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2018 - 2019

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Mathematics

Advanced Stream

United Arab Emirates Edition

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2018 - 2019

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"Extensive knowledge and modern science must be acquired. The educational process we see today is in an ongoing and escalating challenge which requires hard work.

We succeeded in entering the third millennium, while we are more confident in ourselves."

H.H. Sheikh Khalifa Bin Zayed Al Nahyan

President of the United Arab Emirates

Contents in Brief

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CHAPTER 1

Linear Systems and Matrices

CHAPTER 2 Quadratic Functions and Relations

CHAPTER 4 Inverses and Radical Functions and Relations

CHAPTER 5 Circles

CHAPTER 6

Transformations and Symmetry

CHAPTER 7

Probability and Measurement

CHAPTER 8

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Rational Functions and Relations

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CHAPTER 11

Trigonometric Identities and Equations

CHAPTER 12

Sequences and Series

Student Handbook

Symbols, Formulas, and Key Concepts

Glossary is available in the electronic version

CHAPTER 9

Rational Functions and Relations

Then

You used factoring to solve quadratic equations and you graphed quadratic equations.

Now

You will:

- Simplify rational expressions.
- Graph rational functions.
- Solve direct, joint, and inverse variation problems.
- Solve rational equations and inequalities.

Why?

TRAVEL Whether you travel by boat, car, bicycle, or airplane, rational functions can be used to find distance traveled, time spent traveling, and speed. If you want to arrive at a destination on time, rational relations can tell you at what speed you need to travel to reach your goal. When graphing rational functions you see clearly how the speed at which you travel affects the time it takes to get there.

Get Ready for the Chapter

Diagnose Readiness

1 Textbook Option Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Solve each equation. Write in simplest form.

1. $\frac{5}{14} = \frac{1}{3}x$

2. $\frac{1}{8}m = \frac{7}{3}$

3. $\frac{8}{5} = \frac{1}{4}k$

4. $\frac{10}{9}p = 7$

5. **TRUCKS** Mazen used $\frac{1}{3}$ of a tank of gas in his truck to get to work. He began with a full tank of gas. If he has 68 liters of gas left, how many liters does his tank hold?

Simplify each expression.

6. $\frac{3}{4} - \frac{7}{8}$

7. $\frac{8}{9} - \frac{7}{6} + \frac{1}{3}$

8. $\frac{9}{10} - \frac{4}{15} + \frac{1}{3}$

9. $\frac{10}{3} + \frac{5}{6} + 3$

10. **BAKING** Alia baked cookies for a bake sale. She used $\frac{2}{3}$ cup of flour for one recipe and $4\frac{1}{2}$ cups of flour for the other recipe. How many cups did she use in all?

Solve each proportion.

11. $\frac{9}{12} = \frac{p}{36}$

12. $\frac{9}{18} = \frac{6}{m}$

13. $\frac{2}{7} = \frac{5}{k}$

14. **SALES TAX** Suhaila pays AED 4.40 tax on AED 55 worth of clothes. What amount of tax will she pay on AED 35 worth of clothes?

QuickReview

Example 1

Solve $\frac{9}{11} = \frac{7}{8}r$. Write in simplest form.

$\frac{9}{11} = \frac{7}{8}r$

$\frac{72}{11} = 7r$ Multiply each side by 8.

$\frac{72}{77} = r$ Divide each side by 7.

Since the GCF of 72 and 77 is 1, the solution is in simplest form.

Example 2

Simplify $\frac{1}{3} + \frac{3}{4} - \frac{5}{6}$.

$\frac{1}{3} + \frac{3}{4} - \frac{5}{6}$

$= \frac{1}{3}\left(\frac{4}{4}\right) + \frac{3}{4}\left(\frac{3}{3}\right) - \frac{5}{6}\left(\frac{2}{2}\right)$ The GCF of 3, 4, and 6 is 12.

$= \frac{4}{12} + \frac{9}{12} - \frac{10}{12}$ Simplify.

$= \frac{3}{12}$ Add and subtract.

$= \frac{3 \div 3}{12 \div 3}$ or $\frac{1}{4}$ Simplify.

Example 3

Solve $\frac{5}{8} = \frac{u}{11}$.

$\frac{5}{8} = \frac{u}{11}$ Write the equation.

$5(11) = 8u$ Find the cross products.

$55 = 8u$ Simplify.

$\frac{55}{8} = u$ Divide each side by 8.

Since the GCF of 55 and 8 is 1, the answer is in simplified form.

$u = \frac{55}{8}$ or $6\frac{7}{8}$.

Get Started on the Chapter

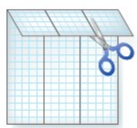
You will learn several new concepts, skills, and vocabulary terms as you study Chapter 9. To get ready, identify important terms and organize your resources.

FOLDABLES StudyOrganizer

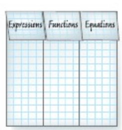
Rational Functions and Relations Make this Foldable to help you organize your Chapter 9 notes about rational functions and relations. Begin with an $8\frac{1}{2}'' \times 11''$ sheet of grid paper.

1 **Fold** in thirds along the height.

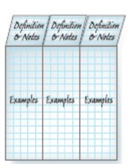
2 **Fold** the top edge down making a 2″ tab at the top. Cut along the folds.



3 **Label** the outside tabs *Expressions*, *Functions*, and *Equations*. Use the inside tabs for definitions and notes.



4 **Write** examples of each topic in the space below each tab.



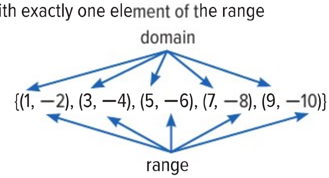
NewVocabulary

English

rational expression
complex fraction
reciprocal function
hyperbola
rational function
vertical asymptote
horizontal asymptote
oblique asymptote
point discontinuity
direct variation
constant of variation
joint variation
inverse variation
combined variation
rational equation
weighted average
rational inequality

ReviewVocabulary

function a relation in which each element of the domain is paired with exactly one element of the range



least common multiple the least number that is a common multiple of two or more numbers

rational number a number expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$

LESSON 9-1

Multiplying and Dividing Rational Expressions

Then

- You factored polynomials.

Now

1. Simplify rational expressions.
2. Simplify complex fractions.

Why?

- If a scuba diver goes to depths greater than 10 meters, the rational function $T(d) = \frac{1700}{d - 10}$ gives the maximum time a diver can remain at those depths and still surface at a steady rate with no stops. $T(d)$ represents the dive time in minutes, and d represents the depth in meters.

New Vocabulary

rational expression
complex fraction

Mathematical Practices

8 Look for and express regularity in repeated reasoning.

1 Simplify Rational Expressions

A ratio of two polynomial expressions such as $\frac{1700}{d - 10}$ is called a **rational expression**.

Because variables in algebra often represent real numbers, operations with rational numbers and rational expressions are similar. Just as with reducing fractions, to simplify a rational expression, you divide the numerator and denominator by their greatest common factor (GCF).

$$\frac{8}{12} = \frac{2 \cdot \cancel{4}^{1}}{3 \cdot \cancel{4}_{1}} = \frac{2}{3} \qquad \boxed{\text{GCF} = 4}$$

$$\frac{x^2 - 4x + 3}{x^2 - 6x + 5} = \frac{(x-3)\cancel{(x-1)}^{1}}{(x-5)\cancel{(x-1)}_{1}} = \frac{(x-3)}{(x-5)} \qquad \boxed{\text{GCF} = (x-1)}$$

Example 1 Simplify a Rational Expression

a. Simplify $\frac{5x(x^2 + 4x + 3)}{(x - 6)(x^2 - 9)}$.

$$\frac{5x(x^2 + 4x + 3)}{(x - 6)(x^2 - 9)} = \frac{5x(x+3)(x+1)}{(x-6)(x+3)(x-3)}$$ Factor numerator and denominator.

$$= \frac{5x(x+1)}{(x-6)(x-3)} \cdot \frac{\cancel{(x+3)}^{1}}{\cancel{(x+3)}_{1}}$$ Eliminate common factors.

$$= \frac{5x(x+1)}{(x-6)(x-3)}$$ Simplify.

b. Under what conditions is this expression undefined?

The original factored denominator is $(x - 6)(x + 3)(x - 3)$.
Determine the values that would make the denominator equal to 0.
These values are 6, -3, or 3, so the expression is undefined when $x = 6, -3$ or 3.

Guided Practice

Simplify each expression. Under what conditions is the expression undefined?

1A. $\frac{4y(y-3)(y+4)}{y(y^2 - y - 6)}$

1B. $\frac{2z(z+5)(z^2 + 2z - 8)}{(z-1)(z+5)(z-2)}$

Standardized Test Example 2 Use Elimination

For what value(s) is $\dfrac{x^2(x^2 - 5x - 14)}{4x(x^2 + 6x + 8)}$ undefined?

A $-2, -4$

B $-2, 7$

C $0, -2, -4$

D $0, -2, -4, 7$

Read the Test Item

You want to determine which values of x make the denominator equal to 0.

Solve the Test Item

With $4x$ in the denominator, x cannot equal 0. So, choices A and B can be eliminated. Next, factor the denominator.

$x^2 + 6x + 8 = (x + 2)(x + 4)$, so the denominator is $4x(x + 2)(x + 4)$.

Because the denominator equals 0 when $x = 0$, -2, and -4, the answer is C.

> **Test-Taking Tip**
>
> **Eliminating Choices** Sometimes you can save time by looking at the possible answers and eliminating choices.

Guided Practice

2. For what value(s) of x is $\dfrac{x(x^2 + 8x + 12)}{-6(x^2 - 3x - 10)}$ undefined?

F $0, 5, -2$ **G** $5, -2$ **H** $0, -2, -6$ **J** $5, -2, -6$

Sometimes you can factor out -1 in the numerator or denominator to help simplify a rational expression.

Example 3 Simplify Using −1

Simplify $\dfrac{(4w^2 - 3wy)(w + y)}{(3y - 4w)(5w + y)}$.

$$\frac{(4w^2 - 3wy)(w + y)}{(3y - 4w)(5w + y)} = \frac{w(4w - 3y)(w + y)}{(3y - 4w)(5w + y)}$$ Factor.

$$= \frac{w(-1)\overset{1}{\cancel{(3y - 4w)}}(w + y)}{\underset{1}{\cancel{(3y - 4w)}}(5w + y)}$$ $4w - 3y = -1(3y - 4w)$

$$= \frac{(-w)(w + y)}{5w + y}$$ Simplify.

Guided Practice

Simplify each expression.

3A. $\dfrac{(xz - 4z)}{z^2(4 - x)}$

3B. $\dfrac{ab^2 - 5ab}{(5 + b)(5 - b)}$

The method for multiplying and dividing fractions also works with rational expressions. Remember that to multiply two fractions, you multiply the numerators and multiply the denominators. To divide two fractions, you multiply by the multiplicative inverse, or the reciprocal, of the divisor.

Multiplication

$$\frac{2}{9} \cdot \frac{15}{4} = \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{3}} \cdot 5}{\underset{1}{\cancel{3}} \cdot 3 \cdot \underset{1}{\cancel{2}} \cdot 2} = \frac{5}{3 \cdot 2} = \frac{5}{6}$$

Division

$$\frac{3}{5} \div \frac{6}{35} = \frac{3}{5} \cdot \frac{35}{6} = \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{5}} \cdot 7}{\underset{1}{\cancel{5}} \cdot 2 \cdot \underset{1}{\cancel{3}}} = \frac{7}{2}$$

The following table summarizes the rules for multiplying and dividing rational expressions.

KeyConcept

Multiplying Rational Expressions	
Words	To multiply rational expressions, multiply the numerators and multiply the denominators.
Symbols	For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
Dividing Rational Expressions	
Words	To divide rational expressions, multiply by the reciprocal of the divisor.
Symbols	For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$, $c \neq 0$, and $d \neq 0$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

StudyTip

Eliminating Common Factors Be sure to eliminate factors from both the numerator and denominator.

Example 4 Multiply and Divide Rational Expressions

Simplify each expression.

a. $\frac{6c}{5d} \cdot \frac{15cd^2}{8a}$

$\frac{6c}{5d} \cdot \frac{15cd^2}{8a} = \frac{2 \cdot 3 \cdot c \cdot 5 \cdot 3 \cdot c \cdot d \cdot d}{5 \cdot d \cdot 2 \cdot 2 \cdot 2 \cdot a}$ Factor.

$= \frac{\overset{1}{\cancel{2}} \cdot 3 \cdot c \cdot \overset{1}{\cancel{5}} \cdot 3 \cdot c \cdot \overset{1}{\cancel{d}} \cdot d}{\underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{d}} \cdot \underset{1}{\cancel{2}} \cdot 2 \cdot 2 \cdot a}$ Eliminate common factors.

$= \frac{3 \cdot 3 \cdot c \cdot c \cdot d}{2 \cdot 2 \cdot a}$ Simplify.

$= \frac{9c^2d}{4a}$ Simplify.

b. $\frac{18xy^3}{7a^2b^2} \div \frac{12x^2y}{35a^2b}$

$\frac{18xy^3}{7a^2b^2} \div \frac{12x^2y}{35a^2b} = \frac{18xy^3}{7a^2b^2} \cdot \frac{35a^2b}{12x^2y}$ Multiply by reciprocal of the divisor.

$= \frac{2 \cdot 3 \cdot 3 \cdot x \cdot y \cdot y \cdot y \cdot 5 \cdot 7 \cdot a \cdot a \cdot b}{7 \cdot a \cdot a \cdot b \cdot b \cdot 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot y}$ Factor.

$= \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{3}} \cdot 3 \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{y}} \cdot y \cdot y \cdot 5 \cdot \overset{1}{\cancel{7}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{b}}}{\underset{1}{\cancel{7}} \cdot \underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{b}} \cdot b \cdot \underset{1}{\cancel{2}} \cdot 2 \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{x}} \cdot x \cdot \underset{1}{\cancel{y}}}$ Eliminate common factors.

$= \frac{3 \cdot 5 \cdot y \cdot y}{2 \cdot b \cdot x}$ Simplify.

$= \frac{15y^2}{2bx}$ Simplify.

GuidedPractice

4A. $\frac{12c^3d^2}{21ab} \cdot \frac{14a^2b}{8c^2d}$

4B. $\frac{6xy}{15ab^2} \cdot \frac{21a^3}{18x^4y}$

4C. $\frac{16mt^2}{21a^4b^3} \div \frac{24m^3}{7a^2b^2}$

4D. $\frac{12x^4y^2}{40a^4b^4} \div \frac{6x^2y^4}{16a^2x}$

Sometimes you must factor the numerator and/or the denominator first before you can simplify a product or a quotient of rational expressions.

StudyTip

Regularity When simplifying rational expressions, factors in one polynomial will often reappear in other polynomials. In Example 5a, $x - 8$ appears four times. Use this as a guide when factoring challenging polynomials.

Example 5 Polynomials in the Numerator and Denominator

Simplify each expression.

a. $\dfrac{x^2 - 6x - 16}{x^2 - 16x + 64} \cdot \dfrac{x - 8}{x^2 + 5x + 6}$

$$\frac{x^2 - 6x - 16}{x^2 - 16x + 64} \cdot \frac{x - 8}{x^2 + 5x + 6} = \frac{(x-8)(x+2)}{(x-8)(x-8)} \cdot \frac{x-8}{(x+3)(x+2)} \quad \text{Factor.}$$

$$= \frac{\overset{1}{\cancel{(x-8)}}\overset{1}{\cancel{(x+2)}}}{\underset{1}{\cancel{(x-8)}}\underset{1}{\cancel{(x-8)}}} \cdot \frac{\overset{1}{\cancel{x-8}}}{(x+3)\underset{1}{\cancel{(x+2)}}} \quad \text{Eliminate common factors.}$$

$$= \frac{1}{x+3} \quad \text{Simplify.}$$

b. $\dfrac{x^2 - 16}{12y + 36} \div \dfrac{x^2 - 12x + 32}{y^2 - 3y - 18}$

$$\frac{x^2 - 16}{12y + 36} \div \frac{x^2 - 12x + 32}{y^2 - 3y - 18} = \frac{x^2 - 16}{12y + 36} \cdot \frac{y^2 - 3y - 18}{x^2 - 12x + 32} \quad \text{Multiply by reciprocal.}$$

$$= \frac{(x+4)(x-4)}{12(y+3)} \cdot \frac{(y-6)(y+3)}{(x-4)(x-8)} \quad \text{Factor.}$$

$$= \frac{(x+4)\overset{1}{\cancel{(x-4)}}}{12\underset{1}{\cancel{(y+3)}}} \cdot \frac{(y-6)\overset{1}{\cancel{(y+3)}}}{\underset{1}{\cancel{(x-4)}}(x-8)} \quad \text{Eliminate common factors.}$$

$$= \frac{(x+4)(y-6)}{12(x-8)} \quad \text{Simplify.}$$

GuidedPractice

5A. $\dfrac{8x - 20}{x^2 + 2x - 35} \cdot \dfrac{x^2 - 7x + 10}{4x^2 - 16}$

5B. $\dfrac{x^2 - 9x + 20}{x^2 + 10x + 21} \div \dfrac{x^2 - x - 12}{6x + 42}$

2 Simplify Complex Fractions

A **complex fraction** is a rational expression with a numerator and/or denominator that is also a rational expression. The following expressions are complex fractions.

$$\frac{\frac{c}{6}}{5d} \qquad \frac{\frac{8}{x}}{x-2} \qquad \frac{\frac{x-3}{8}}{\frac{x-2}{x+4}} \qquad \frac{\frac{4}{a}+6}{\frac{12}{a}-3}$$

To simplify a complex fraction, first rewrite it as a division expression.

Example 6 Simplify Complex Fractions

Simplify each expression.

a. $\dfrac{\frac{a+b}{4}}{\frac{a^2+b^2}{4}}$

$$\frac{\frac{a+b}{4}}{\frac{a^2+b^2}{4}} = \frac{a+b}{4} \div \frac{a^2+b^2}{4} \quad \text{Express as a division expression.}$$

$$= \frac{a+b}{4} \cdot \frac{4}{a^2+b^2} \quad \text{Multiply by the reciprocal.}$$

$$= \frac{a+b}{\underset{1}{\cancel{4}}} \cdot \frac{\overset{1}{\cancel{4}}}{a^2+b^2} \text{ or } \frac{a+b}{a^2+b^2} \quad \text{Simplify.}$$

b. $\dfrac{\frac{x^2}{x^2 - y^2}}{\frac{4x}{y - x}}$

$$\frac{\frac{x^2}{x^2 - y^2}}{\frac{4x}{y - x}} = \frac{x^2}{x^2 - y^2} \div \frac{4x}{y - x}$$ Express as a division expression.

$$= \frac{x^2}{x^2 - y^2} \cdot \frac{y - x}{4x}$$ Multiply by the reciprocal.

$$= \frac{x \cdot x}{(x + y)(x - y)} \cdot \frac{(-1)(x - y)}{4x}$$ Factor.

$$= \frac{x \cdot \overset{1}{\cancel{x}}}{(x + y)\underset{1}{\cancel{(x - y)}}} \cdot \frac{(-1)\overset{1}{\cancel{(x - y)}}}{4\underset{1}{\cancel{x}}}$$ Eliminate Factors.

$$= \frac{-x}{4(x + y)}$$ Simplify.

GuidedPractice

Simplify each expression.

6A. $\dfrac{\frac{(x - 2)^2}{2(x^2 - 5x + 4)}}{\frac{x^2 - 4}{4x - 10}}$

6B. $\dfrac{\frac{x^2 - y^2}{y^2 - 49}}{\frac{y - x}{y + 7}}$

Check Your Understanding

Example 1 **Simplify each expression.**

1. $\dfrac{x^2 - 5x - 24}{x^2 - 64}$

2. $\dfrac{c + d}{3c^2 - 3d^2}$

Example 2

3. MULTIPLE CHOICE Identify all values of x for which $\dfrac{x + 7}{x^2 - 3x - 28}$ is undefined.

A $-7, 4$ **B** $7, 4$ **C** $4, -7, 7$ **D** $-4, 7$

Examples 3–6 **Simplify each expression.**

4. $\dfrac{y^2 + 3y - 40}{25 - y^2}$

5. $\dfrac{a^2x - b^2x}{by - ay}$

6. $\dfrac{27x^2y^4}{16yz^3} \cdot \dfrac{8z}{9xy^3}$

7. $\dfrac{12x^3y}{13ab^2} \div \dfrac{36xy^3}{26b}$

8. $\dfrac{x^2 - 4x - 21}{x^2 - 6x + 8} \cdot \dfrac{x - 4}{x^2 - 2x - 35}$

9. $\dfrac{a^2 - b^2}{3a^2 - 6a + 3} \div \dfrac{4a + 4b}{a^2 - 1}$

10. $\dfrac{\frac{a^3b^3}{xy^4}}{\frac{a^2b}{x^2y}}$

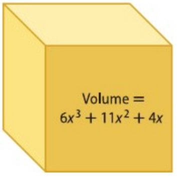
11. $\dfrac{\frac{4x}{x + 6}}{\frac{x^2 - 3x}{x^2 + 3x - 18}}$

12. SENSE-MAKING The volume of a shipping container in the shape of a rectangular prism can be represented by the polynomial $6x^3 + 11x^2 + 4x$, where the height is x.

a. Find the length and width of the container.

b. Find the ratio of the three dimensions of the container when $x = 2$.

c. Will the ratio of the three dimensions be the same for all values of x?



Practice and Problem Solving

Example 1 **Simplify each expression.**

13. $\frac{x(x-3)(x+6)}{x^2+x-12}$

14. $\frac{y^2(y^2+3y+2)}{2y(y-4)(y+2)}$

15. $\frac{(x^2-9)(x^2-z^2)}{4(x+z)(x-3)}$

16. $\frac{(x^2-16x+64)(x+2)}{(x^2-64)(x^2-6x-16)}$

17. $\frac{x^2(x+2)(x-4)}{6x(x^2+x-20)}$

18. $\frac{3y(y-8)(y^2+2y-24)}{15y^2(y^2-12y+32)}$

Example 2 19. **MULTIPLE CHOICE** Identify all values of x for which $\frac{(x-3)(x+6)}{(x^2-7x+12)(x^2-36)}$ is undefined.

F 3, −6 **G** 4, 6 **H** −6, 6 **J** −6, 3, 4, 6

Example 3 **Simplify each expression.**

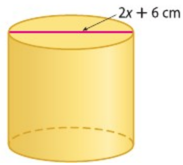
20. $\frac{x^2-5x-14}{28+3x-x^2}$

21. $\frac{x^3-9x^2}{x^2-3x-54}$

22. $\frac{(x-4)(x^2+2x-48)}{(36-x^2)(x^2+4x-32)}$

23. $\frac{16-c^2}{c^2+c-20}$

24. **GEOMETRY** The cylinder at the right has a volume of $(x+3)(x^2-3x-18)\pi$ cubic centimeters. Find the height of the cylinder.



Examples 4–6 **Simplify each expression.**

25. $\frac{3ac^3f^3}{8a^2bcf^4} \cdot \frac{12ab^2c}{18ab^3c^2f}$

26. $\frac{14xy^2z^3}{21w^4x^2yz} \cdot \frac{7wxyz}{12w^2y^3z}$

27. $\frac{64a^2b^5}{35b^2c^3f^4} \div \frac{12a^4b^3c}{70abcf^2}$

28. $\frac{9x^2yz}{5z^4} \div \frac{12x^4y^2}{50xy^4z^2}$

29. $\frac{15a^2b^2}{21ac} \cdot \frac{14a^4c^2}{6ab^3}$

30. $\frac{14c^2f^5}{9a^2} \div \frac{35cf^4}{18ab^3}$

31. $\frac{y^2+8y+15}{y-6} \cdot \frac{y^2-9y+18}{y^2-9}$

32. $\frac{c^2-6c-16}{c^2-d^2} \div \frac{c^2-8c}{c+d}$

33. $\frac{x^2+9x+20}{8x+16} \cdot \frac{4x^2+16x+16}{x^2-25}$

34. $\frac{3a^2+6a+3}{a^2-3a-10} \div \frac{12a^2-12}{a^2-4}$

35. $\dfrac{\frac{x^2-9}{6x-12}}{\frac{x^2+10x+21}{x^2-x-2}}$

36. $\dfrac{\frac{y-x}{z^3}}{\frac{x-y}{6z^2}}$

37. $\dfrac{\frac{a^2-b^2}{b^3}}{\frac{b^2-ab}{a^2}}$

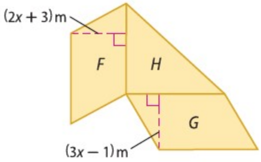
38. $\dfrac{\frac{x-y}{a+b}}{\frac{x^2-y^2}{b^2-a^2}}$

39. **REASONING** At the end of her high school soccer career, Muna had made 33 goals out of 121 attempts.

a. Write a ratio to represent the ratio of the number of goals made to goals attempted by Muna at the end of her high school career.

b. Suppose Muna attempted a goals and made m goals during her first year at college. Write a rational expression to represent the ratio of the number of career goals made to the number of career goals attempted at the end of her first year in college.

B **40.** **GEOMETRY** Parallelogram F has an area of $8x^2 + 10x - 3$ square meters and a height of $2x + 3$ meters. Parallelogram G has an area of $6x^2 + 13x - 5$ square meters and a height of $3x - 1$ meters. Find the area of right triangle H.



41. **POLLUTION** The thickness of an oil spill from a ruptured pipe on a rig is modeled by the function $T(x) = \dfrac{0.4(x^2 - 2x)}{x^3 + x^2 - 6x}$, where T is the thickness of the oil slick in meters and x is the distance from the rupture in meters.

a. Simplify the function.

b. How thick is the slick 100 meters from the rupture?

Simplify each expression.

42. $\dfrac{x^2 - 16}{3x^3 + 18x^2 + 24x} \cdot \dfrac{x^3 - 4x}{2x^2 - 7x - 4}$

43. $\dfrac{3x^2 - 17x - 6}{4x^2 - 20x - 24} \div \dfrac{6x^2 - 7x - 3}{2x^2 - x - 3}$

44. $\dfrac{9 - x^2}{x^2 - 4x - 21} \cdot \left(\dfrac{2x^2 + 7x + 3}{2x^2 - 15x + 7}\right)^{-1}$

45. $\left(\dfrac{2x^2 + 2x - 12}{x^2 + 4x - 5}\right)^{-1} \cdot \dfrac{2x^3 - 8x}{x^2 - 2x - 35}$

46. $\left(\dfrac{3xy^3z}{2a^2bc^2}\right)^3 \cdot \dfrac{16a^4b^3c^5}{15x^7yz^3}$

47. $\dfrac{20x^2y^6z^{-2}}{3a^3c^2} \cdot \left(\dfrac{16x^3y^3}{9acz}\right)^{-1}$

48. $\left(\dfrac{2xy^3}{3abc}\right)^{-2} \div \dfrac{6a^2b}{x^2y^4}$

49. $\dfrac{\dfrac{8x^2 - 10x - 3}{10x^2 + 35x - 20}}{\dfrac{2x^2 + x - 6}{4x^2 + 18x + 8}}$

50. $\dfrac{\dfrac{2x^2 + 7x - 30}{-6x^2 + 13x + 5}}{\dfrac{4x^2 + 12x - 72}{3x^2 - 11x - 4}}$

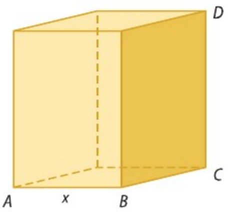
51. $\dfrac{\dfrac{4x^2 - 1}{3x^3 - 6x^2 - 24x}}{\dfrac{12x^2 + 12x - 9}{-2x^2 + 5x + 12}}$

52. **GEOMETRY** The area of the base of the rectangular prism at the right is 20 square centimeters.

a. Find the length of $\overline{BC}$ in terms of x.

b. If $DC = 3BC$, determine the area of the shaded region in terms of x.

c. Determine the volume of the prism in terms of x.



C **Simplify each expression.**

53. $\dfrac{x^2 + 4x - 32}{2x^2 + 9x - 5} \cdot \dfrac{3x^2 - 75}{3x^2 - 11x - 4} \div \dfrac{6x^2 - 18x - 60}{x^3 - 4x}$

54. $\dfrac{8x^2 + 10x - 3}{3x^2 - 12x - 36} \div \dfrac{2x^2 - 5x - 12}{3x^2 - 17x - 6} \cdot \dfrac{4x^2 + 3x - 1}{4x^2 - 40x + 24}$

55. $\dfrac{4x^2 - 9x - 9}{3x^2 + 6x - 18} \div \dfrac{-2x^2 + 5x + 3}{x^2 - 4x - 32} \div \dfrac{8x^2 + 10x + 3}{6x^2 - 6x - 12}$

56. **PERSEVERANCE** Use the formula $d = rt$ and the following information. An airplane is traveling at a rate r of 500 kilometers per hour for a time t of $(6 + x)$ hours. A second airplane travels at the rate r of $(540 + 90x)$ kilometers per hour for a time t of 6 hours.

a. Write a rational expression to represent the ratio of the distance d traveled by the first airplane to the distance d traveled by the second airplane.

b. Simplify the rational expression. What does this expression tell you about the distances traveled by the two airplanes?

c. Under what condition is the rational expression undefined? Describe what this condition would tell you about the two airplanes.

57. **TRAINS** Trying to get into a train yard one evening, all of the trains are backed up for 2 miles along a system of tracks. Assume that each car occupies an average of 75 feet of space on a track and that the train yard has 5 tracks.

a. Write an expression that could be used to determine the number of train cars involved isn the backup.

b. How many train cars are involved in the backup?

c. Suppose that there are 8 attendants doing safety checks on each car, and it takes each vehicle an average of 45 seconds for each check. Approximately how many hours will it take for all the vehicles in the backup to exit?

58. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the graph of a rational function.

a. **Algebraic** Simplify $\frac{x^2 - 5x + 4}{x - 4}$.

b. **Tabular** Let $f(x) = \frac{x^2 - 5x + 4}{x - 4}$. Use the expression you wrote in part **a** to write the related function $g(x)$. Use a graphing calculator to make a table for both functions for $0 \leq x \leq 10$.

c. **Analytical** What are $f(4)$ and $g(4)$? Explain the significance of these values.

d. **Graphical** Graph the functions on the graphing calculator. Use the **TRACE** function to investigate each graph, using the ▲ and ▼ keys to switch from one graph to the other. Compare and contrast the graphs.

e. **Verbal** What conclusions can you draw about the expressions and the functions?

H.O.T. Problems Use Higher-Order Thinking Skills

59. **REASONING** Compare and contrast $\frac{(x-6)(x+2)(x+3)}{x+3}$ and $(x-6)(x+2)$.

60. **CRITIQUE** Humaid and Hamad are simplifying $\frac{x+y}{x-y} \div \frac{4}{y-x}$. Is either of them correct? Explain your reasoning.

Humaid

$$\frac{x+y}{x-y} \div \frac{4}{y-x} = \frac{x-y}{x+y} \cdot \frac{4}{y-x}$$
$$= \frac{-4}{x+y}$$

Hamad

$$\frac{x+y}{x-y} \div \frac{4}{y-x} = \frac{x+y}{x-y} \cdot \frac{y-x}{4}$$
$$= -\frac{x+y}{4}$$

61. **CHALLENGE** Find the expression that makes the following statement true.

$$\frac{x-6}{x+3} \cdot \frac{?}{x-6} = x - 2$$

62. **WHICH ONE DOESN'T BELONG?** Identify the expression that does not belong with the other three. Explain your reasoning.

$\frac{1}{x-1}$	$\frac{x^2+3x+2}{x-5}$	$\frac{x+1}{\sqrt{x+3}}$	$\frac{x^2+1}{3}$

63. **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

A rational function that has a variable in the denominator is defined for all real values of x.

64. **OPEN ENDED** Write a rational expression that simplifies to $\frac{x-1}{x+4}$.

65. **WRITING IN MATH** The rational expression $\frac{x^2+3x}{4x}$ is simplified to $\frac{x+3}{4}$. Explain why this new expression is not defined for all values of x.

Standardized Test Practice

66. SAT/ACT Fahd's family wants to drive an average of 250 kilometers per day on their vacation. On the first five days, they travel 220 kilometers, 300 kilometers, 210 kilometers, 275 kilometers, and 240 kilometers. How many kilometers must they travel on the sixth day to meet their goal?

A 235 kilometers
B 251 kilometers
C 255 kilometers
D 275 kilometers
E 315 kilometers

67. Which of the following equations gives the relationship between N and T in the table?

N	1	2	3	4	5	6
T	1	4	7	10	13	16

F $T = 2 - N$
G $T = 4 - 3N$
H $T = 3N + 1$
J $T = 3N - 2$

68. A monthly cell phone plan costs AED 39.99 for up to 300 minutes and 20 fils per minute thereafter. Which of the following represents the total monthly bill (in dirhams) to talk for x minutes if x is greater than 300?

A $39.99 + 0.20(300 - x)$
B $39.99 + 0.20(x - 300)$
C $39.99 + 0.20x$
D $39.99 + 20x$

69. SHORT RESPONSE The area of a circle 6 meters in diameter exceeds the combined areas of a circle 4 meters in diameter and a circle 2 meters in diameter by how many square meters?

Spiral Review

70. ANTHROPOLOGY An anthropologist studying the bones of a prehistoric person finds there is so little remaining Carbon-14 in the bones that instruments cannot measure it. This means that there is less than 0.5% of the amount of Carbon-14 the bones would have contained when the person was alive. The half-life of Carbon-14 is 5760 years. How long ago did the person die?

Solve each equation. Round to the nearest ten-thousandth.

71. $3e^x + 1 = 5$ **72.** $2e^x - 1 = 0$ **73.** $-3e^{4x} + 11 = 2$ **74.** $8 + 3e^{3x} = 26$

75. NOISE ORDINANCE A proposed city ordinance will make it illegal in a residential area to create sound that exceeds 72 decibels during the day and 55 decibels during the night. How many times as intense is the noise level allowed during the day than at night?

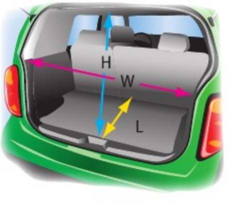
Simplify.

76. $\sqrt{50x^4}$ **77.** $\sqrt[3]{16y^3}$ **78.** $\sqrt{18x^2y^3}$ **79.** $\sqrt{40a^3b^4}$

80. AUTOMOBILES The length of the cargo space in a sport-utility vehicle is 10.2 centimeters greater than the height of the space. The width is 40.6 centimeters less than twice the height. The cargo space has a total volume of 906,139.1 cubic centimeters.

a. Write a polynomial function that represents the volume of the cargo space.

b. Will a package 86.4 centimeters long, 111.8 centimeters wide, and 86.4 centimeters tall fit in the cargo space? Explain.



Skills Review

Simplify.

81. $(2a + 3b) + (8a - 5b)$ **82.** $(x^2 - 4x + 3) - (4x^2 + 3x - 5)$ **83.** $(5y + 3y^2) + (-8y - 6y^2)$

84. $2x(3y + 9)$ **85.** $(x + 6)(x + 3)$ **86.** $(x + 1)(x^2 - 2x + 3)$

LESSON 9-2

Adding and Subtracting Rational Expressions

Then

- You added and subtracted polynomial expressions.

Now

1. Determine the LCM of polynomials.
2. Add and subtract rational expressions.

Why?

- As a fire engine moves toward a person, the pitch of the siren sounds higher to that person than it would if the fire engine were at rest. This is because the sound waves are compressed closer together, referred to as the *Doppler effect*. The Doppler effect can be represented by the rational expression $P_0\left(\frac{s_0}{s_0 - v}\right)$, where P_0 is the actual pitch of the siren, v is the speed of the fire truck, and s_0 is the speed of sound in air.

Mathematical Practices
3 Construct viable arguments and critique the reasoning of others.

1 LCM of Polynomials

Just as with rational numbers in fractional form, to add or subtract two rational expressions that have unlike denominators, you must first find the least common denominator (LCD). The LCD is the least common multiple (LCM) of the denominators.

To find the LCM of two or more numbers or polynomials, factor them. The LCM contains each factor the greatest number of times it appears as a factor.

Numbers	Polynomials
$\frac{5}{6} + \frac{4}{9}$	$\frac{3}{x^2 - 3x + 2} + \frac{5}{2x^2 - 2}$
LCM of 6 and 9	**LCM of $x^2 - 3x + 2$ and $2x^2 - 2$**
$6 = 2 \cdot 3$	$x^2 - 3x + 2 = (x - 1)(x - 2)$
$9 = 3 \cdot 3$	$2x^2 - 2 = 2 \cdot (x - 1)(x + 1)$
LCM $= 2 \cdot 3 \cdot 3$ or 18	LCM $= 2(x - 1)(x - 2)(x + 1)$

Example 1 LCM of Monomials and Polynomials

Find the LCM of each set of polynomials.

a. $6xy$, $15x^2$, and $9xy^4$

$6xy = 2 \cdot 3 \cdot x \cdot y$ — Factor the first monomial.

$15x^2 = 3 \cdot 5 \cdot x^2$ — Factor the second monomial.

$9xy^4 = 3 \cdot 3 \cdot x \cdot y^4$ — Factor the third monomial.

$\text{LCM} = 2 \cdot 3 \cdot 3 \cdot 5 \cdot x^2 \cdot y^4$ — Use each factor the greatest number of times it appears.

$= 90x^2y^4$ — Then simplify.

b. $y^4 + 8y^3 + 15y^2$ and $y^2 - 3y - 40$

$y^4 + 8y^3 + 15y^2 = y^2(y + 5)(y + 3)$ — Factor the first polynomial.

$y^2 - 3y - 40 = (y + 5)(y - 8)$ — Factor the second polynomial.

$\text{LCM} = y^2(y + 5)(y + 3)(y - 8)$ — Use each factor the greatest number of times it appears as a factor.

GuidedPractice

1A. $12a^2b$, $15abc$, $8b^3c^4$

1B. $4a^2 - 12a - 16$ and $a^3 - 9a^2 + 20a$

2 Add and Subtract Rational Expressions

As with fractions, rational expressions must have common denominators in order to be added or subtracted.

KeyConcept

Adding Rational Expressions	
Words	To add rational expressions, find the least common denominator (LCD). Rewrite each expression with the LCD. Then add.
Symbols	For all $\frac{a}{b}$ and $\frac{c}{d}$, with $b \neq 0$ and $d \neq 0$, $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$.
Subtracting Rational Expressions	
Words	To subtract rational expressions, find the least common denominator (LCD). Rewrite each expression with the LCD. Then subtract.
Symbols	For all $\frac{a}{b}$ and $\frac{c}{d}$, with $b \neq 0$ and $d \neq 0$, $\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}$.

Example 2 Monomial Denominators

Simplify $\frac{3y}{2x^3} + \frac{5z}{8xy^2}$.

$\frac{3y}{2x^3} + \frac{5z}{8xy^2} = \frac{3y}{2x^3} \cdot \frac{4y^2}{4y^2} + \frac{5z}{8xy^2} \cdot \frac{x^2}{x^2}$ The LCD is $8x^3y^2$.

$= \frac{12y^3}{8x^3y^2} + \frac{5x^2z}{8x^3y^2}$ Multiply fractions.

$= \frac{12y^3 + 5x^2z}{8x^3y^2}$ Add the numerators.

GuidedPractice

Simplify each expression.

2A. $\frac{4}{5a^3b^2} + \frac{9c}{10ab}$

2B. $\frac{3a^2}{16b^2} - \frac{8x}{5a^3b}$

The LCD is also used to combine rational expressions with polynomial denominators.

Example 3 Polynomial Denominators

Simplify $\frac{5}{6x - 18} - \frac{x - 1}{4x^2 - 14x + 6}$.

$\frac{5}{6x - 18} - \frac{x - 1}{4x^2 - 14x + 6} = \frac{5}{6(x - 3)} - \frac{x - 1}{2(2x - 1)(x - 3)}$ Factor denominators.

$= \frac{5(2x - 1)}{6(x - 3)(2x - 1)} - \frac{(x - 1)(3)}{2(2x - 1)(x - 3)(3)}$ Multiply by missing factors.

$= \frac{10x - 5 - 3x + 3}{6(x - 3)(2x - 1)}$ Subtract numerators.

$= \frac{7x - 2}{6(x - 3)(2x - 1)}$ Simplify.

StudyTip

Simplifying Rational Expressions After you add or subtract rational expressions, it is possible that the resulting expression can be further simplified.

GuidedPractice

Simplify each expression.

3A. $\frac{x - 1}{x^2 - x - 6} - \frac{4}{5x + 10}$

3B. $\frac{x - 8}{4x^2 + 21x + 5} + \frac{6}{12x + 3}$

One way to simplify a complex fraction is to simplify the numerator and the denominator separately, and then simplify the resulting expressions.

StudyTip

Undefined Terms Remember that there are restrictions on variables in the denominator.

Example 4 Complex Fractions with Different LCDs

Simplify $\dfrac{1+\frac{1}{x}}{1-\frac{x}{y}}$.

$\dfrac{1+\frac{1}{x}}{1-\frac{x}{y}} = \dfrac{\frac{x}{x}+\frac{1}{x}}{\frac{y}{y}-\frac{x}{y}}$ — The LCD of the numerator is x. The LCD of the denominator is y.

$= \dfrac{\frac{x+1}{x}}{\frac{y-x}{y}}$ — Simplify the numerator and denominator.

$= \dfrac{x+1}{x} \div \dfrac{y-x}{y}$ — Write as a division expression.

$= \dfrac{x+1}{x} \cdot \dfrac{y}{y-x}$ — Multiply by the reciprocal of the divisor.

$= \dfrac{xy+y}{xy-x^2}$ — Simplify.

GuidedPractice

Simplify each expression.

4A. $\dfrac{1-\frac{y}{x}}{\frac{1}{y}+\frac{1}{x}}$

4B. $\dfrac{\frac{c}{d}-\frac{d}{c}}{\frac{d}{c}+2}$

Another method of simplifying complex fractions is to find the LCD of all of the denominators. Then, the denominators are all eliminated by multiplying by the LCD.

Example 5 Complex Fractions with Same LCDs

Simplify $\dfrac{1+\frac{1}{x}}{1-\frac{x}{y}}$.

$\dfrac{1+\frac{1}{x}}{1-\frac{x}{y}} = \dfrac{\left(1+\frac{1}{x}\right)}{\left(1-\frac{x}{y}\right)} \cdot \dfrac{xy}{xy}$ — The LCD of all of the denominators is xy. Multiply by $\frac{xy}{xy}$.

$= \dfrac{xy+y}{xy-x^2}$ — Distribute xy.

Notice that the same problem is solved in Examples 4 and 5 using different methods, but both produce the same answer. So, how you solve problems similar to these is left up to your own discretion.

GuidedPractice

Simplify each expression.

5A. $\dfrac{1+\frac{2}{x}}{\frac{3}{y}-\frac{4}{x}}$

5B. $\dfrac{\frac{1}{d}-\frac{d}{c}}{\frac{1}{c}+6}$

5C. $\dfrac{\frac{1}{y}+\frac{1}{x}}{\frac{1}{y}-\frac{1}{x}}$

5D. $\dfrac{\frac{a}{b}+1}{1-\frac{b}{a}}$

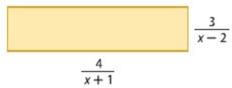
Check Your Understanding

Example 1 **Find the LCM of each set of polynomials.**

1. $16x$, $8x^2y^3$, $5x^3y$
2. $7a^2$, $9ab^3$, $21abc^4$
3. $3y^2 - 9y$, $y^2 - 8y + 15$
4. $x^3 - 6x^2 - 16x$, $x^2 - 4$

Examples 2–3 **Simplify each expression.**

5. $\dfrac{12y}{5x} + \dfrac{5x}{4y^3}$
6. $\dfrac{5}{6ab} + \dfrac{3b^2}{14a^3}$
7. $\dfrac{7b}{12a} - \dfrac{1}{18ab^3}$
8. $\dfrac{y^2}{8c^2d^2} - \dfrac{3x}{14c^4d}$
9. $\dfrac{4x}{x^2 + 9x + 18} + \dfrac{5}{x + 6}$
10. $\dfrac{8}{y - 3} + \dfrac{2y - 5}{y^2 - 12y + 27}$
11. $\dfrac{4}{3x + 6} - \dfrac{x + 1}{x^2 - 4}$
12. $\dfrac{3a + 2}{a^2 - 16} - \dfrac{7}{6a + 24}$
13. **GEOMETRY** Find the perimeter of the rectangle.



Examples 4–5 **Simplify each expression.**

14. $\dfrac{4 + \frac{2}{x}}{3 - \frac{2}{x}}$
15. $\dfrac{6 + \frac{4}{y}}{2 + \frac{6}{y}}$
16. $\dfrac{\frac{3}{x} + \frac{2}{y}}{1 + \frac{4}{y}}$
17. $\dfrac{\frac{2}{b} + \frac{5}{a}}{\frac{3}{a} - \frac{8}{b}}$

Practice and Problem Solving

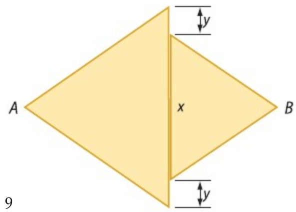
Example 1 **Find the LCM of each set of polynomials.**

18. $24cd$, $40a^2c^3d^4$, $15abd^3$
19. $4x^2y^3$, $18xy^4$, $10xz^2$
20. $x^2 - 9x + 20$, $x^2 + x - 30$
21. $6x^2 + 21x - 12$, $4x^2 + 22x + 24$

Examples 2–3 **PERSEVERANCE** **Simplify each expression.**

22. $\dfrac{5a}{24cf^4} + \dfrac{a}{36bc^4f^3}$
23. $\dfrac{4b}{15x^3y^2} - \dfrac{3b}{35x^2y^4z}$
24. $\dfrac{5b}{6a} + \dfrac{3b}{10a^2} + \dfrac{2}{ab^2}$
25. $\dfrac{4}{3x} + \dfrac{8}{x^3} + \dfrac{2}{5xy}$
26. $\dfrac{8}{3y} + \dfrac{2}{9} - \dfrac{3}{10y^2}$
27. $\dfrac{1}{16a} + \dfrac{5}{12b} - \dfrac{9}{10b^3}$
28. $\dfrac{8}{x^2 - 6x - 16} + \dfrac{9}{x^2 - 3x - 40}$
29. $\dfrac{6}{y^2 - 2y - 35} + \dfrac{4}{y^2 + 9y + 20}$
30. $\dfrac{12}{3y^2 - 10y - 8} - \dfrac{3}{y^2 - 6y + 8}$
31. $\dfrac{6}{2x^2 + 11x - 6} - \dfrac{8}{x^2 + 3x - 18}$
32. $\dfrac{2x}{4x^2 + 9x + 2} + \dfrac{3}{2x^2 - 8x - 24}$
33. $\dfrac{4x}{3x^2 + 3x - 18} - \dfrac{2x}{2x^2 + 11x + 15}$
34. **BIOLOGY** After a person eats something, the pH or acid level A of his or her mouth can be determined by the formula $A = \dfrac{20.4t}{t^2 + 36} + 6.5$, where t is the number of minutes that have elapsed since the food was eaten.
    a. Simplify the equation.
    b. What would the acid level be after 30 minutes?

35. **GEOMETRY** Both triangles in the figure at the right are equilateral. If the area of the smaller triangle is 200 square centimeters and the area of the larger triangle is 300 square centimeters, find the minimum distance from A to B in terms of x and y and simplify.



Examples 4–5 **Simplify each expression.**

36. $\dfrac{\frac{2}{x-3}+\frac{3x}{x^2-9}}{\frac{3}{x+3}-\frac{4x}{x^2-9}}$

37. $\dfrac{\frac{4}{x+5}+\frac{9}{x-6}}{\frac{5}{x-6}-\frac{8}{x+5}}$

38. $\dfrac{\frac{5}{x+6}-\frac{2x}{2x-1}}{\frac{x}{2x-1}+\frac{4}{x+6}}$

39. $\dfrac{\frac{8}{x-9}-\frac{x}{3x+2}}{\frac{3}{3x+2}+\frac{4x}{x-9}}$

B 40. **OIL PRODUCTION** Managers of an oil company have estimated that oil will be pumped from a certain well at a rate based on the function $R(x) = \frac{20}{x} + \frac{200x}{3x^2 + 20}$, where $R(x)$ is the rate of production in thousands of barrels per year x years after pumping begins.

a. Simplify $R(x)$.

b. At what rate will oil be pumping from the well in 50 years?

Find the LCM of each set of polynomials.

41. $12xy^4, 14x^4y^2, 5xyz^3, 15x^5y^3$

42. $-6abc^2, 18a^2b^2, 15a^4c, 8b^3$

43. $x^2 - 3x - 28, 2x^2 + 9x + 4, x^2 - 16$

44. $x^2 - 5x - 24, x^2 - 9, 3x^2 + 8x - 3$

Simplify each expression.

45. $\frac{1}{12a} + 6 - \frac{3}{5a^2}$

46. $\frac{5}{16y^2} - 4 - \frac{8}{3x^2y}$

47. $\frac{5}{6x^2 + 46x - 16} + \frac{2}{6x^2 + 57x + 72}$

48. $\frac{1}{8x^2 - 20x - 12} + \frac{4}{6x^2 + 27x + 12}$

49. $\frac{x^2 + y^2}{x^2 - y^2} + \frac{y}{x + y} - \frac{x}{x - y}$

50. $\frac{x^2 + x}{x^2 - 9x + 8} + \frac{4}{x - 1} - \frac{3}{x - 8}$

51. $\dfrac{\frac{2}{a-1}+\frac{3}{a-4}}{\frac{6}{a^2-5a+4}}$

52. $\dfrac{\frac{1}{x}+\frac{1}{y}}{\left(\frac{1}{x}-\frac{1}{y}\right)(x+y)}$

53. **GEOMETRY** An expression for the length of one rectangle is $\frac{x^2 - 9}{x - 2}$. The length of a similar rectangle is expressed as $\frac{x + 3}{x^2 - 4}$. What is the scale factor of the lengths of the two rectangles? Write in simplest form.

54. **MODELING** Saeed is taking a 20-kilometer kayaking trip. He travels half the distance at one rate. The rest of the distance he travels 2 kilometers per hour slower.

a. If x represents the faster pace in kilometers per hour, write an expression that represents the time spent at that pace.

b. Write an expression for the amount of time spent at the slower pace.

c. Write an expression for the amount of time Saeed needed to complete the trip.

Find the slope of the line that passes through each pair of points.

55. $A\left(\frac{2}{p}, \frac{1}{2}\right)$ and $B\left(\frac{1}{3}, \frac{3}{p}\right)$

56. $C\left(\frac{1}{4}, \frac{4}{q}\right)$ and $D\left(\frac{5}{q}, \frac{1}{5}\right)$

57. $E\left(\frac{7}{w}, \frac{1}{7}\right)$ and $F\left(\frac{1}{7}, \frac{7}{w}\right)$

58. $G\left(\frac{6}{n}, \frac{1}{6}\right)$ and $H\left(\frac{1}{6}, \frac{6}{n}\right)$

59. **PHOTOGRAPHY** The focal length of a lens establishes the field of view of the camera. The shorter the focal length is, the larger the field of view. For a camera with a fixed focal length of 70 mm to focus on an object x mm from the lens, the film must be placed a distance y from the lens. This is represented by $\frac{1}{x} + \frac{1}{y} = \frac{1}{70}$.

a. Express y as a function of x.

b. What happens to the focusing distance when the object is 70 mm away?

60. **PHARMACOLOGY** Two drugs are administered to a patient. The concentrations in the bloodstream of each are given by $f(t) = \frac{2t}{3t^2 + 9t + 6}$ and $g(t) = \frac{3t}{2t^2 + 6t + 4}$ where t is the time, in hours, after the drugs are administered.

a. Add the two functions together to determine a function for the total concentration of drugs in the patient's bloodstream.

b. What is the concentration of drugs after 8 hours?

61. **DOPPLER EFFECT** Refer to the application at the beginning of the lesson. Ahmed is equidistant from two fire engines traveling toward him from opposite directions.

a. Let x be the speed of the faster fire engine and y be the speed of the slower fire engine. Write and simplify a rational expression representing the difference in pitch between the two sirens according to Ahmed.

b. If one is traveling at 45 meters per second and the other is traveling at 70 meters per second, what is the difference in their pitches according to Ahmed? The speed of sound in air is 332 meters per second, and both engines have a siren with a pitch of 500 Hz.

62. **RESEARCH** A student studying learning behavior performed an experiment in which a rat was repeatedly sent through a maze. It was determined that the time it took the rat to complete the maze followed the rational function $T(x) = 4 + \frac{10}{x}$, where x represented the number of trials.

a. What is the domain of the function?

b. Graph the function for $0 \leq x \leq 10$.

c. Make a table of the function for $x = 20, 50, 100, 200$, and 400.

d. If it were possible to have an infinite number of trials, what do you think would be the rat's best time? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

C 63. **CHALLENGE** Simplify $\dfrac{5x^{-2} - \frac{x+1}{x}}{\frac{4}{3 - x^{-1}} + 6x^{-1}}$.

64. **ARGUMENTS** The sum of any two rational numbers is always a rational number. So, the set of rational numbers is said to be closed under addition. Determine whether the set of rational expressions is closed under addition, subtraction, multiplication, and division by a nonzero rational expression. Justify your reasoning.

65. **OPEN ENDED** Write three monomials with an LCM of $180a^4b^6c$.

66. **WRITING IN MATH** Write a how-to manual for adding rational expressions that have unlike denominators. How does this compare to adding rational numbers?

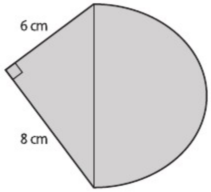
Standardized Test Practice

67. PROBABILITY A drawing is to be held to select the winner of a new bike. There are 100 Grade 12 students, 150 Grade 10 students, and 200 Grade 11 students who had correct entries. The drawing will contain 3 tickets for each Grade 12 name, 2 for each Grade 10, and 1 for each Grade 11. What is the probability that a Grade 12 student's ticket will be chosen?

A $\frac{1}{8}$ C $\frac{2}{7}$

B $\frac{2}{9}$ D $\frac{3}{8}$

68. SHORT RESPONSE Find the area of the figure.



69. SAT/ACT If Nasser receives b books in addition to the number of books he had, he will have t times as many as he had originally. In terms of b and t, how many books did Nasser have at the beginning?

F $\frac{b}{t-1}$ J $\frac{b}{t}$

G $\frac{b}{t+1}$ K $\frac{t}{b}$

H $\frac{t+1}{b}$

70. If $\frac{2a}{a} + \frac{1}{a} = 4$, then $a =$ ____.

A 2 C $\frac{1}{8}$

B $\frac{1}{2}$ D $-\frac{1}{8}$

Spiral Review

Simplify each expression. (Lesson 9-1)

71. $\frac{-4ab}{21c} \cdot \frac{14c^2}{22a^2}$

72. $\frac{x^2 - y^2}{6y} \div \frac{x + y}{36y^2}$

73. $\frac{n^2 - n - 12}{n + 2} \div \frac{n - 4}{n^2 - 4n - 12}$

74. BIOLOGY Bacteria usually reproduce by a process known as *binary fission*. In this type of reproduction, one bacterium divides, forming two bacteria. Under ideal conditions, some bacteria reproduce every 20 minutes.

a. Find the constant k for this type of bacteria under ideal conditions.

b. Write the equation for modeling the exponential growth of this bacterium.

Graph each function. State the domain and range of each function.

75. $y = -\sqrt{2x + 1}$

76. $y = \sqrt{5x - 3}$

77. $y = \sqrt{x + 6} - 3$

78. $y = 5 - \sqrt{x + 4}$

79. $y = \sqrt{3x - 6} + 4$

80. $y = 2\sqrt{3 - 4x} + 3$

Solve each equation. State the number and type of roots.

81. $3x + 8 = 0$

82. $2x^2 - 5x + 12 = 0$

83. $x^3 + 9x = 0$

84. $x^4 - 81 = 0$

Skills Review

Graph each function.

85. $y = 4(x + 3)^2 + 1$

86. $y = -(x - 5)^2 - 3$

87. $y = \frac{1}{4}(x - 2)^2 + 4$

88. $y = \frac{1}{2}(x - 3)^2 - 5$

89. $y = x^2 + 6x + 2$

90. $y = x^2 - 8x + 18$

LESSON 9-3 Graphing Reciprocal Functions

Then

- You graphed polynomial functions.

Now

1. Determine properties of reciprocal functions.
2. Graph transformations of reciprocal functions.

Why?

- The Al Khaldiya Secondary School Chorale wants to raise AED 5,000 to fund a trip to a national competition in Dubai. They have decided to sell candy bars. They will make a AED 1 profit on each candy bar they sell, so they need to sell 5000 candy bars.

If c represents the number of candy bars each student has to sell and n represents the number of students, then $c = \frac{5000}{n}$.

New Vocabulary

reciprocal function
hyperbola

Mathematical Practices

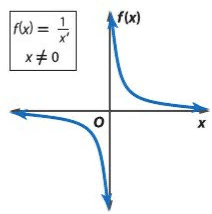
2 Reason abstractly and quantitatively.

1 Vertical and Horizontal Asymptotes

The function $c = \frac{5000}{n}$ is a reciprocal function. A **reciprocal function** has an equation of the form $f(x) = \frac{1}{a(x)}$, where $a(x)$ is a linear function and $a(x) \neq 0$.

Key Concept Parent Function of Reciprocal Functions

Parent function:	$f(x) = \frac{1}{x}$
Type of graph:	**hyperbola**
Domain and range:	all nonzero real numbers
Asymptotes:	$x = 0$ and $f(x) = 0$
Intercepts:	none
Not defined:	$x = 0$



The domain of a reciprocal function is limited to values for which the function is defined.

Functions:	$f(x) = \frac{-3}{x+2}$	$g(x) = \frac{4}{x-5}$	$h(x) = \frac{3}{x}$
Not defined at:	$x = -2$	$x = 5$	$x = 0$

Example 1 Limitations on Domain

Determine the value of x for which $f(x) = \frac{3}{2x+5}$ is not defined.

Find the value for which the denominator of the expression equals 0.

$$\frac{3}{2x+5} \rightarrow 2x + 5 = 0$$

$$x = -\frac{5}{2}$$ The function is undefined for $x = -\frac{5}{2}$.

Guided Practice

Determine the value of x for which each function is not defined.

1A. $f(x) = \frac{2}{x-1}$

1B. $f(x) = \frac{7}{3x+2}$

The graphs of reciprocal functions may have breaks in continuity for excluded values. Some may have an asymptote, which is a line that the graph of the function approaches.

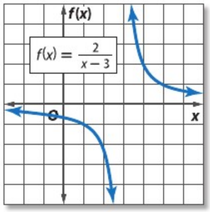
StudyTip

Structure Vertical asymptotes show where a function is undefined, while horizontal asymptotes show the end behavior of a graph.

Example 2 Determine Properties of Reciprocal Functions

Identify the asymptotes, domain, and range of each function.

a.



Identify x-values for which $f(x)$ is undefined.

$x - 3 = 0$

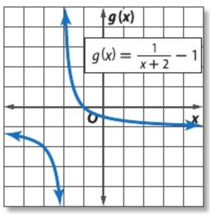
$x = 3$

$f(x)$ is not defined when $x = 3$. So there is an asymptote at $x = 3$.

From $x = 3$, as x-values decrease, $f(x)$-values approach 0, and as x-values increase, $f(x)$-values approach 0. So there is an asymptote at $f(x) = 0$.

The domain is all real numbers not equal to 3 and the range is all real numbers not equal to 0.

b.



Identify x-values for which $g(x)$ is undefined.

$x + 2 = 0$

$x = -2$

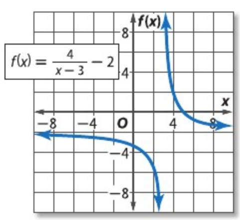
$g(x)$ is not defined when $x = -2$. So there is an asymptote at $x = -2$.

From $x = -2$, as x-values decrease, $g(x)$-values approach -1, and as x-values increase, $g(x)$-values approach -1. So there is an asymptote at $g(x) = -1$.

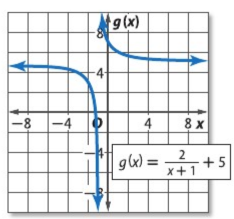
The domain is all real numbers not equal to -2 and the range is all real numbers not equal to -1.

GuidedPractice

2A.



2B.



2 Transformations of Reciprocal Functions

The same techniques used to transform the graphs of other functions you have studied can be applied to the graphs of reciprocal functions. In Example 2, note that the asymptotes have been moved along with the graphs of the functions.

StudyTip

Asymptotes The asymptotes of a reciprocal function move with the graph of the function and intersect at (h, k).

KeyConcept Transformations of Reciprocal Functions

$$f(x) = \frac{a}{x-h} + k$$

h – Horizontal Translation	k – Vertical Translation
h units right if h is positive	k units up if k is positive
$\lvert h \rvert$ units left if h is negative	$\lvert k \rvert$ units down if k is negative
The *vertical* asymptote is at $x = h$.	The *horizontal* asymptote is at $f(x) = k$.

a – Orientation and Shape

If $a < 0$, the graph is reflected across the x-axis.

If $\lvert a \rvert > 1$, the graph is stretched vertically. If $0 < \lvert a \rvert < 1$, the graph is compressed vertically.

Example 3 Graph Transformations

Graph each function. State the domain and range.

a. $f(x) = \frac{2}{x-4} + 2$

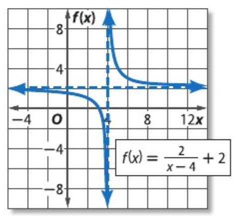
This represents a transformation of the graph of $f(x) = \frac{1}{x}$.

$a = 2$: The graph is stretched vertically.

$h = 4$: The graph is translated 4 units right. There is an asymptote at $x = 4$.

$k = 2$: The graph is translated 2 units up. There is an asymptote at $f(x) = 2$.

Domain: $\{x \mid x \neq 4\}$ Range: $\{f(x) \mid f(x) \neq 2\}$



b. $f(x) = \frac{-3}{x+1} - 4$

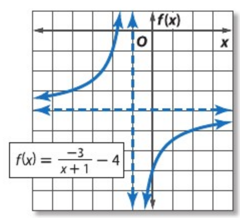
This represents a transformation of the graph of $f(x) = \frac{1}{x}$.

$a = -3$: The graph is stretched vertically and reflected across the x-axis.

$h = -1$: The graph is translated 1 unit left. There is an asymptote at $x = -1$.

$k = -4$: The graph is translated 4 units down. There is an asymptote at $f(x) = -4$.

Domain: $\{x \mid x \neq -1\}$ Range: $\{f(x) \mid f(x) \neq -4\}$



GuidedPractice

3A. $f(x) = \frac{-2}{x+4} + 1$

3B. $g(x) = \frac{1}{3(x-1)} - 2$

Real-World Career

Travel Agent Travel agents assess individual and business needs to help make the best possible travel arrangements. They may specialize by type of travel, such as leisure or business, or by destination, such as Europe or Africa. A high school diploma is required, and vocational training is preferred.

Real-World Example 4 Write Equations

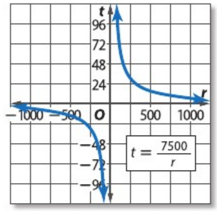
TRAVEL **An airline has a daily nonstop flight between California, and Australia. A one-way trip is about 7500 kilometers.**

a. Write an equation to represent the travel time as a function of flight speed. Then graph the equation.

Solve $rt = d$ for t.

$rt = d$	Original formula
$t = \frac{d}{r}$	Divide each side by r.
$t = \frac{7500}{r}$	$d = 7500$

Graph $t = \frac{7500}{r}$.



b. Explain any limitations to the range or domain in this situation.

In this situation, the range and domain are limited to all real numbers greater than zero because negative values do not make sense. There will be further restrictions to the domain because the aircraft has minimum and maximum speeds at which it can travel.

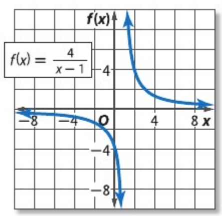
Guided Practice

4. **GRADUATION** The Grade 10 and Grade 12 class officers are sponsoring a graduation celebration. The total cost for the facilities and catering is AED 45 per person plus a AED 2,500 deposit. Write and graph an equation to represent the average cost per person. Then explain any limitations to the domain and range.

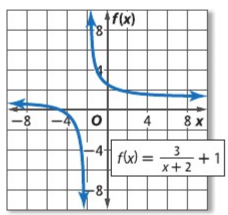
Check Your Understanding

Examples 1–2 **Identify the asymptotes, domain, and range of each function.**

1.



2.



Example 3 **Graph each function. State the domain and range.**

3. $f(x) = \frac{5}{x}$

4. $f(x) = \frac{2}{x+3}$

5. $f(x) = \frac{-1}{x-2} + 4$

Example 4

6. **SENSE-MAKING** A group of friends plans to get their youth group leader a gift certificate for a day at a spa. The certificate costs AED 150.

a. If c represents the cost for each friend and f represents the number of friends, write an equation to represent the cost to each friend as a function of how many friends give.

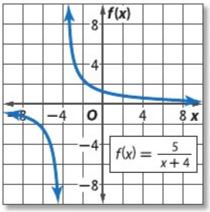
b. Graph the function.

c. Explain any limitations to the range or domain in this situation.

Practice and Problem Solving

Examples 1–2 **Identify the asymptotes, domain, and range of each function.**

7.



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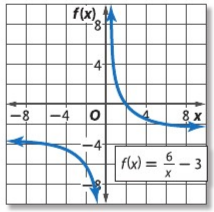

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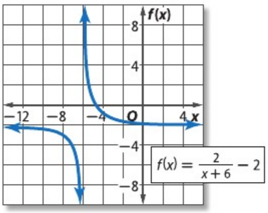
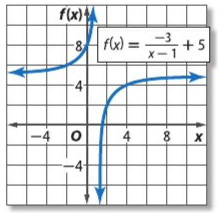


10.



Example 3 **Graph each function. State the domain and range.**

11. $f(x) = \frac{3}{x}$

12. $f(x) = \frac{-4}{x+2}$

13. $f(x) = \frac{2}{x-6}$

14. $f(x) = \frac{6}{x} - 5$

15. $f(x) = \frac{2}{x} + 3$

16. $f(x) = \frac{8}{x}$

17. $f(x) = \frac{-2}{x-5}$

18. $f(x) = \frac{3}{x-7} - 8$

19. $f(x) = \frac{9}{x+3} + 6$

20. $f(x) = \frac{8}{x+3}$

21. $f(x) = \frac{-6}{x+4} - 2$

22. $f(x) = \frac{-5}{x-2} + 2$

Example 4

23. **CYCLING** Manal's New Year's resolution is to ride her bike 5000 kilometers.

a. If m represents the mileage Manal rides each day and d represents the number of days, write an equation to represent the mileage each day as a function of the number of days that she rides.

b. Graph the function.

c. If she rides her bike every day of the year, how many kilometers should she ride each day to meet her goal?

24. **MODELING** Omar has 200 grams of an unknown liquid. Knowing the density will help him discover what type of liquid this is.

a. Density of a liquid is found by dividing the mass by the volume. Write an equation to represent the density of this unknown as a function of volume.

b. Graph the function.

c. From the graph, identify the asymptotes, domain, and range of the function.

B **Graph each function. State the domain and range.**

25. $f(x) = \frac{3}{2x-4}$

26. $f(x) = \frac{5}{3x}$

27. $f(x) = \frac{2}{4x+1}$

28. $f(x) = \frac{1}{2x+3}$

29. **BASEBALL** The distance from the pitcher's mound to home plate is 18.4 meters.

a. If r represents the speed of the pitch and t represents the time it takes the ball to get to the plate, write an equation to represent the speed as a function of time.

b. Graph the function.

c. If a two-seam fastball reaches the plate in 0.48 second, what was its speed?

Graph each function. State the domain and range, and identify the asymptotes.

30. $f(x) = \frac{-3}{x+7} - 1$

31. $f(x) = \frac{-4}{x+2} - 5$

32. $f(x) = \frac{6}{x-1} + 2$

33. $f(x) = \frac{2}{x-4} + 3$

34. $f(x) = \frac{-7}{x-8} - 9$

35. $f(x) = \frac{-6}{x-7} - 8$

C 36. **FINANCIAL LITERACY** Lamis' car went 708 kilometers on one tank of gas.

a. If g represents the number of kilometers to the liter that the car gets and t represents the size of the gas tank, write an equation to represent the kilometers to the liter as a function of tank size.

b. Graph the function.

c. How many kilometers does the car get per liter if it has a 59-liter tank?

37. **MULTIPLE REPRESENTATIONS** In this problem you will investigate the similarities and differences between power functions with positive and negative exponents.

a. TABULAR Make a table of values for $a(x) = x^2$, $b(x) = x^{-2}$, $c(x) = x^3$, and $d(x) = x^{-3}$.

b. GRAPHICAL Graph $a(x)$ and $b(x)$ on the same coordinate plane.

c. VERBAL Compare the domain, range, end behavior, and behavior at $x = 0$ for $a(x)$ and $b(x)$.

d. GRAPHICAL Graph $c(x)$ and $d(x)$ on the same coordinate plane.

e. VERBAL Compare the domain, range, end behavior, and behavior at $x = 0$ for $c(x)$ and $d(x)$.

f. ANALYTICAL What conclusions can you make about the similarities and differences between power functions with positive and negative exponents?

H.O.T. Problems Use Higher-Order Thinking Skills

38. **OPEN ENDED** Write a reciprocal function for which the graph has a vertical asymptote at $x = -4$ and a horizontal asymptote at $f(x) = 6$.

39. **REASONING** Compare and contrast the graphs of each pair of equations.

a. $y = \frac{1}{x}$ and $y - 7 = \frac{1}{x}$

b. $y = \frac{1}{x}$ and $y = 4\left(\frac{1}{x}\right)$

c. $y = \frac{1}{x}$ and $y = \frac{1}{x+5}$

d. Without making a table of values, use what you observed in parts **a–c** to sketch a graph of $y - 7 = 4\left(\frac{1}{x+5}\right)$.

40. **ARGUMENTS** Find the function that does not belong. Explain.

$f(x) = \frac{3}{x+1}$

$g(x) = \frac{x+2}{x^2+1}$

$h(x) = \frac{5}{x^2+2x+1}$

$j(x) = \frac{20}{x-7}$

41. **CHALLENGE** Write two different reciprocal functions with graphs having the same vertical and horizontal asymptotes. Then graph the functions.

42. **WRITING IN MATH** Refer to the beginning of the lesson. Explain how rational functions can be used in fundraising. Explain why only part of the graph is meaningful in the context of the problem.

Standardized Test Practice

43. SHORT RESPONSE What is the value of $(x + y)(x + y)$ if $xy = -3$ and $x^2 + y^2 = 10$?

44. GRIDDED RESPONSE If $x = 2y$, $y = 4z$, $2z = w$, and $w \neq 0$, then $\frac{x}{w} = $ ___.

45. If $c = 1 + \frac{1}{d}$ and $d > 1$, then c could equal ___.

A $\frac{5}{7}$ C $\frac{15}{7}$

B $\frac{9}{7}$ D $\frac{19}{7}$

46. SAT/ACT A car travels m kilometers at the rate of t kilometers per hour. How many hours does the trip take?

F $\frac{m}{t}$ J $\frac{t}{m}$

G $m - t$ K $t - m$

H mt

47. If $-1 < a < b < 0$, then which of the following has the greatest value?

A $a - b$ C $a + b$

B $b - a$ D $2b - a$

Spiral Review

48. BUSINESS A small corporation decides that 8% of its profits will be divided among its six managers. There are two sales managers and four nonsales managers. Fifty percent will be split equally among all six managers. The other 50% will be split among the four nonsales managers. Let p represent the profits. (Lesson 9-2)

a. Write an expression to represent the share of the profits each nonsales manager will receive.

b. Simplify this expression.

c. Write an expression in simplest form to represent the share of the profits each sales manager will receive.

Simplify each expression. (Lesson 9-1)

49. $\dfrac{\frac{p^3}{2n}}{-\frac{p^2}{4n}}$ **50.** $\dfrac{\frac{m+q}{5}}{\frac{m^2+q^2}{5}}$ **51.** $\dfrac{\frac{x+y}{2x-y}}{\frac{x+y}{2x+y}}$

Graph each function. State the domain and range.

52. $y = 2(3)^x$ **53.** $y = 5(2)^x$ **54.** $y = 0.5(4)^x$ **55.** $y = 4\left(\frac{1}{3}\right)^x$

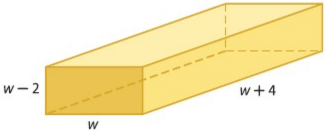
Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

56. $f(x) = x + 9$
$g(x) = x - 9$

57. $f(x) = 2x - 3$
$g(x) = 4x + 9$

58. $f(x) = 2x^2$
$g(x) = 8 - x$

59. GEOMETRY The width of a rectangular prism is w centimeters. The height is 2 centimeters less than the width. The length is 4 centimeters more than the width. If the volume of the prism is 8 times the measure of the length, find the dimensions of the prism.



Skills Review

Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. State the domain and range for each function.

60. $f(x) = x^3 + 2x^2 - 3x - 5$ **61.** $f(x) = x^4 - 8x^2 + 10$

CHAPTER 9

Mid-Chapter Quiz

Lessons 9-1 through 9-3

Simplify each expression. (Lesson 9-1)

1. $\dfrac{2x^2y^5}{7x^3yz} \cdot \dfrac{14xyz^2}{18x^4y}$

2. $\dfrac{24a^4b^6}{35ab^3} \div \dfrac{12abc}{7a^2c}$

3. $\dfrac{3x-3}{x^2+x-2} \cdot \dfrac{4x+8}{6x+18}$

4. $\dfrac{m^2+3m+2}{9} \div \dfrac{m+1}{3m+15}$

5. $\dfrac{\frac{r^2+3r}{r+1}}{\frac{3r}{3r+3}}$

6. $\dfrac{\frac{2y}{y^2-4}}{\frac{3}{y^2-4y+4}}$

7. MULTIPLE CHOICE For all $r \neq \pm 2$, $\dfrac{r^2+6r+8}{r^2-4} =$ ___. (Lesson 9-1)

A $\dfrac{r-2}{r+4}$

B $\dfrac{r+4}{r-2}$

C $\dfrac{r+2}{r-4}$

D $\dfrac{r+4}{r+2}$

8. MULTIPLE CHOICE Identify all values of x for which $\dfrac{x^2-16}{(x^2-6x-27)(x+1)}$ is undefined. (Lesson 9-1)

F $-3, -1$

G $3, 1, -9$

H $-3, -1, 9$

J -1

9. What is the LCM of $x^2 - x$ and $3 - 3x$? (Lesson 9-2)

Simplify each expression. (Lesson 9-2)

10. $\dfrac{2x}{4x^2y} + \dfrac{x}{3xy^3}$

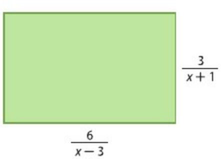
11. $\dfrac{3}{4m} + \dfrac{2}{3mn^2} - \dfrac{4}{n}$

12. $\dfrac{6}{r^2-3r-18} - \dfrac{1}{r^2+r-6}$

13. $\dfrac{3x+6}{x+y} + \dfrac{6}{-x-y}$

14. $\dfrac{x-4}{x^2-3x-4} + \dfrac{x+1}{2x-8}$

15. Determine the perimeter of the rectangle. (Lesson 9-2)



16. TRAVEL Wafa is going to a beach 100 kilometers away. She travels half the distance at one rate. The rest of the distance, she travels 15 kilometers per hour slower. (Lesson 9-2)

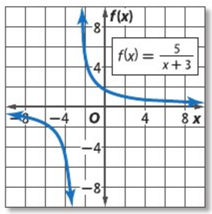
a. If x represents the faster pace in kilometers per hour, write an expression that represents the time spent at that pace.

b. Write an expression for the amount of time spent at the slower pace.

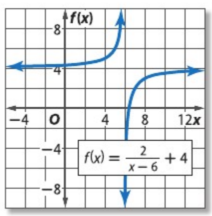
c. Write an expression for the amount of time Wafa needs to complete the trip.

Identify the asymptotes, domain, and range of each function. (Lesson 9-3)

17.



18.



Graph each reciprocal function. State the domain and range. (Lesson 9-3)

19. $f(x) = \dfrac{4}{x}$

20. $f(x) = \dfrac{1}{3x}$

21. $f(x) = \dfrac{6}{x-1}$

22. $f(x) = \dfrac{-2}{x} + 4$

23. $f(x) = \dfrac{3}{x+2} - 5$

24. $f(x) = -\dfrac{1}{x-3} + 2$

25. SANDWICHES A group makes 45 sandwiches to take on a picnic. The number of sandwiches a person can eat depends on how many people go on the trip. (Lesson 9-3)

a. Write a function to represent this situation.

b. Graph the function.

LESSON 9-4

Graphing Rational Functions

Then

- You graphed reciprocal functions.

Now

1. Graph rational functions with vertical and horizontal asymptotes.
2. Graph rational functions with oblique asymptotes and point discontinuity.

Why?

- Rana bought a digital SLR camera and a photo printer for AED 350. The manufacturer claims that ink and photo paper cost AED 0.47 per photo. The rational function $C(p) = \frac{0.47p + 350}{p}$ can be used to determine the average cost $C(p)$ for printing p photos.

NewVocabulary

rational function
vertical asymptote
horizontal asymptote
oblique asymptote
point discontinuity

Mathematical Practices

7 Look for and make use of structure.

1 Vertical and Horizontal Asymptotes

A **rational function** has an equation of the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$.

In order to graph a rational function, it is helpful to locate the zeros and asymptotes. A zero of a rational function $f(x) = \frac{a(x)}{b(x)}$ occurs at every value of x for which $a(x) = 0$.

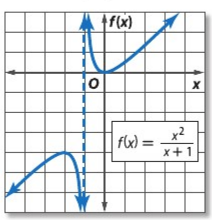
KeyConcept Vertical and Horizontal Asymptotes

Words If $f(x) = \frac{a(x)}{b(x)}$, $a(x)$ and $b(x)$ are polynomial functions with no common factors other than 1, and $b(x) \neq 0$, then:

- $f(x)$ has a **vertical asymptote** whenever $b(x) = 0$.
- $f(x)$ has at most one **horizontal asymptote.**
  - If the degree of $a(x)$ is greater than the degree of $b(x)$, there is no horizontal asymptote.
  - If the degree of $a(x)$ is less than the degree of $b(x)$, the horizontal asymptote is the line $y = 0$.
  - If the degree of $a(x)$ equals the degree of $b(x)$, the horizontal asymptote is the line $y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$.

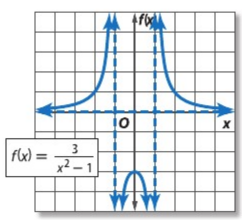
Examples

No horizontal asymptote

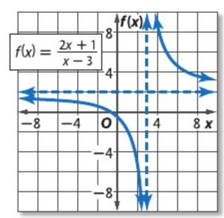


Vertical asymptote: $x = -1$

One horizontal asymptote



Vertical asymptotes: $x = -1, x = 1$
Horizontal asymptote: $f(x) = 0$



Vertical asymptote: $x = 3$
Horizontal asymptote: $f(x) = 2$

WatchOut!

Zeros vs. Vertical Asymptotes Zeros of rational functions occur at the values that make the numerator equal to zero. Vertical asymptotes occur at the values that make the denominator equal to zero.

The asymptotes of a rational function can be used to draw the graph of the function. Additionally, the asymptotes can be used to divide a graph into regions to find ordered pairs on the graph.

Example 1 Graph with no Horizontal Asymptote

Graph $f(x) = \frac{x^3}{x - 1}$.

Step 1 Find the zeros.

$x^3 = 0$ Set $a(x) = 0$.

$x = 0$ Take the cube root of each side.

There is a zero at $x = 0$.

Step 2 Draw the asymptotes.

Find the vertical asymptote.

$x - 1 = 0$ Set $b(x) = 0$.

$x = 1$ Add 1 to each side.

There is a vertical asymptote at $x = 1$.

The degree of the numerator is greater than the degree of the denominator. So, there is no horizontal asymptote.

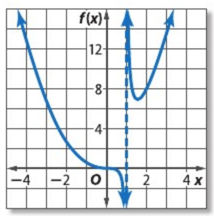
StudyTip

Graphing Calculator The TABLE feature of a graphing calculator can be used to calculate decimal values for x and y.

Step 3 Draw the graph.

Use a table to find ordered pairs on the graph. Then connect the points.

x	$f(x)$
−3	6.75
−2	2.67
−1	0.5
0	0
0.5	−0.25
1.5	6.75
2	8
3	13.5



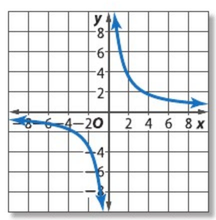
GuidedPractice

Graph each function.

1A. $f(x) = \frac{x^2 - x - 6}{x + 1}$

1B. $f(x) = \frac{(x + 1)^3}{(x + 2)^2}$

In the real world, sometimes values on the graph of a rational function are not meaningful. In the graph at the right, x-values such as time, distance, and number of people cannot be negative in the context of the problem. So, you do not even need to consider that portion of the graph.



Real-World Career

U.S. Coast Guard Boatswain's Mate The most versatile member of the U.S. Coast Guard's operational team is the boatswain's mate. BMs are capable of performing almost any task. Training for BMs is accomplished through 12 weeks of intensive training.

Real-World Example 2 Use Graphs of Rational Functions

AVERAGE SPEED **A boat traveled upstream at r_1 kilometers per hour. During the return trip to its original starting point, the boat traveled at r_2 kilometers per hour. The average speed for the entire trip R is given by the formula $R = \frac{2r_1r_2}{r_1 + r_2}$.**

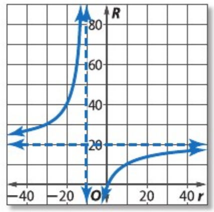
a. Let r_1 be the independent variable, and let R be the dependent variable. Draw the graph if $r_2 = 10$ kilometers per hour.

The function is $R = \frac{2r_1(10)}{r_1 + (10)}$ or $R = \frac{20r_1}{r_1 + 10}$.

The vertical asymptote is $r_1 = -10$.

Graph the vertical asymptote and the function.

Notice that the horizontal asymptote is $R = 20$.



b. What is the R-intercept of the graph?

The R-intercept is 0.

c. What domain and range values are meaningful in the context of the problem?

In the problem context, speeds are nonnegative values. Therefore, only values of r_1 greater than or equal to 0 and values of R between 0 and 20 are meaningful.

Guided Practice

2. SALARIES A company uses the formula $S(x) = \frac{45x + 25}{x + 1}$ to determine the salary in thousands of dirhams of an employee during his xth year. Graph $S(x)$. What domain and range values are meaningful in the context of the problem? What is the meaning of the horizontal asymptote for the graph?

2 Oblique Asymptotes and Point Discontinuity

An **oblique asymptote**, sometimes called a *slant asymptote*, is an asymptote that is neither horizontal nor vertical.

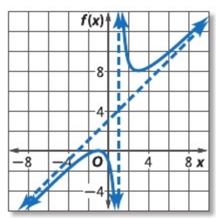
Key Concept Oblique Asymptotes

Words If $f(x) = \frac{a(x)}{b(x)}$, $a(x)$ and $b(x)$ are polynomial functions with no common factors other than 1 and $b(x) \neq 0$, then $f(x)$ has an oblique asymptote if the degree of $a(x)$ minus the degree of $b(x)$ equals 1. The equation of the asymptote is $f(x) = \frac{a(x)}{b(x)}$ with no remainder.

Example $f(x) = \frac{x^4 + 3x^3}{x^3 - 1}$

Vertical asymptote: $x = 1$

Oblique asymptote: $f(x) = x + 3$



StudyTip

Oblique Asymptotes Oblique asymptotes occur for rational functions that have a numerator polynomial that is one degree higher than the denominator polynomial.

Example 3 Determine Oblique Asymptotes

Graph $f(x) = \dfrac{x^2 + 4x + 4}{2x - 1}$.

Step 1 Find the zeros.

$$x^2 + 4x + 4 = 0 \qquad \text{Set } a(x) = 0.$$

$$(x + 2)^2 = 0 \qquad \text{Factor.}$$

$$x + 2 = 0 \qquad \text{Take the square root of each side.}$$

$$x = -2 \qquad \text{Subtract 2 from each side.}$$

There is a zero at $x = -2$.

Step 2 Find the asymptotes.

$$2x - 1 = 0 \qquad \text{Set } b(x) = 0.$$

$$2x = 1 \qquad \text{Add 1 to each side.}$$

$$x = \frac{1}{2} \qquad \text{Divide each side by 2.}$$

There is a vertical asymptote at $x = \frac{1}{2}$.

The degree of the numerator is greater than the degree of the denominator, so there is no horizontal asymptote.

The difference between the degree of the numerator and the degree of the denominator is 1, so there is an oblique asymptote.

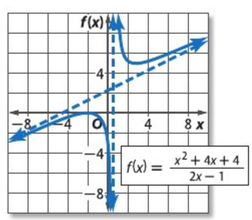
Divide the numerator by the denominator to determine the equation of the oblique asymptote.

The equation of the asymptote is the quotient excluding any remainder.

$$\begin{array}{r} \frac{1}{2}x + \frac{9}{4} \\ 2x - 1 \enclose{longdiv}{x^2 + 4x + 4} \\ (-)x^2 - \frac{1}{2}x \\ \hline \frac{9}{2}x + 4 \\ (-)\frac{9}{2}x - \frac{9}{4} \\ \hline \frac{25}{4} \end{array}$$

Thus, the oblique asymptote is the line $f(x) = \frac{1}{2}x + \frac{9}{4}$.

Step 3 Draw the asymptotes, and then use a table of values to graph the function.



GuidedPractice

Graph each function.

3A. $f(x) = \dfrac{x^2}{x - 2}$

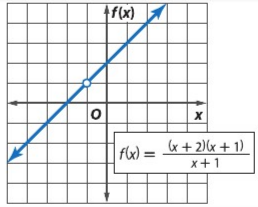
3B. $f(x) = \dfrac{x^3 - 1}{x^2 - 4}$

In some cases, graphs of rational functions may have **point discontinuity**, which looks like a hole in the graph. This is because the function is undefined at that point.

KeyConcept Point Discontinuity

Words If $f(x) = \frac{a(x)}{b(x)}$, $b(x) \neq 0$, and $x - c$ is a factor of both $a(x)$ and $b(x)$, then there is a point discontinuity at $x = c$.

Example $f(x) = \frac{(x+2)(x+1)}{x+1}$
$= x + 2;\ x \neq -1$



WatchOut!

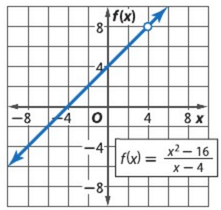
Holes Remember that a common factor in the numerator and denominator can signal a hole.

Example 4 Graph with Point Discontinuity

Graph $f(x) = \frac{x^2 - 16}{x - 4}$.

Notice that $\frac{x^2 - 16}{x - 4} = \frac{(x+4)\overset{1}{\cancel{(x-4)}}}{\underset{1}{\cancel{x-4}}}$ or $x + 4$.

Therefore, the graph of $f(x) = \frac{x^2 - 16}{x - 4}$ is the graph of $f(x) = x + 4$ with a hole at $x = 4$.



GuidedPractice

Graph each function.

4A. $f(x) = \frac{x^2 + 4x - 5}{x + 5}$

4B. $f(x) = \frac{x^3 + 2x^2 - 9x - 18}{x^2 - 9}$

Check Your Understanding

Example 1 **Graph each function.**

1. $f(x) = \frac{x^4 - 2}{x^2 - 1}$

2. $f(x) = \frac{x^3}{x + 2}$

Example 2

3. REASONING Hasan is a forward for his high school football team. So far this season, he has made 7 out of 11 goals. He would like to improve his goal percentage. If he can make x consecutive goals, his goal percentage can be determined using the function $P(x) = \frac{7 + x}{11 + x}$.

a. Graph the function.

b. What part of the graph is meaningful in the context of this problem?

c. Describe the meaning of the intercept of the vertical axis.

d. What is the equation of the horizontal asymptote? Explain its meaning with respect to Hasan's goal percentage.

Examples 3–4 **Graph each function.**

4. $f(x) = \frac{6x^2 - 3x + 2}{x}$

5. $f(x) = \frac{x^2 + 8x + 20}{x + 2}$

6. $f(x) = \frac{x^2 - 4x - 5}{x + 1}$

7. $f(x) = \frac{x^2 + x - 12}{x + 4}$

Practice and Problem Solving

Example 1 Graph each function.

8. $f(x) = \dfrac{x^4}{6x + 12}$

9. $f(x) = \dfrac{x^3}{8x - 4}$

10. $f(x) = \dfrac{x^4 - 16}{x^2 - 1}$

11. $f(x) = \dfrac{x^3 + 64}{16x - 24}$

Example 2

12. **SCHOOL SPIRIT** As president of Student Council, Badria is getting T-shirts made for a student gathering. Each T-shirt costs AED 9.50, and there is a set-up fee of AED 75. The student council plans to sell the shirts, but each of the 15 council members will get one for free.

a. Write a function for the average cost of a T-shirt to be sold. Graph the function.

b. What is the average cost if 200 shirts are ordered? if 500 shirts are ordered?

c. How many T-shirts must be ordered to bring the average cost under AED 9.75?

Examples 2–3 Graph each function.

13. $f(x) = \dfrac{x}{x + 2}$

14. $f(x) = \dfrac{5}{(x - 1)(x + 4)}$

15. $f(x) = \dfrac{4}{(x - 2)^2}$

16. $f(x) = \dfrac{x - 3}{x + 1}$

17. $f(x) = \dfrac{1}{(x + 4)^2}$

18. $f(x) = \dfrac{2x}{(x + 2)(x - 5)}$

19. $f(x) = \dfrac{(x - 4)^2}{x + 2}$

20. $f(x) = \dfrac{(x + 3)^2}{x - 5}$

21. $f(x) = \dfrac{x^3 + 1}{x^2 - 4}$

22. $f(x) = \dfrac{4x^3}{2x^2 + x - 1}$

23. $f(x) = \dfrac{3x^2 + 8}{2x - 1}$

24. $f(x) = \dfrac{2x^2 + 5}{3x + 4}$

25. $f(x) = \dfrac{x^4 - 2x^2 + 1}{x^3 + 2}$

26. $f(x) = \dfrac{x^4 - x^2 - 12}{x^3 - 6}$

27. **PERSEVERANCE** The current in amperes in an electrical circuit with three resistors in a series is given by the equation $I = \dfrac{V}{R_1 + R_2 + R_3}$, where V is the voltage in volts in the circuit and R_1, R_2, and R_3 are the resistances in ohms of the three resistors.

a. Let R_1 be the independent variable, and let I be the dependent variable. Graph the function if $V = 120$ volts, $R_2 = 25$ ohms, and $R_3 = 75$ ohms.

b. Give the equation of the vertical asymptote and the R_1- and I-intercepts of the graph.

c. Find the value of I when the value of R_1 is 140 ohms.

d. What domain and range values are meaningful in the context of the problem?

Example 4 Graph each function.

28. $f(x) = \dfrac{x^2 - 2x - 8}{x - 4}$

29. $f(x) = \dfrac{x^2 + 4x - 12}{x - 2}$

30. $f(x) = \dfrac{x^2 - 25}{x + 5}$

31. $f(x) = \dfrac{x^2 - 64}{x - 8}$

32. $f(x) = \dfrac{(x - 4)(x^2 - 4)}{x^2 - 6x + 8}$

33. $f(x) = \dfrac{(x + 5)(x^2 + 2x - 3)}{x^2 + 8x + 15}$

34. $f(x) = \dfrac{3x^4 + 6x^3 + 3x^2}{x^2 + 2x + 1}$

35. $f(x) = \dfrac{2x^4 + 10x^3 + 12x^2}{x^2 + 5x + 6}$

B **36.** **BUSINESS** Ali purchased a snow plow for AED 4,500 and plows the parking lots of local businesses. Each time he plows a parking lot, he incurs a cost of AED 50 for gas and maintenance.

a. Write and graph the rational function representing his average cost per customer as a function of the number of parking lots.

b. What are the asymptotes of the graph?

c. Why is the first quadrant in the graph the only relevant quadrant?

d. How many total parking lots does Ali need to plow for his average cost per parking lot to be less than AED 80?

37. **FINANCIAL LITERACY** Hana bought a new cell phone with Internet access. The phone cost AED 150, and her monthly usage charge is AED 30 plus AED 10 for the Internet access.

a. Write and graph the rational function representing her average monthly cost as a function of the number of months Hana uses the phone.

b. What are the asymptotes of the graph?

c. Why is the first quadrant in the graph the only relevant quadrant?

d. After how many months will the average monthly charge be AED 45?

C **38.** **SENSE-MAKING** Sally plays softball for Al Barsha Secondary School. So far this season she has gotten a hit 4 out of 12 times at bat. She is determined to improve her batting average. If she can get x consecutive hits, her batting average can be determined using $B(x) = \frac{4 + x}{12 + x}$.

a. Graph the function.

b. What part of the graph is meaningful in the context of the problem?

c. Describe the meaning of the intercept of the vertical axis.

d. What is the equation of the horizontal asymptote? Explain its meaning with respect to Sally's batting average.

Graph each function.

39. $f(x) = \frac{x + 1}{x^2 + 6x + 5}$

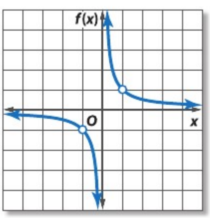
40. $f(x) = \frac{x^2 - 10x - 24}{x + 2}$

41. $f(x) = \frac{6x^2 + 4x + 2}{x + 2}$

H.O.T. Problems Use Higher-Order Thinking Skills

42. **OPEN ENDED** Sketch the graph of a rational function with a horizontal asymptote $y = 1$ and a vertical asymptote $x = -2$.

43. **CHALLENGE** Compare and contrast $g(x) = \frac{x^2 - 1}{x(x^2 - 2)}$ and $f(x)$ shown at the right.



44. **REASONING** What is the difference between the graphs of $f(x) = x - 2$ and $g(x) = \frac{(x + 3)(x - 2)}{x + 3}$?

45. **PROOF** A rational function has an equation of the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$. Show that $f(x) = \frac{x}{a - b} + c$ is a rational function.

46. **WRITING IN MATH** How can factoring be used to determine the vertical asymptotes or point discontinuity of a rational function?

Standardized Test Practice

47. PROBABILITY Of the 6 courses offered by the music department at her school, Khawla must choose exactly 2 of them. How many different combinations of 2 courses are possible for Khawla if there are no restrictions on which 2 courses she can choose?

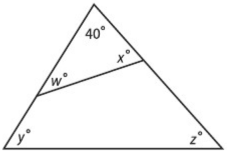
A 48 **C** 15
B 18 **D** 12

48. The projected sales of a game cartridge is given by the function $S(p) = \frac{3000}{2p + a}$, where $S(p)$ is the number of cartridges sold, in thousands, p is the price per cartridge, in dirhams, and a is a constant. If 100,000 cartridges are sold at AED 10 per cartridge, how many cartridges will be sold at AED 20 per cartridge?

F 20,000 **H** 60,000
G 50,000 **J** 150,000

49. GRIDDED RESPONSE Five distinct points lie in a plane such that 3 of the points are on line ℓ and 3 of the points are on a different line m. What is the total number of lines that can be drawn so that each line passes through exactly 2 of these 5 points?

50. GEOMETRY In the figure below, what is the value of $w + x + y + z$?



A 140 **C** 320
B 280 **D** 360

Spiral Review

Graph each function. State the domain and range. (Lesson 9-3)

51. $f(x) = \frac{-5}{x+2}$

52. $f(x) = \frac{4}{x-1} - 3$

53. $f(x) = \frac{1}{x+6} + 1$

Simplify each expression. (Lesson 9-2)

54. $\frac{m}{m^2 - 4} + \frac{2}{3m + 6}$

55. $\frac{y}{y + 3} - \frac{6y}{y^2 - 9}$

56. $\frac{5}{x^2 - 3x - 28} + \frac{7}{2x - 14}$

57. $\frac{d - 4}{d^2 + 2d - 8} - \frac{d + 2}{d^2 - 16}$

Simplify each expression.

58. $y^{\frac{5}{3}} \cdot y^{\frac{7}{3}}$

59. $x^{\frac{3}{4}} \cdot x^{\frac{9}{4}}$

60. $\left(b^{\frac{1}{3}}\right)^{\frac{3}{5}}$

61. $\left(a^{-\frac{2}{3}}\right)^{-\frac{1}{6}}$

Skills Review

62. TRAVEL Mr. and Mrs. Fahd are taking their daughter to college. The table shows their distances from home after various amounts of time.

a. Find the average rate of change in their distances from home between 1 and 3 hours after leaving home.

b. Find the average rate of change in their distances from home between 0 and 5 hours after leaving home.

Time (h)	Distance (km)
0	0
1	55
2	110
3	165
4	165
5	225

EXTEND

9-4 Graphing Technology Lab: Graphing Rational Functions

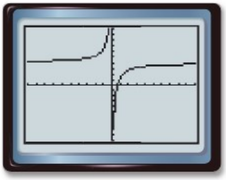
A graphing calculator can be used to explore graphs of rational functions. These graphs have some features that never appear in the graphs of polynomial functions.

Activity 1 Graph with Asymptotes

Graph $y = \frac{8x - 5}{2x}$ in the standard viewing window. Find the equations of any asymptotes. State the domain and range of the function.

Step 1 Enter the equation in the **Y=** list, and then graph.

KEYSTROKES: [Y=] [(] 8 [X,T,θ,n] [−] 5 [)] [÷] [(] 2 [X,T,θ,n] [)] [ZOOM] 6



[−10, 10] scl: 1 by [−10, 10] scl: 1

Step 2 Examine the graph.

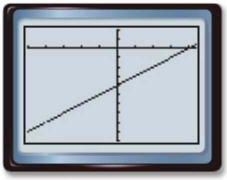
By looking at the equation, we can determine that if $x = 0$, the function is undefined. The equation of the vertical asymptote is $x = 0$. Notice what happens to the y-values as x grows larger and as x gets smaller. The y-values approach 4. So, the equation for the horizontal asymptote is $y = 4$. The domain is $\{x \mid x \neq 0\}$, and the range is all real numbers.

Activity 2 Graph with Point Discontinuity

Graph $y = \frac{x^2 - 16}{x + 4}$ in the window [−5, 4.4] by [−10, 2] with scale factors of 1.

Step 1 Because the function is not continuous, put the calculator in dot mode.

KEYSTROKES: [MODE] [▼] [▼] [▼] [▼] [▶] [ENTER]



[−5, 4.4] scl: 1 by [−20, 2] scl: 1

Step 2 Examine the graph.

This graph looks like a line with a break in continuity at $x = -4$. This happens because the denominator is 0 when $x = -4$. Therefore, the function is undefined when $x = -4$.

If you **TRACE** along the graph, when you come to $x = -4$, you will see that there is no corresponding y-value.

Exercises

Use a graphing calculator to graph each function. Write the x-coordinates of any points of discontinuity and/or the equations of any asymptotes. State the domain and range.

1. $f(x) = \frac{1}{x}$

2. $f(x) = \frac{x}{x + 2}$

3. $f(x) = \frac{2}{x - 4}$

4. $f(x) = \frac{2x}{3x - 6}$

5. $f(x) = \frac{4x + 2}{x - 1}$

6. $f(x) = \frac{x^2 - 9}{x + 3}$

LESSON 9-5 Variation Functions

Then

- You wrote and graphed linear equations.

Now

1. Recognize and solve direct and joint variation problems.
2. Recognize and solve inverse and combined variation problems.

Why?

- While building skateboard ramps, Mazen determined that the best ramps were the ones in which the length of the top of the ramp was 1.5 times as long as the height of the ramp.

As shown in the table, the length of the top of the ramp depends on the height of a ramp. The length increases as the height increases, but the ratio remains the same, or is *constant*.

The equation $\frac{\ell}{h} = 1.5$ can be written as $\ell = 1.5h$.

The length *varies directly* with the height of the ramp.

Length (ℓ)	Height (h)	Ratio $\frac{\ell}{h}$
3	2	1.5
6	4	1.5
9	6	1.5
12	8	1.5

NewVocabulary

direct variation
constant of variation
joint variation
inverse variation
combined variation

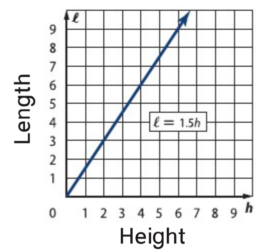
Mathematical Practices

1 Make sense of problems and persevere in solving them.
4 Model with mathematics.

1 Direct Variation and Joint Variation

The relationship given by $\ell = 1.5h$ is an example of direct variation. A **direct variation** can be expressed in the form $y = kx$. In this equation, k is called the **constant of variation**.

Notice that the graph of $\ell = 1.5h$ is a straight line through the origin. A direct variation is a special case of an equation written in slope-intercept form, $y = mx + b$. When $m = k$ and $b = 0$, $y = mx + b$ becomes $y = kx$. So the slope of a direct variation equation is its constant of variation.



To express a direct variation, we say that y varies directly as x. In other words, as x increases, y increases or decreases at a constant rate.

KeyConcept Direct Variation

Words	y varies directly as x if there is some nonzero constant k such that $y = kx$. k is called the *constant of variation.*
Example	If $y = 3x$ and $x = 7$, then $y = 3(7)$ or 21.

If you know that y varies directly as x and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = kx_1 \quad \text{and} \quad y_2 = kx_2$$

$$\frac{y_1}{x_1} = k \qquad \frac{y_2}{x_2} = k \qquad \text{Therefore, } \frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

Using the properties of equality, you can find many other proportions that relate these same x- and y-values.

Example 1 Direct Variation

If y varies directly as x and $y = 15$ when $x = -5$, find y when $x = 7$.

Use a proportion that relates the values.

$\frac{y_1}{x_1} = \frac{y_2}{x_2}$	Direct variation
$\frac{15}{-5} = \frac{y_2}{7}$	$y_1 = 15$, $x_1 = -5$, and $x_2 = 7$
$15(7) = -5(y_2)$	Cross multiply.
$105 = -5y_2$	Simplify.
$-21 = y_2$	Divide each side by -5.

GuidedPractice

1. If r varies directly as t and $r = -20$ when $t = 4$, find r when $t = -6$.

Another type of variation is joint variation. **Joint variation** occurs when one quantity varies directly as the product of two or more other quantities.

StudyTip

Joint Variation Some mathematicians consider joint variation a special type of combined variation.

KeyConcept Joint Variation

Words	y varies jointly as x and z if there is some nonzero constant k such that $y = kxz$.
Example	If $y = 5xz$, $x = 6$, and $z = -2$, then $y = 5(6)(-2)$ or -60.

If you know that y varies jointly as x and z and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = kx_1z_1 \quad \text{and} \quad y_2 = kx_2z_2$$

$$\frac{y_1}{x_1z_1} = k \qquad \frac{y_2}{x_2z_2} = k \qquad \text{Therefore, } \frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}.$$

Example 2 Joint Variation

Suppose y varies jointly as x and z. Find y when $x = 9$ and $z = 2$, if $y = 20$ when $z = 3$ and $x = 5$.

Use a proportion that relates the values.

$\frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}$	Joint variation
$\frac{20}{5(3)} = \frac{y_2}{9(2)}$	$y_1 = 20$, $x_1 = 5$, $z_1 = 3$, $x_2 = 9$, and $z_2 = 2$
$20(9)(2) = 5(3)(y_2)$	Cross multiply.
$360 = 15y_2$	Simplify.
$24 = y_2$	Divide each side by 15.

GuidedPractice

2. Suppose r varies jointly as v and t. Find r when $v = 2$ and $t = 8$, if $r = 70$ when $v = 10$ and $t = 4$.

2 Inverse Variation and Combined Variation

Another type of variation is inverse variation. If two quantities x and y show **inverse variation**, their product is equal to a constant k.

Inverse variation is often described as one quantity increasing while the other quantity is decreasing. For example, speed and time for a fixed distance vary inversely with each other; the faster you go, the less time it takes you to get there.

KeyConcept Inverse Variation

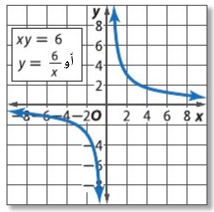
Words y varies inversely as x if there is some nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

Example If $xy = 2$, and $x = 6$, then $y = \frac{2}{6}$ or $\frac{1}{3}$.

StudyTip

Direct and Inverse Variation You can identify the type of variation by looking at a table of values for x and y. If the quotient $\frac{y}{x}$ has a constant value, y varies directly as x. If the product xy has a constant value, y varies inversely as x.

Suppose y varies inversely as x such that $xy = 6$ or $y = \frac{6}{x}$. The graph of this equation is shown at the right. Since k is a positive value, as the values of x increase, the values of y decrease.



Notice that the graph of an inverse variation is a reciprocal function.

A proportion can be used with inverse variation to solve problems in which some quantities are known. The following proportion is only one of several that can be formed.

$$x_1y_1 = k \text{ and } x_2y_2 = k$$

$$x_1y_1 = x_2y_2 \quad \text{Substitution Property of Equality}$$

$$\frac{x_1}{y_2} = \frac{x_2}{y_1} \quad \text{Divide each side by } y_1y_2.$$

Example 3 Inverse Variation

If a varies inversely as b and $a = 28$ when $b = -2$, find a when $b = -10$.

Use a proportion that relates the values.

$$\frac{a_1}{b_2} = \frac{a_2}{b_1} \quad \text{Inverse Variation}$$

$$\frac{28}{-10} = \frac{a_2}{-2} \quad a_1 = 28,\ b_1 = -2, \text{ and } b_2 = -10$$

$$28(-2) = -10(a_2) \quad \text{Cross multiply.}$$

$$-56 = -10(a_2) \quad \text{Simplify.}$$

$$5\tfrac{3}{5} = a_2 \quad \text{Divide each side by } -10.$$

GuidedPractice

3. If x varies inversely as y and $x = 24$ when $y = 4$, find x when $y = 12$.

Inverse variation is often used in real-world situations.

Real-WorldLink

When you pluck a string, it vibrates back and forth. This causes mechanical energy to travel through the air in waves. The number of times per second these waves hit our ear is called the *frequency*. The more waves per second, the higher the pitch.

Real-World Example 4 Write and Solve an Inverse Variation

MUSIC The length of a violin string varies inversely as the frequency of its vibrations. A violin string 25.4 centimeters long vibrates at a frequency of 512 cycles per second. Find the frequency of an 20.3-centimeter violin string.

Let $v_1 = 25.4, f_1 = 512$, and $v_2 = 20.3$. Solve for f_2.

$v_1 f_1 = v_2 f_2$ — Original equation

$25.4 \cdot 512 = 20.3 \cdot f_2$ — $v_1 = 10, f_1 = 512$, and $v_2 = 8$

$\frac{13004.8}{20.3} = f_2$ — Divide each side by 8.

$640.6 = f_2$ — Simplify.

The 20.3-centimeter violin string vibrates at a frequency of 640 cycles per second.

GuidedPractice

4. The apparent length of an object is inversely proportional to one's distance from the object. Earth is about 150 million kilometers from the Sun. Jupiter is about 778.3 million kilometers from the Sun. Find how many times as large the diameter of the Sun would appear on Earth as on Jupiter.

Another type of variation is combined variation. **Combined variation** occurs when one quantity varies directly and/or inversely as two or more other quantities.

If you know that y varies directly as x, y varies inversely as z, and one set of values, you can use a proportion to find the other set of corresponding values.

$y_1 = \frac{kx_1}{z_1}$ and $y_2 = \frac{kx_2}{z_2}$

$\frac{y_1 z_1}{x_1} = k$ and $\frac{y_2 z_2}{x_2} = k$

Therefore, $\frac{y_1 z_1}{x_1} = \frac{y_2 z_2}{x_2}$.

Example 5 Combined Variation

Suppose f varies directly as g, and f varies inversely as h. Find g when $f = 18$ and $h = -3$, if $g = 24$ when $h = 2$ and $f = 6$.

First set up a correct proportion for the information given.

$f_1 = \frac{kg_1}{h_1}$ and $f_2 = \frac{kg_2}{h_2}$ — g varies directly as f, so g goes in the numerator. h varies inversely as f, so h goes in the denominator.

$k = \frac{f_1 h_1}{g_1}$ and $k = \frac{f_2 h_2}{g_2}$ — Solve for k.

$\frac{f_1 h_1}{g_1} = \frac{f_2 h_2}{g_2}$ — Set the two proportions equal to each other.

$\frac{6(2)}{24} = \frac{18(-3)}{g_2}$ — $f_1 = 6, g_1 = 24, h_1 = 2, f_2 = 18$, and $h_2 = -3$

$24(18)(-3) = 6(2)(g_2)$ — Cross multiply.

$-1296 = 12g_2$ — Simplify.

$-108 = g_2$ — Divide each side by 12.

When $f = 18$ and $h = -3$, the value of g is -108.

GuidedPractice

5. Suppose p varies directly as r, and p varies inversely as t. Find t when $r = 10$ and $p = -5$, if $t = 20$ when $p = 4$ and $r = 2$.

StudyTip

Combined Variation Quantities that vary directly appear in the numerator. Quantities that vary inversely appear in the denominator.

Check Your Understanding

Examples 1–3

1. If y varies directly as x and $y = 12$ when $x = 8$, find y when $x = 14$.

2. Suppose y varies jointly as x and z. Find y when $x = 9$ and $z = -3$, if $y = -50$ when z is 5 and x is -10.

3. If y varies inversely as x and $y = -18$ when $x = 16$, find x when $y = 9$.

Example 4

4. **TRAVEL** A map of Illinois is scaled so that 2 centimeters represents 15 kilometers. How far apart are Chicago and Rockford if they are 12 centimeters apart on the map?

Example 5

5. Suppose a varies directly as b, and a varies inversely as c. Find b when $a = 8$ and $c = -3$, if $b = 16$ when $c = 2$ and $a = 4$.

6. Suppose d varies directly as f, and d varies inversely as g. Find g when $d = 6$ and $f = -7$, if $g = 12$ when $d = 9$ and $f = 3$.

Practice and Problem Solving

Example 1 **If x varies directly as y, find x when $y = 8$.**

7. $x = 6$ when $y = 32$
8. $x = 11$ when $y = -3$
9. $x = 14$ when $y = -2$
10. $x = -4$ when $y = 10$

11. **MOON** Astronaut Neil Armstrong, the first man on the Moon, weighed 163.3 kilograms on Earth with all his equipment on, but weighed only 27.2 kilograms on the Moon. Write an equation that relates weight on the Moon m with weight on Earth w.

Example 2 **If a varies jointly as b and c, find a when $b = 4$ and $c = -3$.**

12. $a = -96$ when $b = 3$ and $c = -8$
13. $a = -60$ when $b = -5$ and $c = 4$
14. $a = -108$ when $b = 2$ and $c = 9$
15. $a = 24$ when $b = 8$ and $c = 12$

16. **MODELING** According to the A.C. Nielsen Company, the average American watches 4 hours of television a day.
    a. Write an equation to represent the average number of hours spent watching television by m household members during a period of d days.
    b. Assume that members of your household watch the same amount of television each day as the average American. How many hours of television would the members of your household watch in a week?

Example 3 **If f varies inversely as g, find f when $g = -6$.**

17. $f = 15$ when $g = 9$
18. $f = 4$ when $g = 28$
19. $f = -12$ when $g = 19$
20. $f = 0.6$ when $g = -21$

21. **COMMUNITY SERVICE** Every year students at Al Qassimiya Secondary School collect canned goods for a local food pantry. They plan to distribute flyers to homes in the community asking for donations. Last year, 12 students were able to distribute 1000 flyers in four hours.
    a. Write an equation that relates the number of students s to the amount of time t it takes to distribute 1000 flyers.
    b. How long would it take 15 students to hand out the same number of flyers this year?

Example 4 **22. BIRDS** When a group of snow geese migrate, the distance that they fly varies directly with the amount of time they are in the air.

a. A group of snow geese migrated 375 kilometers in 7.5 hours. Write a direct variation equation that represents this situation.

b. Every year, geese migrate 3000 kilometers from their winter home in the southwest United States to their summer home in the Canadian Arctic. Estimate the number of hours of flying time that it takes for the geese to migrate.

Example 5 **23.** Suppose a varies directly as b, and a varies inversely as c. Find b when $a = 5$ and $c = -4$, if $b = 12$ when $c = 3$ and $a = 8$.

24. Suppose x varies directly as y, and x varies inversely as z. Find z when $x = 10$ and $y = -7$, if $z = 20$ when $x = 6$ and $y = 14$.

B **Determine whether each relation shows *direct* or *inverse* variation, or *neither*.**

25.

x	y
4	12
8	24
16	48
32	96

26.

x	y
8	2
4	4
−2	−8
−8	−2

27.

x	y
2	4
3	9
4	16
5	25

28. If y varies inversely as x and $y = 6$ when $x = 19$, find y when $x = 2$.

29. If x varies inversely as y and $x = 16$ when $y = 5$, find x when $y = 20$.

30. Suppose a varies directly as b, and a varies inversely as c. Find b when $a = 7$ and $c = -8$, if $b = 15$ when $c = 2$ and $a = 4$.

31. Suppose x varies directly as y, and x varies inversely as z. Find z when $x = 8$ and $y = -6$, if $z = 26$ when $x = 8$ and $y = 13$.

State whether each equation represents a *direct*, *joint*, *inverse*, or *combined* variation. Then name the constant of variation.

32. $\frac{x}{y} = 2.75$ **33.** $fg = -2$ **34.** $a = 3bc$ **35.** $10 = \frac{xy^2}{z}$

36. $y = -11x$ **37.** $\frac{n}{p} = 4$ **38.** $9n = pr$ **39.** $-2y = z$

40. $a = 27b$ **41.** $c = \frac{7}{d}$ **42.** $-10 = gh$ **43.** $m = 20cd$

44. PRECISION The volume of a gas v varies inversely as the pressure p and directly as the temperature t.

a. Write an equation to represent the volume of a gas in terms of pressure and temperature. Is your equation a *direct*, *joint*, *inverse*, or *combined* variation?

b. A certain gas has a volume of 8 liters, a temperature of 275 Kelvin, and a pressure of 1.25 atmospheres. If the gas is compressed to a volume of 6 liters and is heated to 300 Kelvin, what will the new pressure be?

c. If the volume stays the same, but the pressure drops by half, then what must have happened to the temperature?

45. VACATION The time it takes Salem and his brother to reach Dubai Creek varies inversely with their average rate of speed.

a. If they are 800 miles away, write and graph an equation relating their travel time to their average rate of speed.

b. What minimum average speed will allow them to arrive within 18 hours?

46. **MUSIC** The maximum number of songs that a digital audio player can hold depends on the lengths and the quality of the songs that are recorded. A song will take up more space on the player if it is recorded at a higher quality, like from a CD, than at a lower quality, like from the Internet.

a. If a certain player has 5400 megabytes of storage space, write a function that represents the number of songs the player can hold as a function of the average size of the songs.

b. Is your function a *direct*, *joint*, *inverse*, or *combined* variation?

c. Suppose the average file size for a high-quality song is 8 megabytes and the average size for a low-quality song is 5 megabytes. Determine how many more songs the player can hold if they are low quality than if they are high quality.

C 47. **GRAVITY** According to the Law of Universal Gravitation, the attractive force F in newtons between any two bodies in the universe is directly proportional to the product of the masses m_1 and m_2 in kilograms of the two bodies and inversely proportional to the square of the distance d in meters between the bodies. That is, $F = \frac{Gm_1m_2}{d^2}$. G is the universal gravitational constant. Its value is 6.67×10^{-11} Nm^2/kg^2.

a. The distance between Earth and the Moon is about 3.84×10^8 meters. The mass of the Moon is 7.36×10^{22} kilograms. The mass of Earth is 5.97×10^{24} kilograms. What is the gravitational force that the Moon and Earth exert upon each other?

b. The distance between Earth and the Sun is about 1.5×10^{11} meters. The mass of the Sun is about 1.99×10^{30} kilograms. What is the gravitational force that the Sun and Earth exert upon each other?

c. Find the gravitational force exerted on each other by two 1000-kilogram iron balls at a distance of 0.1 meter apart.

H.O.T. Problems Use Higher-Order Thinking Skills

48. **CRITIQUE** Yousif and Saeed are setting up a proportion to begin solving the combined variation in which z varies directly as x and z varies inversely as y. Who has set up the correct proportion? Explain your reasoning.

Yousif

$z_1 = \frac{kx_1}{y_1}$ and $z_2 = \frac{kx_2}{y_2}$

$k = \frac{z_1y_1}{x_1}$ and $k = \frac{z_2y_2}{x_2}$

$\frac{z_1y_1}{x_1} = \frac{z_2y_2}{x_2}$

Saeed

$z_1 = \frac{kx_1}{y_1}$ and $z_2 = \frac{kx_2}{y_2}$

$k = \frac{z_1x_1}{y_1}$ and $k = \frac{z_2x_2}{y_2}$

$\frac{z_1x_1}{y_1} = \frac{z_2x_2}{y_2}$

49. **CHALLENGE** If a varies inversely as b, c varies jointly as b and f, and f varies directly as g, how are a and g related?

50. **REASONING** Explain why some mathematicians consider every joint variation a combined variation, but not every combined variation a joint variation.

51. **OPEN ENDED** Describe three real-life quantities that vary jointly with each other.

52. **WRITING IN MATH** Determine the type(s) of variation(s) for which 0 cannot be one of the values. Explain your reasoning.

Standardized Test Practice

53. SAT/ACT Eissa left the dorm and drove toward the cabin at an average speed of 40 km/h. Adnan left some time later driving in the same direction at an average speed of 48 km/h. After driving for five hours, Adnan caught up with Eissa. How long did Eissa drive before Adnan caught up?

A 1 hour
B 2 hours
C 4 hours
D 6 hours
E 8 hours

54. 75% of 88 is the same as 60% of what number?

F 100
G 105
H 108
J 110

55. EXTENDED RESPONSE Baby Amani's hair is 7 centimeters long and is expected to grow at an average rate of 3 centimeters per year.

a. Make a table that shows the expected length of Amani's hair after each of the first 4 years.

b. Write a function that can be used to determine the length of her hair after each year.

c. If she does not get a haircut, determine the length of her hair after 9 years.

56. Which of the following is equal to the sum of two consecutive even integers?

A 144
B 146
C 147
D 148

Spiral Review

Determine any vertical asymptotes and holes in the graph of each rational function. (Lesson 9-4)

57. $f(x) = \frac{1}{x^2 + 5x + 6}$

58. $f(x) = \frac{x + 2}{x^2 + 3x - 4}$

59. $f(x) = \frac{x^2 + 4x + 3}{x + 3}$

60. PHOTOGRAPHY The formula $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$ can be used to determine how far the film should be placed from the lens of a camera to create a perfect photograph. The variable q represents the distance from the lens to the film, f represents the focal length of the lens, and p represents the distance from the object to the lens. (Lesson 9-3)

a. Solve the formula for $\frac{1}{p}$.

b. Write the expression containing f and q as a single rational expression.

c. If a camera has a focal length of 8 centimeters and the lens is 10 centimeters from the film, how far should an object be from the lens so that the picture will be in focus?

Solve each equation. Check your solutions.

61. $\log_3 42 - \log_3 n = \log_3 7$

62. $\log_2(3x) + \log_2 5 = \log_2 30$

63. $2 \log_5 x = \log_5 9$

64. $\log_{10} a + \log_{10} (a + 21) = 2$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

65. $2x^3 - 5x^2 - 28x + 15; x - 5$

66. $3x^3 + 10x^2 - x - 12; x + 3$

Skills Review

Find the LCM of each set of polynomials.

67. $a, 2a, a + 1$

68. $x, 4y, x - y$

69. $8, 24x, 12$

70. $x^4, 3x^2, 2xy$

71. $12a, 15, 4b^2$

72. $x + 2, x - 3, x^2 - x - 6$

LESSON 9-6 Solving Rational Equations and Inequalities

Then	Now	Why?
You simplified rational expressions.	1 Solve rational equations. 2 Solve rational inequalities.	A gaming club charges AED 20 per month for membership. Members also have to pay AED 5 each time they visit the club. If a member visits the club x times in one month, then the charge for that month will be $20 + 5x$. The actual cost per visit will be $\frac{20+5x}{x}$. To determine how many visits are needed for the cost per visit to be AED 6, you would need to solve the equation $\frac{20+5x}{x} = 6$.

NewVocabulary
rational equation
weighted average
rational inequality

Mathematical Practices
6 Attend to precision.

1 Solve Rational Equations

Equations that contain one or more rational expressions are called **rational equations**. These equations are often easier to solve once the fractions are eliminated. You can eliminate the fractions by multiplying each side by the least common denominator (LCD).

Example 1 Solve a Rational Equation

Solve $\frac{4}{x+3} + \frac{5}{6} = \frac{23}{18}$. Check your solution.

The LCD for the terms is $18(x + 3)$.

$\frac{4}{x+3} + \frac{5}{6} = \frac{23}{18}$	Original equation
$18(x+3)\left(\frac{4}{x+3}\right) + 18(x+3)\left(\frac{5}{6}\right) = 18(x+3)\frac{23}{18}$	Multiply by LCD.
$\overset{}{\cancel{18(x+3)}}_{1}\left(\frac{4}{\cancel{x+3}_{1}}\right) + \cancel{18}^{3}(x+3)\left(\frac{5}{\cancel{6}_{1}}\right) = \cancel{18}^{1}(x+3)\left(\frac{23}{\cancel{18}_{1}}\right)$	Divide common factors.
$72 + 15x + 45 = 23x + 69$	Multiply.
$15x + 117 = 23x + 69$	Simplify.
$48 = 8x$	Subtract $15x$ and 69.
$6 = x$	Divide.

CHECK

$\frac{4}{x+3} + \frac{5}{6} = \frac{23}{18}$	Original equation
$\frac{4}{6+3} + \frac{5}{6} \stackrel{?}{=} \frac{23}{18}$	$x = 6$
$\frac{4}{9} + \frac{5}{6} \stackrel{?}{=} \frac{23}{18}$	Simplify.
$\frac{8}{18} + \frac{15}{18} \stackrel{?}{=} \frac{23}{18}$	Simplify.
$\frac{23}{18} = \frac{23}{18}$ ✓	Add.

GuidedPractice

Solve each equation. Check your solution.

1A. $\frac{2}{x+3} + \frac{3}{2} = \frac{19}{10}$ **1B.** $\frac{7}{12} + \frac{9}{x-4} = \frac{55}{48}$

Multiplying each side of an equation by the LCD of rational expressions can yield results that are not solutions of the original equation. These are extraneous solutions.

Math HistoryLink

Brook Taylor (1685–1731) English mathematician Taylor developed a theorem used in calculus known as Taylor's Theorem that relies on the remainders after computations with rational expressions.

Example 2 Solve a Rational Equation

Solve $\frac{2x}{x+5} - \frac{x^2 - x - 10}{x^2 + 8x + 15} = \frac{3}{x+3}$. Check your solution.

The LCD for the terms is $(x + 3)(x + 5)$.

$$\frac{2x}{x+5} - \frac{x^2 - x - 10}{x^2 + 8x + 15} = \frac{3}{x+3}$$ Original equation

$$\frac{(x+3)(x+5)(2x)}{x+5} - \frac{(x+3)(x+5)(x^2 - x - 10)}{x^2 + 8x + 15} = \frac{(x+3)(x+5)3}{x+3}$$ Multiply by LCD.

$$\frac{(x+3)\overset{1}{\cancel{(x+5)}}(2x)}{\underset{1}{\cancel{x+5}}} - \frac{\overset{1}{\cancel{(x+3)}}\overset{1}{\cancel{(x+5)}}(x^2 - x - 10)}{\underset{1}{\cancel{x^2 + 8x + 15}}} = \frac{(x+5)\overset{1}{\cancel{(x+3)}}3}{\underset{1}{\cancel{x+3}}}$$ Divide common factors.

$$(x + 3)(2x) - (x^2 - x - 10) = 3(x + 5)$$ Simplify.

$$2x^2 + 6x - x^2 + x + 10 = 3x + 15$$ Distribute.

$$x^2 + 7x + 10 = 3x + 15$$ Simplify.

$$x^2 + 4x - 5 = 0$$ Subtract $3x + 15$.

$$(x + 5)(x - 1) = 0$$ Factor.

$$x + 5 = 0 \quad \text{or} \quad x - 1 = 0$$ Zero Product Property

$$x = -5 \qquad\qquad x = 1$$

CHECK Try $x = -5$.

$$\frac{2x}{x+5} - \frac{x^2 - x - 10}{x^2 + 8x + 15} = \frac{3}{x+3}$$

$$\frac{2(-5)}{-5+5} - \frac{(-5)^2 - (-5) - 10}{(-5)^2 + 8(-5) + 15} \stackrel{?}{=} \frac{3}{-5+3}$$

$$\frac{-10}{0} - \frac{25 + 5 - 10}{25 - 40 + 15} \neq -\frac{3}{2} \ ✗$$

Try $x = 1$.

$$\frac{2x}{x+5} - \frac{x^2 - x - 10}{x^2 + 8x + 15} = \frac{3}{x+3}$$

$$\frac{2(1)}{1+5} - \frac{1^2 - 1 - 10}{1^2 + 8(1) + 15} \stackrel{?}{=} \frac{3}{1+3}$$

$$\frac{2}{6} - \frac{-10}{24} \stackrel{?}{=} \frac{3}{4}$$

$$\frac{8}{24} + \frac{10}{24} \stackrel{?}{=} \frac{3}{4}$$

$$\frac{3}{4} = \frac{3}{4} \ ✓$$

When solving a rational equation, any possible solution that results in a zero in the denominator must be excluded from your list of solutions.

Since $x = -5$ results in a zero in the denominator, it is extraneous. Eliminate -5 from the list of solutions. The solution is 1.

ReviewVocabulary

extraneous solutions solutions that do not satisfy the original equation

GuidedPractice

2A. $\frac{5}{y-2} + 2 = \frac{17}{6}$

2B. $\frac{2}{z+1} - \frac{1}{z-1} = \frac{-2}{z^2 - 1}$

2C. $\frac{7n}{3n+3} - \frac{5}{4n-4} = \frac{3n}{2n+2}$

2D. $\frac{1}{p-2} = \frac{2p+1}{p^2 + 2p - 8} + \frac{2}{p+4}$

The **weighted average** is a method for finding the mean of a set of numbers in which some elements of the set carry more importance, or weight, than others. Many real-world problems involving mixtures, work, distance, and interest can be solved by using rational equations.

Real-World Example 3 Mixture Problem

CHEMISTRY **Maha adds a 70% acid solution to 12 milliliters of a solution that is 15% acid. How much of the 70% acid solution should be added to create a solution that is 60% acid?**

Understand Maha needs to know how much of a solution needs to be added to an original solution to create a new solution.

Plan Each solution has a certain percentage that is acid. The percentage of acid in the final solution must equal the amount of acid divided by the total solution.

	Original	Added	New
Amount of Acid	0.15(12)	0.7(x)	0.15(12) + 0.7x
Total Solution	12	x	12 + x

$$\text{Percentage of acid in solution} = \frac{\text{amount of acid}}{\text{total solution}}$$

Solve

$\frac{\text{percent}}{100} = \frac{\text{amount of acid}}{\text{total solution}}$ Write a proportion.

$\frac{60}{100} = \frac{0.15(12) + 0.7x}{12 + x}$ Substitute.

$\frac{60}{100} = \frac{1.8 + 0.7x}{12 + x}$ Simplify numerator.

$100(12 + x)\frac{60}{100} = 100(12 + x)\frac{1.8 + 0.7x}{12 + x}$ LCD is $100(12 + x)$. Multiply by LCD.

$\overset{}{\underset{1}{\cancel{100}}}(12 + x)\frac{60}{\underset{1}{\cancel{100}}} = 100\underset{1}{\cancel{(12 + x)}}\frac{1.8 + 0.7x}{\underset{1}{\cancel{12 + x}}}$ Divide common factors.

$(12 + x)60 = 100(1.8 + 0.7x)$ Simplify.

$720 + 60x = 180 + 70x$ Distribute.

$540 = 10x$ Subtract $60x$ and 180.

$54 = x$ Divide by 10.

Check

$\frac{60}{100} = \frac{0.15(12) + 0.7x}{12 + x}$ Original equation

$\frac{60}{100} \stackrel{?}{=} \frac{0.15(12) + 0.7(54)}{12 + 54}$ $x = 54$

$\frac{60}{100} \stackrel{?}{=} \frac{37.8}{66}$ Simplify.

$0.6 = 0.6$ ✓ Simplify.

Maha needs to add 54 milliliters of the 70% acid solution.

GuidedPractice

3. Mansour adds a 65% fruit juice solution to 15 milliliters of a drink that is 10% fruit juice. How much of the 65% fruit juice solution must be added to create a fruit punch that is 35% fruit juice?

StudyTip

Modeling Tables like the one in Example 3 are useful in organizing and solving mixture, work, weighted average, and distance problems.

The formula relating distance, rate, and time can also be used to solve rational equations. The most common use is $d = rt$. However, it can also be represented by $r = \frac{d}{t}$ and $t = \frac{d}{r}$.

Real-World Example 4 Distance Problem

ROWING Nasser is rowing a canoe on Dubai Creek. His rate in still water is 6 miles per hour. It takes Nasser 3 hours to travel 10 miles round trip. Assuming that Nasser rowed at a constant rate of speed, determine the rate of the current.

StudyTip

Distance Problems When distances involve round trips, the distance in one direction usually equals the distance in the other direction.

Understand We are given his speed in still water and the time it takes him to travel with the current and against it. We need to determine the speed of the current.

Plan He traveled 5 miles with the current and 5 miles against it. The formula that relates distance, rate, and time is $d = rt$, or $t = \frac{d}{r}$.

Time with the Current	Time Against the Current	Total Time
$\frac{5}{6+r}$	$\frac{5}{6-r}$	3 hours

Solve

$$\frac{5}{6+r} + \frac{5}{6-r} = 3 \quad \text{Write the equation.}$$

$$(6+r)(6-r)\frac{5}{6+r} + (6+r)(6-r)\frac{5}{6-r} = (6+r)(6-r)3 \quad \text{LCD} = (6+r)(6-r) \text{ Multiply by LCD.}$$

$$\overset{}{\underset{1}{\cancel{(6+r)}}}(6-r)\frac{5}{\underset{1}{\cancel{6+r}}} + (6+r)\underset{1}{\cancel{(6-r)}}\frac{5}{\underset{1}{\cancel{6-r}}} = (6+r)(6-r)3 \quad \text{Divide common factors.}$$

$$(6-r)5 + (6+r)5 = (36-r^2)3 \quad \text{Simplify.}$$

$$30 - 5r + 30 + 5r = 108 - 3r^2 \quad \text{Distribute.}$$

$$60 = 108 - 3r^2 \quad \text{Simplify.}$$

$$0 = -3r^2 + 48 \quad \text{Subtract 10}r\text{.}$$

$$0 = -3(r+4)(r-4) \quad \text{Factor.}$$

$$0 = (r+4)(r-4) \quad \text{Divide each side by } -3.$$

$$r = 4 \text{ or } -4 \quad \text{Zero Product Property}$$

Check

$$\frac{5}{6+r} + \frac{5}{6-r} = 3 \quad \text{Original equation}$$

$$\frac{5}{6+4} + \frac{5}{6-4} \stackrel{?}{=} 3 \quad r = 4$$

$$\frac{5}{10} + \frac{5}{2} \stackrel{?}{=} 3 \quad \text{Simplify.}$$

$$\frac{1}{2} + \frac{5}{2} = \frac{6}{2} \checkmark \quad \text{Simplify.}$$

Since speed cannot be negative, the speed of the current is 4 miles per hour.

GuidedPractice

4. FLYING The speed of the wind is 20 miles per hour. If it takes a plane 7 hours to fly 2368 miles round trip, determine the plane's speed in still air.

Real-world problems that involve work can often be solved using rational equations.

Real-WorldLink

Since 1997, students from Rock Point School in Burlington, Vermont, spend a week servicing communities throughout the world. While working with Habitat for Humanity, the students spent the time in rural Tennessee, starting and completing the roof of a Habitat home in one week.

Source: Vermont Community Work

Real-World Example 5 Work Problems

COMMUNITY SERVICE **Every year, the Grade 11 and 12 classes at Al Munther Secondary School build a house for the community. If it takes the Grade 12 class 24 days to complete a house and 18 days if they work with the Grade 11 class, how long would it take the Grade 11 class to complete a house if they worked alone?**

Understand We are given how long it takes the Grade 12 class working alone and when the classes work together. We need to determine how long it would take the Grade 11 class by themselves.

Plan The Grade 12 class can complete 1 house in 24 days, so their rate is $\frac{1}{24}$ of a house per day.

The rate for the Grade 11 class is $\frac{1}{j}$.

The combined rate for both classes is $\frac{1}{18}$.

Grade 12 Rate	Grade 11 Rate	Combined Rate
$\frac{1}{24}$	$\frac{1}{j}$	$\frac{1}{18}$

Solve

$\frac{1}{24} + \frac{1}{j} = \frac{1}{18}$ Write the equation.

$72j\frac{1}{24} + 72j\frac{1}{j} = 72j\frac{1}{18}$ LCD = $72j$ Multiply by LCD.

$\overset{3}{\cancel{72}}j\frac{1}{\underset{1}{\cancel{24}}} + \overset{1}{72\cancel{j}}\frac{1}{\underset{1}{\cancel{j}}} = \overset{4}{\cancel{72}}j\frac{1}{\underset{1}{\cancel{18}}}$ Divide common factors.

$3j + 72 = 4j$ Distribute.

$72 = j$ Subtract $3j$.

Check Two methods are possible.

Method 1 Substitute values.

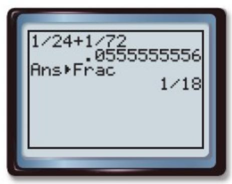
$\frac{1}{24} + \frac{1}{j} = \frac{1}{18}$ Original equation

$\frac{1}{24} + \frac{1}{72} \stackrel{?}{=} \frac{1}{18}$ $j = 72$

$\frac{3}{72} + \frac{1}{72} \stackrel{?}{=} \frac{4}{72}$ LCD = 72

$\frac{4}{72} = \frac{4}{72}$ ✓ Simplify.

Method 2 Use a calculator.



It would take the Grade 11 class 72 days to complete the house by themselves.

GuidedPractice

5A. It took Adnan and Tarek 6 hours to rake the leaves together last year. The previous year it took Tarek 10 hours to do it alone. How long will it take Adnan if he rakes them by himself this year?

5B. Omar and Ali paint houses together. If Omar can paint a particular house in 6 days and Ali can paint the same house in 5 days, how long would it take the two of them if they work together?

2 Solve Rational Inequalities

To solve **rational inequalities**, which are inequalities that contain one or more rational expressions, follow these steps.

KeyConcept Solving Rational Inequalities

Step 1 State the excluded values. These are the values for which the denominator is 0.

Step 2 Solve the related equation.

Step 3 Use the values determined from the previous steps to divide a number line into intervals.

Step 4 Test a value in each interval to determine which intervals contain values that satisfy the inequality.

Example 6 Solve a Rational Inequality

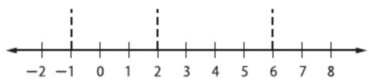
Solve $\frac{x}{3} - \frac{1}{x-2} < \frac{x+1}{4}$.

Step 1 The excluded value for this inequality is 2.

Step 2 Solve the related equation.

$\frac{x}{3} - \frac{1}{x-2} = \frac{x+1}{4}$	Related equation
$\overset{4}{\cancel{12}}(x-2)\frac{x}{\underset{1}{\cancel{3}}} - 12\overset{1}{\cancel{(x-2)}}\frac{1}{\underset{1}{\cancel{x-2}}} = \overset{3}{\cancel{12}}(x-2)\frac{x+1}{\underset{1}{\cancel{4}}}$	LCD is $12(x-2)$. Multiply by LCD.
$4x^2 - 8x - 12 = 3x^2 - 3x - 6$	Distribute.
$x^2 - 5x - 6 = 0$	Subtract $3x^2 - 3x - 6$.
$(x-6)(x+1) = 0$	Factor.
$x = 6$ or -1	Zero Product Property

Step 3 Draw vertical lines at the excluded value and at the solutions to separate the number line into intervals.



Step 4 Now test a sample value in each interval to determine whether the values in the interval satisfy the inequality.

Test $x = -3$.

$$\frac{-3}{3} - \frac{1}{-3-2} \stackrel{?}{<} \frac{-3+1}{4}$$
$$-1 + \frac{1}{5} \stackrel{?}{<} -\frac{2}{4}$$
$$-\frac{4}{5} < -\frac{1}{2} \checkmark$$

Test $x = 0$.

$$\frac{0}{3} - \frac{1}{0-2} \stackrel{?}{<} \frac{0+1}{4}$$
$$0 + \frac{1}{2} \stackrel{?}{<} \frac{1}{4}$$
$$\frac{1}{2} \not< \frac{1}{4}$$

Test $x = 4$.

$$\frac{4}{3} - \frac{1}{4-2} \stackrel{?}{<} \frac{4+1}{4}$$
$$\frac{4}{3} - \frac{1}{2} \stackrel{?}{<} \frac{5}{4}$$
$$\frac{5}{6} < \frac{5}{4} \checkmark$$

Test $x = 8$.

$$\frac{8}{3} - \frac{1}{8-2} \stackrel{?}{<} \frac{8+1}{4}$$
$$\frac{32}{12} - \frac{2}{12} \stackrel{?}{<} \frac{27}{12}$$
$$\frac{30}{12} \not< \frac{27}{12}$$

The statement is true for $x = -3$ and $x = 4$. Therefore, the solution is $x < -1$ or $2 < x < 6$.

GuidedPractice Solve each inequality.

6A. $\frac{5}{x} + \frac{6}{5x} > \frac{2}{3}$

6B. $\frac{4}{3x} + \frac{7}{x} < \frac{5}{9}$

StudyTip

Rational Inequalities It is possible that none or all of the intervals will produce a true statement.

Check Your Understanding

Examples 1–2 **Solve each equation. Check your solution.**

1. $\frac{4}{7} + \frac{3}{x-3} = \frac{53}{56}$

2. $\frac{7}{3} - \frac{3}{x-5} = \frac{19}{12}$

3. $\frac{10}{2x+1} + \frac{4}{3} = 2$

4. $\frac{11}{4} - \frac{5}{y+3} = \frac{23}{12}$

5. $\frac{8}{x-5} - \frac{9}{x-4} = \frac{5}{x^2-9x+20}$

6. $\frac{14}{x+3} + \frac{10}{x-2} = \frac{122}{x^2+x-6}$

7. $\frac{14}{x-8} - \frac{5}{x-6} = \frac{82}{x^2-14x+48}$

8. $\frac{5}{x+2} - \frac{3}{x-2} = \frac{12}{x^2-4}$

Example 3

9. **STRUCTURE** Noura has 4.5 kilograms of dried fruit selling for AED 51 per kilogram. She wants to know how many kilograms of mixed nuts selling for AED 36.73 per kilogram she needs to make a trail mix selling for AED 40.82 per kilogram.

a. Let m = the number of kilograms of mixed nuts. Complete the following table.

	kilograms	Price per kilogram	Total Price
Dried Fruit	4.5	AED 51	51 (4.5)
Mixed Nuts			
Trail Mix			

b. Write a rational equation using the last column of the table.

c. Solve the equation to determine how many kilograms of mixed nuts are needed.

Example 4

10. **DISTANCE** Moza's average speed riding her bike is 11.5 kilometers per hour. She takes a round trip of 40 kilometers. It takes her 1 hour and 20 minutes with the wind and 2 hours and 30 minutes against the wind.

a. Write an expression for Moza's time with the wind.

b. Write an expression for Moza's time against the wind.

c. How long does it take to complete the trip?

d. Write and solve the rational equation to determine the speed of the wind.

Example 5

11. **WORK** Ayoub and Faris wax cars. Ayoub can wax a particular car in 60 minutes and Faris can wax the same car in 80 minutes. They plan on waxing the same car together and want to know how long it will take.

a. How much will Ayoub complete in 1 minute?

b. How much will Ayoub complete in x minutes?

c. How much will Faris complete in 1 minute?

d. How much will Faris complete in x minutes?

e. Write a rational equation representing Ayoub and Faris working together on the car.

f. Solve the equation to determine how long it will take them to finish the car.

Example 6 **Solve each inequality. Check your solutions.**

12. $\frac{3}{5x} + \frac{1}{6x} > \frac{2}{3}$

13. $\frac{1}{4c} + \frac{1}{9c} < \frac{1}{2}$

14. $\frac{4}{3y} + \frac{2}{5y} < \frac{3}{2}$

15. $\frac{1}{3b} + \frac{1}{4b} < \frac{1}{5}$

Practice and Problem Solving

Examples 1–2 **Solve each equation. Check your solutions.**

16. $\frac{9}{x-7} - \frac{7}{x-6} = \frac{13}{x^2 - 13x + 42}$

17. $\frac{13}{y+3} - \frac{12}{y+4} = \frac{18}{y^2 + 7y + 12}$

18. $\frac{14}{x-2} - \frac{18}{x+1} = \frac{22}{x^2 - x - 2}$

19. $\frac{11}{a+2} - \frac{10}{a+5} = \frac{36}{a^2 + 7a + 10}$

20. $\frac{x}{2x-1} + \frac{3}{x+4} = \frac{21}{2x^2 + 7x - 4}$

21. $\frac{2}{y-5} + \frac{y-1}{2y+1} = \frac{2}{2y^2 - 9y - 5}$

Examples 3–5 **22. CHEMISTRY** How many milliliters of a 20% acid solution must be added to 40 milliliters of a 75% acid solution to create a 30% acid solution?

23. GROCERIES Sally bought 1.4 kilograms of bananas for AED 7.35 per kilogram. How many kilograms of apples costing AED 10.20 per kilogram must she purchase so that the total cost for fruit is AED 8.15 per kilogram?

24. BUILDING Badr's volunteer group can build a garage in 12 hours. Shaima's group can build it in 16 hours. How long would it take them if they worked together?

Example 6 **Solve each inequality. Check your solutions.**

25. $3 - \frac{4}{x} > \frac{5}{4x}$

26. $\frac{5}{3a} - \frac{3}{4a} > \frac{5}{6}$

27. $\frac{x-2}{x+2} + \frac{1}{x-2} > \frac{x-4}{x-2}$

28. $\frac{3}{4} - \frac{1}{x-3} > \frac{x}{x+4}$

29. $\frac{x}{5} + \frac{2}{3} < \frac{3}{x-4}$

30. $\frac{x}{x+2} + \frac{1}{x-1} < \frac{3}{2}$

B **31. AIR TRAVEL** It takes a plane 20 hours to fly to its destination against the wind. The return trip takes 16 hours. If the plane's average speed in still air is 500 miles per hour, what is the average speed of the wind during the flight?

32. FINANCIAL LITERACY Huda wants to invest AED 10,000 in two different accounts. The risky account could earn 9% interest, while the other account earns 5% interest. She wants to earn AED 750 interest for the year. Of tables, graphs, or equations, choose the best representation needed and determine how much should be invested in each account.

33. MULTIPLE REPRESENTATIONS Consider $\frac{2}{x-3} + \frac{1}{x} = \frac{x-1}{x-3}$.

a. **Algebraic** Solve the equation for x. Were any values of x extraneous?

b. **Graphical** Graph $y_1 = \frac{2}{x-3} + \frac{1}{x}$ and $y_2 = \frac{x-1}{x-3}$ on the same graph for $0 < x < 5$.

c. **Analytical** For what value(s) of x do they intersect? Do they intersect where x is extraneous for the original equation?

d. **Verbal** Use this knowledge to describe how you can use a graph to determine whether an apparent solution of a rational equation is extraneous.

C **Solve each equation. Check your solutions.**

34. $\frac{2}{y+3} - \frac{3}{4-y} = \frac{2y-2}{y^2 - y - 12}$

35. $\frac{2}{y+2} - \frac{y}{2-y} = \frac{y^2+4}{y^2-4}$

H.O.T. Problems Use Higher-Order Thinking Skills

36. OPEN ENDED Give an example of a rational equation that can be solved by multiplying each side of the equation by $4(x+3)(x-4)$.

37. CHALLENGE Solve $\dfrac{1 + \frac{9}{x} + \frac{20}{x^2}}{1 - \frac{25}{x^2}} = \frac{x+4}{x-5}$.

38. TOOLS While using the table feature on the graphing calculator to explore $f(x) = \frac{1}{x^2 - x - 6}$, the values -2 and 3 say "**ERROR.**" Explain its meaning.

39. WRITING IN MATH Why should you check solutions of rational equations and inequalities?

Standardized Test Practice

40. Nine kilograms of mixed nuts containing 55% peanuts were mixed with 6 kilograms of another kind of mixed nuts that contain 40% peanuts. What percent of the new mixture is peanuts?

A 58% **B** 51% **C** 49% **D** 47%

41. Working alone, Salem can dig a 3-meter by 3-meter hole in five hours. Hasan can dig the same hole in six hours. How long would it take them if they worked together?

F 1.5 hours **H** 2.52 hours

G 2.34 hours **J** 2.73 hours

42. An aircraft carrier made a trip to a city and back. The trip there took three hours and the trip back took four hours. It averaged 6 kilometers per hour on the return trip. Find the average speed of the trip to the city.

A 6 km/h **C** 10 km/h

B 8 km/h **D** 12 km/h

43. SHORT RESPONSE If a line ℓ is perpendicular to a segment CD at point F and $CF = FD$, how many points on line ℓ are the same distance from point C as from point D?

Spiral Review

Determine whether each relation shows *direct* or *inverse* variation, or *neither*. (Lesson 9-5)

44.

x	y
14	3
28	1.5
56	0.75
112	0.375

45.

x	y
0.2	24
0.6	72
1.8	216
5.4	648

46.

x	y
12	18
24	36
36	18
72	9

Graph each function. (Lesson 9-4)

47. $f(x) = \dfrac{x + 4}{x^2 + 7x + 12}$

48. $f(x) = \dfrac{x^2 - 5x - 14}{x - 7}$

49. $f(x) = \dfrac{x^2 + 3x - 6}{x - 2}$

50. WEATHER The atmospheric pressure P, in bars, of a given height on Earth is given by the formula $P = a \cdot e^{-\frac{k}{H}}$. In the formula, a is the surface pressure on Earth, which is approximately 1 bar, k is the altitude for which you want to find the pressure in kilometers, and H is always 7 kilometers.

a. Find the pressure for 2, 4, and 7 kilometers.

b. What do you notice about the pressure as altitude increases?

51. COMPUTERS Since computers have been invented, computational speed has multiplied by a factor of 4 about every three years.

a. If a typical computer operates with a computational speed s today, write an expression for the speed at which you can expect an equivalent computer to operate after x three-year periods.

b. Suppose your computer operates with a processor speed of 2.8 gigahertz and you want a computer that can operate at 5.6 gigahertz. If a computer with that speed is currently unavailable for home use, how long can you expect to wait until you can buy such a computer?

Skills Review

Determine whether the following are possible lengths of the sides of a right triangle.

52. 5, 12, 13

53. 60, 80, 100

54. 7, 24, 25

EXTEND 9-6 Graphing Technology Lab
Solving Rational Equations and Inequalities

You can use a graphing calculator to solve rational equationsby graphing or by using the table feature. Graph both sides of the equation, andlocate the point(s) of intersection.

Mathematical Practices
5 Use appropriate tools strategically.

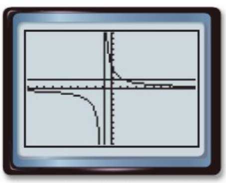
Activity 1 Rational Equation

Solve $\frac{4}{x+1} = \frac{3}{2}$.

Step 1 Graph each side of the equation.

Graph each side of the equation as a separate function. Enter $\frac{4}{x+1}$ as **Y1** and $\frac{3}{2}$ as **Y2**. Then graph the two equations in the standard viewing window.

KEYSTROKES: [Y=] 4 [÷] [(] [X,T,θ,n] [+] 1 [)] [ENTER] 3 [÷] 2 [ZOOM] 6



[−10, 10] scl: 1 by [−10, 10] scl: 1

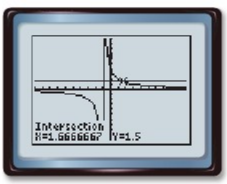
Because the calculator is in connected mode, a vertical line may appear connecting the two branches of the hyperbola. This line is not part of the graph.

Step 2 Use the **intersect** feature.

The **intersect** feature on the **CALC** menu allows you to approximate the ordered pair of the point at which the graphs cross.

KEYSTROKES: [2nd] [CALC] 5

Select one graph and press [ENTER]. Select the other graph, press [ENTER], and press [ENTER] again.



[−10, 10] scl: 1 by [−10, 10] scl: 1

The solution is $1\frac{2}{3}$.

Step 3 Use the **TABLE** feature.

Verify the solution using the **TABLE** feature. Set up the table to show x-values in increments of $\frac{1}{3}$.

KEYSTROKES: [2nd] [TBLSET] 0 [ENTER] 1 [÷] 3 [ENTER] [2nd] [TABLE]

The table displays x-values and corresponding y-values for each graph. At $x = 1\frac{2}{3}$, both functions have a y-value of 1.5. Thus, the solution of the equation is $1\frac{2}{3}$.

(continued on the next page)

Solving Rational Equations and Inequalities *Continued*

You can use a similar procedure to solve rational inequalities using a graphing calculator.

Activity 2 Rational Inequality

Solve $\frac{3}{x} + \frac{7}{x} > 9$.

Step 1 Enter the inequalities.

Rewrite the problem as a system of inequalities.

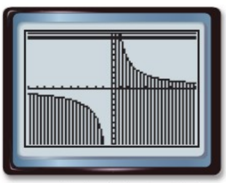
The first inequality is $\frac{3}{x} + \frac{7}{x} > y$ or $y < \frac{3}{x} + \frac{7}{x}$. Since this inequality includes the *less than* symbol, shade below the curve. First enter the boundary and then use the arrow and [ENTER] keys to choose the shade below icon, .

The second inequality is $y > 9$. Shade above the curve since this inequality contains *greater than*.

KEYSTROKES: [Y=] [◀] [◀] [ENTER] [ENTER] [ENTER] [▶] [▶] 3 [÷] [X,T,θ,n] [+] 7 [÷] [X,T,θ,n] [ENTER] [◀] [◀] [ENTER] [ENTER] [▶] [▶] 9

Step 2 Graph the system.

KEYSTROKES: [GRAPH]



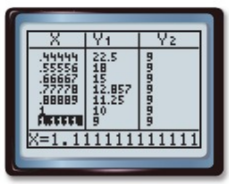
[−10, 10] scl: 1 by [−10, 10] scl: 1

The solution set of the original inequality is the set of x-values of the points in the region where the shadings overlap. Using the calculator's **intersect** feature, you can conclude that the solution set is $\left\{x \middle| 0 < x < 1\frac{1}{9}\right\}$.

Step 3 Use the **TABLE** feature.

Verify using the **TABLE** feature. Set up the table to show x-values in increments of $\frac{1}{9}$.

KEYSTROKES: [2nd] [TBLSET] 0 [ENTER] 1 [÷] 9 [ENTER] [2nd] [TABLE]



Scroll through the table. Notice that for x-values greater than 0 and less than $1\frac{1}{9}$, **Y1** > **Y2**. This confirms that the solution of the inequality is $\left\{x \middle| 0 < x < 1\frac{1}{9}\right\}$.

Exercises

Solve each equation or inequality.

1. $\frac{1}{x} + \frac{1}{2} = \frac{2}{x}$

2. $\frac{1}{x-4} = \frac{2}{x-2}$

3. $\frac{4}{x} = \frac{6}{x^2}$

4. $\frac{1}{1-x} = 1 - \frac{x}{x-1}$

5. $\frac{1}{x+4} = \frac{2}{x^2+3x-4} - \frac{1}{1-x}$

6. $\frac{1}{x} + \frac{1}{2x} > 5$

7. $\frac{1}{x-1} + \frac{2}{x} < 0$

8. $1 + \frac{5}{x-1} \leq 0$

9. $2 + \frac{1}{x-1} \geq 0$

CHAPTER 9

Study Guide and Review

Study Guide

KeyConcepts

Rational Expressions (Lessons 9-1 and 9-2)

- Multiplying and dividing rational expressions is similar to multiplying and dividing fractions.
- To simplify complex fractions, simplify the numerator and the denominator separately, and then simplify the resulting expression.

Reciprocal and Rational Functions (Lessons 9-3 and 9-4)

- A reciprocal function is of the form $f(x) = \frac{1}{a(x)}$, where $a(x)$ is a linear function and $a(x) \neq 0$.
- A rational function is of the form $\frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$.

Direct, Joint, and Inverse Variation (Lesson 9-5)

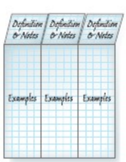
- Direct Variation: There is a nonzero number k such that $y = kx$.
- Joint Variation: There is a nonzero number k such that $y = kxz$.
- Inverse Variation: There is a nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

Rational Equations and Inequalities (Lesson 9-6)

- Eliminate fractions in rational equations by multiplying each side of the equation by the LCD.
- Possible solutions of a rational equation must exclude values that result in zero in the denominator.

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.



KeyVocabulary

combined variation	point discontinuity
complex fraction	rational equation
constant of variation	rational expression
direct variation	rational function
horizontal asymptote	rational inequality
hyperbola	reciprocal function
inverse variation	vertical asymptote
joint variation	weighted average
oblique asymptote	

VocabularyCheck

Choose a term from the list above that best completes each statement or phrase.

1. A(n) _______________ is a rational expression whose numerator and/or denominator contains a rational expression.
2. If two quantities show _______________, their product is equal to a constant k.
3. A(n) _______________ asymptote is a linear asymptote that is neither horizontal nor vertical.
4. A(n) _______________ can be expressed in the form $y = kx$.
5. Equations that contain one or more rational expressions are called _______________.
6. The graph of $y = \frac{x}{x+2}$ has a(n) _______________ at $x = -2$.
7. _______________ occurs when one quantity varies directly as the product of two or more other quantities.
8. A ratio of two polynomial expressions is called a(n) _______________.
9. _______________ looks like a hole in a graph because the graph is undefined at that point.
10. _______________ occurs when one quantity varies directly and/or inversely as two or more other quantities.

Lesson-by-Lesson Review

9-1 Multiplying and Dividing Rational Expressions

Simplify each expression.

11. $\frac{-16xy}{27z} \cdot \frac{15z^3}{8x^2}$

12. $\frac{x^2-2x-8}{x^2+x-12} \cdot \frac{x^2+2x-15}{x^2+7x+10}$

13. $\frac{x^2-1}{x^2-4} \cdot \frac{x^2-5x-14}{x^2-6x-7}$

14. $\frac{x+y}{15x} \div \frac{x^2-y^2}{3x^2}$

15. $\dfrac{\frac{x^2+3x-18}{x+4}}{\frac{x^2+7x+6}{x+4}}$

16. **GEOMETRY** A triangle has an area of $3x^2 + 9x - 54$ square centimeters. If the height of the triangle is $x + 6$ centimeters, find the length of the base.

Example 1

Simplify $\frac{4a}{3b} \cdot \frac{9b^4}{2a^2}$.

$$\frac{4a}{3b} \cdot \frac{9b^4}{2a^2} = \frac{2 \cdot 2 \cdot a \cdot 3 \cdot 3 \cdot b \cdot b \cdot b \cdot b}{3 \cdot b \cdot 2 \cdot a \cdot a}$$

$$= \frac{6b^3}{a}$$

Example 2

Simplify $\frac{r^2+5r}{2r} \div \frac{r^2-25}{6r-12}$.

$$\frac{r^2+5r}{2r} \div \frac{r^2-25}{6r-12} = \frac{r^2+5r}{2r} \cdot \frac{6r-12}{r^2-25}$$

$$= \frac{r(r+5)}{2r} \cdot \frac{6(r-2)}{(r+5)(r-5)}$$

$$= \frac{3(r-2)}{r-5}$$

9-2 Adding and Subtracting Rational Expressions

Simplify each expression.

17. $\frac{9}{4ab} + \frac{5a}{6b^2}$

18. $\frac{3}{4x-8} - \frac{x-1}{x^2-4}$

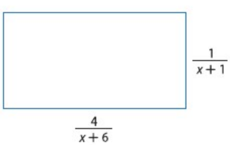
19. $\frac{y}{2x} + \frac{4y}{3x^2} - \frac{5}{6xy^2}$

20. $\frac{2}{x^2-3x-10} - \frac{6}{x^2-8x+15}$

21. $\frac{3}{3x^2+2x-8} + \frac{4x}{2x^2+6x+4}$

22. $\dfrac{\frac{3}{2x+3} - \frac{x}{x+1}}{\frac{2x}{x+1} + \frac{5}{2x+3}}$

23. **GEOMETRY** What is the perimeter of the rectangle?



Example 3

Simplify $\frac{3a}{a^2-4} - \frac{2}{a-2}$.

$$\frac{3a}{a^2-4} - \frac{2}{a-2} = \frac{3a}{(a-2)(a+2)} - \frac{2}{a-2}$$

$$= \frac{3a}{(a-2)(a+2)} - \frac{2(a+2)}{(a-2)(a+2)}$$

$$= \frac{3a-2(a+2)}{(a-2)(a+2)} \quad \text{Subtract numerators.}$$

$$= \frac{3a-2a-4}{(a-2)(a+2)} \quad \text{Distributive Property}$$

$$= \frac{a-4}{(a-2)(a+2)} \quad \text{Simplify.}$$

9-3 Graphing Reciprocal Functions

Graph each function. State the domain and range.

24. $f(x) = \frac{10}{x}$

25. $f(x) = -\frac{12}{x} + 2$

26. $f(x) = \frac{3}{x+5}$

27. $f(x) = \frac{6}{x-9}$

28. $f(x) = \frac{7}{x-2} + 3$

29. $f(x) = -\frac{4}{x+4} - 8$

30. **CONSERVATION** The student council is planting 28 trees for a service project. The number of trees each person plants depends on the number of student council members.

a. Write a function to represent this situation.

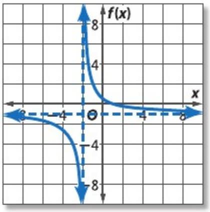
b. Graph the function.

Example 4

Graph $f(x) = \frac{3}{x+2} - 1$. State the domain and range.

$a = 3$: The graph is stretched vertically.
$h = -2$: The graph is translated 2 units left.
There is an asymptote at $x = -2$.
k: $= -1$: The graph is translated 1 unit down.
There is an asymptote at $f(x) = -1$.

Domain: $\{x \mid x \neq -2\}$,
Range: $\{f(x) \mid f(x) \neq -1\}$



9-4 Graphing Rational Functions

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

31. $f(x) = \frac{3}{x^2 + 4x}$

32. $f(x) = \frac{x+2}{x^2 + 6x + 8}$

33. $f(x) = \frac{x^2 - 9}{x^2 - 5x - 24}$

Graph each rational function.

34. $f(x) = \frac{x+2}{(x+5)^2}$

35. $f(x) = \frac{x}{x+1}$

36. $f(x) = \frac{x^2 + 4x + 4}{x+2}$

37. $f(x) = \frac{x-1}{x^2 + 5x + 6}$

38. **SALES** Amal is selling magazine subscriptions. Out of the first 15 houses, she sold subscriptions to 10 of them. Suppose Amal goes to x more houses and sells subscriptions to all of them. The percentage of houses that she sold to out of the total houses can be determined using $P(x) = \frac{10+x}{15+x}$.

a. Graph the function.

b. What domain and range values are meaningful in the context of the problem?

Example 5

Determine the equation of any vertical asymptotes and the values of x for any holes in the graph of $f(x) = \frac{x^2 - 1}{x^2 + 2x - 3}$.

$$\frac{x^2 - 1}{x^2 + 2x - 3} = \frac{(x-1)(x+1)}{(x-1)(x+3)}$$

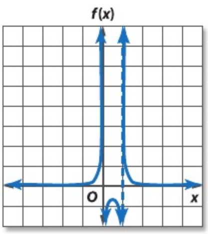
The function is undefined for $x = 1$ and $x = -3$.

Since $\frac{(x-1)(x+1)}{(x-1)(x+3)} = \frac{x+1}{x+3}$, $x = -3$ is a vertical asymptote, and $x = 1$ represents a hole in the graph.

Example 6

Graph $f(x) = \frac{1}{6x(x-1)}$.

The function is undefined for $x = 0$ and $x = 1$. Because $\frac{1}{6x(x-1)}$ is in simplest form, $x = 0$ and $x = 1$ are vertical asymptotes. Draw the two asymptotes and sketch the graph.



Study Guide and Review *Continued*

9-5 Variation Functions

39. If a varies directly as b and $b = 18$ when $a = 27$, find a when $b = 10$.

40. If y varies inversely as x and $y = 15$ when $x = 3.5$, find y when $x = -5$.

41. If y varies inversely as x and $y = -3$ when $x = 9$, find y when $x = 81$.

42. If y varies jointly as x and z, and $x = 8$ and $z = 3$ when $y = 72$, find y when $x = -2$ and $z = -5$.

43. If y varies jointly as x and z, and $y = 18$ when $x = 6$ and $z = 15$, find y when $x = 12$ and $z = 4$.

44. JOBS Laila's earnings vary directly with how many hours she cares for children. If she earns AED 68 for 8 hours of child care, find her earnings after 5 hours of child care.

Example 7

If y varies inversely as x and $x = 24$ when $y = -8$, find x when $y = 15$.

$\frac{x_1}{y_2} = \frac{x_2}{y_1}$	Inverse variation
$\frac{24}{15} = \frac{x_2}{-8}$	$x_1 = 24, y_1 = -8, y_2 = 15$
$24(-8) = 15(x_2)$	Cross multiply.
$-192 = 15x_2$	Simplify.
$-12\frac{4}{5} = x_2$	Divide each side by 15.

When $y = 15$, the value of x is $-12\frac{4}{5}$.

9-6 Solving Rational Equations and Inequalities

Solve each equation or inequality. Check your solutions.

45. $\frac{1}{3} + \frac{4}{x-2} = 6$

46. $\frac{6}{x+5} - \frac{3}{x-3} = \frac{6}{x^2+2x-15}$

47. $\frac{2}{x^2-9} = \frac{3}{x^2-2x-3}$

48. $\frac{4}{2x-3} + \frac{x}{x+1} = \frac{-8x}{2x^2-x-3}$

49. $\frac{x}{x+4} - \frac{28}{x^2+x-12} = \frac{1}{x-3}$

50. $\frac{x}{2} + \frac{1}{x-1} < \frac{x}{4}$

51. $\frac{1}{2x} - \frac{4}{5x} > \frac{1}{3}$

52. YARD WORK Yasmin can plant a garden in 3 hours. Hala can plant the same garden in 4 hours. How long will it take them if they work together?

Example 8

Solve $\frac{3}{x+2} + \frac{1}{x} = 0$.

The LCD is $x(x + 2)$.

$$\frac{3}{x+2} + \frac{1}{x} = 0$$

$$x(x+2)\left(\frac{3}{x+2} + \frac{1}{x}\right) = x(x+2)(0)$$

$$x(x+2)\left(\frac{3}{x+2}\right) + x(x+2)\left(\frac{1}{x}\right) = 0$$

$$3(x) + 1(x+2) = 0$$

$$3x + x + 2 = 0$$

$$4x + 2 = 0$$

$$4x = -2$$

$$x = -\frac{1}{2}$$

CHAPTER 9 Practice Test

Simplify each expression.

1. $\dfrac{r^2 + rt}{2r} \div \dfrac{r + t}{16r^2}$

2. $\dfrac{m^2 - 4}{3m^2} \cdot \dfrac{6m}{2 - m}$

3. $\dfrac{m^2 + m - 6}{n^2 - 9} \div \dfrac{m - 2}{n + 3}$

4. $\dfrac{\dfrac{x^2 + 4x + 3}{x^2 - 2x - 15}}{\dfrac{x^2 - 1}{x^2 - x - 20}}$

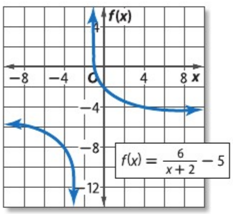
5. $\dfrac{x + 4}{6x + 3} + \dfrac{1}{2x + 1}$

6. $\dfrac{x}{x^2 - 1} - \dfrac{3}{2x + 2}$

7. $\dfrac{1}{y} + \dfrac{2}{7} - \dfrac{3}{2y^2}$

8. $\dfrac{2 + \frac{1}{x}}{5 - \frac{1}{x}}$

9. Identify the asymptotes, domain, and range of the function graphed.



10. MULTIPLE CHOICE What is the equation for the vertical asymptote of the rational function $f(x) = \dfrac{x + 1}{x^2 + 3x + 2}$?

A $x = -2$

B $x = -1$

C $x = 1$

D $x = 2$

Graph each function.

11. $f(x) = -\dfrac{8}{x} - 9$

12. $f(x) = \dfrac{2}{x + 4}$

13. $f(x) = \dfrac{3}{x - 1} + 8$

14. $f(x) = \dfrac{5x}{x + 1}$

15. $f(x) = \dfrac{x}{x - 5}$

16. $f(x) = \dfrac{x^2 + 5x - 6}{x - 1}$

17. Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of the function $f(x) = \dfrac{x + 5}{x^2 - 2x - 35}$.

18. Determine the equations of any oblique asymptotes in the graph of the function $f(x) = \dfrac{x^2 + x - 5}{x + 3}$.

Solve each equation or inequality.

19. $\dfrac{-1}{x + 4} = 6 - \dfrac{x}{x + 4}$

20. $\dfrac{1}{3} = \dfrac{5}{m + 3} + \dfrac{8}{21}$

21. $7 + \dfrac{2}{x} < -\dfrac{5}{x}$

22. $r + \dfrac{6}{r} - 5 = 0$

23. $\dfrac{6}{7} - \dfrac{3m}{2m - 1} = \dfrac{11}{7}$

24. $\dfrac{r + 2}{3r} = \dfrac{r + 4}{r - 2} - \dfrac{2}{3}$

25. If y varies inversely as x and $y = 18$ when $x = -\frac{1}{2}$, find x when $y = -10$.

26. If m varies directly as n and $m = 24$ when $n = -3$, find n when $m = 30$.

27. Suppose r varies jointly as s and t. If $s = 20$ when $r = 140$ and $t = -5$, find s when $r = 7$ and $t = 2.5$.

28. BICYCLING When Suha rides her bike, the distance that she travels varies directly with the amount of time she is biking. Suppose she bikes 50 kilometers in 2.5 hours. At this rate, how many hours would it take her to bike 80 kilometers?

29. PAINTING Omar can paint a house in 10 hours. Amer can paint the same house in 9 hours. How long would it take if they worked together?

30. MULTIPLE CHOICE How many liters of a 25% acid solution must be added to 30 liters of an 80% acid solution to create a 50% acid solution?

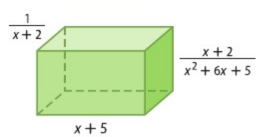
F 18

G 30

H 36

J 66

31. What is the volume of the rectangular prism?



CHAPTER 9

Preparing for Standardized Tests

Guess and Check

It is very important to pace yourself and keep track of how much time you have when taking a standardized test. If time is running short, or if you are unsure how to solve a problem, the guess-and-check strategy may help you determine the correct answer quickly.

Strategies for Guessing and Checking

Step 1

Carefully look over each possible answer choice and evaluate for reasonableness. Eliminate unreasonable answers.

Ask yourself:

- Are there any answer choices that are clearly incorrect?
- Are there any answer choices that are not in the proper format?
- Are there any answer choices that do not have the proper units for the correct answer?

Step 2

For the remaining answer choices, use the guess-and-check method.

- **Equations:** If you are solving an equation, substitute the answer choice for the variable and see if this results in a true number sentence.
- **System of Equations:** For a system of equations, substitute the answer choice for all variables and make sure all equations result in a true number sentence.

Step 3

Choose an answer choice and see if it satisfies the constraints of the problem statement. Identify the correct answer.

- If the answer choice you are testing does not satisfy the problem, move on to the next reasonable guess and check it.
- When you find the correct answer choice, stop.

Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

Solve: $\frac{2}{x-3} - \frac{4}{x+3} = \frac{8}{x^2-9}$.

A -1

B 1

C 5

D 7

The solution of the rational equation will be a real number. Since all four answer choices are real numbers, they are all possible correct answers and must be checked. Begin with the first answer choice and check it in the rational equation. Continue until you find the answer choice that results in a true number sentence.

	Check:
Guess: −1	$\frac{2}{(-1)-3} - \frac{4}{(-1)+3} = \frac{8}{(-1)^2-9}$ $-\frac{5}{2} \neq -1$ ✗

	Check:
Guess: 1	$\frac{2}{1-3} - \frac{4}{1+3} = \frac{8}{(1)^2-9}$ $-2 \neq -1$ ✗

	Check:
Guess: 5	$\frac{2}{5-3} - \frac{4}{5+3} = \frac{8}{(5)^2-9}$ $\frac{1}{2} = \frac{1}{2}$ ✓

If $x = 5$, the result is a true number sentence. So, the correct answer is C.

Exercises

Read each problem. Identify what you need to know. Then use the information in the problem to solve.

1. Solve: $\frac{2}{5x} - \frac{1}{2x} = -\frac{1}{2}$.

A $\frac{1}{10}$

B $\frac{1}{5}$

C $\frac{1}{4}$

D $\frac{1}{2}$

2. The sum of Khalid's, Asma's, and Noura's ages is 40. Asma is 1 year more than twice as old as Noura. Khalid is 3 years older than Asma. How old is Asma?

F 7
G 14
H 15
J 18

3. Determine the point(s) where the following rational function crosses the x-axis.

$$f(x) = \frac{2}{x-1} - \frac{x+4}{3}$$

A −5

B 4

C 2 or 3

D −5 or 2

4. Eissa's Theatre Company sells tickets for AED 10. At this price, they sell 400 tickets. Eissa estimates that they would sell 40 fewer tickets for each AED 2 price increase. What charge would give the most income?

F 10
G 13
H 15
J 20

CHAPTER 9

Standardized Test Practice

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Saleh's father can mow the lawn on his riding mower in 45 minutes. It takes Saleh 1 hour 45 minutes to mow the lawn with a push mower. Which of the following rational equations can be solved for the number of minutes t it would take them to mow the lawn working together?

A $\frac{t}{45} + \frac{t}{1.45} = 1$

B $\frac{t}{150} = 1$

C $\frac{t}{45} + \frac{t}{105} = 1$

D $\frac{t + 45}{t + 105} = 1$

2. The total cost of reserving a campsite varies directly as the number of nights the site is rented, as shown in the table.

Days	Total Cost
1	AED 24
2	AED 48
3	AED 72
4	AED 96

Which equation represents the direct variation?

F $y = x + 24$

G $y = 24x$

H $y = \frac{24}{x}$

J $y = 96x$

3. In which direction must the graph of $y = \frac{1}{x}$ be shifted to produce the graph of $y = \frac{1}{x} + 2$?

A up

B down

C right

D left

Test-Taking Tip

Question 2 Check your answer by substituting 1, 2, 3, and 4 for x and making sure the values in the table are produced.

4. Which of the following is ***not*** an asymptote of the rational function $f(x) = \frac{1}{x^2 - 49}$?

F $f(x) = 0$

G $x = -7$

H $x = 7$

J $f(x) = 1$

5. Simplify the complex fraction.

$$\frac{\frac{(x+3)^2}{x^2 - 16}}{\frac{x+3}{x+4}}$$

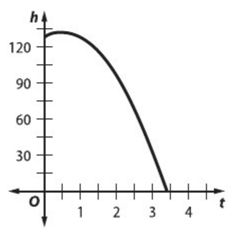
A $\frac{x+3}{x+4}$

B $\frac{1}{x-4}$

C $\frac{x+3}{x-4}$

D $\frac{x-4}{x+3}$

6. A ball was thrown upward with an initial velocity of 16 feet per second from the top of a building 128 feet high. Its height h in feet above the ground t seconds later will be $h = 128 + 16t - 16t^2$.



Which is the best conclusion about the ball's action?

F The ball stayed above 128 feet for more than 3 seconds.

G The ball returned to the ground in less than 4 seconds.

H The ball traveled a greater distance going up than it did going down.

J The ball traveled less than 128 feet in 3.4 seconds.

7. Which of these equations describes a relationship in which every negative real number x corresponds to a nonnegative real number y?

A $y = -x$

B $y = x$

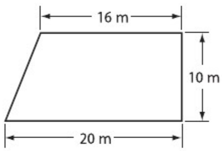
C $y = \sqrt{x}$

D $y = x^3$

Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

8. Mohammad is putting a stone walkway around the garden pictured at the right. About how many meters of stone are needed?



9. **GRIDDED RESPONSE** Huda had some money saved for a week-long vacation. The first day of the vacation she spent AED 125 on food and a hotel. On the second day, she was given AED 80 from her sister for expenses. Huda then had AED 635 left for the rest of the vacation. How much money, in dirhams, did she begin the vacation with?

10. Salem wants to print 800 one-page flyers for his landscaping business. He has a printer that is capable of printing 8 pages per minute. His business partner has another printer that prints 10 pages per minute.

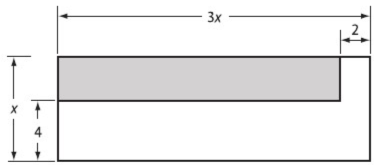
a. How long would it take Salem's printer to print all the flyers? How long would it take his partner's printer?

b. Set up a rational equation that can be used to find the number of minutes t it would take to print all 800 flyers if both printers are used simultaneously.

c. Solve the equation you wrote in part **b**. How long would it take both printers to print all the flyers if they print simultaneously? Round to the nearest minute.

11. **GRIDDED RESPONSE** The population of a country can be modeled by the equation $P(t) = 40e^{0.02t}$, where P is the population in millions and t is the number of years since 2005. When will the population be 400 million?

12. What is the area of the shaded region of the rectangle expressed as a polynomial in simplest form?

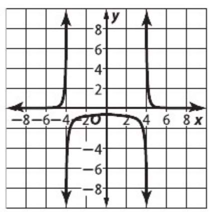


13. **GRIDDED RESPONSE** Suppose y varies inversely as x and $y = 4$ when $x = 12$. What is y when x is 5? Round to the nearest tenth.

Extended Response

Record your answers on a sheet of paper. Show your work.

14. Use the graph of the rational function at the right to answer each question.



a. Describe the vertical and horizontal asymptotes of the graph.

b. Write the equation of the rational function. Explain how you found your answer.

15. Consider the polynomial function

$$f(x) = 3x^4 + 19x^3 + 7x^2 - 11x - 2.$$

a. What is the degree of the function?

b. What is the leading coefficient of the function?

c. Evaluate $f(1)$, $f(-2)$, and $f(2a)$. Show your work.

CHAPTER 10

Trigonometric Functions

Then

You have graphed and analyzed functions.

Now

You will:

- Find values of trigonometric functions.
- Solve problems by using right triangle trigonometry.
- Solve triangles by using the Law of Sines and Law of Cosines.
- Graph trigonometric functions.

Why?

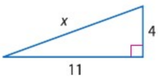
WATER SPORTS Knowing trigonometric functions has practical applications in water sports. For instance, you can use right triangle trigonometry to find the distance a kayak has traveled down river. If you are familiar with angles and angle measures, then you have a better understanding of how impressive it is to be able to do a 540° rotation on a wakeboard.

Get Ready for the Chapter

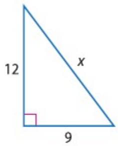
QuickCheck

Find the value of x to the nearest tenth.

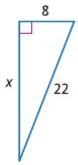
1.



2.



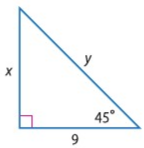
3.



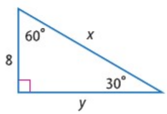
4. Suhaila has a rectangular garden in her backyard that measures 12 meters by 15 meters. She wants to put a rock walkway on the diagonal. How long will the walkway be? Round to the nearest tenth of a meter.

Find each missing measure. Write all radicals in simplest form.

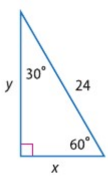
5.



6.



7.

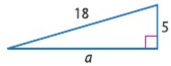


8. A ladder leans against a wall at a 45° angle. If the ladder is 12 meters long, how far up the wall does the ladder reach?

QuickReview

Example 1

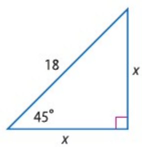
Find the missing measure of the right triangle.



$c^2 = a^2 + b^2$	Pythagorean Theorem
$18^2 = a^2 + 5^2$	Replace c with 18 and b with 5.
$324 = a^2 + 25$	Simplify.
$299 = a^2$	Subtract 25 from each side.
$17.3 \approx a$	Take the positive square root of each side.

Example 2

Find the missing measures. Write all radicals in simplest form.



$x^2 + x^2 = 18^2$	Pythagorean Theorem
$2x^2 = 18^2$	Combine like terms.
$2x^2 = 324$	Simplify.
$x^2 = 162$	Divide each side by 2.
$x = \sqrt{162}$	Take the positive square root of each side.
$x = 9\sqrt{2}$	Simplify.

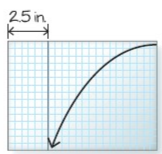
Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 10. To get ready, identify important terms and organize your resources.

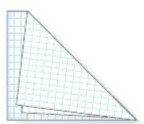
FOLDABLES® StudyOrganizer

Trigonometric Functions Make this Foldable to help you organize your Chapter 10 notes about trigonometric functions. Begin with four pieces of grid paper.

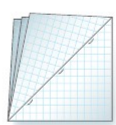
1 **Stack** paper together and measure 2.5 inches from the bottom.



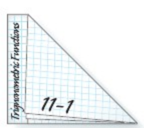
2 **Fold** on the diagonal.



3 **Staple** along the diagonal to form a book.



4 **Label** the edge as Trigonometric Functions.



NewVocabulary

English

trigonometry
sine
cosine
tangent
cosecant
secant
cotangent
angle of elevation
angle of depression
standard position
radian
Law of Sines
ambiguous case
Law of Cosines
unit circle
circular function
periodic function
cycle
period
amplitude
frequency

ReviewVocabulary

function a relation in which each element of the domain is paired with exactly one element in the range

inverse function two functions *f* and *g* are inverse functions if and only if both of their compositions are the identity function

EXPLORE 10-1

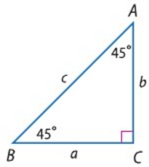
Spreadsheet Lab
Investigating Special Right Triangles

You can use a spreadsheet to investigate side measures of special right triangles.

Mathematical Practices
Look for and express regularity in repeated reasoning.

Activity 45°-45°-90° Triangle

The legs of a 45°-45°-90° triangle, *a* and *b*, are equal in measure. What patterns do you observe in the ratios of the side measures of these triangles?



Step 1 Enter the indicated formulas in the spreadsheet. The formula uses the Pythagorean Theorem in the form $c = \sqrt{a^2 + b^2}$.

=SQRT(A2^2+B2^2) =B2/A2 =B2/C2 =A2/C2

45-45-90 triangles

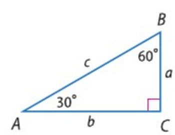
	A	B	C	D	E	F
1	a	b	c	b/a	b/c	a/c
2	1	1	1.414213562	1	0.707106781	0.707106781
3	2	2	2.828427125	1	0.707106781	0.707106781
4	3	3	4.242640687	1	0.707106781	0.707106781
5	4	4	5.656854249	1	0.707106781	0.707106781

Sheet 1 Sheet 2 Sheet 3

Step 2 Examine the results. Because 45°-45°-90° triangles share the same angle measures, these triangles are all similar. The ratios of the sides of these triangles are all the same. The ratios of side *b* to side *a* are 1. The ratios of side *b* to side *c* and of side *a* to side *c* are approximately 0.71.

Model and Analyze

Use the spreadsheet below for 30°-60°-90° triangles.



30-60-90 triangles

	A	B	C	D	E	F
1	a	b	c	b/a	b/c	a/c
2	1		2			
3	2		4			
4	3		6			
5	4		8			

Sheet 1 Sheet 2 Sheet 3

1. Copy and complete the spreadsheet above.
2. Describe the relationship among the 30°-60°-90° triangles with the dimensions given.
3. What patterns do you observe in the ratios of the side measures of these triangles?

LESSON

10-1 Trigonometric Functions in Right Triangles

Then

- You used the Pythagorean Theorem to find side lengths of right triangles.

Now

1. Find values of trigonometric functions for acute angles.
2. Use trigonometric functions to find side lengths and angle measures of right triangles.

Why?

- The altitude of a person parasailing depends on the length of the tow rope ℓ and the angle the rope makes with the horizontal $x°$. If you know these two values, you can use a ratio to find the altitude of the person parasailing.

NewVocabulary

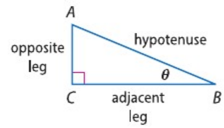
trigonometry
trigonometric ratio
trigonometric function
sine
cosine
tangent
cosecant
secant
cotangent
reciprocal functions
inverse sine
inverse cosine
inverse tangent
angle of elevation
angle of depression

Mathematical Practices
Attend to precision.

1 Trigonometric Functions for Acute Angles

Trigonometry is the study of relationships among the angles and sides of a right triangle. A **trigonometric ratio** compares the side lengths of a right triangle. A **trigonometric function** has a rule given by a trigonometric ratio.

The Greek letter *theta* θ is often used to represent the measure of an acute angle in a right triangle. The *hypotenuse*, the *leg opposite* θ, and the *leg adjacent to* θ are used to define the six trigonometric functions.



KeyConcept Trigonometric Functions in Right Triangles

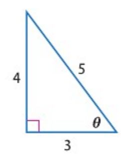
Words If θ is the measure of an acute angle of a right triangle, then the following trigonometric functions involving the opposite side *opp*, the adjacent side *adj*, and the hypotenuse *hyp* are true.

Symbols

sin (**sine**) $\theta = \frac{\text{opp}}{\text{hyp}}$ csc (**cosecant**) $\theta = \frac{\text{hyp}}{\text{opp}}$

cos (**cosine**) $\theta = \frac{\text{adj}}{\text{hyp}}$ sec (**secant**) $\theta = \frac{\text{hyp}}{\text{adj}}$

tan (**tangent**) $\theta = \frac{\text{opp}}{\text{adj}}$ cot (**cotangent**) $\theta = \frac{\text{adj}}{\text{opp}}$



Examples

$\sin\theta = \frac{4}{5}$ $\cos\theta = \frac{3}{5}$ $\tan\theta = \frac{4}{3}$

$\csc\theta = \frac{5}{4}$ $\sec\theta = \frac{5}{3}$ $\cot\theta = \frac{3}{4}$

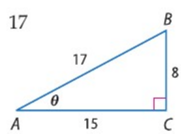
Example 1 Evaluate Trigonometric Functions

Find the values of the six trigonometric functions for angle θ.

leg opposite θ: $BC = 8$ leg adjacent θ: $AC = 15$ hypotenuse: $AB = 17$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{8}{17}$ $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{15}{17}$ $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{8}{15}$

$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{17}{8}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{17}{15}$ $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{15}{8}$



GuidedPractice

1. Find the values of the six trigonometric functions for angle B.

> **StudyTip**
>
> **Memorize Trigonometric Ratios** SOH-CAH-TOA is a mnemonic device for remembering the first letter of each word in the ratios for sine, cosine, and tangent.
>
> $\sin \theta = \frac{\text{opp}}{\text{hyp}}$
>
> $\cos \theta = \frac{\text{adj}}{\text{hyp}}$
>
> $\tan \theta = \frac{\text{opp}}{\text{adj}}$

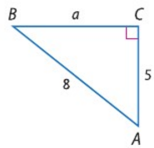
Notice that the cosecant, secant, and cotangent ratios are reciprocals of the sine, cosine, and tangent ratios, respectively. These are called the **reciprocal functions**.

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$

The domain of any trigonometric function is the set of all acute angles θ of a right triangle. So, trigonometric functions depend only on the measures of the acute angles, not on the side lengths of a right triangle.

Example 2 Find Trigonometric Ratios

If $\sin B = \frac{5}{8}$, find the exact values of the five remaining trigonometric functions for B.



Step 1 Draw a right triangle and label one acute angle B. Label the opposite side 5 and the hypotenuse 8.

Step 2 Use the Pythagorean Theorem to find a.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + 5^2 = 8^2$	$b = 5$ and $c = 8$
$a^2 + 25 = 64$	Simplify.
$a^2 = 39$	Subtract 25 from each side.
$a = \pm\sqrt{39}$	Take the square root of each side.
$a = \sqrt{39}$	Length cannot be negative.

Step 3 Find the other values.

Since $\sin B = \frac{5}{8}$, $\csc B = \frac{\text{hyp}}{\text{opp}}$ or $\frac{8}{5}$.

$\cos B = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{39}}{8}$ $\qquad$ $\sec B = \frac{\text{hyp}}{\text{adj}} = \frac{8}{\sqrt{39}}$ or $\frac{8\sqrt{39}}{39}$

$\tan B = \frac{\text{opp}}{\text{adj}} = \frac{5}{\sqrt{39}}$ or $\frac{5\sqrt{39}}{39}$ $\qquad$ $\cot B = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{39}}{5}$

GuidedPractice

2. If $\tan B = \frac{3}{7}$, find exact values of the remaining trigonometric fuctions for B.

> **ReadingMath**
>
> **Labeling Triangles** Throughout this chapter, a capital letter is used to represent both a vertex of a triangle and the measure of the angle at that vertex. The same letter in lowercase is used to represent both the side opposite that angle and the length of the side.

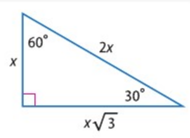
Angles that measure 30°, 45°, and 60° occur frequently in trigonometry.

KeyConcept Trigonometric Values for Special Angles

30°-60°-90°

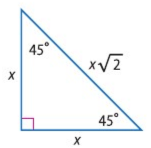
$\sin 30° = \frac{1}{2}$ $\qquad$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\qquad$ $\tan 30° = \frac{\sqrt{3}}{3}$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\qquad$ $\cos 60° = \frac{1}{2}$ $\qquad$ $\tan 60° = \sqrt{3}$



45°-45°-90°

$\sin 45° = \frac{\sqrt{2}}{2}$ $\qquad$ $\cos 45° = \frac{\sqrt{2}}{2}$ $\qquad$ $\tan 45° = 1$

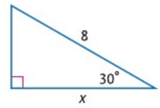


2 Use Trigonometric Functions

You can use trigonometric functions to find missing side lengths and missing angle measures of right triangles.

Example 3 Find a Missing Side Length

Use a trigonometric function to find the value of x. Round to the nearest tenth if necessary.



The length of the hypotenuse is 8. The missing measure is for the side adjacent to the 30° angle. Use the cosine function to find x.

$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$ Cosine function

$$\cos 30^\circ = \frac{x}{8}$$ Replace θ with 30°, adj with x, and hyp with 8.

$$\frac{\sqrt{3}}{2} = \frac{x}{8}$$ $\cos 30^\circ = \frac{\sqrt{3}}{2}$

$$\frac{8\sqrt{3}}{2} = x$$ Multiply each side by 8.

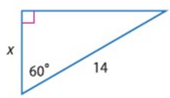
$$6.9 \approx x$$ Use a calculator.

StudyTip

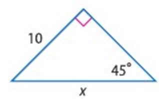
Choose a Function If the length of the hypotenuse is unknown, then either the sine or cosine function must be used to find the missing measure.

GuidedPractice

3A.



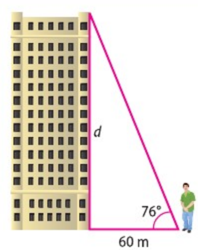
3B.



You can use a calculator to find the missing side lengths of triangles that do not have 30°, 45°, or 60° angles.

Real-World Example 4 Find a Missing Side Length

BUILDINGS To calculate the height of a building, Mazen walked 60 meters from the base of the building and used an inclinometer to measure the angle from his eye to the top of the building. If his eye level is at 2 meters, how tall is the building?



The measured angle is 76°. The side adjacent to the angle is 60 meters. The missing measure is the side opposite the angle. Use the tangent function to find d.

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$ Tangent function

$$\tan 76^\circ = \frac{d}{60}$$ Replace θ with 76°, opp with d, and adj with 60.

$$60 \tan 76^\circ = d$$ Multiply each side by 60.

$$240 \approx d$$ Use a calculator to simplify: 60 [TAN] 76 [ENTER].

Because the inclinometer was 2 meters above the ground, the height of the building is approximately 242 meters.

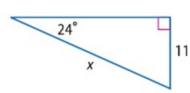
Real-WorldLink

Inclinometers measure the angle of Earth's magnetic field as well as the pitch and roll of vehicles, sailboats, and airplanes. They are also used for monitoring volcanoes and well drilling.

Source: Science Magazine

GuidedPractice

4. Use a trigonometric function to find the value of x. Round to the nearest tenth if necessary.



When solving equations like $3x = -27$, you use the inverse of multiplication to find x. You also can find angle measures by using the inverse of sine, cosine, or tangent.

ReadingMath

Precision The expression $\sin^{-1} x$ is read *the inverse sine of x* and is interpreted as the angle whose sine is x. Be careful not to confuse this notation with the notation for negative exponents; $\sin^{-1} x \neq \frac{1}{\sin x}$. Instead, this notation is similar to the notation for an inverse function, $f^{-1}(x)$.

KeyConcept Inverse Trigonometric Ratios

Words If $\angle A$ is an acute angle and the sine of A is x, then the **inverse sine** of x is the measure of $\angle A$.

Symbols If $\sin A = x$, then $\sin^{-1} x = m\angle A$.

Example $\sin A = \frac{1}{2} \longrightarrow \sin^{-1} \frac{1}{2} = m\angle A \longrightarrow m\angle A = 30°$

Words If $\angle A$ is an acute angle and the cosine of A is x, then the **inverse cosine** of x is the measure of $\angle A$.

Symbols If $\cos A = x$, then $\cos^{-1} x = m\angle A$.

Example $\cos A = \frac{\sqrt{2}}{2} \longrightarrow \cos^{-1} \frac{\sqrt{2}}{2} = m\angle A \longrightarrow m\angle A = 45°$

Words If $\angle A$ is an acute angle and the tangent of A is x, then the **inverse tangent** of x is the measure of $\angle A$.

Symbols If $\tan A = x$, then $\tan^{-1} x = m\angle A$.

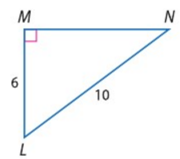
Example $\tan A = \sqrt{3} \longrightarrow \tan^{-1}\sqrt{3} = m\angle A \longrightarrow m\angle A = 60°$

If you know the sine, cosine, or tangent of an acute angle, you can use a calculator to find the measure of the angle, which is the inverse of the trigonometric ratio.

Example 5 Find a Missing Angle Measure

Find the measure of each angle. Round to the nearest tenth if necessary.

a. $\angle N$



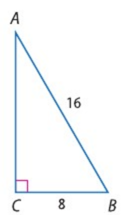
You know the measure of the side opposite $\angle N$ and the measure of the hypotenuse. Use the sine function.

$\sin N = \frac{6}{10}$ $\quad$ $\sin \theta = \frac{\text{opp}}{\text{hyp}}$

$\sin^{-1} \frac{6}{10} = m\angle N$ $\quad$ Inverse sine

$36.9° \approx m\angle N$ $\quad$ Use a calculator.

b. $\angle B$



Use the cosine function.

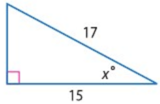
$\cos B = \frac{8}{16}$ $\quad$ $\cos \theta = \frac{\text{adj}}{\text{hyp}}$

$\cos^{-1} \frac{8}{16} = m\angle B$ $\quad$ Inverse cosine

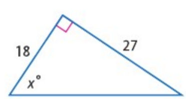
$60° = m\angle B$ $\quad$ Use a calculator.

GuidedPractice **Find x. Round to the nearest tenth if necessary.**

5A.



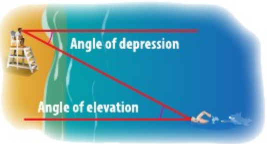
5B.



StudyTip

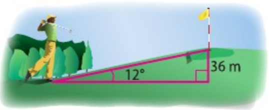
Angles of Elevation and Depression The angle of elevation and the angle of depression are congruent since they are alternate interior angles of parallel lines.

In the figure at the right, the angle formed by the line of sight from the swimmer and a line parallel to the horizon is called the **angle of elevation**. The angle formed by the line of sight from the lifeguard and a line parallel to the horizon is called the **angle of depression**.



Real-World Example 6 Use Angles of Elevation and Depression

a. GOLF A golfer is standing at the tee, looking up to the green on a hill. If the tee is 36 meters lower than the green and the angle of elevation from the tee to the hole is 12°, find the distance from the tee to the hole.



Write an equation using a trigonometric function that involves the ratio of the vertical rise (side opposite the 12° angle) and the distance from the tee to the hole (hypotenuse).

$\sin 12° = \frac{36}{x}$ — $\sin \theta = \frac{\text{opp}}{\text{hyp}}$

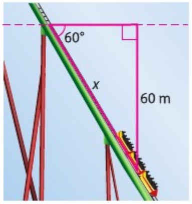
$x \sin 12° = 36$ — Multiply each side by x.

$x = \frac{36}{\sin 12°}$ — Divide each side by sin 12°.

$x \approx 173.2$ — Use a calculator.

So, the distance from the tee to the hole is about 173.2 meters.

b. ROLLER COASTER The hill of the roller coaster has an *angle of descent*, or an angle of depression, of 60°. Its vertical drop is 60 meters. Estimate the length of the hill.



Write an equation using a trigonometric function that involves the ratio of the vertical drop (side opposite the 60° angle) and the length of the hill (hypotenuse).

$\sin 60° = \frac{60}{x}$ — $\sin \theta = \frac{\text{opp}}{\text{hyp}}$

$x \sin 60° = 60$ — Multiply each side by x.

$x = \frac{60}{\sin 60°}$ — Divide each side by sin 60°.

$x \approx 70$ — Use a calculator.

So, the length of the hill is about 70 meters.

Real-WorldLink

The steepest roller coasters in the world have angles of descent that are close to 90°.

Source: Ultimate Roller Coaster

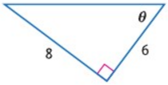
GuidedPractice

6A. MOVING A ramp for unloading a moving truck has an angle of elevation of 32°. If the top of the ramp is 1.2 meters above the ground, estimate the length of the ramp.

6B. LADDERS A 14-m long ladder is placed against a house at an angle of elevation of 72°. How high above the ground is the top of the ladder?

Check Your Understanding

Example 1 **Find the values of the six trigonometric functions for angle θ.**

1.


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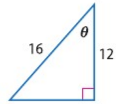
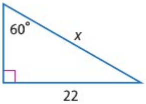


Example 2 **In a right triangle, $\angle A$ is acute. Find the values of the five remaining trigonometric funtions.**

3. $\cos A = \frac{4}{7}$

4. $\tan A = \frac{20}{21}$

Examples 3–4 **Use a trigonometric function to find the value of x. Round to the nearest tenth.**

5.


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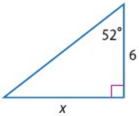

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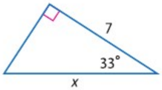
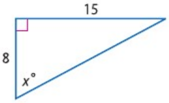


Example 5 **Find the value of x. Round to the nearest tenth.**

8.


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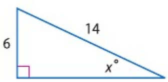


10.

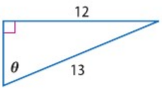

Example 6

11. **SENSE-MAKING** Omar found two trees directly across from each other in a canyon. When he moved 100 meters from the tree on his side (parallel to the edge of the canyon), the angle formed by the tree on his side and the tree on the other side was 70°. Find the distance across the canyon.

12. **LADDERS** The recommended angle of elevation for a ladder used in fire fighting is 75°. At what height on a building does a 21-meter ladder reach if the recommended angle of elevation is used? Round to the nearest tenth.

Practice and Problem Solving

Example 1 **Find the values of the six trigonometric functions for angle θ.**

13.


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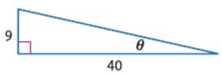

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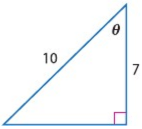

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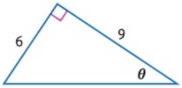


Example 2 **In a right triangle, $\angle A$ and $\angle B$ are acute. Find the values of the five remaining trigonometric funtions.**

17. $\tan A = \frac{8}{15}$

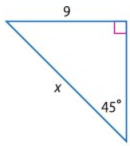
18. $\cos A = \frac{3}{10}$

19. $\tan B = 3$

20. $\sin B = \frac{4}{9}$

Examples 3–4 **Use a trigonometric function to find each value of x. Round to the nearest tenth.**

21.



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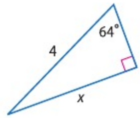

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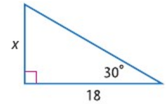

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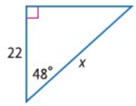

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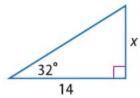
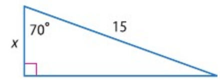
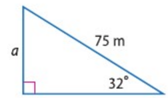


26.



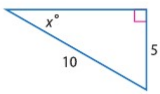
27. PARASAILING Refer to the beginning of the lesson and the figure at the right. Find a, the altitude of a person parasailing, if the tow rope is 75 meters long and the angle formed is 32°. Round to the nearest tenth.



28. MODELING Ali wants to build a rope bridge between his treehouse and Khalid's treehouse. Suppose Ali's treehouse is directly behind Khalid's treehouse. At a distance of 20 meters to the left of Ali's treehouse, an angle of 52° is measured between the two treehouses. Find the length of the rope.

Example 5 **Find the value of x. Round to the nearest tenth.**

29.



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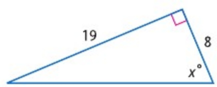

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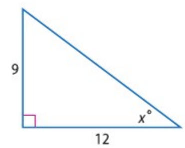

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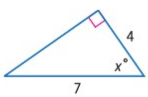

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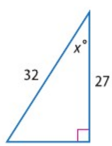
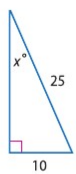


34.

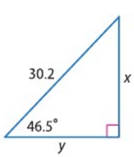


Example 6 **35. SQUIRRELS** Adult flying squirrels can make glides of up to 50 meters. If a flying squirrel glides a horizontal distance of 50 meters and the angle of descent is 9°, find its change in height.

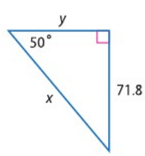
36. HANG GLIDING A hang glider climbs at a 20° angle of elevation. Find the change in altitude of the hang glider when it has flown a horizontal distance of 18 meters.

B **Use trigonometric functions to find the values of x and y. Round to the nearest tenth.**

37.



38.



39.

Solve each equation.

40. $\cos A = \frac{3}{19}$

41. $\sin N = \frac{9}{11}$

42. $\tan X = 15$

43. $\sin T = 0.35$

44. $\tan G = 0.125$

45. $\cos Z = 0.98$

46. **MONUMENTS** A monument casts a shadow 24 meters long. The angle of elevation from the end of the shadow to the top of the monument is 50°.

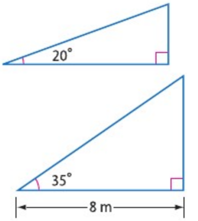
a. Draw and label a right triangle to represent this situation.

b. Write a trigonometric function that can be used to find the height of the monument.

c. Find the value of the function to determine the height of the monument to the nearest tenth.

C 47. **NESTS** Amani's eyes are 1.5 meters above the ground as she looks up to a bird's nest in a tree. If the angle of elevation is 74.5° and she is standing 4 meters from the tree's base, what is the height of the bird's nest? Round to the nearest meter.

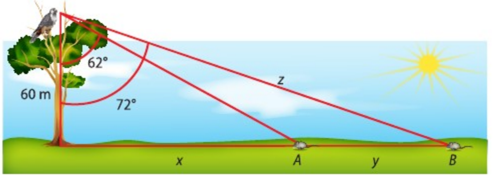
48. **RAMPS** Two bicycle ramps each cover a horizontal distance of 8 meters. One ramp has a 20° angle of elevation, and the other ramp has a 35° angle of elevation, as shown at the right.



a. How much taller is the second ramp than the first? Round to the nearest tenth.

b. How much longer is the second ramp than the first? Round to the nearest tenth.

49. **FALCONS** A falcon at a height of 60 meters sees two mice A and B, as shown in the diagram.



a. What is the approximate distance z between the falcon and mouse B?

b. How far apart are the two mice?

In $\triangle ABC$, $\angle C$ is a right angle. Use the given measurements to find the missing side lengths and missing angle measures of $\triangle ABC$. Round to the nearest tenth if necessary.

50. $m\angle A = 36°$, $a = 12$

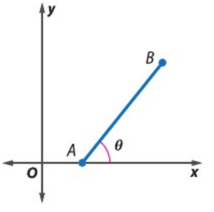
51. $m\angle B = 31°$, $b = 19$

52. $a = 8$, $c = 17$

53. $\tan A = \frac{4}{5}$, $a = 6$

H.O.T. Problems Use Higher-Order Thinking Skills

54. **CHALLENGE** A line segment has endpoints $A(2, 0)$ and $B(6, 5)$, as shown in the figure at the right. What is the measure of the acute angle θ formed by the line segment and the x-axis? Explain how you found the measure.



55. **ARGUMENTS** Determine whether the following statement is *true* or *false*. Explain your reasoning.

For any acute angle, the sine function will never have a negative value.

56. **OPEN ENDED** In right triangle ABC, $\sin A = \sin C$. What can you conclude about $\triangle ABC$? Justify your reasoning.

57. **WRITING IN MATH** A roof has a slope of $\frac{2}{3}$. Describe the connection between the slope and the angle of elevation θ that the roof makes with the horizontal. Then use an inverse trigonometric function to find θ.

Standardized Test Practice

58. EXTENDED RESPONSE Your school needs 5 cases of yearbooks. Neighborhood Yearbooks lists a case of yearbooks at AED 153.85 with a 10% discount on an order of 5 cases. Yearbooks R Us lists a case of yearbooks at AED 157.36 with a 15% discount on 5 cases.

a. Which company would you choose?

b. What is the least amount that you would have to spend for the yearbooks?

59. SHORT RESPONSE As a fundraiser, the marching band sold T-shirts and hats. They sold a total of 105 items and raised AED 1170. If the cost of a hat was AED 10 and the cost of a T-shirt was AED 15, how many T-shirts were sold?

60. A sandwich stand charges price x for a sandwich and price y for a drink. Two sandwiches and one drink cost AED 4.50. Three sandwiches and two drinks cost AED 7.25. Which matrix could be multiplied by $\begin{bmatrix} 4.50 \\ 7.25 \end{bmatrix}$ to find x and y?

A $\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

B $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

61. SAT/ACT The length and width of a rectangle are in the ratio of 5:12. If the rectangle has an area of 240 square centimeters, what is the length, in centimeters, of its diagonal?

F 24 **H** 28 **K** 32

G 26 **J** 30

Spiral Review

62. SWIMMING POOL The number of visits to a community swimming pool per year by a sample of 425 members is normally distributed with a mean of 90 and a standard deviation of 15.

a. About what percent of the members went to the pool at least 45 times?

b. What is the probability that a member selected at random went to the pool more than 120 times?

c. What percent of the members went to the pool between 75 and 105 times?

63. POLLS A polling company wants to estimate how many people are in favor of a new environmental law. The polling company polls 20 people. The probability that a person is in favor of the law is 0.5.

a. What is the probability that exactly 12 people are in favor of the new law?

b. What is the expected number of people in favor of the law?

Skills Review

Find each product. Include the appropriate units with your answer.

64. 4.3 kilometers$\left(\frac{5280 \text{ meters}}{1 \text{ kilometer}}\right)$

65. 8 liters$\left(\frac{8 \text{ milliliters}}{1 \text{ liter}}\right)$

66. $\left(\frac{5 \text{ dirhams}}{3 \text{ meters}}\right)$21 meters

67. $\left(\frac{18 \text{ cubic centimeters}}{5 \text{ seconds}}\right)$24 seconds

68. 65 degrees$\left(\frac{10 \text{ centimeters}}{3 \text{ degrees}}\right)$

69. $\left(\frac{7 \text{ liters}}{30 \text{ minutes}}\right)$10 minutes

LESSON 10-2 Angles and Angle Measure

Then	Now	Why?
You used angles with degree measures.	**1** Draw and find angles in standard position. **2** Convert between degree measures and radian measures.	A sundial is an instrument that indicates the time of day by the shadow that it casts on a surface marked to show hours or fractions of hours. The shadow moves around the dial 15° every hour.

NewVocabulary

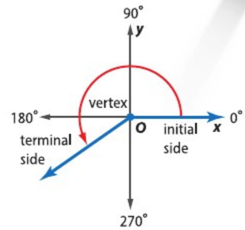
standard position
initial side
terminal side
coterminal angles
radian
central angle
arc length

Mathematical Practices
Reason abstractly and quantitatively.

1 Angles in Standard Position

An angle on the coordinate plane is in **standard position** if the vertex is at the origin and one ray is on the positive x-axis.

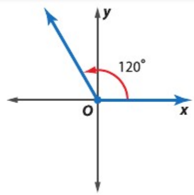
- The ray on the x-axis is called the **initial side** of the angle.
- The ray that rotates about the center is called the **terminal side**.

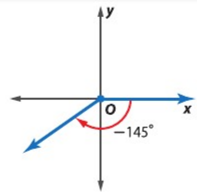


KeyConcept Angle Measures

If the measure of an angle is positive, the terminal side is rotated counterclockwise.

If the measure of an angle is negative, the terminal side is rotated clockwise.



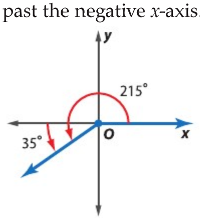


Example 1 Draw an Angle in Standard Position

Draw an angle with the given measure in standard position.

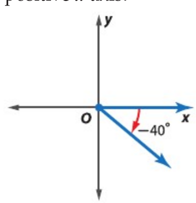
a. 215° $215^\circ = 180^\circ + 35^\circ$

Draw the terminal side of the angle 35° counterclockwise past the negative x-axis.



b. −40°

The angle is negative. Draw the terminal side of the angle 40° clockwise from the positive x-axis.

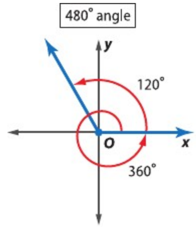


GuidedPractice

1A. 80°

1B. -105°

The terminal side of an angle can make more than one complete rotation. For example, a complete rotation of 360° plus a rotation of 120° forms an angle that measures 360° + 120° or 480°.



Real-WorldLink

Wakeboarding is one of the fastest-growing water sports in the United States. Participation has increased more than 100% in recent years.

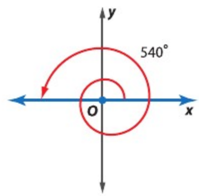
Source: King of Wake

Real-World Example 2 Draw an Angle in Standard Position

WAKEBOARDING ***Wakeboarding*** **is a combination of surfing, skateboarding, snowboarding, and water skiing. One maneuver involves a 540-degree rotation in the air. Draw an angle in standard position that measures 540°.**

540° = 360° + 180°

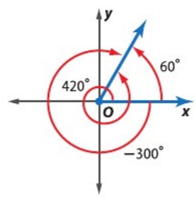
Draw the terminal side of the angle 180° past the positive x-axis.



GuidedPractice

2. Draw an angle in standard position that measures 600°.

Two or more angles in standard position with the same terminal side are called **coterminal angles**. For example, angles that measure 60°, 420°, and −300° are coterminal, as shown in the figure at the right.



An angle that is coterminal with another angle can be found by adding or subtracting a multiple of 360°.

- 60° + 360° = 420°
- 60° − 360° = −300°

ReadingMath

Angle of Rotation

In trigonometry, an angle is sometimes referred to as an *angle of rotation*.

Example 3 Find Coterminal Angles

Find an angle with a positive measure and an angle with a negative measure that are coterminal with each angle.

a. 130°

positive angle: 130° + 360° = 490° — Add 360°.

negative angle: 130° − 360° = −230° — Subtract 360°.

b. −200°

positive angle: −200° + 360° = 160° — Add 360°.

negative angle: −200° − 360° = −560° — Subtract 360°.

GuidedPractice

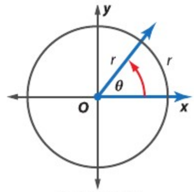
3A. 15°

3B. −45°

> **StudyTip**
>
> **Structure** As with degrees, radians measure the amount of rotation from the initial side to the terminal side.
>
> - The measure of an angle in radians is positive if its rotation is counterclockwise.
> - The measure is negative if the rotation is clockwise.

2 Convert Between Degrees and Radians

Angles can also be measured in units that are based on arc length. One **radian** is the measure of an angle θ in standard position with a terminal side that intercepts an arc with the same length as the radius of the circle.



$\theta = 1$ radian

The circumference of a circle is $2\pi r$. So, one complete revolution around a circle equals 2π radians. Since 2π radians $= 360°$, degree measure and radian measure are related by the following equations.

$$2\pi \text{ radians} = 360° \qquad \pi \text{ radians} = 180°$$

KeyConcept Convert Between Degrees and Radians

Degrees to Radians	Radians to Degrees
To convert from degrees to radians, multiply the number of degrees by $\dfrac{\pi \text{ radians}}{180°}$.	To convert from radians to degrees, multiply the number of radians by $\dfrac{180°}{\pi \text{ radians}}$.

> **ReadingMath**
>
> **Radian Measures** The word *radian* is usually omitted when angles are expressed in radian measure. Thus, when no units are given for an angle measure, radian measure is implied.

Example 4 Convert Between Degrees and Radians

Rewrite the degree measure in radians and the radian measure in degrees.

a. $-30°$

$$-30° = -30\cancel{°} \cdot \frac{\pi \text{ radians}}{180\cancel{°}}$$

$$= \frac{-30\pi}{180} \text{ or } -\frac{\pi}{6} \text{ radians}$$

b. $\dfrac{5\pi}{2}$

$$\frac{5\pi}{2} = \frac{5\cancel{\pi}}{2}\ \cancel{\text{radians}} \cdot \frac{180°}{\cancel{\pi \text{ radians}}}$$

$$= \frac{900°}{2} \text{ or } 450°$$

GuidedPractice

4A. $120°$

4B. $-\dfrac{3\pi}{8}$

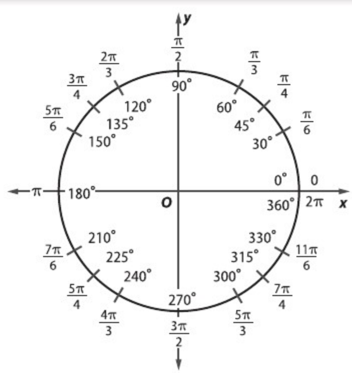
ConceptSummary Degrees and Radians

The diagram shows equivalent degree and radian measures for special angles.

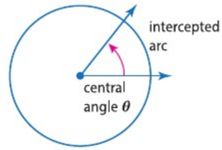
You may find it helpful to memorize the following equivalent degree and radian measures. The other special angles are multiples of these angles.

$30° = \dfrac{\pi}{6}$ $\qquad 45° = \dfrac{\pi}{4}$

$60° = \dfrac{\pi}{3}$ $\qquad 90° = \dfrac{\pi}{2}$



A **central angle** of a circle is an angle with a vertex at the center of the circle. If you know the measure of a central angle and the radius of the circle, you can find the length of the arc that is intercepted by the angle.

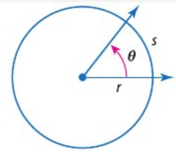


KeyConcept Arc Length

Words For a circle with radius r and central angle θ (in radians), the **arc length** s equals the product of r and θ.

Symbols $s = r\theta$

Model



You will justify this formula in Exercise 52

Real-World Example 5 Find Arc Length

TRUCKS **Monster truck tires have a radius of 82 centimeters. How far does a monster truck travel in meters after just three fourths of a tire rotation?**

Step 1 Find the central angle in radians.

$\theta = \frac{3}{4} \cdot 2\pi$ or $\frac{3\pi}{2}$ The angle is $\frac{3}{4}$ of a complete rotation.

Step 2 Use the radius and central angle to find the arc length.

$s = r\theta$	Write the formula for arc length.
$= 82 \cdot \frac{3\pi}{2}$	Replace r with 82 and θ with $\frac{3\pi}{2}$.
≈ 388.8 cm	Use a calculator to simplify.
≈ 3.9 m	Divide by 100 to convert to meters.

So, the truck travels about 3.9 meters after three fourths of a tire rotation.

GuidedPractice

5. A circle has a diameter of 9 centimeters. Find the arc length if the central angle is 60°. Round to the nearest tenth.

WatchOut!

Arc Length Remember to write the angle measure in radians, not degrees, when finding arc length. Also, recall that the number of radians in a complete rotation is 2π.

Check Your Understanding

Examples 1–2 **Draw an angle with the given measure in standard position.**

1. 140° **2.** −60° **3.** 390°

Example 3 **Find an angle with a positive measure and an angle with a negative measure that are coterminal with each angle.**

4. 25° **5.** 175° **6.** −100°

Example 4 **Rewrite each degree measure in radians and each radian measure in degrees.**

7. $\frac{\pi}{4}$ **8.** 225° **9.** −40°

Example 5 **10.** **REASONING** A tennis player's swing moves along the path of an arc. If the radius of the arc's circle is 1.2 meters and the angle of rotation is 100°, what is the length of the arc? Round to the nearest tenth.

Examples 1–2 **Draw an angle with the given measure in standard position.**

11. $75°$ **12.** $160°$ **13.** $-90°$

14. $-120°$ **15.** $295°$ **16.** $510°$

17. **GYMNASTICS** A gymnast on the uneven bars swings to make a $240°$ angle of rotation.

18. **FOOD** The lid on a jar of pasta sauce is turned $420°$ before it comes off.

Example 3 **Find an angle with a positive measure and an angle with a negative measure that are coterminal with each angle.**

19. $50°$ **20.** $95°$ **21.** $205°$

22. $350°$ **23.** $-80°$ **24.** $-195°$

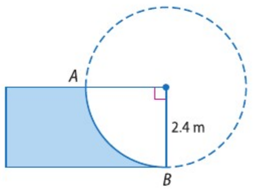
Example 4 **Rewrite each degree measure in radians and each radian measure in degrees.**

25. $330°$ **26.** $\frac{5\pi}{6}$ **27.** $-\frac{\pi}{3}$

28. $-50°$ **29.** $190°$ **30.** $-\frac{7\pi}{3}$

Example 5

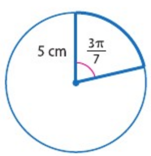
31. **SKATEBOARDING** The skateboard ramp at the right is called a *quarter pipe*. The curved surface is determined by the radius of a circle. Find the length of the curved part of the ramp.



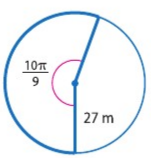
32. **RIVERBOATS** The paddlewheel of a riverboat has a diameter of 7.2 meters. Find the arc length of the circle made when the paddlewheel rotates $300°$.

Find the length of each arc. Round to the nearest tenth.

33.



34.



B

35. **CLOCKS** How long does it take for the minute hand on a clock to pass through 2.5π radians?

36. **PERSEVERANCE** Refer to the beginning of the lesson. A shadow moves around a sundial $15°$ every hour.

a. After how many hours is the angle of rotation of the shadow $\frac{8\pi}{5}$ radians?

b. What is the angle of rotation in radians after 5 hours?

c. A sundial has a radius of 20 centimeters. What is the arc formed by a shadow after 14 hours? Round to the nearest tenth.

Find an angle with a positive measure and an angle with a negative measure that are coterminal with each angle.

37. $620°$ **38.** $-400°$ **39.** $-\frac{3\pi}{4}$ **40.** $\frac{19\pi}{6}$

41. **SWINGS** A swing has a 165° angle of rotation.

a. Draw the angle in standard position.

b. Write the angle measure in radians.

c. If the chains of the swing are 2 meters long, what is the length of the arc that the swing makes? Round to the nearest tenth.

d. Describe how the arc length would change if the lengths of the chains of the swing were doubled.

42. **MULTIPLE REPRESENTATIONS** Consider $A(-4, 0)$, $B(-4, 6)$, $C(6, 0)$, and $D(6, 8)$.

a. **Geometric** Draw $\triangle EAB$ and $\triangle ECD$ with E at the origin.

b. **Algebraic** Find the values of the tangent of $\angle BEA$ and the tangent of $\angle DEC$.

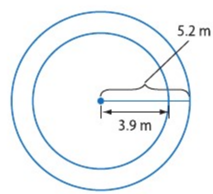
c. **Algebraic** Find the slope of $\overline{BE}$ and $\overline{ED}$.

d. **Verbal** What conclusions can you make about the relationship between slope and tangent?

Rewrite each degree measure in radians and each radian measure in degrees.

43. $\frac{21\pi}{8}$ 44. 124° 45. −200° 46. 5

47. **CAROUSELS** A carousel makes 5 revolutions per minute. The circle formed by riders sitting in the outside row has a radius of 5.2 meters. The circle formed by riders sitting in the inside row has a radius of 3.9 meters.

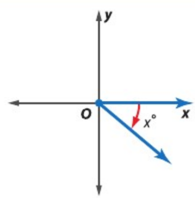


a. Find the angle θ in radians through which the carousel rotates in one second.

b. In one second, what is the difference in arc lengths between the riders sitting in the outside row and the riders sitting in the inside row?

H.O.T. Problems Use Higher-Order Thinking Skills

48. **CRITIQUE** Saeed and Ayoub are writing an expression for the measure of an angle coterminal with the angle shown at the right. Is either of them correct? Explain your reasoning.



Saeed	Ayoub
The measure of a coterminal angle is $(x - 360)°$.	The measure of a coterminal angle is $(360 - x)°$.

49. **CHALLENGE** A line makes an angle of $\frac{\pi}{2}$ radians with the positive x-axis at the point (2, 0). Find an equation for this line.

50. **REASONING** Express $\frac{1}{8}$ of a revolution in degrees and in radians. Explain your reasoning.

51. **OPEN ENDED** Draw and label an acute angle in standard position. Find two angles, one positive and one negative, that are coterminal with the angle.

52. **REASONING** Justify the formula for the length of an arc.

53. **WRITING IN MATH** Use a circle with radius r to describe what one degree and one radian represent. Then explain how to convert between the measures.

Standardized Test Practice

54. SHORT RESPONSE If $(x + 6)(x + 8) - (x - 7)(x - 5) = 0$, find x.

55. Which of the following represents an inverse variation?

A

x	2	5	10	20	25	50
y	50	20	10	5	4	2

B

x	2	4	6	8	10	12
y	−4	−8	−12	−16	−20	−24

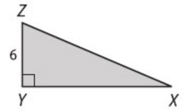
C

x	1	2	3	4	5	6
y	5	10	15	20	25	30

D

x	10	9	8	7	6	5
y	5	6	7	8	9	10

56. GEOMETRY If the area of the figure is 60 square units, what is the length of side $\overline{XZ}$?



F $2\sqrt{34}$ H $4\sqrt{34}$
G $2\sqrt{109}$ J $4\sqrt{109}$

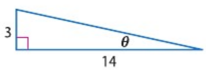
57. SAT/ACT The first term of a sequence is −6, and every term after the first is 8 more than the term immediately preceding it. What is the value of the 101st term?

A 788 D 806
B 794 E 814
C 802

Spiral Review

Find the values of the six trigonometric functions for angle θ. (Lesson 10-1)

58.



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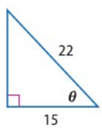
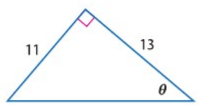


60.



Identify the null and alternative hypotheses for each statement. Then identify the statement that represents the claim.

61. Yousif drinks at least eight glasses of water every day.

62. Suha says that she has two umbrellas in her car.

63. MANUFACTURING The sizes of CDs made by a company are normally distributed with a standard deviation of 1 millimeter. The CDs are supposed to be 120 millimeters in diameter, and they are made for drives that are 122 millimeters wide.

a. What percent of the CDs would you expect to be greater than 120 millimeters?

b. If the company manufactures 1000 CDs per hour, how many of the CDs made in one hour would you expect to be between 119 and 122 millimeters?

c. About how many CDs per hour will be too large to fit in the drives?

64. FINANCIAL LITERACY If the rate of inflation is 2%, the cost of an item in future years can be found by iterating the function $c(x) = 1.02x$. Find the cost of a AED 70 digital audio player in four years if the rate of inflation remains constant.

Skills Review

Use the Pythagorean Theorem to find the length of the hypotenuse for each right triangle with the given side lengths.

65. $a = 12, b = 15$

66. $a = 8, b = 17$

67. $a = 14, b = 11$

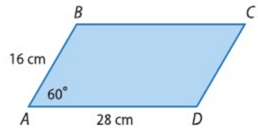
EXTEND 10-2

Geometry Lab
Areas of Parallelograms

The area of any triangle can be found using the sine ratios in the triangle. A similar process can be used to find the area of a parallelogram.

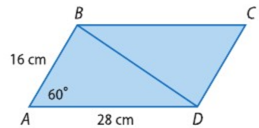
Activity

Find the area of parallelogram *ABCD*.



Step 1 Draw diagonal $\overline{BD}$.

$\overline{BD}$ divides the parallelogram into two congruent triangles, $\triangle ABD$ and $\triangle CDB$.



Step 2 Find the area of $\triangle ABD$.

Use the sine ratio to determine the height h from B to $\overline{AD}$.

$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ Definition of sine

$\sin\theta = \frac{h}{AB}$ $h = \text{opp}$, $AB = \text{hyp}$

$AB \sin\theta = h$ Solve for h.

So, $h = AB \sin\theta$.

$\text{Area} = \frac{1}{2}bh$ Area of a triangle

$= \frac{1}{2}(AD)(AB)\sin A$ $b = AD$, $h = AB \sin A$

$= \frac{1}{2}(28)(16)\sin 60°$ $AD = 28$, $AB = 16$, $A = 60°$

$= 224\left(\frac{\sqrt{3}}{2}\right)$ Multiply and evaluate sin 60°.

$= 112\sqrt{3}$ Simplify.

Step 3 Find the area of $\square ABCD$.

The area of $\square ABCD$ is equal to the sum of the areas of $\triangle ABD$ and $\triangle CDB$. Because $\triangle ABD \cong \triangle CDB$, the areas of $\triangle ABD$ and $\triangle CDB$ are equal. So, the area of $\square ABCD$ equals twice the area of $\triangle ABD$.

$2 \cdot 112\sqrt{3} = 224\sqrt{3}$ or about 387.98 square centimeters.

Exercises

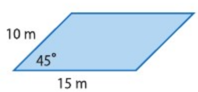
For each of the following,

a. find the area of each parallelogram.

b. find the area of each parallelogram when the included angle is half the given measure.

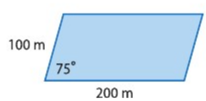
c. find the area of each parallelogram when the included angle is twice the given measure.

1.



2.

3.



LESSON 10-3 Trigonometric Functions of General Angles

Then

- You found values of trigonometric functions for acute angles.

Now

1. Find values of trigonometric functions for general angles.
2. Find values of trigonometric functions by using reference angles.

Why?

- In the ride at the right, the cars rotate back and forth about a central point. The positions of the arms supporting the cars can be described using trigonometric angles in standard position, with the central point of the ride at the origin of a coordinate plane.

NewVocabulary

quadrantal angle
reference angle

Mathematical Practices
Attend to precision.

1 Trigonometric Functions for General Angles

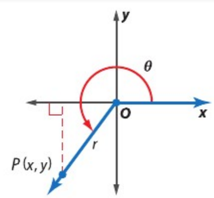
You can find values of trigonometric functions for angles greater than 90° or less than 0°.

KeyConcept Trigonometric Functions of General Angles

Let θ be an angle in standard position and let $P(x, y)$ be a point on its terminal side. Using the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$. The six trigonometric functions of θ are defined below.

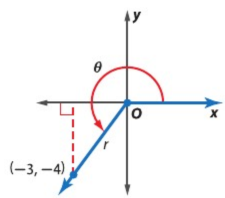
$\sin\theta = \frac{y}{r}$ $\quad\cos\theta = \frac{x}{r}$ $\quad\tan\theta = \frac{y}{x}, x \neq 0$

$\csc\theta = \frac{r}{y}, y \neq 0$ $\quad\sec\theta = \frac{r}{x}, x \neq 0$ $\quad\cot\theta = \frac{x}{y}, y \neq 0$



Example 1 Evaluate Trigonometric Functions Given a Point

The terminal side of θ in standard position contains the point at $(-3, -4)$. Find the exact values of the six trigonometric functions of θ.



Step 1 Draw the angle, and find the value of r.

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-3)^2 + (-4)^2}$$
$$= \sqrt{25} \text{ or } 5$$

Step 2 Use $x = -3$, $y = -4$, and $r = 5$ to write the six trigonometric ratios.

$\sin\theta = \frac{y}{r} = \frac{-4}{5}$ or $-\frac{4}{5}$ $\quad\cos\theta = \frac{x}{r} = \frac{-3}{5}$ or $-\frac{3}{5}$ $\quad\tan\theta = \frac{y}{x} = \frac{-4}{-3}$ or $\frac{4}{3}$

$\csc\theta = \frac{r}{y} = \frac{5}{-4}$ or $-\frac{5}{4}$ $\quad\sec\theta = \frac{r}{x} = \frac{5}{-3}$ or $-\frac{5}{3}$ $\quad\cot\theta = \frac{x}{y} = \frac{-3}{-4}$ or $\frac{3}{4}$

GuidedPractice

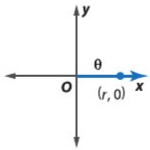
1. The terminal side of θ in standard position contains the point at $(-6, 2)$. Find the exact values of the six trigonometric functions of θ.

If the terminal side of angle θ in standard position lies on the x- or y-axis, the angle is called a **quadrantal angle**.

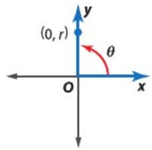
> **StudyTip**
>
> **Quadrantal Angles** The measure of a quadrantal angle is a multiple of $90°$ or $\frac{\pi}{2}$.

KeyConcept Quadrantal Angles

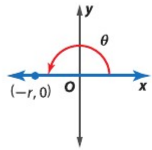
$\theta = 0°$ or 0 radians



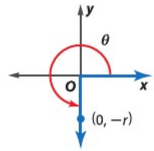
$\theta = 90°$ or $\frac{\pi}{2}$ radians



$\theta = 180°$ or π radians

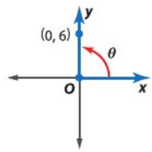


$\theta = 270°$ or $\frac{3\pi}{2}$ radians



Example 2 Quadrantal Angles

The terminal side of θ in standard position contains the point at (0, 6). Find the values of the six trigonometric functions of θ.



The point at (0, 6) lies on the positive y-axis, so the quadrantal angle θ is $90°$. Use $x = 0$, $y = 6$, and $r = 6$ to write the trigonometric functions.

$\sin \theta = \frac{y}{r} = \frac{6}{6}$ or 1 $\qquad \cos \theta = \frac{x}{r} = \frac{0}{6}$ or 0 $\qquad \tan \theta = \frac{y}{x} = \cancel{\frac{6}{0}}$ undefined

$\csc \theta = \frac{r}{y} = \frac{6}{6}$ or 1 $\qquad \sec \theta = \frac{r}{x} = \cancel{\frac{6}{0}}$ undefined $\qquad \cot \theta = \frac{x}{y} = \frac{0}{6}$ or 0

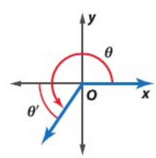
GuidedPractice

2. The terminal side of θ in standard position contains the point at $(-2, 0)$. Find the values of the six trigonometric functions of θ.

> **ReadingMath**
>
> **Theta Prime** θ' is read *theta prime.*

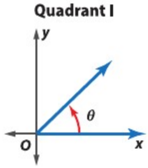
2 Trigonometric Functions with Reference Angles

If θ is a nonquadrantal angle in standard position, its **reference angle** θ' is the acute angle formed by the terminal side of θ and the x-axis. The rules for finding the measures of reference angles for $0° < \theta < 360°$ or $0° < \theta < 2\pi$ are shown below.



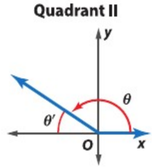
KeyConcept Reference Angles

Quadrant I



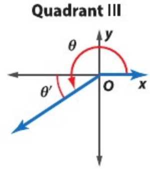
$\theta' = \theta$

Quadrant II



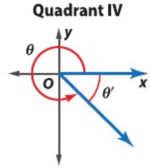
$\theta' = 180° - \theta$
$\theta' = \pi - \theta$

Quadrant III



$\theta' = \theta - 180°$
$\theta' = \theta - \pi$

Quadrant IV



$\theta' = 360° - \theta$
$\theta' = 2\pi - \theta$

If the measure of θ is greater than 360° or less than 0°, then use a coterminal angle with a positive measure between 0° and 360° to find the reference angle.

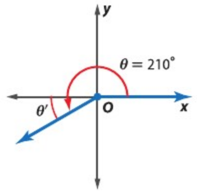
StudyTip

Graphing Angles You can refer to the diagram in the Lesson 10-2 Concept Summary to help you sketch angles.

Example 3 Find Reference Angles

Sketch each angle. Then find its reference angle.

a. 210°



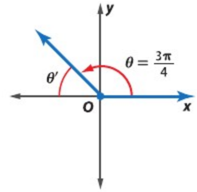
The terminal side of 210° lies in Quadrant III.

$\theta' = \theta - 180^\circ$

$= 210^\circ - 180^\circ$ or 30°

b. $-\frac{5\pi}{4}$

coterminal angle: $-\frac{5\pi}{4} + 2\pi = \frac{3\pi}{4}$



The terminal side of $\frac{3\pi}{4}$ lies in Quadrant III.

$\theta' = \pi - \theta$

$= \pi - \frac{3\pi}{4}$ or $\frac{\pi}{4}$

GuidedPractice

3A. -110°

3B. $\frac{2\pi}{3}$

You can use reference angles to evaluate trigonometric functions for any angle θ. The sign of a function is determined by the quadrant in which the terminal side of θ lies. Use these steps to evaluate a trigonometric function for any angle θ.

KeyConcept Evaluate Trigonometric Functions

Step 1 Find the measure of the reference angle θ'.

Step 2 Evaluate the trigonometric function for θ'.

Step 3 Determine the sign of the trigonometric function value. Use the quadrant in which the terminal side of θ lies.

Quadrant II	Quadrant I
$\sin\theta, \csc\theta$: +	$\sin\theta, \csc\theta$: +
$\cos\theta, \sec\theta$: −	$\cos\theta, \sec\theta$: +
$\tan\theta, \cot\theta$: −	$\tan\theta, \cot\theta$: +
Quadrant III	**Quadrant IV**
$\sin\theta, \csc\theta$: −	$\sin\theta, \csc\theta$: −
$\cos\theta, \sec\theta$: −	$\cos\theta, \sec\theta$: +
$\tan\theta, \cot\theta$: +	$\tan\theta, \cot\theta$: −

You can use the trigonometric values of angles measuring 30°, 45°, and 60° that you learned in Lesson 10-1.

Trigonometric Values for Special Angles

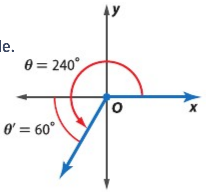
Sine	Cosine	Tangent	Cosecant	Secant	Cotangent
$\sin 30^\circ = \frac{1}{2}$	$\cos 30^\circ = \frac{\sqrt{3}}{2}$	$\tan 30^\circ = \frac{\sqrt{3}}{3}$	$\csc 30^\circ = 2$	$\sec 30^\circ = \frac{2\sqrt{3}}{3}$	$\cot 30^\circ = \sqrt{3}$
$\sin 45^\circ = \frac{\sqrt{2}}{2}$	$\cos 45^\circ = \frac{\sqrt{2}}{2}$	$\tan 45^\circ = 1$	$\csc 45^\circ = \sqrt{2}$	$\sec 45^\circ = \sqrt{2}$	$\cot 45^\circ = 1$
$\sin 60^\circ = \frac{\sqrt{3}}{2}$	$\cos 60^\circ = \frac{1}{2}$	$\tan 60^\circ = \sqrt{3}$	$\csc 60^\circ = \frac{2\sqrt{3}}{3}$	$\sec 60^\circ = 2$	$\cot 60^\circ = \frac{\sqrt{3}}{3}$

Example 4 Use a Reference Angle to Find a Trigonometric Value

Find the exact value of each trigonometric function.

a. cos 240°

The terminal side of 240° lies in Quadrant III.



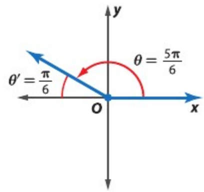
$\theta' = \theta - 180°$ — Find the measure of the reference angle.

$= 240° - 180°$ or $60°$ — $\theta = 240°$

$\cos 240° = -\cos 60°$ or $-\frac{1}{2}$ — The cosine function is negative in Quadrant III.

b. csc $\frac{5\pi}{6}$

The terminal side of $\frac{5\pi}{6}$ lies in Quadrant II.



$\theta' = \pi - \theta$ — Find the measure of the reference angle.

$= \pi - \frac{5\pi}{6}$ or $\frac{\pi}{6}$ — $\theta = \frac{5\pi}{6}$

$\csc \frac{5\pi}{6} = \csc \frac{\pi}{6}$ — The cosecant function is positive in Quadrant II.

$= \csc 30°$ — $\frac{\pi}{6}$ radians $= 30°$

$= 2$ — $\csc 30° = \frac{1}{\sin 30}$

Guided Practice

4A. $\cos 135°$

4B. $\tan \frac{5\pi}{6}$

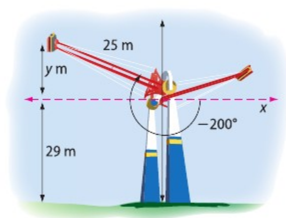
Real-World Link

On a swing ride, riders experience weightlessness just like the drop side of a roller coaster. The ride lasts one minute and reaches speeds of 96 kilometers per hour in both directions.

Source: Cedar Point

Real-World Example 5 Use Trigonometric Functions

RIDES **The swing arms of the ride at the right are 25 meters long and the height of the axis from which the arms swing is 29 meters. What is the total height of the ride at the peak of the arc?**



coterminal angle: $-200° + 360° = 160°$

reference angle: $180° - 160° = 20°$

$\sin \theta = \frac{y}{r}$ — Sine function

$\sin 20° = \frac{y}{25}$ — $\theta = 20°$ and $r = 25$

$25 \sin 20° = y$ — Multiply each side by 25.

$8.6 \approx y$ — Use a calculator to solve for y.

Since y is approximately 8.6 meters, the total height of the ride at its peak is 8.6 + 29 or about 37.6 meters.

Guided Practice

5. RIDES A similar ride that is smaller has swing arms that are 22 meters long. The height of the axis from which the arms swing is 26 meters, and the angle of rotation from the standard position is −195°. What is the total height of the ride at the peak of the arc?

Check Your Understanding

Examples 1–2 **The terminal side of θ in standard position contains each point. Find the exact values of the six trigonometric functions of θ.**

1. $(1, 2)$ **2.** $(-8, -15)$ **3.** $(0, -4)$

Example 3 **Sketch each angle. Then find its reference angle.**

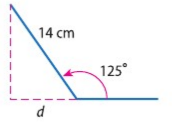
4. $300°$ **5.** $115°$ **6.** $-\frac{3\pi}{4}$

Example 4 **Find the exact value of each trigonometric function.**

7. $\sin \frac{3\pi}{4}$ **8.** $\tan \frac{5\pi}{3}$ **9.** $\sec 120°$ **10.** $\sin 300°$

Example 5

11. **ENTERTAINMENT** Maysa opens her portable DVD player so that it forms a $125°$ angle. The screen is 14 centimeters long.



a. Redraw the diagram so that the angle is in standard position on the coordinate plane.

b. Find the reference angle. Then write a trigonometric function that can be used to find the distance to the wall d that she can place the DVD player.

c. Use the function to find the distance. Round to the nearest tenth.

Practice and Problem Solving

Examples 1–2 **The terminal side of θ in standard position contains each point. Find the exact values of the six trigonometric functions of θ.**

12. $(5, 12)$ **13.** $(-6, 8)$ **14.** $(3, 0)$

15. $(0, -7)$ **16.** $(4, -2)$ **17.** $(-9, -3)$

Example 3 **Sketch each angle. Then find its reference angle.**

18. $195°$ **19.** $285°$ **20.** $-250°$

21. $\frac{7\pi}{4}$ **22.** $-\frac{\pi}{4}$ **23.** $400°$

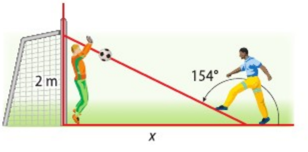
Example 4 **Find the exact value of each trigonometric function.**

24. $\sin 210°$ **25.** $\tan 315°$ **26.** $\cos 150°$ **27.** $\csc 225°$

28. $\sin \frac{4\pi}{3}$ **29.** $\cos \frac{5\pi}{3}$ **30.** $\cot \frac{5\pi}{4}$ **31.** $\sec \frac{11\pi}{6}$

Example 5

32. **REASONING** A soccer player x meters from the goalie kicks the ball toward the goal, as shown in the figure. The goalie jumps up and catches the ball 2 meters in the air.

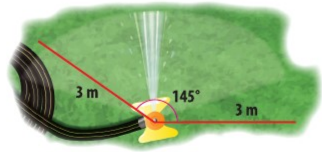


a. Find the reference angle. Then write a trigonometric function that can be used to find how far from the goalie the soccer player was when he kicked the ball.

b. About how far away from the goalie was the soccer player?

B

33. **SPRINKLER** A sprinkler rotating back and forth shoots water out a distance of 3 meters. From the horizontal position, it rotates 145° before reversing its direction.At a 145° angle, about how far to the left of the sprinkler does the water reach?



34. **BASKETBALL** The formula $R = \dfrac{v_0{}^2 \sin 2\theta}{9.8}$ gives the distance of a basketball shot with an initial velocity of v_0 meters per second at an angle θ with the ground.

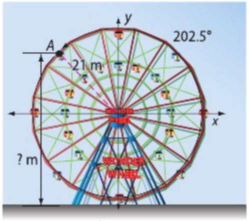
   a. If the basketball was shot with an initial velocity of 7 meters per second at an angle of 75°, how far will the basketball travel?

   b. If the basketball was shot at an angle of 65° and traveled 3 meters, what was its initial velocity?

   c. If the basketball was shot with an initial velocity of 9 meters per second and traveled 4 meters, at what angle was it shot?

35. **PHYSICS** A rock is shot off the edge of a ravine with a slingshot at an angle of 65° and with an initial velocity of 6 meters per second. The equation that represents the horizontal distance of the rock x is $x = v_0 (\cos \theta)t$, where v_0 is the initial velocity, θ is the angle at which it is shot, and t is the time in seconds. About how far does the rock travel after 4 seconds?

36. **FERRIS WHEELS** The Wonder Wheel Ferris wheel at Candidate for change to local topic? has a radius of about 21 meters and is 4.5 meters off the ground. After a person gets on the bottom car, the Ferris wheel rotates 202.5° counterclockwise before stopping. How high above the ground is this car when it has stopped?



C

Suppose θ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

37. $\sin \theta = \frac{4}{5}$, Quadrant II

38. $\tan \theta = -\frac{2}{3}$, Quadrant IV

39. $\cos \theta = -\frac{8}{17}$, Quadrant III

40. $\cot \theta = -\frac{12}{5}$, Quadrant IV

Find the exact value of each trigonometric function.

41. $\cot 270°$

42. $\csc 180°$

43. $\sin 570°$

44. $\tan \left(-\frac{7\pi}{6}\right)$

45. $\cos \left(-\frac{11\pi}{6}\right)$

46. $\cot \frac{9\pi}{4}$

H.O.T. Problems Use Higher-Order Thinking Skills

47. **CHALLENGE** For an angle θ in standard position, $\sin \theta = \frac{\sqrt{2}}{2}$ and $\tan \theta = -1$. Can the value of θ be 225°? Justify your reasoning.

48. **ARGUMENTS** Determine whether $3 \sin 60° = \sin 180°$ is *true* or *false*. Explain your reasoning.

49. **REASONING** Use the sine and cosine functions to explain why cot 180° is undefined.

50. **OPEN ENDED** Give an example of a negative angle θ for which $\sin \theta > 0$ and $\cos \theta < 0$.

51. **WRITING IN MATH** Describe the steps for evaluating a trigonometric function for an angle θ that is greater than 90°. Include a description of a reference angle.

Standardized Test Practice

52. GRIDDED RESPONSE If the sum of two numbers is 21 and their difference is 3, what is their product?

53. GEOMETRY D is the midpoint of $\overline{BC}$, and A and E are the midpoints of $\overline{BD}$ and $\overline{DC}$, respectively. If the length of $\overline{AE}$ is 12, what is the length of $\overline{BC}$?

A 6 **C** 24
B 12 **D** 48

54. The expression $(-6 + i)^2$ is equivalent to which of the following expressions?

F $-12i$ **H** $36 - 12i$
G $36 - i$ **J** $35 - 12i$

55. SAT/ACT Of the following, which is least?

A $1 + \frac{1}{4}$ **D** $1 \times \frac{1}{4}$
B $1 - \frac{1}{4}$ **E** $\frac{1}{4} - 1$
C $1 \div \frac{1}{4}$

Spiral Review

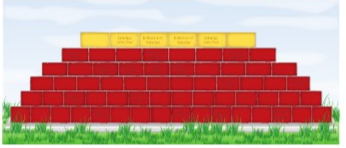
Rewrite each radian measure in degrees. (Lesson 10-2)

56. $\frac{4}{3}\pi$ **57.** $\frac{11}{6}\pi$ **58.** $-\frac{17}{4}\pi$

Solve each equation. (Lesson 10-1)

59. $\cos a = \frac{13}{17}$ **60.** $\sin 30 = \frac{b}{6}$ **61.** $\tan c = \frac{9}{4}$

62. ARCHITECTURE A memorial being constructed in a city park will be a brick wall, with a top row of six gold-plated bricks engraved with the names of six local famous people. Each row has two more bricks than the row above it. Prove that the number of bricks in the top n rows is $n^2 + 5n$.



63. LEGENDS There is a legend of a king who wanted to reward a boy for a good deed. The king gave the boy a choice. He could have AED 1,000,000 at once, or he could be rewarded daily for a 30-day month, with one fils on the first day, two fils on the second day, and so on, receiving twice as many fils each day as the previous day. How much would the second option be worth?

Write an equation for each circle given the endpoints of a diameter.

64. $(2, -4), (10, 2)$ **65.** $(-1, -10), (-7, 6)$ **66.** $(9, 0), (4, -7)$

Simplify each expression.

67. $\frac{5}{x^2 + 6x + 8} + \frac{x}{x^2 - 3x - 28}$ **68.** $\frac{3x}{x^2 + 8x - 20} - \frac{6}{x^2 + 7x - 18}$ **69.** $\frac{4}{3x^2 + 12x} + \frac{2x}{x^2 - 2x - 24}$

Solve each equation or inequality. Round to the nearest ten-thousandth.

70. $8^x = 30$ **71.** $5^x = 64$ **72.** $3^{x+2} = 41$

Evaluate each expression.

73. $16^{-\frac{1}{4}}$ **74.** $27^{\frac{4}{3}}$ **75.** $25^{-\frac{5}{2}}$

Skills Review

Solve for x.

76. $\frac{x + 2}{18} = \frac{x - 2}{9}$ **77.** $\frac{x + 5}{x - 1} = \frac{7}{4}$ **78.** $\frac{5}{x + 8} = \frac{15}{2x + 20}$

LESSON 10-4 Law of Sines

Then

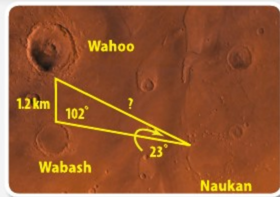
- You found side lengths and angle measures of right triangles.

Now

1. Find the area of a triangle using two sides and an included angle.
2. Use the Law of Sines to solve triangles.

Why?

- Mars has hundreds of thousands of craters. These craters are named after famous scientists, science fiction authors, and towns on Earth. The craters named Wahoo, Wabash, and Naukan are shown in the figure. You can use trigonometry to find the distance between Wahoo and Naukan.

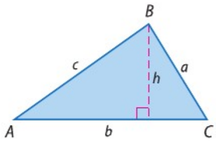


NewVocabulary
Law of Sines
solving a triangle
ambiguous case

Mathematical Practices
Make sense of problems and persevere in solving them.
Construct viable arguments and critique the reasoning of others.

1 Find the Area of a Triangle

In the triangle at the right, $\sin A = \frac{h}{c}$, or $h = c \sin A$.



$\text{Area} = \frac{1}{2}bh$ — Formula for area of a triangle

$\text{Area} = \frac{1}{2}b(c \sin A)$ — Replace h with $c \sin A$.

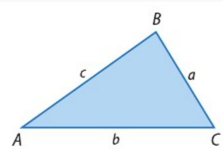
$\text{Area} = \frac{1}{2}bc \sin A$ — Simplify.

You can use this formula or two other formulas to find the area of a triangle if you know the lengths of two sides and the measure of the included angle.

KeyConcept Area of a Triangle

Words The area of a triangle is one half the product of the lengths of two sides and the sine of their included angle.

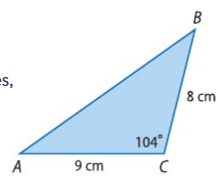
Symbols $\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$



Example 1 Find the Area of a Triangle

Find the area of $\triangle ABC$ to the nearest tenth.

In $\triangle ABC$, $a = 8$, $b = 9$, and $C = 104°$.



$\text{Area} = \frac{1}{2}ab \sin C$ — Based on the known measures, use the third area formula.

$= \frac{1}{2}(8)(9) \sin 104°$ — Substitution

$\approx 34.9 \text{ cm}^2$ — Simplify.

MENTAL CHECK Round the sin 104° to sin 90° because the sin of 90° is 1.
$\frac{1}{2}(8)(9)\sin 90° = \frac{1}{2}(8)(9)(1) = 36$
This is close to the answer of 34.9 square centimeters.

GuidedPractice

1. Find the area of $\triangle ABC$ to the nearest tenth if $A = 31°$, $b = 18$ meters, and $c = 22$ meters.

Math HistoryLink

Pauline Sperry (1885–1967) During the 1920s, Pauline Sperry wrote two textbooks, *Short Course in Spherical Trigonometry* and *Plane Trigonometry*. In 1923, she became the first woman to be promoted to assistant professor in the mathematics department at the University of California, Berkeley.

2 Use the Law of Sines to Solve Triangles

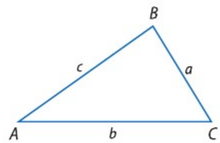
You can use the area formulas to derive the **Law of Sines**, which shows the relationships between side lengths of a triangle and the sines of the angles opposite them.

$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$	Set the area formulas equal to each other.
$bc \sin A = ac \sin B = ab \sin C$	Multiply each expression by 2.
$\frac{bc \sin A}{abc} = \frac{ac \sin B}{abc} = \frac{ab \sin C}{abc}$	Divide each expression by *abc*.
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	Simplify.

KeyConcept Law of Sines

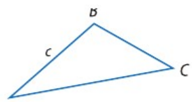
In $\triangle ABC$, if sides with lengths a, b, and c are opposite angles with measures A, B, and C, respectively, then the following is true.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

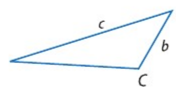


You can use the Law of Sines to solve a triangle if you know either one of the following.

- the measures of two angles and any side (angle-angle-side AAS or angle-side-angle ASA cases)



- the measures of two sides and the angle opposite one of the sides (side-side-angle SSA case)



Using given measures to find all unknown side lengths and angle measures of a triangle is called **solving a triangle**.

StudyTip

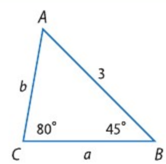
Reasoning The Law of Sines may also be written as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

So, the expressions below could also be used to solve the triangle in Example 2.

- $\frac{a}{\sin 55°} = \frac{3}{\sin 80°}$
- $\frac{b}{\sin 45°} = \frac{3}{\sin 80°}$

Example 2 Solve a Triangle Given Two Angles and a Side

Solve $\triangle ABC$. Round to the nearest tenth if necessary.



Step 1 Find the measure of the third angle.

$m\angle A = 180 - (80 + 45)$ or $55°$

Step 2 Use the Law of Sines to find side lengths a and b. Write an equation to find each variable.

$\frac{\sin A}{a} = \frac{\sin C}{c}$	Law of Sines	$\frac{\sin B}{b} = \frac{\sin C}{c}$
$\frac{\sin 55°}{a} = \frac{\sin 80°}{3}$	Substitution	$\frac{\sin 45°}{b} = \frac{\sin 80°}{3}$
$a = \frac{3 \sin 55°}{\sin 80°}$	Solve for each variable.	$b = \frac{3 \sin 45°}{\sin 80°}$
$a \approx 2.5$	Use a calculator.	$b \approx 2.2$

So, $A = 55°$, $a \approx 2.5$, and $b \approx 2.2$.

GuidedPractice

2. Solve $\triangle NPQ$ if $P = 42°$, $Q = 65°$, and $n = 5$.

StudyTip

Two Solutions A situation in which two solutions for a triangle exist is called the *ambiguous case*.

If you are given the measures of two angles and a side, exactly one triangle is possible. However, if you are given the measures of two sides and the angle opposite one of them, zero, one, or two triangles may be possible. This is known as the **ambiguous case**. So, when solving a triangle using the SSA case, zero, one, or two solutions are possible.

StudyTip

A is Acute In the figures at the right, the altitude h is compared to a because h is the minimum distance from C to $\overline{AB}$ when A is acute.

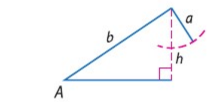
$\sin A = \frac{\text{opp}}{\text{hyp}}$

$\sin A = \frac{h}{b}$

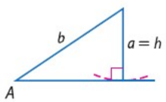
KeyConcept Possible Triangles in SSA Case

Consider a triangle in which a, b, and $m\angle A$ are given.

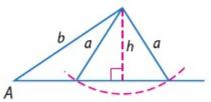
$\angle A$ is Acute.



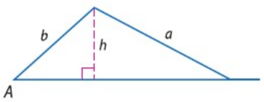
$a < h$
no solution



$a = h$
one solution

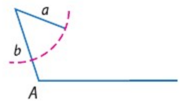


$h < a < b$
two solutions

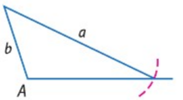


$a \geq b$
one solution

$\angle A$ is Right or Obtuse.



$a \leq b$
no solution



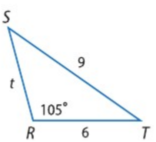
$a > b$
one solution

Since $\sin A = \frac{h}{b}$, you can use $h = b \sin A$ to find h in acute triangles.

Example 3 Solve a Triangle Given Two Sides and an Angle

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve the triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree.

a. In $\triangle RST$, $R = 105°$, $r = 9$, and $s = 6$.



Because $\angle R$ is obtuse and $9 > 6$, you know that one solution exists.

Step 1 Use the Law of Sines to find $m\angle S$.

$\frac{\sin S}{6} = \frac{\sin 105°}{9}$ — Law of Sines

$\sin S = \frac{6 \sin 105°}{9}$ — Multiply each side by 6.

$\sin S \approx 0.6440$ — Use a calculator.

$S \approx 40°$ — Use the $\sin^{-1}$ function.

Step 2 Find $m\angle T$.

$m\angle T \approx 180 - (105 + 40)$ or $35°$

Step 3 Use the Law of Sines to find t.

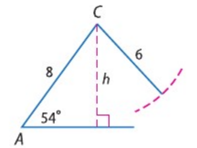
$\frac{\sin 35°}{t} \approx \frac{\sin 105°}{9}$ — Law of Sines

$t \approx \frac{9 \sin 35°}{\sin 105°}$ — Solve for t.

$t \approx 5.3$ — Use a calculator.

So, $S \approx 40°$, $T \approx 35°$, and $t \approx 5.3$.

b. In $\triangle ABC$, $A = 54°$, $a = 6$, and $b = 8$.



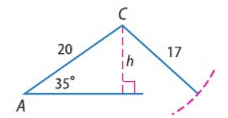
Since $\angle A$ is acute and $6 < 8$, find h and compare it to a.

$b \sin A = 8 \sin 54°$ — $b = 8$ and $A = 54°$

≈ 6.5 — Use a calculator.

Since $6 \leq 6.5$ or $a \leq h$, there is no solution.

c. In $\triangle ABC$, $A = 35°$, $a = 17$, and $b = 20$.



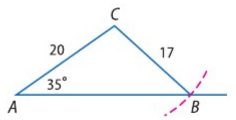
Since $\angle A$ is acute and $17 < 20$, find h and compare it to a.

$b \sin A = 20 \sin 35°$ — $b = 20$ and $A = 35°$

≈ 11.5 — Use a calculator.

Since $11.5 < 17 < 20$ or $h < a < b$, there are two solutions. So, there are two triangles to be solved

Case 1 $\angle B$ **is acute.**



Step 1 Find $m\angle B$.

$\frac{\sin B}{20} = \frac{\sin 35°}{17}$ — Law of Sines

$\sin B = \frac{20 \sin 35°}{17}$ — Solve for sin B.

$\sin B \approx 0.6748$ — Use a calculator.

$B \approx 42°$ — Find $\sin^{-1} 0.6748$.

Step 2 Find $m\angle C$.

$m\angle C \approx 180 - (35 + 42)$ or $103°$

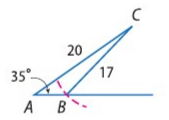
Step 3 Find c.

$\frac{\sin 103°}{c} = \frac{\sin 35°}{17}$ — Law of Sines

$c = \frac{17 \sin 103°}{\sin 35°}$ — Solve for c.

$c \approx 28.9$ — Simplify.

Case 2 $\angle B$ **is obtuse.**



Step 1 Find $m\angle B$.

The sine function also has a positive value in Quadrant II. So, find an obtuse angle B for which $\sin B \approx 0.6748$.

$m\angle B \approx 180° - 42°$ or $138°$

Step 2 Find $m\angle C$.

$m\angle C \approx 180 - (35 + 138)$ or $7°$

Step 3 Find c.

$\frac{\sin 7°}{c} \approx \frac{\sin 35°}{17}$ — Law of Sines

$c \approx \frac{17 \sin 7°}{\sin 35°}$ — Solve for c.

$c \approx 3.6$ — Simplify.

So, one solution is $B \approx 42°$, $C \approx 103°$, and $c \approx 28.9$, and another solution is $B \approx 138°$, $C \approx 7°$, and $c \approx 3.6$.

StudyTip

Reference Angle In the triangle in Case 2, you are using the reference angle 42° to find the other value of B.

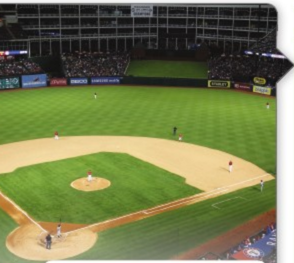
GuidedPractice

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve the triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree.

3A. In $\triangle RST$, $R = 95°$, $r = 10$, and $s = 12$.

3B. In $\triangle MNP$, $N = 32°$, $n = 7$, and $p = 4$.

3C. In $\triangle ABC$, $A = 47°$, $a = 15$, and $b = 18$.



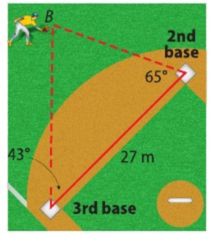
Real-WorldLink

High school and college baseball fields share the same infield dimensions as professional baseball fields. The outfield dimensions vary greatly.

Source: *Baseball Digest Magazine*

Real-World Example 4 Use the Law of Sines to Solve a Problem

BASEBALL **A baseball is hit between second and third bases and is caught at point B, as shown in the figure. How far away from second base was the ball caught?**



$\frac{\sin 72^\circ}{27} = \frac{\sin 43^\circ}{x}$	Law of Sines
$x \sin 72^\circ = 27 \sin 43^\circ$	Cross products
$x = \frac{27 \sin 43^\circ}{\sin 72^\circ}$	Solve for x.
$x \approx 19.4$	Use a calculator.

So, the distance is about 19.4 meters.

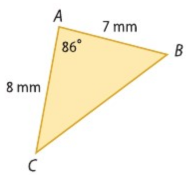
GuidedPractice

4. How far away from third base was the ball caught?

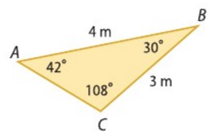
Check Your Understanding

Example 1 **Find the area of $\triangle ABC$ to the nearest tenth, if necessary.**

1.



2.

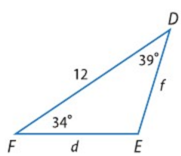


3. $A = 40^\circ$, $b = 11$ cm, $c = 6$ cm

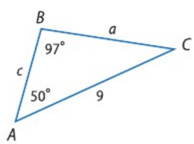
4. $B = 103^\circ$, $a = 20$ cm, $c = 18$ cm

Example 2 **Solve each triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree.**

5.



6.



7. Solve $\triangle FGH$ if $G = 80^\circ$, $H = 40^\circ$, and $g = 14$.

Example 3 **PERSEVERANCE** **Determine whether each $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve the triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree.**

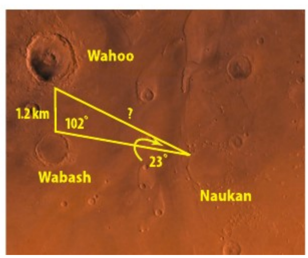
8. $A = 95^\circ$, $a = 19$, $b = 12$

9. $A = 60^\circ$, $a = 15$, $b = 24$

10. $A = 34^\circ$, $a = 8$, $b = 13$

11. $A = 30^\circ$, $a = 3$, $b = 6$

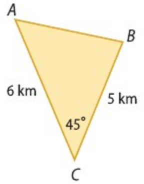
Example 4 **12.** **SPACE** Refer to the beginning of the lesson. Find the distance between the Wahoo Crater and the Naukan Crater on Mars.



Practice and Problem Solving

Example 1 **Find the area of $\triangle ABC$ to the nearest tenth.**

13.



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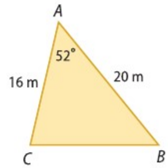

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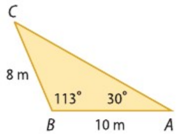

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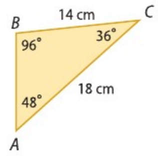


17. $C = 25°$, $a = 4$ m, $b = 7$ m

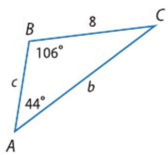
18. $A = 138°$, $b = 10$ cm., $c = 20$ cm.

19. $B = 92°$, $a = 14.5$ m, $c = 9$ m

20. $C = 116°$, $a = 2.7$ cm, $b = 4.6$ cm

Example 2 **REASONING** **Solve each triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree.**

21.



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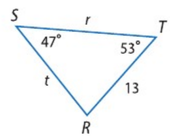

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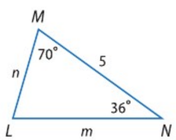

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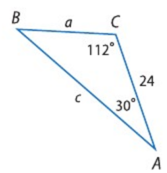


25. Solve $\triangle HJK$ if $H = 53°$, $J = 20°$, and $h = 31$.

26. Solve $\triangle NPQ$ if $P = 109°$, $Q = 57°$, and $n = 22$.

27. Solve $\triangle ABC$ if $A = 50°$, $a = 2.5$, and $C = 67°$.

28. Solve $\triangle ABC$ if $B = 18°$, $C = 142°$, and $b = 20$.

Example 3 **Determine whether each $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve the triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree.**

29. $A = 100°$, $a = 7$, $b = 3$

30. $A = 75°$, $a = 14$, $b = 11$

31. $A = 38°$, $a = 21$, $b = 18$

32. $A = 52°$, $a = 9$, $b = 20$

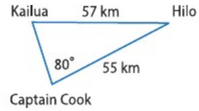
33. $A = 42°$, $a = 5$, $b = 6$

34. $A = 44°$, $a = 14$, $b = 19$

35. $A = 131°$, $a = 15$, $b = 32$

36. $A = 30°$, $a = 17$, $b = 34$

Example 4 **GEOGRAPHY** **In Hawaii, the distance from Hilo to Kailua is 57 kilometers, and the distance from Hilo to Captain Cook is 55 kilometers.**



37. What is the measure of the angle formed at Hilo?

38. What is the distance between Kailua and Captain Cook?

B **39.** **TORNADOES** Tornado sirens A, B, and C form a triangular region in one area of a city. Sirens A and B are 8 miles apart. The angle formed at siren A is 112°, and the angle formed at siren B is 40°. How far apart are sirens B and C?

40. **MYSTERIES** The Bermuda Triangle is a region of the Atlantic Ocean between Bermuda, Miami, Florida, and San Juan, Puerto Rico. It is an area where ships and airplanes have been rumored to mysteriously disappear.

a. What is the distance between Miami and Bermuda?

b. What is the approximate area of the Bermuda Triangle?

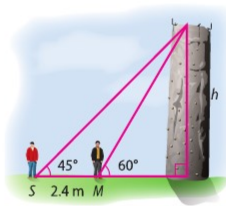
41. **BICYCLING** One side of a triangular cycling path is 4 kilometers long. The angle opposite this side is 64°. Another angle formed by the triangular path measures 66°.

a. Sketch a drawing of the situation. Label the missing sides a and b.

b. Write equations that could be used to find the lengths of the missing sides.

c. What is the perimeter of the path?

42. **ROCK CLIMBING** Saeed S and Majed M are standing 2.4 meters apart in front of a rock climbing wall, as shown at the right. What is the height of the wall? Round to the nearest tenth.



H.O.T. Problems Use Higher-Order Thinking Skills

C **43.** **CRITIQUE** In $\triangle RST$, $R = 56°$, $r = 24$, and $t = 12$. Maysoun and Maha are using the Law of Sines to find T. Is either of them correct? Explain your reasoning.

Maysoun

$$\frac{\sin T}{12} = \frac{\sin 56°}{24}$$

$$\sin T \approx 0.4145$$

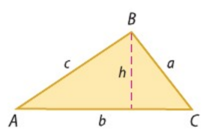
$$T \approx 24.5°$$

Maha

Since $r > t$, there is no solution.

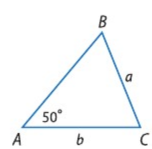
44. **OPEN ENDED** Create an application problem involving right triangles and the Law of Sines. Then solve your problem, drawing diagrams if necessary.

45. **CHALLENGE** Using the figure at the right, derive the formula Area $= \frac{1}{2}bc \sin A$.



46. **REASONING** Find the side lengths of two different triangles ABC that can be formed if $A = 55°$ and $C = 20°$.

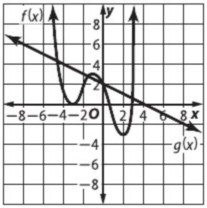
47. **WRITING IN MATH** Use the Law of Sines to explain why a and b do not have unique values in the figure shown.



48. **OPEN ENDED** Given that $E = 62°$ and $d = 38$, find a value for e such that no triangle DEF can exist. Explain your reasoning.

Standardized Test Practice

49. SHORT RESPONSE Given the graphs of $f(x)$ and $g(x)$, what is the value of $f(g(4))$?



50. STATISTICS If the average of seven consecutive odd integers is n, what is the median of these seven integers?

A 0
B 7
C n
D $n - 2$

51. One zero of $f(x) = x^3 - 7x^2 - 6x + 72$ is 4. What is the factored form of the expression $x^3 - 7x^2 - 6x + 72$?

F $(x - 6)(x + 3)(x + 4)$
G $(x - 6)(x + 3)(x - 4)$
H $(x + 6)(x + 3)(x - 4)$
J $(x + 12)(x - 1)(x - 4)$

52. SAT/ACT Three people are splitting AED 48,000 using the ratio 5 : 4 : 3. What is the amount of the greatest share?

A AED 12,000
B AED 16,000
C AED 20,000
D AED 24,000
E AED 30,000

Spiral Review

Find the exact value of each trigonometric function. (Lesson 10-3)

53. $\sin 210^\circ$ **54.** $\cos \frac{3}{4}\pi$ **55.** $\cot 60^\circ$

Find an angle with a positive measure and an angle with a negative measure that are coterminal with each angle. (Lesson 10-2)

56. 125° **57.** -32° **58.** $\frac{2}{3}\pi$

59. CLOCKS Buthaina's grandfather clock is broken. When she sets the pendulum in motion by holding it against the side of the clock and letting it go, it swings 24 centimeters to the other side, then 18 centimeters back, then 13.5 centimeters, and so on. What is the total distance that the pendulum swings before it stops?

Find the sum of each infinite series, if it exists.

60. $64 + 48 + 36 + \dots$ **61.** $27 + 36 + 48 + \dots$ **62.** $\sum_{n=1}^{\infty} 0.5(1.1)^n$

63. ASTRONOMY At its closest point, Earth is 146.9 million kilometers from the center of the Sun. At its farthest point, Earth is 151.8 million kilometers from the center of the Sun. Write an equation for the orbit of Earth, assuming that the center of the orbit is the origin and the Sun lies on the x-axis.

Simplify.

64. $\sqrt{(x - 4)^2}$ **65.** $\sqrt{(y + 2)^4}$ **66.** $\sqrt[3]{(a - b)^6}$

Skills Review

Evaluate each expression if $w = 6$, $x = -4$, $y = 1.5$, and $z = \frac{3}{4}$.

67. $w^2 + y^2 - 6xz$ **68.** $x^2 + z^2 + 5wy$ **69.** $wy + xz + w^2 - x^2$

EXTEND 10-4

Geometry Lab
Regular Polygons

You can use central angles of circles to investigate characteristics of regular polygons inscribed in a circle. Recall that a regular polygon is inscribed in a circle if each of its vertices lies on the circle.

Activity Collect the Data

Step 1 Use a compass to draw a circle with a radius of one centimeter.

Step 2 Inscribe an equilateral triangle inside the circle. To do this, use a protractor to measure three angles of $120°$ at the center of the circle, since $\frac{360°}{3} = 120°$.

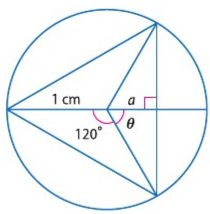
Then connect the points where the sides of the angles intersect the circle using a straightedge.

Step 3 The **apothem** of a regular polygon is a segment that is drawn from the center of the polygon perpendicular to a side of the polygon. Use the cosine of angle θ to find the length of an apothem, labeled a in the diagram.

Model and Analyze

1. Make a table like the one shown below and record the length of the apothem of the equilateral triangle. Inscribe each regular polygon named in the table in a circle with radius one centimeter. Complete the table.

Number of Sides, n	θ	a	Number of Sides, n	θ	a
3	60		7		
4	45		8		
5			9		
6			10		



2. What do you notice about the measure of θ as the number of sides of the inscribed polygon increases?
3. What do you notice about the value of a?
4. **MAKE A CONJECTURE** Suppose you inscribe a 30-sided regular polygon inside a circle. Find the measure of angle θ.
5. Write a formula that gives the measure of angle θ for a polygon with n sides.
6. Write a formula that gives the length of the apothem of a regular polygon inscribed in a circle with radius one centimeter.
7. How would the formula you wrote in Exercise 6 change if the radius of the circle was not one centimeter?