33 Two-Way Layout with Decomposition

33.1. Introduction

When a factor in an experiment is a continuous variable, such as temperature or time, it is often necessary to know the linear or quadratic effects of the factor. For this purpose, the orthogonal polynomial equation is used. In this chapter, methods of analysis for one or two continuous variables in two-way layout experiments are shown. This chapter is based on Genichi Taguchi et al., *Design of Experiments*. Tokyo: Japanese Standards Association, 1973.

33.2. One Continuous Variable

To observe changes in the elongation of plastics, three plastics are prepared with three types of additives (A_1, A_2, and A_3), and four levels of temperature to be used in the experiment are as follows: $B_1 = -15°C$, $B_2 = 0°C$, $B_3 = 15°C$, $B_4 = 30°C$. Elongation is then measured for each combination of factors A and B. The results are shown in Table 33.1.

The most desirable data we want would be high elongation that will not change with temperature. This type of problem of wanting to produce a product whose characteristics do not change with temperature is very familiar to engineers and researchers. Temperature is considered an indicative factor. The indicative factor, in this case temperature, would indicate the temperature when the products are used by the customer. An indicative factor has levels like a control factor, which can be controlled in the stage of production to achieve optimum results. When an indicative factor is a condition such as temperature, humidity, or electric current, it is usually decomposed into linear and residual effects.

Table 33.1
Percent elongation

	B_1	B_2	B_3	B_4
A_1	15	31	47	62
A_2	11	20	37	45
A_3	34	42	49	54

The procedure for this analysis is as follows: First, subtract a working mean of 40 from all data (Table 33.2). The factorial effect, B, is decomposed into components of linear, quadratic, and cubic effects. For this purpose, coefficients of the four-level polynomial are used.

$$S_{B_l} = \frac{(W_1B_1 + W_2B_2 + W_3B_3 + W_4B_4)^2}{r(\lambda^2 S)}$$

$$= \frac{[(-3)(-60)-(1)(-27)+(1)(13) + (3)(41)]^2}{(3)(20)}$$

$$= \frac{343^2}{(3)(20)} = \frac{117{,}649}{60} = 1961 \tag{33.1}$$

$$S_{B_q} = \frac{(-60 + 27 - 13 + 41)^2}{(3)(4)} = \frac{(-5)^2}{12}$$

$$= \frac{25}{12} = 2 \tag{33.2}$$

$$S_{B_c} = \frac{[(-1)(-60) + (3)(-27) - (3)(13) + 41]^2}{(3)(20)}$$

$$= \frac{(-19)^2}{60} = \frac{361}{60} = 6.0 \tag{33.3}$$

Table 33.2
Data after subtracting working mean, 40

	B_1	B_2	B_3	B_4	Total
A_1	−25	−9	7	22	−5
A_2	−29	−20	−3	5	−47
A_3	−6	2	9	14	19
Total	−60	−27	13	41	−33

The main effect of B is calculated as

$$S_B = \frac{(-60)^2 + (-27)^2 + 13^2 + 41^2}{3} - \frac{(-33)^2}{12}$$
$$= \frac{3600 + 729 + 169 + 1681}{3} - 91$$
$$= \frac{6179}{3} - 19 = 1969 \qquad (f = 3) \tag{33.4}$$

The total of S_{B_l}, S_{B_q}, and S_{B_c} is calculated as

$$S_{B_l} + S_{B_q} + S_{B_C} = 1961 + 2 + 6 = 1969 \tag{33.5}$$

which coincides with S_B.

When there are two factors, as in this example, it is also necessary to investigate their interaction, which signifies that the effect of one factor will change at different levels of the other factor. In our case it means discovering whether the effect of temperature (factor B) changes if a plastic is made from different additives.

The linear coefficient of temperature at A_1, denoted by $b_1(A_1)$, is calculated as

$$b_1(A_1) = \frac{(-3)(-25) - (1)(-9) + 7 + (3)(22)}{r(\lambda S)\,h}$$
$$= \frac{157}{(1)(10)(15)} = 1.05 \tag{33.6}$$

Linear coefficients of temperature at A_2 and A_3, denoted by $b_1(A_2)$ and $b_1(A_3)$, are calculated similarly. The variation of linear coefficients of B at different conditions of A, denoted by S_{AB_l}, is obtained as

$$S_{AB_l} = \frac{L(A_1)^2 + L(A_2)^2 + L(A_3)^2}{20} - \frac{[L(A_1) + L(A_2) + L(A_3)^2]}{(20)(3)} \tag{33.7}$$

where

$$L(A_1) = (-3)(-25) - (1)(-9) + 7 + (3)(22) = 157$$
$$L(A_2) = (-3)(-29) - (1)(-20) - 3 + (3)(5) = 119 \tag{33.8}$$
$$L(A_3) = (-3)(-6) - (1)(2) + 9 + (3)(14) = 67$$

Hence,

$$S_{AB_l} = \frac{157^2 + 119^2 + 67^2}{20} - \frac{(157 + 119 + 67)^2}{60}$$
$$= 2165 - 1961 = 204 \qquad (f = 2) \tag{33.9}$$

The effect of A is obtained from the totals of A_1, A_2, and A_3:

$$S_A = \frac{A_1^2 + A_2^2 + A_3^2}{4} - \text{CF}$$

$$= \frac{(-5)^2 + (-47)^2 + 19^2}{4} - \frac{(-33)^2}{12}$$

$$= \frac{2595}{4} - 91 = 558 \qquad (f = 2) \tag{33.10}$$

The total variation, S_T, and error variation, S_e, are calculated.

$$S_T = (-25)^2 + (-9)^2 + \cdots + 14^2 - \frac{(-33)^2}{12}$$

$$= 2831 - 91 = 2740 \qquad (f = 11) \tag{33.11}$$

$$S_e = S_T - S_A - S_B - S_{AB_l}$$

$$= 2740 - 558 - 1969 - 204$$

$$= 9 \qquad (f = 4) \tag{33.12}$$

The analysis of variance is shown in Table 33.3.

B_q and B_c are not significant and are pooled with e. This composite is shown as (e). A, B_l, and AB_l are significant. The significance of A shows that elongation does change with different types of additives. The significance of B_l shows that the change of elongation due to temperature has a linear tendency. Since AB_l is significant in this case, the linear tendency changes due to the different types of additive.

Table 33.3
Analysis of variance

Source	f	S	V	S'	ρ (%)
A	2	558	279	552	20.1
B					
l	1	1961	1961	1958	71.5
q	1	2	2		
c	1	6	6		
AB_l	2	204	102	198	7.3
e	4	9	2.25		
(e)	(6)	(17)	(2.83)	32	1.1
Total	11	2740		2740	100.0

Next, an estimation is made against the significant effects. The main effect of A is calculated as

$$\overline{A}_1 = \frac{-5}{4} + 40 = 38.75 \tag{33.13}$$

$$\overline{A}_2 = \frac{-47}{4} + 40 + 2.06 = 28.25 \tag{33.14}$$

$$\overline{A}_3 = \frac{19}{4} + 40 \pm 2.06 = 44.75 \tag{33.15}$$

This shows that elongation of A_3 is the highest. Next, the change of elongation due to changes in temperature is calculated. B is significant, as is AB_l, so the linear coefficient at each level of A is calculated.

$$\begin{aligned}\hat{b}_1(A_1) &= \frac{-3A_1B_1 - A_1B_2 + A_1B_3 + 3A_1B_4}{r(\lambda S)\,h} \\ &= \frac{(-3)(-25) - (1)(-9) + 7 + (3)(22)}{(1)(10)(15)} = 1.05\end{aligned} \tag{33.16}$$

$$\hat{b}_1(A_2) = \frac{(-3)(-29) - (1)(-20) + (-3) + (3)(5)}{(1)(10)(15)} = 0.79 \tag{33.17}$$

$$\hat{b}_1(A_3) = \frac{(-3)(-6) - 2 + 9 + (3)(14)}{(1)(10)(15)} = 0.45 \tag{33.18}$$

From the results above, it is known that the change due to temperature is the smallest at A_3.

We can now conclude that the plastic-using additive, A_3, has the highest elongation, the least change of elongation, and the least change of elongation due to a change in temperature. If cost is not considered, A_3 is the best additive.

33.3. Two Continuous Variables

Now we describe an analysis with two continuous variable factors. In the production of 8% phosphor bronze (containing 0.45% phosphorus), an experiment was planned for the purpose of obtaining a material to substitute an alloy for the current material used in making a spring. To increase the tensile strength of the material, factor A, the extent of processing, and factor B, the annealing temperature, are analyzed. Each factor has four levels.

There are four types of material that can be used when making a spring. Their specifications are:

- ❑ *Type 1:* over 75 kg/cm^2
- ❑ *Type 2:* over 70 kg/cm^2
- ❑ *Type 3:* over 65 kg/cm^2
- ❑ *Type 4:* over 60 kg/cm^2

The levels of A and B are

Extent of processing, A (%): $A_1 = 30$, $A_2 = 40$, $A_3 = 50$, $A_4 = 60$

Annealing temperature, B (°C): $B_1 = 150$, $B_2 = 200$, $B_3 = 250$, $B_4 = 300$

Sixteen experiments were performed, combining the factors in random order.

In such experiments, two test pieces are usually chosen from each of the 16 combinations, and then their strengths are measured. Generally, r pieces are taken, so there are r data for each combination. But for the analysis of variance, averages are not adequate. Instead, the analysis must be done starting from the original data of r repetitions. In this example, let us assume that $r = 1$; the following problems are discussed:

1. What are the relationships between extent of processing (A), annealing temperature (B), and tensile strength, and how can we express them in the form of a polynomial?
2. What is the range of the extent of processing and annealing temperature that will satisfy the tensile strength specifications for each of the four products?

The observed data are shown in Table 33.4. From each item, subtract a working mean of 70 and multiply by 10 to get the data in Table 33.5.

To get an observational equation, variables A and B are expanded up to cubic terms, and the product of A and B is also calculated. Interaction may be considered as the product of the orthogonal polynomial of one variable.

$$\begin{aligned} y = m &+ a_1(A - \overline{A}) + a_2\left[(A - \overline{A})^2 - \frac{a^2 - 1}{12} h_A^2\right] \\ &+ a_3\left[(A - \overline{A})^3 - \frac{3a^2 - 7}{20}(A - \overline{A})h_A^2\right] + b_1(B - \overline{B}) \\ &+ b_2\left[(B - \overline{B})^2 - \frac{b^2 - 1}{12} h_B^2\right] + b_3\left[(B - \overline{B})^3 - \frac{3b^2 - 7}{20}(B - \overline{B})h_B^2\right] \\ &+ c_{11}(A - \overline{A})(B - \overline{B}) \end{aligned} \qquad (33.19)$$

where a and b are the numbers of levels of A and B, and h_A and h_B are the intervals of the levels of A and B.

The magnitude of each term is evaluated in the analysis of variance; then a polynomial is written only from the necessary terms. Letting the linear, quadratic,

Table 33.4
Tensile strength data (kg/cm²)

	B_1	B_2	B_3	B_4
A_1	64.9	62.6	61.1	59.2
A_2	69.1	70.1	66.8	63.6
A_3	76.1	74.0	71.3	67.2
A_4	82.9	80.0	76.0	72.3

Table 33.5
Supplementary table

	B_1	B_2	B_3	B_4	Total
A_1	−51	−74	−89	−108	−322
A_2	−9	1	−32	−64	−104
A_3	61	40	13	−28	86
A_4	129	100	60	23	312
Total	130	67	−48	−177	−28

and cubic terms of A be S_{a_l}, S_{A_q}, and S_{a_c}, and defining the term of the product of linear effects of A and B as $S_{A_l}B_l$, the calculations are then made as follows:

$$S_{A_l} = \frac{(-3A_1 - A_2 + A_3 + 3A_4)^2}{\gamma(\lambda^2 S)}$$

$$= \frac{[-3(-322)\ -1(-104)\ + 86 + 3(312)]^2}{(4)(20)}$$

$$= \frac{(2092)^2}{80} = 54{,}706 \tag{33.20}$$

$$S_{A_q} = \frac{(A_1 - A_2 - A_3 + A_4)^2}{r(\lambda^2 S)}$$

$$= \frac{(-322 + 104 - 86 + 312)^2}{(4)(4)}$$

$$= \frac{8^2}{16} = 4 \tag{33.21}$$

$$S_{A_c} = \frac{(-A_1 + 3A_2 - 3A_3 + A_4)^2}{(4)(20)}$$

$$= \frac{[-(-322) + 3(-104)\ -3(86) + 312]^2}{80}$$

$$= \frac{64^2}{80} = 51 \tag{33.22}$$

$$S_{B_l} = \frac{(-3B_1 - B_2 + B_3 + 3B_4)^2}{r(\lambda^2 S)}$$

$$= \frac{[-3(130) - (67) + (-48) + 3(-177)]^2}{(4)(20)}$$

$$= \frac{(-1036)^2}{80} = 13{,}416 \tag{33.23}$$

$$S_{B_q} = \frac{(130 - 67 + 48 - 177)^2}{(4)(4)}$$

$$= 272 \qquad (33.24)$$

$$S_{B_q} = \frac{[-1(130) + 3(67) - 3(-48) + (-177)]^2}{(4)(20)}$$

$$= 18 \qquad (33.25)$$

To calculate $S_{A_1B_1}$, comparisons of A_l at each level of B (L_1, L_2, L_3, and L_4) are written:

$$L_1 = (-3)(-51) + (-1)(-9) + (1)(61) + (3)(129) = 610$$

$$L_2 = (-3)(-74) + (-1)(1) + (1)(40) + (3)(100) = 561$$

$$L_3 = (-3)(-89) + (-1)(-32) + (1)(13) + (3)(60) = 492$$

$$L_4 = (-3)(-108) + (-1)(-64) + (1)(-28) + (3)(23) = 429 \qquad (33.26)$$

The number of units of these levels is 20. The values of L_1, L_2, L_3, and L_4, divided by $r(\lambda S)h_A = (1)(10)(10) = 100$, are the increments of tensile strength when the extent of processing is increased by 1% for each annealing temperature. From the values of L_1, L_2, L_3, and L_4, it seems that the effect of processing decreases linearly when annealing temperature increases. $S_{A_1B_1}$, the extent of the processing effect, decreases linearly as the annealing temperature increases. This is calculated by multiplying the linear coefficients of the four levels: -3, -1, 1, and 3:

$$S_{A_lB_l} = \frac{(-3L_1 - L_2 + L_3 + 3L_4)^2}{[(-3)^2 + (-1)^2 + 1^2 + 3^2](20)}$$

$$= \frac{(-612)^2}{400} = 936 \qquad (33.27)$$

The total variation is

$$S_T = (-51)^2 + (-74)^2 + \cdots + 23^2 - \frac{(-28)^2}{16}$$

$$= 69859 \qquad (33.28)$$

Residual variation is obtained by subtracting the variations of factorial effects, which are being analyzed from the total variation, S_T.

$$S_e = S_T - (S_{A_l} + S_{A_q} + S_{A_c} + S_{B_l} + S_{B_q} + S_{B_c} + S_{A_lB_l})$$

$$= 69{,}859 - (54706 + 4 + 51 + 13416 + 272 + 18 + 936)$$

$$= 456 \qquad (33.29)$$

The result of the calculation of S_A is checked by adding S_{A_l}, S_{A_q}, and S_{A_c}. It means that the differences between A_1, A_2, A_3, and A_4 are decomposed to the components of a polynomial to achieve more analytical meaning. If necessary, the terms $S_{A_qB_1}$ corresponding to A^2B can be calculated. If it were calculated, we would find it to be very small, since it is merely a component of S_e.

Table 33.6
Analysis of variance

Source	f	S	V	ρ (%)
A				
Linear	1	54,706	54,706	78.3
Quadratic	1	4	4	
Cubic	1	51	51	
B				
Linear	1	13,416	13,416	19.1
Quadratic	1	272	272	
Cubic	1	18	18	
A_1B_1	1	936	936	1.2
e	8	456	57.0	
Total	15	69,859		100.0
(e)	(12)	(801)	(66.7)	(1.4)

The result of the decomposition of variation and the analysis of variance are summarized in Table 33.6. The degree of contribution of A is calculated after pooling the insignificant effects, S_{A_q}, S_{A_c}, S_{B_q}, and S_{B_c}, with S_e.

$$\rho_A = \frac{S_A - V_e}{S_T} = \frac{54{,}706 - 66.7}{69{,}859} = 78.2\% \tag{33.30}$$

However, it is a general rule to pool insignificant factorial effects with error. In some special cases, insignificant factorial effects are not pooled, and in other cases, significant factorial effects are pooled. If the product of A and B is pooled with the residue, the degrees of contribution of error increases from 1.4% to 2.6%.

Now we can just consider four terms in the observational equation. These terms are the linear term of A, the linear term of B, the product of A and B, and the general mean. The observational equation is as follows:

$$y = m + a_1(A - \overline{A}) + b_1(B - \overline{B}) + c_{11}\,(A - \overline{A})(B - \overline{B}) \tag{33.31}$$

In this calculation, remember to convert the values of the four unknowns, m, a_1, b_1, and c_{11}, to their original units.

$$\begin{aligned}\hat{m} &= \text{average of total} = \text{working mean} + \frac{1}{10}\left(\frac{T}{16}\right)\\ &= 70.0 + \frac{1}{10}\left(\frac{-28}{16}\right) = 69.82\end{aligned} \tag{33.32}$$

$$\begin{aligned}\hat{a} &= \frac{-3A_1 - A_2 + A_3 + 3A_4}{10r(\lambda S)h_A} = \frac{(-3)(-322) - (-104) + 86 + (3)(312)}{(10)(4)(10)(10)}\\ &= 0.523\end{aligned} \tag{33.33}$$

In the denominator of equation (33.33), the first 10 is for the conversion to the original unit, since the original data were multiplied by 10; r is the number of replications; λS is obtained from the table of orthogonal polynomials; and h_A is the interval between the levels of A. Similarly,

$$\hat{b}_1 = \frac{-3B_1 - B_2 + B_3 + 3B_4}{10rSh_B} = \frac{-3(130) - 67 - 48 + (3)(-177)}{(10)(4)(10)(50)}$$

$$= -0.0518 \tag{33.34}$$

$$\hat{c}_{11} = \frac{-3L_1 - L_2 + L_3 + 3L_4}{10r(\lambda S)_A h_A (\lambda S)_B h_B} = \frac{(-3)(610) - (561) + 492 + (3)(429)}{(10)(1)(10)(10)(10)(50)}$$

$$= -0.00122 \tag{33.35}$$

The population mean shown as y in equation (33.31) is written as $\hat{\mu}$, since it is the estimation for any values of A and B:

$$\hat{\mu} = 69.82 + 0.523(A - 45) - 0.0518(B - 225) - 0.00122(A - 45)(B - 225) \tag{33.36}$$

Equation (33.36) is the result of orthogonal expansion, so the estimates of the coefficient of each term are independent of each other.