{ m} \\ (h_t)_B &= \Delta H_t - \Delta h \times (N_d)_B = (8+8) - 1 \times 1 = 15 \text{ m} \\ (h_t)_C &= \Delta H_t - \Delta h \times (N_d)_C = (8+8) - 1 \times 7 = 9 \text{ m} \\ (h_t)_D &= \Delta H_t - \Delta h \times (N_d)_D = (8+8) - 1 \times 8 = 8 \text{ m} \end{aligned}$$

Hydraulic gradient between points A and B:

$$i = \frac{\Delta h}{L_{AB}} = \frac{16 - 15}{3.5} = 0.29$$

Hydraulic gradient between points C and D:

$$i = \frac{\Delta h}{L_{CD}} = \frac{9 - 8}{1.5} = 0.67$$

f. The factor of safety against a quick condition on the exit face of the cofferdam (Figure 13.3s*c*) by the exit gradient method and the critical block method

Exit Gradient

The critical hydraulic gradient (i_c):

$$i_c = \frac{\gamma_{sat} - \gamma_w}{\gamma_w} = \frac{20 - 9.81}{9.81} = 1.04$$

Determine the factor of safety against quicksand and explain it

$$FOS = \frac{i_c}{i} = \frac{1.04}{0.67} = 1.55 < 4$$

Heave and Critical Block

Taking flow field CD as the critical block

$$FOS = \frac{W'}{S} = \frac{(\gamma_{sat} - \gamma_w)d}{(h_{t(C)} - h_{t(D)})\gamma_w} = \frac{(20 - 9.81) \times 1.5}{(9 - 8) \times 9.81} = 1.6$$

g. The seepage force applied by the water on a soil grain on the exit face of the slope if the grain has a volume of 1 mm^3 (Figure 13.3s*d*) The drop in total head in the last flow field on the exit face is

$$\Delta h_t = \frac{4}{16} = 0.25 \text{ m}$$

The corresponding hydraulic gradient and the force on the soil grain are

$$i = \frac{\Delta h_t}{l} = \frac{0.25}{0.625} = 0.4$$
$$F_s = i\,\gamma_w V = 0.4 \times 9.81 \times 1 \times 10^{-9} = 3.92 \times 10^{-9}\ \text{kN}$$

h. The water pressure distribution behind the retaining wall of Figure 13.3s*e*
   The head loss between each equipotential line:

$$\Delta h = \frac{\Delta H}{N_d} = \frac{9.25}{11} = 0.841\ \text{m}$$

The total head:

$$\begin{aligned} h_{t(A)} &= H_{t(beg)} - \Delta h \times (N_d)_A = 16.5 - 0.841 \times 0 = 16.5\ \text{m} \\ h_{t(C)} &= 16.5 - 0.841 \times 2 = 14.82\ \text{m} \\ h_{t(D)} &= 16.5 - 0.841 \times 3.5 = 13.56\ \text{m} \\ h_{t(bottom\ of\ wall)} &= 16.5 - 0.841 \times 5.8 = 11.62\ \text{m} \end{aligned}$$

The water pressure:

$$\begin{aligned} u_{(A)} &= (h_{t(A)} - h_{e(A)}) \times \gamma_w = (16.5 - 16.5) \times 9.81 = 0\ \text{kPa} \\ u_{(C)} &= (14.82 - 9.37) \times 9.81 = 53.46\ \text{kPa} \\ u_{(D)} &= (13.56 - 5.87) \times 9.81 = 75.44\ \text{kPa} \\ u_{(bottom\ of\ wall)} &= (11.62 - 4.10) \times 9.81 = 73.77\ \text{kPa} \end{aligned}$$

Problem 13.6

A tube is filled with a relatively dry soil at a water tension corresponding to a pressure head h_0 and a volumetric water content θ_0. Water is made available at one end of the tube (Figure 13.4s). As a result, a wetting front is created and advances from left to right on the figure. The wetted soil has a water tension corresponding to a pressure head of h_1 and a volumetric water content of θ_1. How fast will the wetting front propagate across the sample?

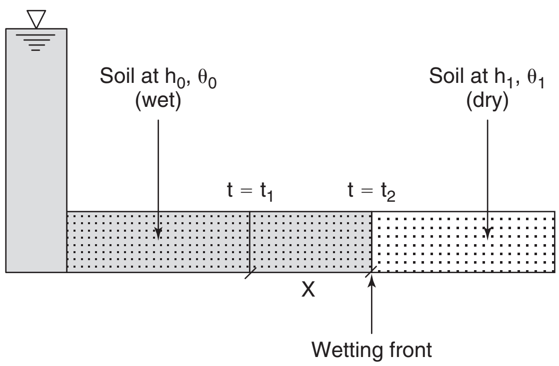

Figure 13.4s Horizontal wetting front propagation.

Solution 13.6

Consider the position of the wetting front at time t and then at time $t + dt$. The volume of water dV_w which has filled the voids during that interval of time is:

$$dV_w = (\theta_o - \theta_1) dV_t = (\theta_o - \theta_1) A dx$$

where A is the tube cross section and dx is the advance of the wetting front over the time dt. The corresponding flow rate is:

$$Q = \frac{dV_w}{dt} = \frac{(\theta_o - \theta_1)Adx}{dt}$$

The velocity can be obtained from the flow rate and also from Darcy's law:

$$v = \frac{Q}{A} = \frac{(\theta_o - \theta_1)dx}{dt} = k_o \frac{h_o - h_1}{x}$$

or

$$xdx = k_o \frac{h_o - h_1}{\theta_o - \theta_1} dt$$

Then the distance x is given as a function of time, as:

$$x = \sqrt{2k_o \frac{h_o - h_1}{\theta_o - \theta_1} t}$$

Problem 13.7

A soil sample has a saturated hydraulic conductivity k_{sat} equal to 10^{-8} m/s. Estimate the hydraulic conductivity of the sample if it dries to a degree of saturation equal to 0.9 and then 0.5. Use Figure 13.5s to estimate k_{unsat}.

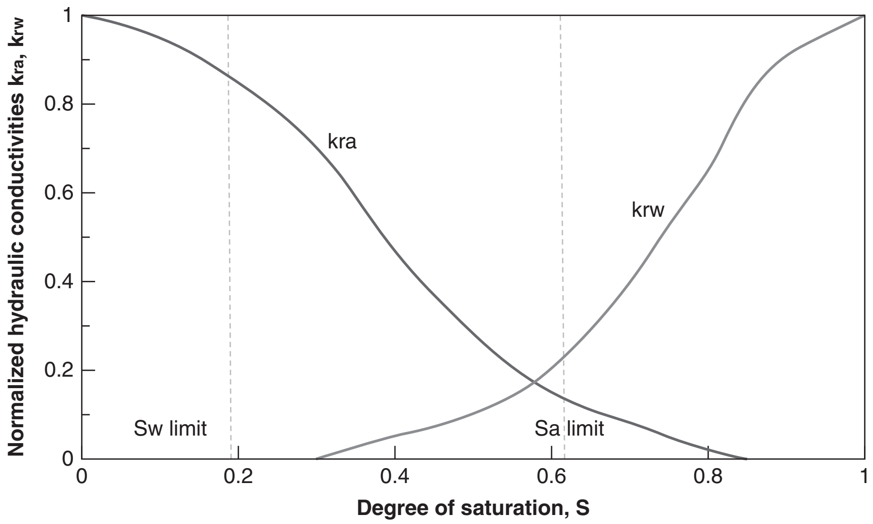

Figure 13.5s 5s Relative hydraulic conductivity of water and air as a function of degree of saturation.

Solution 13.7

From Figure 13.5s, we can find k_{rw} and use it in Eq. 13.62 to calculate the hydraulic conductivity of the soil sample in unsaturated conditions.

For $S = 0.9$, $k_{rw} = 0.93$, $k_{unsat} = k_{rw} x\ k_{sat}. = 0.93 \times 10^{-8}$ m/s $= 9.3 \times 10^{-9}$ m/s

For $S = 0.5$, $k_{rw} = 0.1$, $k_{unsat} = k_{rw} x\ k_{sat}. = 0.1 \times 10^{-8}$ m/s $= 1 \times 10^{-9}$ m/s