Appendix A

EQUIVALENCE BETWEEN PLANE WAVE PROPAGATION IN SOURCE-FREE, LINEAR, ISOTROPIC, AND HOMOGENEOUS MEDIA; TEM WAVE PROPAGATION IN TRANSMISSION LINES; AND WAVE PROPAGATION IN TRANSMISSION LINES DESCRIBED BY ITS DISTRIBUTED CIRCUIT MODEL

Let us consider a source-free, linear, isotropic, and homogeneous dielectric medium. Assuming a time dependence of the form $e^{j\omega t}$, the Maxwell's curl equations can be expressed as follows:

$$\nabla \times \vec{E} = -j\omega\mu \vec{H} \tag{A.1a}$$

$$\nabla \times \vec{H} = j\omega\varepsilon \vec{E} \tag{A.1b}$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic field intensities, respectively, and μ and ε are the permeability and permittivity, respectively, of the considered medium. Taking the curl of (A.1a), and using (A.1b), gives

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$$\nabla \times \nabla \times \vec{E} = -j\omega\mu\nabla \times \vec{H} = \omega^2\mu\varepsilon\vec{E} \tag{A.2}$$

Equation (A.2) can be simplified by using the following vector identity (where $\vec{A}$ is an arbitrary vector):

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} \tag{A.3}$$

Using (A.3), and taking into account that in a source-free region $\nabla \cdot \vec{E} = 0$, the equation for the electric field (known as wave equation or Helmholtz equation) is found to be

$$\nabla^2\vec{E} + \omega^2\mu\varepsilon\vec{E} = 0 \tag{A.4}$$

Similarly, the wave equation for the magnetic field is

$$\nabla^2\vec{H} + \omega^2\mu\varepsilon\vec{H} = 0 \tag{A.5}$$

In the previous expressions, we can define the wavenumber, or propagation constant, as follows:

$$k = \omega\sqrt{\varepsilon\mu} \tag{A.6}$$

If the medium is lossless, μ and ε are real and k is also real. Note the analogy between the wavenumber and the phase constant of a transmission line (given by expression 1.6) for the lossless case. If we now consider plane wave propagation in the z-direction, with the electric field polarized in the x-direction, (A.4) reduces to

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \tag{A.7}$$

and the general solution of this equation is formally identical to (1.7), that is,

$$E_x(z) = E_o^+ e^{-jkz} + E_o^- e^{jkz} \tag{A.8}$$

Introducing (A.8) in (A.1a), the magnetic field intensity is found to be

$$H_x = H_z = 0 \tag{A.9a}$$

$$H_y = -\frac{1}{j\omega\mu}\frac{\partial E_x}{\partial z} = \frac{k}{\omega\mu}\left(E_o^+ e^{-jkz} - E_o^- e^{jkz}\right) \tag{A.9b}$$

Defining the ratio of the $\vec{E}$ and $\vec{H}$ fields as the wave impedance of the medium, η, it is apparent from (A.8) and (A.9) that

$$\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}} \tag{A.10}$$

On the other hand, the phase velocity is given by

$$v_{\mathrm{p}} = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}} \tag{A.11}$$

In a lossy medium, with real and imaginary part of the permittivity and small (but finite) conductivity, the wave equation that results from the Ampere–Maxwell law (1.62) is as follows:

$$\nabla^2 \vec{E} + \omega^2 \mu \left(\varepsilon' - j\frac{\sigma + \omega\varepsilon''}{\omega} \right) \vec{E} = 0 \tag{A.12}$$

and the complex propagation constant is defined in this case as

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\left(\varepsilon' - j\frac{\sigma + \omega\varepsilon''}{\omega} \right)} \tag{A.13}$$

The solutions of (A.12) for the electric field are similar to those given by (A.8), that is,

$$E_x(z) = E_{\mathrm{o}}^{+} e^{-\gamma z} + E_{\mathrm{o}}^{-} e^{\gamma z} \tag{A.14}$$

corresponding to exponentially decaying travelling waves with phase velocity $v_{\mathrm{p}} = \omega/\beta$. The associated magnetic field is

$$H_y = -\frac{1}{j\omega\mu}\frac{\partial E_x}{\partial z} = \frac{\gamma}{j\omega\mu}\left(E_{\mathrm{o}}^{+} e^{-\gamma z} - E_{\mathrm{o}}^{-} e^{\gamma z} \right) \tag{A.15}$$

and the wave impedance is

$$\eta = \frac{j\omega\mu}{\gamma} \tag{A.16}$$

Plane wave propagation in a source-free, linear, isotropic, and homogeneous medium is a particular case of TEM wave propagation. The field solutions given earlier (A.8 and A.9 for the lossless case, or A.14 and A.15 for the lossy case) are identical to those that result by considering a hypothetical parallel plate transmission line with infinitely wide plates (or at least very wide as compared to their separation) oriented with the line axis in the z-direction and the plates parallel to the z-y plane, and filled with a source-free, linear, isotropic, and homogeneous dielectric.

For a general transmission line supporting TEM wave propagation[1] along the z-axis (positive direction), the fields can be expressed as follows:

$$\vec{E}(x,y,z) = \vec{E}_t(x,y) \cdot e^{-j\beta z} \quad \text{(A.17a)}$$

$$\vec{H}(x,y,z) = \vec{H}_t(x,y) \cdot e^{-j\beta z} \quad \text{(A.17b)}$$

where $\vec{E}_t(x,y)$ and $\vec{H}_t(x,y)$ are the transverse electric and magnetic field components, respectively, and β is the propagation constant[2]. Introducing (A.17) in (A.1), the following equations result:

$$j\beta E_y = -j\omega\mu H_x \quad \text{(A.18a)}$$

$$-j\beta E_x = -j\omega\mu H_y \quad \text{(A.18b)}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \quad \text{(A.18c)}$$

$$j\beta H_y = j\omega\varepsilon E_x \quad \text{(A.18d)}$$

$$-j\beta H_x = j\omega\varepsilon E_y \quad \text{(A.18e)}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \quad \text{(A.18f)}$$

where it has been taken into account that $E_z = H_z = 0$ and the $e^{-j\beta z}$ z-dependence of the field components. Combining (A.18a) and (A.18e), or (A.18b) and (A.18d), one obtains the following:

$$\beta = \omega\sqrt{\varepsilon\mu} = k \quad \text{(A.19)}$$

and the wave impedance for the TEM mode is found to be

$$Z_{\mathrm{TEM}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\varepsilon}} = \eta \quad \text{(A.20)}$$

as results from (A.18a) and (A.18b), or (A.18d) and (A.18e). Notice that from (A.20), it follows that the transverse fields are related by

$$\vec{H}_t(x,y) = \frac{1}{Z_{\mathrm{TEM}}} \vec{z} \times \vec{E}_t(x,y) \quad \text{(A.21)}$$

[1] For TEM wave propagation, the line must have at least two conductors.
[2] If losses are present, then $j\beta$ must be replaced with γ.

where $\vec{z}$ is the unit vector in the z-direction. Thus, the propagation constant and the wave impedance in TEM transmission lines are given by identical expressions as those for plane waves in a source-free, linear, isotropic, and homogeneous media. By mapping the material parameters with the line parameters according to

$$\varepsilon = C' \tag{A.22a}$$

$$\mu = L' \tag{A.22b}$$

the wave equation (A.7) for the electric field is equivalent to Equation 1.4a for the line voltage, considering a lossless line (i.e., $\gamma^2 = -\beta^2$). Analogously, the wave equation for the magnetic field (formally identical to (A.7)) is equivalent to Equation 1.4b for the line current. Hence, there is a clear link between plane wave propagation in source-free, linear, isotropic, and homogeneous media, TEM wave propagation in transmission lines and wave propagation in transmission lines as described by the distributed approach (this link is pointed out in Section 1.1).

The solutions for the transverse field components depend on the specific geometry of the line and are given by the Helmholtz equations (A.4) and (A.5). Both equations can be expressed as a pair of order-2 differential equations, that is,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0 \tag{A.23a}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E_y = 0 \tag{A.23b}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_x = 0 \tag{A.23c}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_y = 0 \tag{A.23d}$$

Taking into account the $e^{-j\beta z}$ z-dependence of the field components, the previous equations can be simplified and expressed in a compact form as follows:[3]

$$\nabla_t^2 \vec{E}_t(x,y) = 0 \tag{A.24a}$$

$$\nabla_t^2 \vec{H}_t(x,y) = 0 \tag{A.24b}$$

[3] Notice that $\partial^2 E_x/\partial z^2 = -\beta^2 E_x = -k^2 E_x$ and $\partial^2 E_y/\partial z^2 = -\beta^2 E_y = -k^2 E_y$. The same applies to the transverse components of the magnetic field.

where $\nabla_t^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian operator. The solutions of (A.24) are determined by the transverse geometry of the line and boundary conditions. Nevertheless, since the transverse fields satisfy the Laplace equation, these fields are the same as the static fields between the two conductors of the TEM line. In particular, the electric field can be expressed as the gradient of a scalar potential, that is,[4]

$$\vec{E}_t(x,y) = -\nabla_t V(x,y) \tag{A.25}$$

Since the divergence of the electric field is zero ($\nabla_t \cdot \vec{E}_t(x,y) = 0$), the scalar potential also satisfies the Laplace equation:

$$\nabla_t^2 V(x,y) = 0 \tag{A.26}$$

and the voltage between the two conductors is found as the path integral of the field

$$V_{12} = -\int_1^2 \vec{E}_t(x,y) \cdot d\vec{l} \tag{A.27}$$

Finally, the current in the conductor can be found from the Ampere law as follows:

$$I = \oint_C \vec{H}_t(x,y) \cdot d\vec{l} \tag{A.28}$$

where C is the cross-sectional contour of the conductor.

For the particular case of a parallel plate transmission line (Fig. A.1) with width W and height h (and $W >> h$ to neglect the fringing fields at the lateral sides), the solution

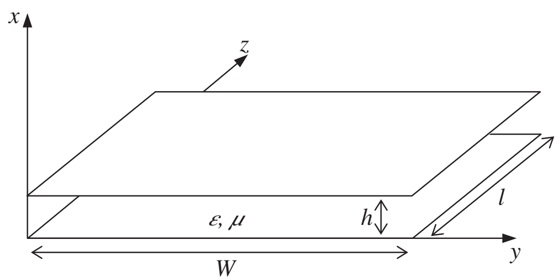

FIGURE A.1 Section of a parallel-plate transmission line of length l (the other relevant dimensions are also indicated). TEM waves propagate in the z-direction.

[4] This is valid if the two-dimensional curl of $\vec{E}_t(x,y)$, defined as $\partial E_y/\partial x - \partial E_x/\partial y$, is null. Equation A.18c reveals that this is the case.

of (A.26) subjected to the boundary conditions $V(x=0,\ y)=0$ and $V(x=h,\ y)=V_o$ gives

$$V(x,y)=\frac{V_o x}{h} \tag{A.29}$$

The total voltage and electric field are thus

$$V(x,y,z)=\frac{V_o x}{h}e^{-jkz} \tag{A.30}$$

$$\vec{E}(x,y,z)=-\vec{x}\frac{V_o}{h}e^{-jkz} \tag{A.31}$$

and the voltage between the two conductors is simply

$$V_{12}\equiv V=V_o e^{-jkz} \tag{A.32}$$

Using (A.21), the magnetic field is found to be

$$\vec{H}(x,y,z)=-\vec{y}\frac{V_o}{\eta h}e^{-jkz} \tag{A.33}$$

and the current is obtained from (A.28), where the z-dependence is introduced

$$I=\int_0^W \vec{y}\frac{V_o}{\eta h}e^{-jkz}\cdot\vec{y}\,dy=\frac{V_o W}{\eta h}e^{-jkz} \tag{A.34}$$

Finally the characteristic impedance is given by

$$Z_o=\frac{V}{I}=\eta\frac{h}{W} \tag{A.35}$$

It is worth mentioning that (1.9) can be obtained by calculating the per unit length capacitance C' and inductance L' of the parallel plate transmission line. Using the general expression of the time-averaged density of energy[5]

$$U_{nd}=\frac{1}{4}\left(\varepsilon\left|\vec{E}\right|^2+\mu\left|\vec{H}\right|^2\right) \tag{A.36}$$

it follows that the electric and magnetic energy stored in a volume V are given by

[5] Notice that this expression in only valid for nondispersive media. The general expression for both dispersive and nondispersive media is given in Chapter 3.

$$E_e = \frac{1}{4}\int \varepsilon \left|\vec{E}\right|^2 dV \tag{A.37a}$$

and

$$E_m = \frac{1}{4}\int \mu \left|\vec{H}\right|^2 dV \tag{A.37b}$$

respectively. Introducing (A.31) in (A.37a) and (A.33) in (A.37b), and integrating over a volume of the parallel plate transmission line delimited by its width W, its height h and an arbitrary length l, the stored electric and magnetic energy are found to be:

$$E_e = \frac{\varepsilon}{4} V_o^2 \frac{Wl}{h} \tag{A.38a}$$

$$E_m = \frac{\mu}{4} \frac{V_o^2}{\eta^2} \frac{Wl}{h} \tag{A.38b}$$

On the other hand, the energy stored by the capacitance C and inductance L of the line corresponding to a section of length l are, according to circuit theory, $CV_o^2/4$ and $LI_o^2/4$, respectively. Hence, the per-unit-length inductance and capacitance of the line are found to be

$$C' = \varepsilon \frac{W}{h} \tag{A.39a}$$

$$L' = \mu \frac{1}{\eta^2} \frac{V_o^2}{I_o^2} \frac{W}{h} = \varepsilon \frac{V_o^2}{I_o^2} \frac{W}{h} \tag{A.39b}$$

Combining (A.39a) and (A.39b), the characteristic impedance $Z_o = V_o/I_o$ as expressed by (1.9) is obtained.

The calculation of the fields, characteristic impedance, and propagation constant for other TEM transmission lines (e.g., stripline and coaxial line) and quasi-TEM lines (e.g., microstrip and CPW) is out of the scope of this appendix (see Ref. [1–8] of Chapter 1 for an in-depth study of these lines), which is only devoted to demonstrating the equivalence between the field theory and the distributed approach for the analysis of TEM transmission lines. For quasi-TEM lines, wave propagation can be approximated by TEM wave propagation (as described in this appendix) by simply replacing the permeability μ and permittivity ε (the parameters of the homogeneous dielectric material) with effective parameters (μ_{eff} and ε_{eff}) that take into account the transverse nonuniformity (nonhomogeneity) of the dielectric or the presence of air (in open lines).[6]

[6] The effective permeability or permittivity of an ordinary quasi-TEM transmission line depends on the material composition of the substrate and transverse geometry and should not be confused with the effective permeability and permittivity of metamaterials and metamaterial transmission lines (introduced in Chapter 3). It is also worth mentioning that in Chapter 2 nonuniform transmission lines are defined as transmission lines where the transverse geometry and/or material composition of the substrate change along the propagation direction z. In these lines, the effective permittivity and permeability are a function of z.