16 Tolerance Design

16.1. Introduction 340
16.2. Analysis of Variance 342
16.3. Tolerance Design of a Wheatstone Bridge 346
16.4. Summary of Parameter and Tolerance Design 349
Reference 351

This chapter is based on Yuin Wu et al., *Quality Engineering: Product and Process Optimization*. Livonia, Michigan. American Supplier Institute, 1986.

16.1. Introduction

Two specific sets of characteristics determine the quality level of a given product. These are the characteristics of the product's subsystems and/or component parts, called *low-rank characteristics,* and the characteristics of the product itself, called *high-rank characteristics*. Often, specifications and characteristic come from different sources. The end-item manufacturer often provides the specifications for component parts, while the marketing or product planning activity might specify the end item itself. In any case, one function is responsible for determining the specification of low-rank quality characteristics (or cause parameters), which will eventually affect high-rank quality characteristics. Determination of these low-rank specifications includes parameter design and tolerance design.

Parameter design, which was the subject of Chapter 15, is the determination of optimum component or parameter midvalues in such a way that variability is reduced. If, after these midvalues are determined, the level of variability is still not acceptable, components and materials must be upgraded (usually at a higher cost) to reduce variability further. The determination of acceptable levels of variability after midvalues have been established is *tolerance design*, the subject of this chapter.

 Taguchi's Quality Engineering Handbook. Genichi Taguchi, Subir Chowdhury and Yuin Wu

❐ Example

A 100-V, 50-Hz power supply across a resistance A in series with an inductor B results in a current of y amperes.

$$y = \frac{100}{\sqrt{A^2 + [(2\pi)(50B)]^2}} \tag{16.1}$$

The customer's tolerance of this circuit is $y = 10.0 \pm 4.0$ A. When the current is out of tolerance, the loss (including after services), denoted by A, is $150. The annual production of this circuit is 200,000 units.

Currently, second-grade resistors, A, and inductors, B, are used. For resistor A (Ω), we have

Mean:	$m_A = 9.92$
Standard deviation:	$\sigma_A = 0.300$

For inductor B (mH), we have

Mean:	$m_B = 4.0$
Standard deviation:	$\sigma_B = 0.80$

The variation in the components due to 10 years of deterioration is included in the standard deviations σ_A and σ_B. The problem is to reduce the variability of the current by upgrading the resistor, the inductance coil, or both. If first-grade components were used, one-half would reduce the varying ranges, including the deterioration. The cost increases would be 12 cents for a resistor and $1 for an inductor. Taylor's expansion and Monte Carlo simulation have frequently been used to solve such problems of component upgrade to reduce variability.

Here the experimental design approach will be discussed. The noise factor levels are set in the following way:

Three-Level Factors	**Two-Level Factors**
First level = second level $- \sqrt{\frac{3}{2}}\,\sigma$	First level = nominal $- \sigma$
Second level = nominal	Second level = nominal $+ \sigma$
Third level = second level $+ \sqrt{\frac{3}{2}}\,\sigma$	

The three levels of these components are set as follows:

$$\begin{aligned} A_1 &= 9.920 - \sqrt{\tfrac{3}{2}}(0.300) = 9.552 \\ A_2 &= 9.920 \\ A_3 &= 9.920 + \sqrt{\tfrac{3}{2}}(0.300) = 10.288 \end{aligned} \tag{16.2}$$

$$\begin{aligned} B_1 &= 0.00400 - \sqrt{\tfrac{3}{2}}(0.00080) = 0.00302 \\ B_2 &= 0.00400 \\ B_3 &= 0.00400 + \sqrt{\tfrac{3}{2}}(0.00080) = 0.00498 \end{aligned} \tag{16.3}$$

The units for B are changed from millihenries to henries. Using the nine possible combinations of A and B, the output current is computed from equation (16.1). For example, the current that results when the resistor/inductor combination A_1B_1 is used is

$$y_{11} = \frac{100}{A_1^2 + [(2\pi)(50B_1)]^2} = \frac{100}{9.552^2 + [(314.16)(0.00302)]^2}$$

$$= 10.42 \tag{16.4}$$

Similar computations are made for each configuration to obtain Table 16.1. In the table, the target value of 10 A has been subtracted from each observation.

16.2. Analysis of Variance

From equations (16.2) and (16.3) the quantitative contribution of each of the two component parts to the variability of output current is difficult to assess, so an analysis of variance is performed.

❒ Example [1]

From Table 16.1, the following factorial effects (constituents of the orthogonal polynomial equation) are computed:

m = general mean

A_l = linear term of A

A_q = quadratic term of A

B_l = linear term of B

B_q = quadratic term of B

A_1B_1 = interaction of linear terms of A and B

e = error term (including all higher-order terms)

Sum of squares of these terms are calculated as follows:

$$S_m = \frac{0.06^2}{9} = 0.00040 \tag{16.5}$$

Table 16.1
Results of two-way layout

	B_1	B_2	B_3	Total
A_1	0.42	0.38	0.33	1.13
A_2	0.03	0.00	−0.04	−0.01
A_3	−0.32	−0.35	−0.39	−1.06
Total	0.13	0.03	−0.10	0.06

S_{Al}, and S_{Aq}, and so on, are calculated.

$$S_{A_l} = \frac{(W_1A_1 + W_2A_2 + W_3A_3)^2}{r(\lambda^2S)} = \frac{[(-1)A_1 + (0)A_2 + (1)A_3]^2}{(3)(2)}$$

$$= \frac{(-1.13 - 1.06)^2}{(3)(2)} = 0.79935 \tag{16.6}$$

$$S_{A_q} = \frac{[(1)(1.13) - (2)(-0.01) + (1)(-1.06)]^2}{(3)(6)} = 0.00045 \tag{16.7}$$

$$S_{B_l} = \frac{[(-1)(0.13) + (1)(-0.10)]^2}{(3)(2)} = 0.00882 \tag{16.8}$$

$$S_{B_q} = \frac{[(1)(0.13) - (2)(0.03) + (1)(-0.10)]^2}{(3)(6)} = 0.00005 \tag{16.9}$$

To obtain the interaction $A_1 \times B_1$, the linear effects of A at each level of B are written using coefficients $W_1 = -1$, $W_2 = 0$, and $W_3 = 1$.

$$\begin{aligned} L(B_1) &= (-1)(0.42) + (0)(0.03) + (1)(-0.32) = -0.74 \\ L(B_2) &= (-1)(0.38) + (0)(0.00) + (1)(-0.35) = -0.73 \\ L(B_3) &= (-1)(0.33) + (0)(-0.04) + (1)(-0.39) = -0.72 \end{aligned} \tag{16.10}$$

and

$$\begin{aligned} S_{A_lB_l} &= \frac{[W_1L(B_1) + W_2L(B_2) + W_3L(B_3)]^2}{r(\lambda^2S)_A(\lambda^2S)_B} \\ &= \frac{[(-1)(-0.74) + (0)(-0.73) + (1)(-0.72)]^2}{(1)(2)(2)} \\ &= 0.00010 \end{aligned} \tag{16.11}$$

The total variation, S_T, is

$$S_T = 0.42^2 + 0.38^2 + \cdots + -0.39^2 = 0.8092 \tag{16.12}$$

The error sum of squares is

$$S_e = S_T - (S_m + S_{A_l} + \cdots + S_{A_lB_l})$$
$$= 0.8092 - (0.00040 + 0.79935 + \cdots + 0.00010)$$
$$= 0.00003 \tag{16.13}$$

Table 16.2 is the ANOVA table. From an ANOVA table, a design engineer can decide what should be considered as the error variance. It is exactly the same as the case of the expansion of the exponential function. In the latter case, one can include terms up to those of a certain order as factors and assign the remainder as the error. However, it is important to evaluate the magnitude of the error base on such a consideration. In other words, there is freedom to select a model, but the error of the model has to be evaluated.

After comparing the variances, B_q and A_lB_l are pooled with error to get (*e*). The pooled error sum of squares is 0.00018, with 5 degrees of freedom. The pooled error variance is

$$V_{(e)} = \frac{0.00018}{5} = 0.000036 \tag{16.14}$$

This variance is used to calculate the contribution percentages; for example,

$$\rho_m = \frac{S_m - V_e}{S_T}(100) = \frac{0.00040 - 0.000036}{0.80920}(100) = 0.045\% \tag{16.15}$$

The total variance when second-grade component parts are used is

$$V_T = \frac{S_T}{9} = \frac{0.89020}{9} = 0.08991 \tag{16.16}$$

Table 16.2
ANOVA table

Source	*f*	*S*	*V*	ρ (%)
m	1	0.00040	0.00040	0.045
A *l*	1	0.79935	0.79935	98.778
q	1	0.00045	0.00045	0.051
B *l*	1	0.00882	0.00882	1.086
q	1	0.00005	0.00005°	—
A_lB_l	1	0.00010	0.00010°	—
e	3	0.00003	0.00001	—
(*e*)	(5)	(0.00018)	(0.000036)	0.040
Total	9	0.80920	0.08991	100.00

Resulting Loss

Tolerance design is done at the design stage where the grades of the individual component parts are determined. For this purpose, it is necessary to acquire information regarding the loss function and the magnitude of the variability of each component part (standard deviation including deterioration). In the current example, the loss function is

$$L = k(y - m)^2$$
$$= \frac{150}{4.0^2}\sigma^2 = \$9.38(\sigma^2) \qquad (16.17)$$

Letting $V_T = \sigma^2$ in equation (16.17), the loss is

$$L = 9.38V_T = (9.38)(0.08991) = 84 \text{ cents} \qquad (16.18)$$

The annual loss is (0.84)(200,000) = \$1680. When first-grade component parts were used, the varying range would be reduced to one-half, that is, the standard deviation would be reduced to one-half. The variance would be reduced to $(\frac{1}{2})^2$ or $\frac{1}{4}$ for a linear term and to $(\frac{1}{2})^4$ or $\frac{1}{16}$ for a quadratic term.

The variance after upgrading component parts is calculated using the equation

$$V_N = V_C\left[\rho_{A_l}\left(\frac{H_N}{H_C}\right)^2 + \rho_{A_q}\left(\frac{H_N}{H_C}\right)^4 + \rho_{B_l}\frac{H'_N}{H'_C} + \rho_e\right] \qquad (16.19)$$

where

V_N = variance after upgrading (new)

V_C = variance before upgrading (current)

ρ_{A_l} = percent contribution of linear term of A

ρ_{A_q} = percent contribution of quadratic term of A

ρ_{B_l} = percent contribution of linear term of B

H_N = varying range of new component A

H_C = varying range of current component A

H'_N = varying range of new component B

H'_C = varying range of current component B

In the case of component A, the loss due to using second-grade resistors is

$$L = \frac{9.375V_T(\rho_{A_l} + \rho_{A_q})}{100}$$
$$= (9.375)(0.08991)(0.98778 + 0.00051)$$
$$= \$0.8330 \qquad (16.20)$$

The loss due to using first-grade resistors is

$$L = (9.375)(0.08991)[(0.98778)(\tfrac{1}{4}) + (0.00051)(\tfrac{1}{16})]$$
$$= \$0.2082 \qquad (16.21)$$

The quality improvement is

$$0.8330 - 0.2082 = \$0.6248 \qquad (16.22)$$

Since there is a 12-cent cost increase for upgrading, the net gain would be

$$\$0.6248 - 12 = \$0.5048 \qquad (16.23)$$

The annual improvement would be

$$(0.5048)(200{,}000) = \$100{,}000 \qquad (16.24)$$

Similar calculations can be made for the inductor.
Table 16.3 is the summary of the calculation results.

The first-grade resistor and second-grade inductor should be used. In this example, the tolerances of only two factors are explained. Normally, there are many system elements in a product, and orthogonal arrays are used to facilitate analysis.

16.3. Tolerance Design of a Wheatstone Bridge

After parameter design is completed, further research may be conducted by widening the ranges of levels or by citing additional control factors and conducting additional experiments.

❒ Example

Consider the parameter design of the Wheatstone bridge example described in Chapter 15. It is assumed that the final optimum configuration is $A_1C_3D_2E_3F_1$. To start the tolerance design, lower-price elements are again considered, and the three

Table 16.3
Summary of tolerance design

Element	Grade	Price	Quality	Total
Resistor	Second	Base	83.30	83.30
	First	+12	20.82	32.82
	Second	Base	0.92	0.92
Inductor	First	+100	0.23	100.23

levels for each element are set closest to the optimum condition (with variations such as those shown in Table 15.2). In this step we usually cite more error factors than in the parameter design stage. In this example, however, the levels and error factors shown in Table 15.2 are investigated. Let the midvalue and the standard deviation of each element be m and σ, respectively. The three levels are prepared as follows:

$$\begin{aligned} &\text{First level:} && m - \sqrt{\tfrac{3}{2}}\,\sigma \\ &\text{Second level:} && m \\ &\text{Third level:} && m + \sqrt{\tfrac{3}{2}}\,\sigma \end{aligned} \tag{16.25}$$

These are assigned in the orthogonal array L_{36}. For this optimum configuration, the measurement errors of the 36 combinations of the outer array factors are given in the last column of Table 15.6. To facilitate the analysis of variance, multiply each of the 36 data points and the percent contribution of each is calculated. The main effects of these error factors are decomposed into linear and quadratic components using the orthogonal polynomial equation.

Table 16.4
ANOVA for tolerance design

Source		f	S	V
m		1	0°	
A	l	1	1°	1
	q	1	0°	
B	l	1	86,482	86,482
	q	1	2°	1
C	l	1	87,102	87,102
	q	1	0°	
D	l	1	87,159	87,159
	q	1	0°	
E	l	1	0°	
	q	1	0°	
F	l	1	0°	
	q	1	1°	1
X	l	1	28,836	28,836
	q	1	0°	
e		21	41°	1.95
Total		36	289,623	
(e)		32	44	1.38

°: Pooled into error.
(e): Error after pooling.

From column 2 of Table 15.6, each datum is multiplied by 10,000.

$$A_1 = -24 - 60 + \cdots + 87 = -3$$
$$A_2 = 0 - 33 + \cdots + 36 = 3 \qquad (16.26)$$
$$A_3 = 27 + 97 + \cdots -120 = 3$$

$$S_m = \frac{(A_1 + A_2 + A_3)^2}{36} = \frac{(-3 + 3 + 3)^2}{36} = \frac{9}{36} = 0.25 \qquad (f = 1) \qquad (16.27)$$

$$S_{A_l} = \frac{(W_1A_1 + W_2A_2 + W_3A_3)^2}{r(\lambda^2 S)} = \frac{(-A_1 + A_3)^2}{r(\lambda^2 S)}$$
$$= \frac{[-(-3) + 3]^2}{(12)(2)} = \frac{36}{24} = 1.5 \qquad (f = 1) \qquad (16.28)$$

$$S_{A_q} = \frac{(A_1 - 2A_2 + A_3)^2}{r(\lambda^2 S)} = \frac{[(-3) - (2)(3) + 3]^2}{(12)(6)} = \frac{36}{72}$$
$$= 0.5 \qquad (f = 1) \qquad (16.29)$$

$\lambda^2 S$ is the sum of squares of W_i, which can be obtained from the orthogonal polynomial table.

Other factors are calculated similarly. Table 16.4 shows the results. Notice that the four linear effects, B, C, D, and X, are significantly large. Pool the small effects with the error to obtain Table 16.5. Because B, C, and D have large contributions, higher-quality (less varying) resistors must be used for these elements. Assume that premium-quality resistors with one-tenth of the original standard deviations are used for B, C, and D. For the ammeter, the first-grade product with one-fifth of the original reading error is used. As a result, the percents of contribution of B, C, and D are one one-hundredth of, and X is one twenty-fifth of, the original. The error variance, V_e, is given as (see Table 16.5)

$$V_e = V_T[(\tfrac{1}{10})^2(\rho_B + \rho_C + \rho_D)] + (\tfrac{1}{5})^2\,\rho_X + \rho_e$$
$$= (8045.1)[(0.01)(0.29860 + 0.30074 + 0.30093)]$$
$$+ (0.04)(0.09956) + 0.00017$$
$$= (8045.1)(0.013155)$$
$$= 105.8 \qquad (16.30)$$

After reconverting the measurement, the error variance, σ^2, of equation (12.7) is now reduced by 1/8176:

$$\sigma^2 = 0.000001058 \qquad (16.31)$$

The allowance, which is three times that of the standard deviation, becomes

$$\pm 3\sigma = \pm 3(\sqrt{0.000001058})$$
$$= \pm 0.0031 \qquad (16.32)$$

Table 16.5
Rearranged ANOVA table

Source	f	S	V	ρ (%)
B	1	86,482	86,482	29.86
C	1	87,102	87,102	30.07
D	1	87,159	87,159	30.09
X	1	28,836	28,836	9.96
e	32	44	1.38	0.02
Total	36	289,623	8045.1	100.000

If only resistors *B*, *C*, and *D* are upgraded, the resulting error variance will be

$$V_e = (8045)[(0.01)(0.29860 + 0.30074 + 0.30093) + 0.09956 + 0.00017] = (8045)(0.108733) = 874.8 \quad (16.33)$$

Converting back to the original units (dividing by $1/10^8$) gives 0.00000875.

16.4. Summary of Parameter and Tolerance Design

In the Wheatstone bridge example, the midvalues of the parameters first considered were those in column (1) of Table 16.6. When these midvalues were widely changed as shown in Table 15.1, the optimum midvalues were obtained and are shown in column (2) of Table 16.6. The information in the table also reveals that the error variance was reduced to less than 1/100 of the original value without upgrading components.

Through parameter design investigation, a method of increasing the measurement precision using low-priced and widely varying elements was sought. This method is effective and applicable to all product design situations. On the other hand, the tolerance design method demands that higher-priced and less-varying elements be used whenever these elements have a large influence on measurement results. Therefore, it is necessary to evaluate carefully the profitability of variation reduction. The countermeasures taken to minimize error effects due to resistors *A*, *C*, *D*, and ammeter *X* must be evaluated for economic impact.

In the preceding section, the conclusion was to use the highest-grade resistors for *A*, *C*, and *D* and the first-grade ammeter *X*. However, such a conclusion might not be correct from an economic viewpoint.

❒ Example

Wheatstone bridge is used in a resistor manufacturing plant to control the quality of resistors. The plant is producing 120,000 resistors annually. The tolerance of manufactured resistors is 2 ± 0.5 Ω. When a resistor is out of specification, the loss is $3. At the stage of parameter design, there is no cost increase for optimization. In the tolerance design stage, the cost for upgrading components is $4 for a resistor and $20 for an ammeter. The annual cost for interest and depreciation combined is 50% of the upgrading cost.

Losses are calculated using the following:

1. Before parameter design [column (1) of Table 16.6]
2. After parameter design [column (2) of Table 16.6]
3. After tolerance design by upgrading only resistors to the premium grade
4. After tolerance design by upgrading resistors to the premium grade and the ammeter to the first grade

The loss due to measurement before parameter design is

$$L = \frac{3}{0.5^2}\sigma^2 = (12)(0.00865036) = \$0.103804 \qquad (16.34)$$

The loss after parameter design is

$$L = \frac{3}{0.5^2}(0.00008045) = \$0.000965 \qquad (16.35)$$

The loss after tolerance design by upgrading only three resistors to the premium grade is

$$L = \frac{(4)(0.5)(3)}{120{,}000} + \frac{3}{0.5^2}(0.00000875) = \$0.000155 \qquad (16.36)$$

Table 16.6
Initial and optimum midvalues and variance of parameters

	(1) Initial Midvalue	(2) Optimum Midvalue
A (Ω)	100	20
C (Ω)	10	50
D (Ω)	10	10
E (V)	6	30
F (Ω)	10	2
Error variance	0.00865036	0.00008045
3σ	0.2790	0.0269

Table 16.7
Product design stages and costs

Design Stage	σ^2 Out	Quality *L*	Upgrading Cost	Total	Annual Loss
System design only	0.00805636	$0.103804	(base)	$0.103804	$12,456.00
After parameter design	0.00008045	0.00965	+0	0.000965	115.80
Tolerance design Premium-grade resistors only	0.00000875	0.000105	$0.000050	0.000155	18.60
Premium-grade resistors only plus first-grade ammeter	0.00000106	0.000013	0.000883	0.000896	107.52

The loss after tolerance design by upgrading only resistors to the premium grade and the ammeter to the first grade is

$$L = \frac{(4)(0.5)(3) + (20)(0.5)}{120{,}000} + (1200)(0.000001058)$$

$$= \$0.0000896 \qquad (16.37)$$

The best solution is therefore (3) above, that is, equation (16.36), as shown in Table 16.7.

Reference

1. Genichi Taguchi, 1987. *System of Experimental Design*. Livonia, Michigan: Unipub/American Supplier Institute.