Chapter Project

Sound the Alarm

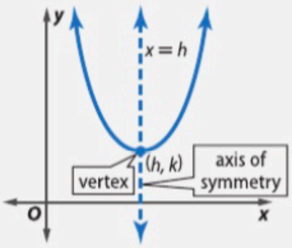
Students use what they have learned about conic sections to complete a project.

This chapter project addresses environmental literacy, as well as several specific skills identified as being essential to student success by the Framework for 21st Century Learning.

KeyVocabulary Introduce the key vocabulary in the chapter using the routine below.

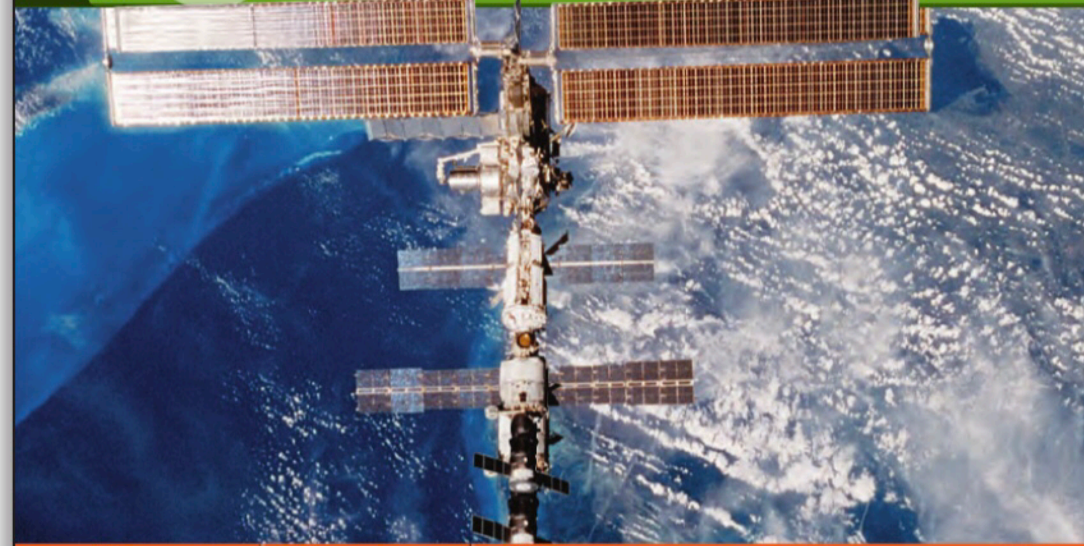
Define: A parabola is the set of all points in a plane that are the same distance from a point, called the focus, and a line, called the directrix.

Example: The diagram shows a parabola.



Ask: What are the coordinates of the vertex of the parabola? *(h, k)*

CHAPTER 6 Conic Sections and Parametric Equations



Then

You solved systems of linear equations algebraically and graphically.

Now

You will:

- Use the Midpoint and Distance Formulas.
- Write and graph equations of parabolas, circles, ellipses, and hyperbolas.
- Identify conic sections.
- Solve systems of quadratic equations and inequalities.

Why?

SPACE Conic sections are evident in many aspects of space. Equations of circles are used to pilot spacecraft and satellites in circular orbits around Earth and the Moon. Planets travel in elliptical paths, not circular ones as previously thought. Comets travel along one branch of a hyperbola, which can help us to predict when they will appear again.

LESSON 6-1 Parabolas

Then

- You graphed quadratic functions.

Now

1. Write equations of parabolas in standard form.
2. Graph parabolas.

Why?

- Satellite dishes can be used to send and receive signals and can be seen attached to residential homes and businesses.

A satellite dish is a type of antenna constructed to receive signals from orbiting satellites. The signals are reflected off of the dish's parabolic surface to a common collection point.

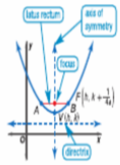
NewVocabulary
parabola
focus
directrix
latus rectum
standard form
general form

Mathematical Practices
1 Make sense of problems and persevere in solving them.

1 Equations of Parabolas

A **parabola** can be defined as the set of all points in a plane that are the same distance from a given point called the **focus** and a given line called the **directrix**.

The line segment through the focus of a parabola and perpendicular to the axis of symmetry is called the **latus rectum**. The endpoints of the latus rectum lie on the parabola.



KeyConcept Equations of Parabolas

Form of Equation	$y = a(x - h)^2 + k$	$x = a(y - k)^2 + h$
Direction of Opening	upward if $a > 0$, downward if $a < 0$	right if $a > 0$, left if $a < 0$
Vertex	(h, k)	(h, k)
Axis of Symmetry	$x = h$	$y = k$
Focus	$\left(h, k + \frac{1}{4a}\right)$	$\left(h + \frac{1}{4a}, k\right)$
Directrix	$y = k - \frac{1}{4a}$	$x = h - \frac{1}{4a}$
Length of Latus Rectum	$\left\|\frac{1}{a}\right\|$ units	$\left\|\frac{1}{a}\right\|$ units

The **standard form** of the equation of a parabola with vertex (h, k) and axis of symmetry $x = h$ is $y = a(x - h)^2 + k$.

- If $a > 0$, k is the minimum value of the related function and the parabola opens upward.
- If $a < 0$, k is the maximum value of the related function and the parabola opens downward.

An equation of a parabola in the form $y = ax^2 + bx + c$ is the **general form**. Any equation in general form can be written in standard form. The shape of a parabola and the distance between the focus and directrix depend on the value of a in the equation.

1 Focus

VerticalAlignment

Before Lesson 6-1 Graph quadra functions.

Lesson 6-1 Write equations of parabolas in standard form. Graph parabolas.

After Lesson 6-1 Identify the con section from a given equation.

2 Teach

Scaffolding Questions

Have students read the **Why?** secti the lesson.

Ask:

- What does it mean to say that a satellite dish's surface is *paraboli* cross-sections of its surface taken perpendicular to its base are parabolas
- Why is a parabola the perfect sha for a satellite dish? The parabola reflects all parallel incoming rays the same point.

1 Equations of Parabolas

Example 1 shows how to write an equation of a parabola in standard form and then analyze the equation to identify the vertex, axis of symmetry, and direction of opening of the parabola.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Example

1 Write $y = -x^2 - 2x + 3$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola. $y = -(x + 1)^2 + 4$; vertex: $(-1, 4)$; axis of symmetry: $x = -1$; opens downward

2 Graph Parabolas

Example 2 shows how to use the parent function, $y = ax^2$, symmetry, and translations to graph the equation of a parabola with a vertical axis of symmetry. **Example 3** shows how to graph the equation of a parabola that is not in standard form and has a horizontal axis of symmetry. **Example 4** shows how to write and graph the equation of a parabola given the vertex and directrix. **Example 5** shows how to write and graph the equation of a parabola to solve a real-world problem.

Teaching the Mathematical Practices

Structure Mathematically proficient students look closely to discern a pattern or structure. Encourage students to identify all important values before beginning their graphs.

ReviewVocabulary

Completing the Square rewriting a quadratic expression as a perfect square trinomial

Example 1 Analyze the Equation of a Parabola

Write $y = 2x^2 - 12x + 6$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

$y = 2x^2 - 12x + 6$	Original equation
$= 2(x^2 - 6x) + 6$	Factor 2 from the x- and x^2-terms.
$= 2(x^2 - 6x + ■) + 6 - 2(■)$	Complete the square on the right side.
$= 2(x^2 - 6x + 9) + 6 - 2(9)$	The 9 added when you complete the square is multiplied by 2.
$= 2(x - 3)^2 - 12$	Factor.

The vertex of this parabola is located at $(3, -12)$, and the equation of the axis of symmetry is $x = 3$. The parabola opens upward.

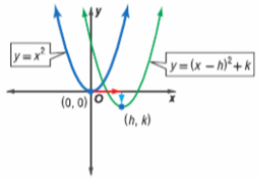
GuidedPractice

1. Write $y = 4x^2 + 16x + 34$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

$y = 4(x + 2)^2 + 18$; vertex: $(h, k) = (-2, 18)$; axis of symmetry: $x = -2$; opens upward

2 Graph Parabolas

Previously you learned that the graph of the quadratic equation $y = a(x - h)^2 + k$ is a transformation of the parent graph of $y = x^2$ translated h units horizontally and k units vertically, and reflected and/or dilated depending on the value of a.



WatchOut!

Structure Carefully examine the values for h and k before beginning to graph an equation.

- If h is positive, translate the graph h units to the right.
- If h is negative, translate the graph $|h|$ units to the left.
- If k is positive, translate the graph k units up.
- If k is negative, translate the graph $|k|$ units down.

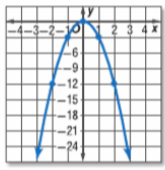
Example 2 Graph Parabolas

Graph each equation.

a. $y = -3x^2$

For this equation, $h = 0$ and $k = 0$. The vertex is at the origin. Since the equation of the axis of symmetry is $x = 0$, substitute some small positive integers for x and find the corresponding y-values.

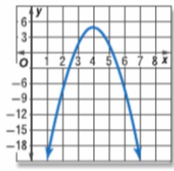
x	y
1	−3
2	−12
3	−27



Since the graph is symmetric about the y-axis, the points at $(-1, -3)$, $(-2, -12)$, and $(-3, -27)$ are also on the parabola. Use all of these points to draw the graph.

b. $y = -3(x - 4)^2 + 5$

The equation is of the form $y = a(x - h)^2 + k$, where $h = 4$ and $k = 5$. The graph of this equation is the graph of $y = -3x^2$ in part **a** translated 4 units to the right and up 5 units. The vertex is now at $(4, 5)$.



GuidedPractice

2A. $y = 2x^2$ **2B.** $y = 2(x - 1)^2 - 4$

2A, 2B. See Chapter 6 Answer Appendix.

DifferentiatedInstruction AL OL

If some students think that any curve can be called a parabola,

Then explain that only curves with a certain well-defined shape meet the definition of a parabola. Have students use the Internet to research objects in the real world that are parabolas. Ask them to make a poster to display their findings.

StudyTip

Graphing When graphing these functions, it may be helpful to sketch the graph of the parent function.

Equations of parabolas with vertical axes of symmetry have the parent function $y = x^2$ and are of the form $y = a(x - h)^2 + k$. These are functions. Equations of parabolas with horizontal axes of symmetry are of the form $x = a(y - k)^2 + h$ and are not functions. The parent graph for these equations is $x = y^2$.

Example 3 Graph an Equation in General Form

Graph each equation.

a. $2x - y^2 = 4y + 10$

Step 1 Write the equation in the form $x = a(y - k)^2 + h$.

$2x - y^2 = 4y + 10$	Original equation
$2x = y^2 + 4y + 10$	Add y^2 to each side to isolate the x-term.
$2x = (y^2 + 4y + ■) + 10 - ■$	Complete the square.
$2x = (y^2 + 4y + 4) + 10 - 4$	Add and subtract 4, since $\left(\frac{4}{2}\right)^2 = 4$.
$2x = (y + 2)^2 + 6$	Factor and subtract.
$x = \frac{1}{2}(y + 2)^2 + 3$	$(h, k) = (3, -2)$

Step 2 Use the equation to find information about the graph. Then draw the graph based on the parent graph, $x = y^2$.

vertex: $(3, -2)$

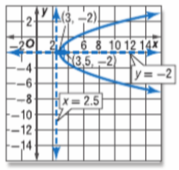
axis of symmetry: $y = -2$

focus: $\left(3 + \frac{1}{4\left(\frac{1}{2}\right)}, -2\right)$ or $(3.5, -2)$

directrix: $x = 3 - \frac{1}{4\left(\frac{1}{2}\right)}$ or 2.5

direction of opening: right, since $a > 0$

length of latus rectum: $\left|\frac{1}{\left(\frac{1}{2}\right)}\right|$ or 2 units



ReadingMath

latus rectum from the Latin *latus*, meaning side, and *rectum*, meaning straight

b. $y + 2x^2 + 32 = -16x - 1$

Step 1

$y + 2x^2 + 32 = -16x - 1$	Original equation
$y = -2x^2 - 16x - 33$	Solve for y.
$y = -2(x^2 + 8x + ■) - 33 - ■$	Complete the square.
$y = -2(x^2 + 8x + 16) - 33 - (-32)$	Add and subtract −32.
$y = -2(x + 4)^2 - 1$	Factor and simplify.

Step 2 vertex: $(-4, -1)$

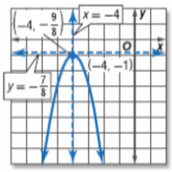
axis of symmetry: $x = -4$

focus: $\left(-4, -\frac{9}{8}\right)$

directrix: $y = -\frac{7}{8}$

length of latus rectum: $\frac{1}{2}$ unit

opens downward



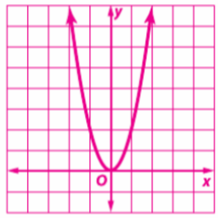
GuidedPractice **3A, 3B. See Chapter 6 Answer Appendix.**

3A. $3x - y^2 = 4x + 25$ **3B.** $y = x^2 + 6x - 4$

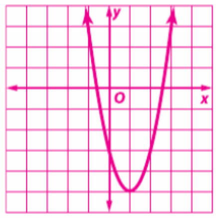
Additional Examples

2 Graph each equation.

a. $y = 2x^2$

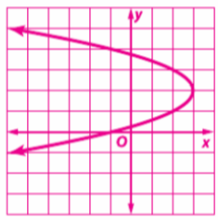


b. $y = 2(x - 1)^2 - 5$

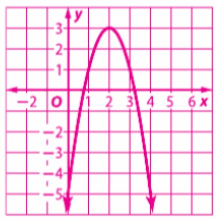


3 Graph each equation.

a. $x + y^2 = 4y - 1$



b. $y + 2x^2 + 10 = 8x + 5$



Teach with Tech

Video Recording Break the class into groups, and give each group an equation of a different parabola. Have the group create a video showing how to find all of the properties and information about the parabola. Share each group's video with the entire class.

Tips for New Teachers

Building on Prior Knowledge Rem students that the distance from a poi a line is measured on the perpendicu from the point to the line.

Additional Examples

4 Write an equation for a parabola with vertex at (8, 6) and focus (2, 6). Then graph the equation.

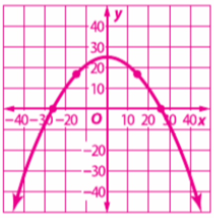
$x = -\frac{1}{24}(y - 6)^2 + 8$

5 **BRIDGES** The Hulme Arch Bridge in Manchester, England, is supported by cables suspended from a parabolic steel arch. The highest point of the arch is 25 meters above the bridge, and the focus of the arch is 18 meters above the bridge.

a. Let the bridge be the x–axis, and let the y-axis pass through the vertex of the arch. Write an equation that models the arch.

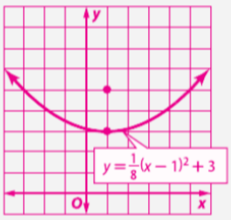
$y = -\frac{1}{28}x^2 + 25$

b. Graph the equation.



Additional Answers (Guided Practice)

4A. $y = \frac{1}{8}(x - 1)^2 + 3$



You can use specific information about a parabola to write an equation and draw a graph.

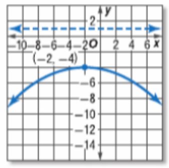
Example 4 Write an Equation of a Parabola

Write an equation for a parabola with vertex at (−2, −4) and directrix $y = 1$. Then graph the equation.

The directrix is a horizontal line, so the equation of the parabola is of the form $y = a(x - h)^2 + k$. Find a, h, and k.

- The vertex is at (−2, −4), so $h = -2$ and $k = -4$.
- Use the equation of the directrix to find a.

$y = k - \frac{1}{4a}$	Equation of directrix
$1 = -4 - \frac{1}{4a}$	Replace y with 1 and k with −4.
$5 = -\frac{1}{4a}$	Add 4 to each side.
$20a = -1$	Multiply each side by $4a$.
$a = -\frac{1}{20}$	Divide each side by 20.



So, the equation of the parabola is $y = -\frac{1}{20}(x + 2)^2 - 4$.

GuidedPractice **4A, 4B. See margin.**

Write an equation for each parabola described below. Then graph the equation.

4A. vertex (1, 3), focus (1, 5)

4B. focus (5, 6), directrix $x = -2$

Parabolas are often used in the real world.

Real-WorldLink
In California's Mojave Desert, parabolic mirrors are used to heat oil that flows through tubes placed at the focus. The heated oil is used to produce electricity.
Source: Solel

Real-World Example 5 Write an Equation for a Parabola

ENVIRONMENT **Solar energy may be harnessed by using parabolic mirrors. The mirrors reflect the rays from the Sun to the focus of the parabola. The focus of each parabolic mirror at the facility described at the left is 1.9 meters above the vertex. The latus rectum is 7.6 meters long.**

a. Assume that the focus is at the origin. Write an equation for the parabola formed by each mirror.

In order for the mirrors to collect the Sun's energy, the parabola must open upward. Therefore, the vertex must be below the focus.

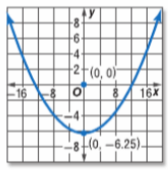
focus: (0, 0) vertex: (0, −1.9)

The measure of the latus rectum is 7.6. So $7.6 = \left|\frac{1}{a}\right|$, and $a = \frac{1}{7.6}$.

Using the form $y = a(x - h)^2 + k$, an equation for the parabola formed by each mirror is $y = \frac{1}{7.6}x^2 - 1.9$.

b. Graph the equation.

Now use all of the information to draw a graph.



GuidedPractice **5. See margin.**

5. Write and graph an equation for a parabolic mirror that has a focus 1.4 meters above the vertex and a latus rectum that is 5.5 meters long, when the focus is at the origin.

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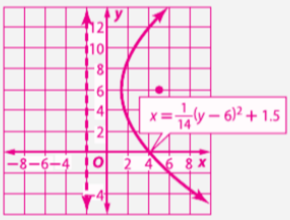
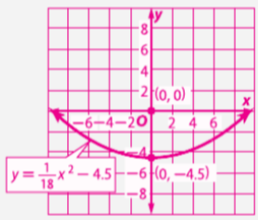
4B. $x = \frac{1}{14}(y - 6)^2 + 1.5$



5. $y = \frac{1}{5.5}x^2 - 1.4$



Uncorrected first proof - for training purposes only

Check Your Understanding

Example 1 Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola. **1–4. See margin.**

1. $y = 2x^2 - 24x + 40$
2. $y = 3x^2 - 6x - 4$
3. $x = y^2 - 8y - 11$
4. $x + 3y^2 + 12y = 18$

Examples 2–3 Graph each equation. **5–8. See Chapter 6 Answer Appendix.**

5. $y = (x - 4)^2 - 6$
6. $y = 4(x + 5)^2 + 3$
7. $y = -3x^2 - 4x - 8$
8. $x = 3y^2 - 6y + 9$

9–12. See Chapter 6 Answer Appendix.

Example 4 Write an equation for each parabola described below. Then graph the equation.

9. vertex (0, 2), focus (0, 4)
10. vertex (−2, 4), directrix $x = -1$
11. focus (3, 2), directrix $y = 8$
12. vertex (−1, −5), focus (−5, −5)

Example 5

13. **ASTRONOMY** Consider a parabolic mercury mirror like the one described at the beginning of the lesson. The focus is 1.8 meters above the vertex and the latus rectum is 7.3 meters long.
    a. Assume that the focus is at the origin. Write an equation for the parabola formed by the parabolic microphone. $y = \frac{1}{7.3}x^2 - 1.8$
    b. Graph the equation. **See margin.**

Practice and Problem Solving

Example 1 Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola. **14–19. See margin.**

14. $y = x^2 - 8x + 13$
15. $y = 3x^2 + 42x + 149$
16. $y = -6x^2 - 36x - 8$
17. $y = -3x^2 - 9x - 6$
18. $x = \frac{1}{3}y^2 - 3y + 4$
19. $x = \frac{2}{3}y^2 - 4y + 12$

Examples 2–3 Graph each equation. **20–25. See Chapter 6 Answer Appendix.**

20. $y = \frac{1}{3}x^2$
21. $y = -2x^2$
22. $y = -2(x - 2)^2 + 3$
23. $y = 3(x - 3)^2 - 5$
24. $x = \frac{1}{2}y^2$
25. $4x - y^2 = 2y + 13$

Example 4 Write an equation for each parabola described below. Then graph the equation. **26–31. See Chapter 6 Answer Appendix.**

26. vertex (0, 1), focus (0, 4)
27. vertex (1, 8), directrix $y = 3$
28. focus (−2, −4), directrix $x = -6$
29. focus (2, 4), directrix $x = 10$
30. vertex (−6, 0), directrix $x = 2$
31. vertex (9, 6), focus (9, 5)

Example 5

32. **BASEBALL** When a ball is thrown, the path it travels is a parabola. Suppose a baseball is thrown from ground level, reaches a maximum height of 15.2 meters, and hits the ground 61 meters from where it was thrown. Assuming this situation could be modeled on a coordinate plane with the focus of the parabola at the origin, find the equation of the parabolic path of the ball. Assume the focus is on ground level. $y = -\frac{1}{61}x^2 + 15.2$

33. **PERSEVERANCE** Ground antennas and satellites are used to relay signals between the NASA Mission Operations Center and the spacecraft it controls. One such parabolic dish is 146 feet in diameter. Its focus is 48 feet from the vertex. **a. See Chapter 6 Answer Appendix.**
    a. Sketch two options for the dish, one that opens up and one that opens left.
    b. Write two equations that model the sketches in part a. $y = \frac{x^2}{192}$ and $x = \frac{y^2}{-192}$
    c. If you wanted to find the depth of the dish, does it matter which equation you use? Why or why not? **Sample answer: No; except for the direction in which the graphs are identical. they open,**

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	14–33, 37–61	15–33 odd, 41–44	14–32 even, 37–40, 45–61
OL Core	15–30 odd, 32–61	14–33, 41–44	34–40, 45–61
BL Advanced	34–57, (optional: 58–61)		

3 Practice

Formative Assessment

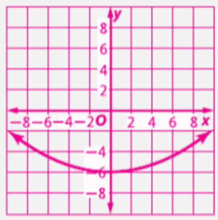
Use Exercises 1–13 to check for understanding.

Use the chart at the bottom of this p to customize assignments for your students.

Additional Answers

1. $y = 2(x - 6)^2 - 32$; vertex (6, −32); axis of symmetry: $x =$ opens upward
2. $y = 3(x - 1)^2 - 7$; vertex (1, −7); axis of symmetry: $x = 1$; opens upward
3. $x = (y - 4)^2 - 27$; vertex (−27, 4); axis of symmetry: $y =$ opens right
4. $x = -3(y + 2)^2 + 30$; vertex (30, −2); axis of symmetry: $y = -2$; opens left

13b.



14. $y = (x - 4)^2 - 3$; vertex = (4, − axis of symmetry: $x = 4$; opens upward
15. $y = 3(x + 7)^2 + 2$; vertex = (−7, 2); axis of symmetry: $x =$ opens upward
16. $y = -6(x + 3)^2 + 46$; vertex = (−3, 46); axis of symmetry: $x = -3$; opens downward
17. $y = -3\left(x + \frac{3}{2}\right)^2 + \frac{3}{4}$; vertex = $\left(-\frac{3}{2}, \frac{3}{4}\right)$; axis of symmetry: $x = -\frac{3}{2}$; opens downward
18. $x = \frac{1}{3}(y - 4.5)^2 - 2.75$; verte (−2.75, 4.5); axis of symmetry: $y = 4.5$; opens right
19. $x = \frac{2}{3}(y - 3)^2 + 6$; vertex = (6, 3); axis of symmetry: $y = 3$; opens right

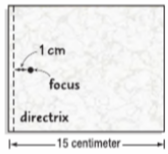
Teaching the Mathematical Practices

Critique Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is.

34. **UMBRELLAS** A beach umbrella has an arch in the shape of a parabola that opens downward. The umbrella spans 1.8 meters across and is 0.45 meters high. Write an equation of a parabola to model the arch, assuming that the origin is at the point where the pole and umbrella meet at the vertex of the arch. $y = -\frac{1}{1.8}x^2 + 0.45$

35. **AUTOMOBILES** An automobile headlight contains a parabolic reflector. The light coming from the source bounces off the parabolic reflector and shines out the front of the headlight. The equation of the cross section of the reflector is $y = \frac{1}{12}x^2$. How far from the vertex should the filament for the high beams be placed? **3 units**

36. **MULTIPLE REPRESENTATIONS** Start with a sheet of wax paper that is about 15 centimeters long and 12 centimeters wide. **a–c. See students' work.**



a. **Concrete** Make a line that is perpendicular to the sides of the sheet by folding the sheet near one end. Open up the paper again. This line is the directrix. Mark a point about midway between the sides of the sheet so that the distance from the directrix is about 1 inch. This is the focus.

b. **Concrete** Start with a new sheet of wax paper. Form another outline of a parabola with a focus that is about 3 centimeter from the directrix.

c. **Concrete** On a new sheet of a wax paper, form a third outline of a parabola with a focus that is about 5 centimeter from the directrix.

d. **Verbal** Compare the shapes of the three parabolas. How does the distance between the focus and the directrix affect the shape of a parabola?
As the distance between the directrix and the focus increases, the parabola becomes wider.

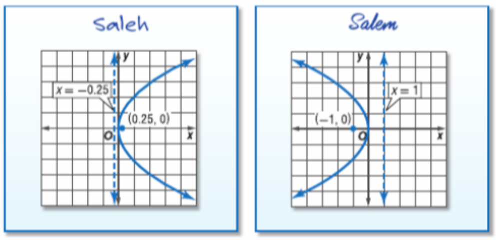
H.O.T. Problems Use Higher-Order Thinking Skills

37. **REASONING** How do you change the equation of the parent function $y = x^2$ to shift the graph to the right? **Rewrite it as $y = (x - h)^2$, where $h > 0$.**

38. **OPEN ENDED** Two different parabolas have their vertex at $(-3, 1)$ and contain the point with coordinates $(-1, 0)$. Write two possible equations for these parabolas.
Sample answers: $y = -\frac{1}{4}(x + 3)^2 + 1$ and $x = 2(y - 1)^2 - 3$

39. **CRITIQUE** Saleh and Salem are graphing $\frac{1}{4}y^2 + x = 0$. Is either of them correct? Explain your reasoning.

39. Salem; the parabola should open to the left rather than to the right.



40. **WRITING IN MATH** Why are parabolic shapes used in the real world? **See margin.**

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Standardized Test Practice

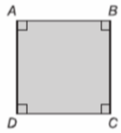
41. A gardener is placing a fence around a 1320-square-meter rectangular garden. He ordered 148 meters of fencing. If he uses all the fencing, what is the length of the longer side of the garden? **C**

A 30 m C 44 m
B 34 m D 46 m

42. SAT/ACT When a number is divided by 5, the result is 7 more than the number. Find the number. **F**

F $-\frac{35}{4}$ J $\frac{28}{4}$
G $-\frac{35}{6}$ K $\frac{35}{4}$
H $\frac{35}{6}$

43. GEOMETRY What is the area of the following square, if the length of $\overline{BD}$ is $2\sqrt{2}$? **D**



A 1
B 2
C 3
D 4

44. SHORT RESPONSE The measure of the smallest angle of a triangle is two thirds the measure of the middle angle. The measure of the middle angle is three sevenths of the measure of the largest angle. Find the largest angle's measure. **105°**

Spiral Review

45. GEOMETRY Find the perimeter of a triangle with vertices at (2, 4), (−1, 3), and (1, −3). (Lesson 6-1) **$5\sqrt{2} + 3\sqrt{10}$ units**

46. WORK A worker can powerwash a wall of a certain size in 5 hours. Another worker can do the same job in 4 hours. If the workers work together, how long would it take to do the job? Determine whether your answer is reasonable.

46. $2\frac{2}{9}$ h; The answer is reasonable. The time to complete the job when working together must be less than the time it would take either person working alone.

Solve each equation or inequality. Round to the nearest ten-thousandth.

47. $\ln(x + 1) = 1$ **1.7183**

48. $\ln(x - 7) = 2$ **14.3891**

49. $e^x > 1.6$ **$x > 0.4700$**

50. $e^{5x} \geq 25$ **$x \geq 0.6438$**

Simplify.

51. $\sqrt{0.25}$ **0.5**

52. $\sqrt[3]{-0.064}$ **−0.4**

53. $\sqrt[4]{z^8}$ **z^2**

54. $-\sqrt[6]{x^6}$ **$-|x|$**

List all of the possible rational zeros of each function.

55. $h(x) = x^3 + 8x + 6$ **$\pm1, \pm2, \pm3, \pm6$**

56. $p(x) = 3x^3 - 5x^2 - 11x + 3$ **$\pm1, \pm\frac{1}{3}, \pm3$**

57. $h(x) = 9x^6 - 5x^3 + 27$ **$\pm1, \pm\frac{1}{3}, \pm\frac{1}{9}, \pm3, \pm9, \pm27$**

Skills Review

Simplify each expression.

58. $\sqrt{24}$ **$2\sqrt{6}$**

59. $\sqrt{45}$ **$3\sqrt{5}$**

60. $\sqrt{252}$ **$6\sqrt{7}$**

61. $\sqrt{512}$ **$16\sqrt{2}$**

Exercise Alert

Wax Paper For Exercise 36, stude will need three 15″ by 12″ sheets of paper.

WatchOut!

Error Analysis In Exercise 39, remind students that they must rewrite the equation in standard form before graphing.

4 Assess

Yesterday's News Have students how understanding how to complet the square helped them in today's lesson with graphing equations of a parabola.

Additional Answer

40. Sample answer: When rays wh are parallel to the axis of symm of a parabolic mirror are reflec off the mirror, they are directe the focus. This focuses all of th rays at one specific point. Also, parabolic microphone can be u to make capturing sound more effective, because reflected so waves are focused at the partic point where the microphone is located.

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Extension Remind students that if they know two points on a line, they can write the equation of the line. Tell students that to write the equation of a parabola, they need three points on the graph. Ask students to write the equation for a parabola through (0, 3), (1, 4), and (−2, 7). Using the standard quadratic equation $y = ax^2 + bx + c$, substitute each of the given values for x and y. Solve the system of three equations in a, b, and c. Substituting (0, 3): $c = 3$; (1, 4): $a + b + c = 4$; (−2, 7): $4a - 2b + c = 7$. Solving simultaneously, $a = 1$, $b = 0$, and $c = 3$; the equation is $y = x^2 + 3$.

LESSON **6-2** Circles

∴ Then	∴ Now	∴ Why?
• You graphed and wrote equations of parabolas.	• 1 Write equations of circles. 2 Graph circles.	• When an object is thrown into water, ripples move out from the center forming concentric circles. If the point where the object entered the water is assigned coordinates, each ripple can be modeled by an equation of a circle.

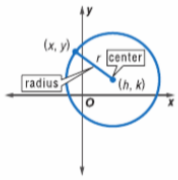
NewVocabulary
circle
center
radius

Mathematical Practices
4 Model with mathematics.

1 Equations of Circles

A **circle** is the set of all points in a plane that are equidistant from a given point in the plane, called the **center**. Any segment with endpoints at the center and a point on the circle is a **radius** of the circle.

Assume that (x, y) are the coordinates of a point on the circle at the right. The center is at (h, k), and the radius is r. You can find an equation of the circle by using the Distance Formula.



$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d$$ Distance Formula

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$ $(x_1, y_1) = (h, k)$, $(x_2, y_2) = (x, y)$, $d = r$

$$(x - h)^2 + (y - k)^2 = r^2$$ Square each side.

KeyConcept Equations of Circles

Standard Form of Equation	$x^2 + y^2 = r^2$	$(x - h)^2 + (y - k)^2 = r^2$
Center	(0, 0)	(h, k)
Radius	r	r

You can use the standard form of the equation of a circle to write an equation for a circle given the center and the radius or diameter.

Real-World Example 1 Write an Equation Given the Radius

DELIVERY **Appliances + More offers free delivery within 35 kilometers of the store. The Abu Dhabi store is located 100 kilometers north and 45 kilometers east of the corporate office. Write an equation to represent the delivery boundary of the Abu Dhabi store if the origin of the coordinate system is the corporate office.**

Since the corporate office is at (0, 0), the Abu Dhabi store is at (45, 100). The boundary of the delivery region is the circle centered at (45, 100) with radius 35 kilometers.

$(x - h)^2 + (y - k)^2 = r^2$ Equation of a circle

$(x - 45)^2 + (y - 100)^2 = 35^2$ $(h, k) = (45, 100)$ and $r = 35$

$(x - 45)^2 + (y - 100)^2 = 1225$ Simplify.

GuidedPractice

1. $(x + 3)^2 + (y + 4)^2 = 900$

1. **WI-FI** A certain wi-fi phone has a range of 30 kilometers in any direction. If the phone is 4 kilometers south and 3 kilometers west of headquarters, write an equation to represent the area within which the phone can operate via the Wi-Fi system.

1 Focus

VerticalAlignment

Before Lesson 6-2 Graph and wr equations of parabolas.

Lesson 6-2 Write equations of circles. Graph circles.

After Lesson 6-2 Write and graph equations of ellipses.

2 Teach

Scaffolding Questions

Have students read the **Why?** secti the lesson.

Ask:

- What are concentric circles? circ with a common center
- What part of a circle is the point where the rock hits the water? center
- Are all the circles similar? Explain Yes; they are the same shape, bu different sizes.

1 Equations of Circles

Example 1 shows how to write an equation for a circle for a real-world problem given the center and radius. **Example 2** shows how to write an equation for a circle given the graph of the circle. **Example 3** shows how to write an equation for a circle given the endpoints of a diameter.

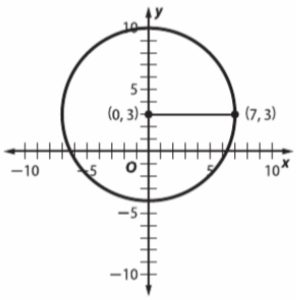
Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 **LANDSCAPING** The plan for a park puts the center of a circular pond of radius 0.6 kilometer at 2.5 kilometers east and 3.8 kilometers south of the park headquarters. Write an equation to represent the border of the pond, using the headquarters as the origin.
$(x - 2.5)^2 + (y + 3.8)^2 = 0.36$

2 Write an equation for the graph.
$x^2 + (y - 3)^2 = 49$



3 Write an equation for a circle if the endpoints of the diameter are at (2, 8) and (2, −2).
$(x - 2)^2 + (y - 3)^2 = 25$

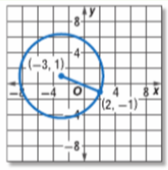
StudyTip

Center-Radius Form Standard form is sometimes referred to as *center-radius form* because the center and radius of the circle are apparent in the equation.

You can write the equation of a circle when you know the location of the center and a point on the circle.

Example 2 Write an Equation from a Graph

Write an equation for the graph.

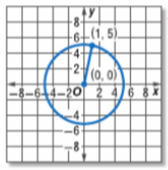


$(x - h)^2 + (y - k)^2 = r^2$	Standard form
$(2 + 3)^2 + (-1 - 1)^2 = r^2$	$x = 2, y = -1, h = -3, k = 1$
$(5)^2 + (-2)^2 = r^2$	Simplify.
$25 + 4 = r^2$	Evaluate the exponents.
$29 = r^2$	Add.

So, the equation of the circle is $(x + 3)^2 + (y - 1)^2 = 29$.

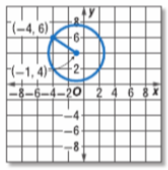
GuidedPractice **2B.** $(x + 1)^2 + (y - 4)^2 = 13$

2A.



$x^2 + y^2 = 26$

2B.



You can use the Midpoint and Distance Formulas when you know the endpoints of the radius or diameter of a circle.

Example 3 Write an Equation Given a Diameter

Write an equation for a circle if the endpoints of a diameter are at (7, 6) and (−1, −8).

Step 1 Find the center.

$(h, k) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	Midpoint Formula
$= \left(\frac{7 + (-1)}{2}, \frac{6 + (-8)}{2}\right)$	$(x_1, y_1) = (7, 6), (x_2, y_2) = (-1, -8)$
$= \left(\frac{6}{2}, \frac{-2}{2}\right)$	Add.
$= (3, -1)$	Simplify.

Step 2 Find the radius.

$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{(3 - 7)^2 + (-1 - 6)^2}$	$(x_1, y_1) = (7, 6), (x_2, y_2) = (3, -1)$
$= \sqrt{(-4)^2 + (-7)^2}$	Subtract.
$= \sqrt{65}$	Simplify.

The radius of the circle is $\sqrt{65}$ units, so $r^2 = 65$. Substitute h, k, and r^2 into the standard form of the equation of a circle. An equation of the circle is $(x - 3)^2 + (y + 1)^2 = 65$.

GuidedPractice **3.** $(x - 2)^2 + (y - 1)^2 = 17$

3. Write an equation for a circle if the endpoints of a diameter are at (3, −3) and (1, 5).

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If students need study tools,

Then have students write the Key Concepts for each conic section on a notecard.

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StudyTip

Axis of Symmetry Every diameter in a circle is an axis of symmetry. There are infinitely many axes of symmetry in a circle.

2 Graph Circles

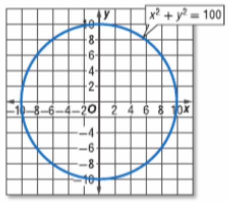
You can use symmetry to help you graph circles.

Example 4 Graph an Equation in Standard Form

Find the center and radius of the circle with equation $x^2 + y^2 = 100$. Then graph the circle.

- The center of the circle is at (0, 0), and the radius is 10.
- The table lists some integer values for x and y that satisfy the equation.
- Because the circle is centered at the origin, it is symmetric about the y-axis. Therefore, the points at (−6, 8), (−8, 6), and (−10, 0) lie on the graph.
- The circle is also symmetric about the x-axis, so the points (−6, −8), (−8, −6), (0, −10), (6, −8), and (8, −6) lie on the graph.
- Plot all of these points and draw the circle that passes through them.

x	y
0	10
6	8
8	6
10	0



GuidedPractice

4. Find the center and radius of the circle with equation $x^2 + y^2 = 81$. Then graph the circle. **See margin.**

Circles with centers that are not (0, 0) can be graphed by using translations. The graph of $(x - h)^2 + (y - k)^2 = r^2$ is the graph of $x^2 + y^2 = r^2$ translated h units horizontally and k units vertically.

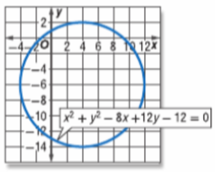
Example 5 Graph an Equation Not in Standard Form

Find the center and radius of the circle with equation $x^2 + y^2 - 8x + 12y - 12 = 0$. Then graph the circle.

Complete the squares.

$$x^2 + y^2 - 8x + 12y - 12 = 0$$
$$x^2 - 8x + ■ + y^2 + 12y + ■ = 12 + ■ + ■$$
$$x^2 - 8x + 16 + y^2 + 12y + 36 = 12 + 16 + 36$$
$$(x - 4)^2 + (y + 6)^2 = 64$$

The center of the circle is at (4, −6), and the radius is 8. The graph of $(x - 4)^2 + (y + 6)^2 = 64$ is the same as $x^2 + y^2 = 64$ translated 4 units to the right and down 6 units.



GuidedPractice

5. Find the center and radius of the circle with equation $x^2 + y^2 + 4x - 10y - 7 = 0$. Then graph the circle. **See margin.**

Teach with Tech

Blog Divide the class into two grou Have the first group write blog entr containing the coordinates of the ce of a circle and its radius. Have the second group of students reply to t postings with the equation of the ci Have students check each others' answers.

WatchOut!

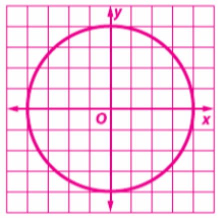
Preventing Errors Suggest that students draw sketches showing the circle and the endpoints of the diameter to check their work

2 Graph Circles

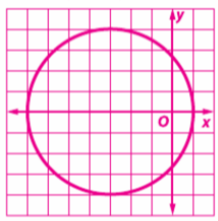
Example 4 shows how to find the center and radius of a circle and ho graph the circle given its equation i standard form. **Example 5** shows h to use translations to graph a circle an equation not in standard form.

Additional Examples

4 Find the center and radius of th circle with equation $x^2 + y^2 =$ 16. Then graph the circle. (0, 0); 4

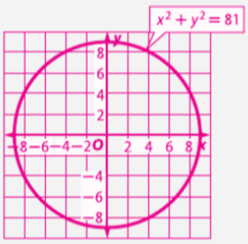


5 Find the center and radius of th circle with equation $x^2 + y^2 +$ − 7 = 0. Then graph the circle. (−3, 0); 4

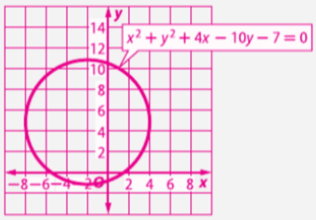


Additional Answers (Guided Practice)

4. center: (0, 0); radius: 9



5. center: (−2, 5); radius: 6



3 Practice

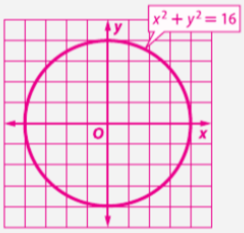
Formative Assessment

Use Exercises 1–11 to check for understanding.

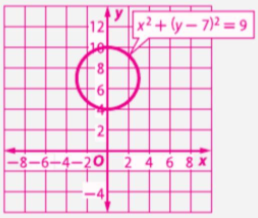
Use the chart at the bottom of the next page to customize assignments for your students.

Additional Answers

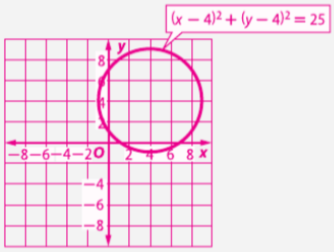
8. center: (0, 0); radius: 4



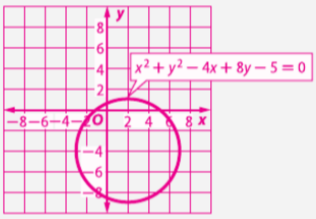
9. center: (0, 7); radius: 3



10. center: (4, 4); radius: 5



11. center: (2, −4); radius: 5



Check Your Understanding

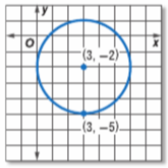
Example 1

1. **WEATHER** On average, the eye of a tornado is about 200 feet across. Suppose the center of the eye is at the point (72, 39). Write an equation to represent the boundary of the eye. $(x - 72)^2 + (y - 39)^2 = 10{,}000$

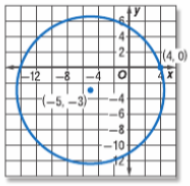
Write an equation for each circle given the center and radius.

2. center: (−2, −6), $r = 4$ units $(x + 2)^2 + (y + 6)^2 = 16$
3. center: (1, −5), $r = 3$ units $(x - 1)^2 + (y + 5)^2 = 9$

Example 2 Write an equation for each graph.

4. 

$(x - 3)^2 + (y + 2)^2 = 9$

5. 

$(x + 5)^2 + (y + 3)^2 = 90$

Example 3 Write an equation for each circle given the endpoints of a diameter.

6. (−1, −7) and (0, 0) $\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{7}{2}\right)^2 = \frac{25}{2}$
7. (4, −2) and (−4, −6) $x^2 + (y + 4)^2 = 20$

Examples 4–5 Find the center and radius of each circle. Then graph the circle. **8–11. See margin.**

8. $x^2 + y^2 = 16$
9. $x^2 + (y - 7)^2 = 9$
10. $(x - 4)^2 + (y - 4)^2 = 25$
11. $x^2 + y^2 - 4x + 8y - 5 = 0$

12. $(x - 4)^2 + (y - 9)^2 = 36$ **13.** $(x + 3)^2 + (y - 1)^2 = 16$ **14.** $(x + 7)^2 + (y + 3)^2 = 169$
15. $(x + 2)^2 + (y + 1)^2 = 81$ **16.** $(x - 1)^2 + y^2 = 15$ **17.** $x^2 + (y + 6)^2 = 35$

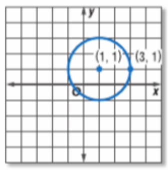
Practice and Problem Solving

Example 1 Write an equation for each circle given the center and radius.

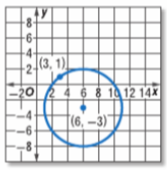
12. center: (4, 9), $r = 6$
13. center: (−3, 1), $r = 4$
14. center: (−7, −3), $r = 13$
15. center: (−2, −1), $r = 9$
16. center: (1, 0), $r = \sqrt{15}$
17. center: (0, −6), $r = \sqrt{35}$

18. **MODELING** The radar for an airport control tower is located at (5, 10) on a map. It can detect a plane up to 20 kilometers away. Write an equation for the outer limits of the detection area. $(x - 5)^2 + (y - 10)^2 = 400$

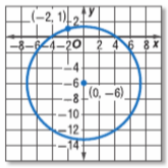
Example 2 Write an equation for each graph.

19. 

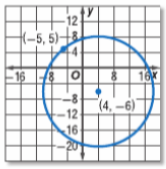
$(x - 1)^2 + (y - 1)^2 = 4$

20. 

$(x - 6)^2 + (y + 3)^2 = 25$

21. 

$x^2 + (y + 6)^2 = 53$

22. 

$(x - 4)^2 + (y + 6)^2 = 202$

Differentiated Instruction BL

Extension Does your school have a track? Students can be challenged to develop a scale model of the track using equations and graph paper. Scaffold the task as needed using the following:

- What lines and/or curves make up a track?
- How can we locate the center so it can be placed at (0, 0)?
- What measurements do we need?
- What scale factor should be used to fit the model on a sheet of graph paper?

If there is no track at the school, the necessary information can be found using the Internet.

Example 3 **Write an equation for each circle given the endpoints of a diameter.**

23. (2, 1) and (2, −4) **24.** (−4, −10) and (4, −10) **25.** (5, −7) and (−2, −9)

26. (−6, 4) and (4, 8) **27.** (2, −5) and (6, 3) **28.** (18, 11) and (−19, −13)

29. **LAWN CARE** A sprinkler waters a circular section of lawn.

a. Write an equation to represent the boundary of the sprinkler area if the endpoints of a diameter are at (−12, 16) and (12, −16). $x^2 + y^2 = 400$

b. What is the area of the lawn that the sprinkler waters? approximately 1256.64 units^2

30. **SPACE** Apollo 8 was the first manned spacecraft to orbit the Moon at an average altitude of 185 kilometers above the Moon's surface. Write an equation to model a single circular orbit of the command module if the endpoints of a diameter of the Moon are at (1740, 0) and (−1740, 0). Let the center of the Moon be at the origin of the coordinate system measured in kilometers. $x^2 + y^2 = 3{,}705{,}625$

31–46. See Chapter 6 Answer Appendix.

Examples 4–5 **Find the center and radius of each circle. Then graph the circle.**

31. $x^2 + y^2 = 75$ **32.** $(x - 3)^2 + y^2 = 4$

33. $(x - 1)^2 + (y - 4)^2 = 34$ **34.** $x^2 + (y - 14)^2 = 144$

35. $(x - 5)^2 + (y + 2)^2 = 16$ **36.** $x^2 + y^2 = 256$

37. $(x - 4)^2 + y^2 = \frac{8}{9}$ **38.** $\left(x + \frac{2}{3}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{16}{25}$

39. $x^2 + y^2 + 4x = 9$ **40.** $x^2 + y^2 - 6y + 8x = 0$

41. $x^2 + y^2 + 2x + 4y = 9$ **42.** $x^2 + y^2 - 3x + 8y = 20$

43. $x^2 + y^2 + 6y = -50 - 14x$ **44.** $x^2 - 18x + 53 = 18y - y^2$

45. $2x^2 + 2y^2 - 4x + 8y = 32$ **46.** $3x^2 + 3y^2 - 6y + 12x = 24$

23. $(x - 2)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{25}{4}$
24. $x^2 + (y + 10)^2 = 16$
25. $\left(x - \frac{3}{2}\right)^2 + (y + 8)^2 = \frac{53}{4}$
26. $(x + 1)^2 + (y - 6)^2 = 29$
27. $(x - 4)^2 + (y + 1)^2 = 20$
28. $\left(x + \frac{1}{2}\right)^2 + (y + 1)^2 = 486\frac{1}{4}$

B **47.** **SPACE** A satellite is in a circular orbit 25,000 miles above Earth.

a. Write an equation for the orbit of this satellite if the origin is at the center of Earth. Use 8000 miles as the diameter of Earth. $x^2 + y^2 = 841{,}000{,}000$

b. Draw a sketch of Earth and the orbit to scale. Label your sketch. See margin.

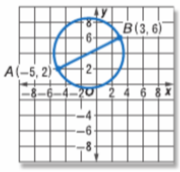
48. **SENSE-MAKING** Suppose an unobstructed radio station broadcast could travel 120 kilometers. Assume the station is centered at the origin.

a. Write an equation to represent the boundary of the broadcast area with the origin as the center. $x^2 + y^2 = 14{,}400$

b. If the transmission tower is relocated 40 kilometers east and 10 kilometers south of the current location, and an increased signal will transmit signals an additional 80 kilometers, what is an equation to represent the new broadcast area? $(x - 40)^2 + (y + 10)^2 = 40{,}000$

49a. $(x + 1)^2 + (y - 4)^2 = 36 + 16\sqrt{5}$

49. **GEOMETRY** Concentric circles are circles with the same center but different radii. Refer to the graph at the right where $\overline{AB}$ is a diameter of the circle.



a. Write an equation of the circle concentric with the circle at the right, with radius 4 units greater.

b. Write an equation of the circle concentric with the circle at the right, with radius 2 units less.

c. Graph the circles from parts a and b on the same coordinate plane. See margin.

49b. $(x + 1)^2 + (y - 4)^2 = 24 - 8\sqrt{5}$

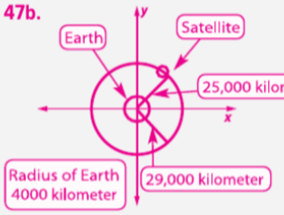
50. **EARTHQUAKES** A stadium is located about 35 kilometers west and 40 kilometers north of a city. Suppose an earthquake occurs with its epicenter about 55 kilometers from the stadium. Assume that the origin of a coordinate plane is located at the center of the city. Write an equation for the set of points that could be the epicenter of the earthquake. **50.** $(x + 35)^2 + (y - 40)^2 = 3025$

Teaching the Mathematical Practices

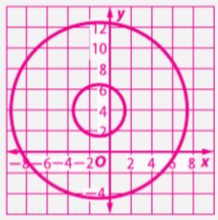
Sense-Making Mathematically proficient students start by explaini to themselves the meaning of a problem and looking for entry poin its solution. They analyze givens, constraints, relationships, and goal They check their answers to proble using a different method, and they continually ask themselves, "Does make sense?"

Additional Answers

47b.



49c.



Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	12–46, 62–88	13–45 odd, 68–71	12–46 even, 62–67, 72–88
OL Core	13–45 odd, 47–50, 51–61 odd, 62–88	12–46, 68–71	47–67, 72–88
BL Advanced	47–85, (optional: 86–88)		

Teaching the Mathematical Practices

Precision Mathematically proficient students try to use clear definitions in their reasoning, calculate accurately and efficiently, and make explicit use of definitions.

C **PRECISION** **Write an equation for the circle that satisfies each set of conditions.**

51. center (9, −8), passes through (19, 22) $(x-9)^2+(y+8)^2=1000$

52. center $\left(-\sqrt{15}, 30\right)$, passes through the origin $\left(x+\sqrt{15}\right)^2+(y-30)^2=915$

53. center at (8, −9), tangent to y-axis $(x-8)^2+(y+9)^2=64$

54. center at (2, 4), tangent to x-axis $(x-2)^2+(y-4)^2=16$

55. center in the first quadrant; tangent to $x = 5$, the x-axis, and the y-axis

56. center in the second quadrant; tangent to $y = 1$, $y = 5$, and the y-axis

55. $(x-2.5)^2+(y-2.5)^2=6.25$
56. $(x+2)^2+(y-3)^2=4$

57. **MULTIPLE REPRESENTATIONS** Graph $y=\sqrt{9-x^2}$ and $y=-\sqrt{9-x^2}$ on the same graphing calculator screen.

a. **Verbal** Describe the graph formed by the union of these two graphs. **circle**

b. **Algebraic** Write an equation for the union of the two graphs. $x^2+y^2=9$

c. **Verbal** Most graphing calculators cannot graph the equation $x^2 + y^2 = 49$ directly. Describe a way to use a graphing calculator to graph the equation. Then graph the equation. **See margin.**

d. **Analytical** Solve $(x-2)^2+(y+1)^2=4$ for y. Why do you need two equations to graph a circle on a graphing calculator?

e. **Verbal** Do you think that it is easier to graph the equation in part **d** using graph paper and a pencil or using a graphing calculator? Explain. **See students' work.**

Find the center and radius of each circle. Then graph the circle. **58–61. See Chapter 6 Answer Appendix.**

58. $x^2-12x+84=-y^2+16y$

59. $4x^2+4y^2+36y+5=0$

60. $\left(x+\sqrt{5}\right)^2+y^2-8y=9$

61. $x^2+2\sqrt{7}x+7+\left(y-\sqrt{11}\right)^2=11$

57d. $y=\pm\sqrt{4-(x-2)^2}-1$; **When you solve for y you must take the square root resulting in both a positive and negative answer, so you have to enter the positive equation as Y1 and the negative equation as Y2.**

H.O.T. Problems Use Higher-Order Thinking Skills

62. **ERROR ANALYSIS** Hana says that $(x-2)^2+(y+3)^2=36$ and $(x-2)+(y+3)=6$ are equivalent equations. Samira says that the equations are *not* equivalent. Is either of them correct? Explain your reasoning.
Samira; the square root of $(x-2)^2+(y+3)^2$ is not $(x-2)+(y+3)$.

63. **OPEN ENDED** Consider graphs with equations of the form $(x-3)^2+(y-a)^2=64$. Assign three different values for a, and graph each equation. Describe all graphs with equations of this form.
See students' work; circles with a radius of 8 and centers on the graph of $x = 3$.

64. **REASONING** Explain why the phrase "in a plane" is included in the definition of a circle. What would be defined if the phrase were *not* included?
If the phrase is not included, the figure would be a sphere.

65. **OPEN ENDED** Concentric circles have the same center, but most often, not the same radius. Write equations of two concentric circles. Then graph the circles. **See margin.**

66. **REASONING** Assume that (x, y) are the coordinates of a point on a circle. The center is at (h, k), and the radius is r. Find an equation of the circle by using the Distance Formula. **See margin.**

67. **WRITING IN MATH** The circle with equation $(x-a)^2+(y-b)^2=r^2$ lies in the first quadrant and is tangent to both the x-axis and the y-axis. Sketch the circle. Describe the possible values of a, b, and r. Do the same for a circle in Quadrants II, III, and IV. Discuss the similarities among the circles. **See Chapter 6 Answer Appendix.**

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Standardized Test Practice

68. **GRIDDED RESPONSE** Two circles, both with a radius of 6, have exactly one point in common. If *A* is a point on one circle and *B* is a point on the other circle, what is the maximum possible length for the line segment $\overline{AB}$? **24**

69. In a movie theatre, there are 20% more girls than boys. If there are 180 girls, how many more girls than boys are there? **A**

- **A** 30
- **B** 36
- **C** 90
- **D** 144

70. A AED 1,000 deposit is made at a bank that pays 2% compounded weekly. How much will you have in your account at the end of 10 years? **H**

- **F** AED 1,200.00
- **G** AED 1,218.99
- **H** AED 1,221.36
- **J** AED 1,224.54

71. The mean of six numbers is 20. If one of the numbers is removed, the average of the remaining numbers is 15. What is the number that was removed? **C**

- **A** 42
- **B** 43
- **C** 45
- **D** 48

75. $\left(1, \frac{7}{22}\right)$; $\sqrt{65}$ units **76.** $\left(-\frac{\sqrt{3}}{2}, 2\right)$; $\sqrt{271}$ units

Spiral Review

Graph each equation. (Lesson 6-2) **72–74. See Chapter 6 Answer Appendix.**

72. $y = -\frac{1}{2}(x-1)^2 + 4$

73. $4(x-2) = (y+3)^2$

74. $(y-8)^2 = -4(x-4)$

Find the midpoint of the line segment with endpoints at the given coordinates. Then find the distance between the points. (Lesson 6-1)

75. $\left(-3, -\frac{2}{11}\right), \left(5, \frac{9}{11}\right)$

76. $(2\sqrt{3}, -5), (-3\sqrt{3}, 9)$

77. $(2.5, 4), (-2.5, 2)$ **(0, 3); $\sqrt{29}$ units**

78. If *y* varies directly as *x* and $y = 8$ when $x = 6$, find *y* when $x = 15$. **20**

79. If *y* varies jointly as *x* and *z* and $y = 80$ when $x = 5$ and $z = 8$, find *y* when $x = 16$ and $z = 2$. **64**

80. If *y* varies inversely as *x* and $y = 16$ when $x = 5$, find *y* when $x = 20$. **4**

Evaluate each expression.

81. $\log_9 243$ **$\frac{5}{2}$**

82. $\log_2 \frac{1}{32}$ **−5**

83. $\log_3 \frac{1}{81}$ **−4**

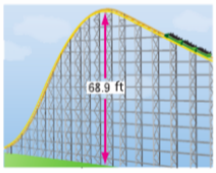
84. $\log_{10} 0.001$ **−3**

85. **AMUSEMENT PARKS** The velocity *v* in feet per second of a roller coaster at the bottom of a hill is related to the vertical drop *h* in feet and the velocity v_0 in feet per second of the coaster at the top of the hill by the formula $v_0 = \sqrt{v^2 - 64h}$.

a. Explain why $v_0 = v - 8\sqrt{h}$ is not equivalent to the given formula.

b. What velocity must the coaster have at the top of the hill to achieve a velocity of 38.1 feet per second at the bottom? **10.4 ft/s**

85a. The square root of a difference is not the difference of the square roots.



Skills Review

Solve each equation by completing the square.

86. $x^2 + 3x - 18 = 0$ **{−6, 3}**

87. $2x^2 - 3x - 3 = 0$ $\left\{\frac{3 \pm \sqrt{33}}{4}\right\}$

88. $x^2 + 2x + 6 = 0$ $\{-1 \pm i\sqrt{5}\}$

WatchOut!

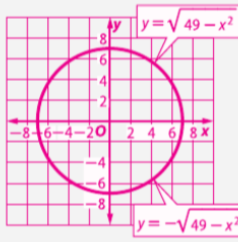
Error Analysis In Exercise 62, ask students if the square root o $(x-2)^2 + (y+3)^2$ is $(x-2) +$ $(y+3)$. If they think it is, ask them to check to see if the square root of $4^2 + 3^2$ is equal $4 + 3$.

4 Assess

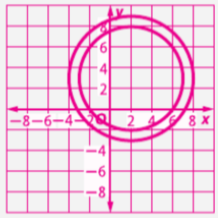
Ticket Out the Door Write five different equations for circles on separate sheets of paper. Make sev copies of each. Give one equation t each student. As students leave the room, ask them to tell you either th center or the radius of the circles formed by the equations.

Additional Answers

57c. Solve the equation for *y*: $y = \pm\sqrt{49 - x^2}$. Then graph positive and negative answer



65. Sample answer: $(x-2)^2 +$ $(y-3)^2 = 25$ and $(x-2)^2 +$ $(y-3)^2 = 36$



66. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} =$
$\sqrt{(x-h)^2 + (y-k)^2} =$
$(x-h)^2 + (y-k)^2 =$
This is the standard form of th equation of a circle.

DifferentiatedInstruction OL BL

Extension Write $x + 2y = 2$ and $x^2 + y^2 = 25$ on the board. Ask students to determine how many points the graphs of these two equations have in common. Then ask students to explain whether all lines intersect a circle in two points. There are two points that the graphs of the equations have in common: (−4, 3) and (4.8, −1.4). Not all lines intersect a circle in two points. A tangent line intersects a circle at one point. Some lines do not intersect a circle at all. Hence, a line can intersect a circle in 0, 1, or 2 points.

LESSON 6-3 Ellipses

Then

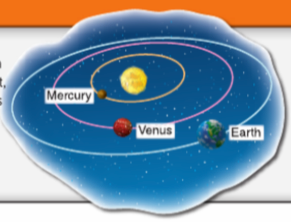
- You graphed and wrote equations for circles.

Now

1. Write equations of ellipses.
2. Graph ellipses.

Why?

- Mercury, like all of the planets of our solar system, does not orbit the Sun in a perfect circular path. At its farthest point, Mercury is about 69.2 million kilometers from the Sun. At its closest point, it is only about 45.9 million kilometers from the Sun. This orbit is in the shape of an ellipse with the Sun at a focus.



New Vocabulary

ellipse
foci
major axis
minor axis
center
vertices
co-vertices
constant sum

Mathematical Practices

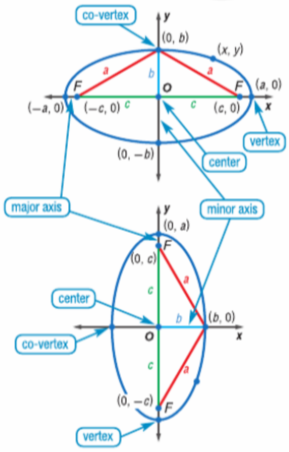
7 Look for and make use of structure.

1 Equations of Ellipses

An **ellipse** is the set of all points in a plane such that the sum of the distances from two fixed points is constant. These two points are called the **foci** of the ellipse.

Every ellipse has two axes of symmetry, the **major axis** and the **minor axis**. The axes are perpendicular at the **center** of the ellipse.

The foci of an ellipse always lie on the major axis. The endpoints of the major axis are the **vertices** of the ellipse and the endpoints of the minor axis are the **co-vertices** of the ellipse.



Key Concept Equations of Ellipses Centered at the Origin

Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
Orientation	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Length of Major Axis	$2a$ units	$2a$ units
Length of Minor Axis	$2b$ units	$2b$ units

There are several important relationships among the many parts of an ellipse.

- The length of the major axis, $2a$ units, equals the sum of the distances from the foci to any point on the ellipse.
- The values of a, b, and c are related by the equation $c^2 = a^2 - b^2$.
- The distance from a focus to either co-vertex is a units.

1 Focus

Vertical Alignment

Before Lesson 8-4 Graph and wr equations for circles.

Lesson 8-4 Write equations of ellipses. Graph ellipses.

After Lesson 8-4 Write and graph equations of hyperbolas.

2 Teach

Scaffolding Questions

Have students read the **Why?** secti of the lesson.

Ask:

- What makes up our solar system? The Sun and the group of objects orbiting around the Sun.
- Is Earth always the same distanc from the Sun? No; it is closest in January and farthest in July.
- What does it mean when it says t the Sun is at a focus? The Sun is the center of the ellipse. It is at a point off to one side.

Teach with Tech

Document Camera Work on the document camera to show students how to construct an ellipse using two pins, string, and a pencil. Repeat several times to show how the shape of the ellipse changes when the foci are closer and farther away from each other.

1 Equation of Ellipses

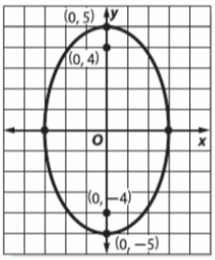
Example 1 shows how to write an equation for an ellipse shown on a graph. **Example 2** shows how to write an equation for an ellipse when all four vertices are given. **Example 3** shows how to write the equation of an ellipse that models a real-world situation.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Example

1 Write an equation for the ellipse.



$\frac{y^2}{25} + \frac{x^2}{9} = 1$

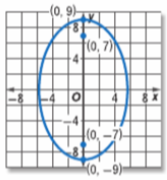
StudyTip

Major Axis In standard form, if the x^2-term has the greater denominator, then the major axis is horizontal. If the y^2-term has the greater denominator, then it is vertical.

The sum of the distances from the foci to any point on the ellipse, or the **constant sum**, must be greater than the distance between the foci.

Example 1 Write an Equation Given Vertices and Foci

Write an equation for the ellipse.



Step 1 Find the center.
The foci are equidistant from the center.
The center is at (0, 0).

Step 2 Find the value of a.
The vertices are (0, 9) and (0, −9), so the length of the major axis is 18.
The value of a is 18 ÷ 2 or 9, and $a^2 = 81$.

Step 3 Find the value of b.
We can use $c^2 = a^2 - b^2$ to find b.
The foci are 7 units from the center, so $c = 7$.

$c^2 = a^2 - b^2$ — Equation relating a, b, and c
$49 = 81 - b^2$ — $a = 9$ and $c = 7$
$b^2 = 32$ — Solve for b^2.

Step 4 Write the equation.
Because the major axis is vertical, a^2 goes with y and b^2 goes with x.
The equation for the ellipse is $\frac{y^2}{81} + \frac{x^2}{32} = 1$.

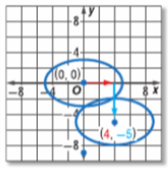
GuidedPractice $\frac{x^2}{16} + \frac{y^2}{12} = 1$

1. Write an equation for an ellipse with vertices at (−4, 0) and (4, 0) and foci at (2, 0) and (−2, 0).

Like other graphs, the graph of an ellipse can be translated. When the graph is translated h units right and k units up, the center of the translation is (h, k). This is equivalent to replacing x with $x - h$ and replacing y with $y - k$ in the parent function.

KeyConcept Equations of Ellipses Centered at (h, k)

Standard Form	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Orientation	horizontal	vertical
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Co-vertices	$(h, k \pm b)$	$(h \pm b, k)$



We can use this information to determine the equations for ellipses. The original ellipse at the right is horizontal and has a major axis of 10 units, so $a = 5$.

The length of the minor axis is 6 units, so $b = 3$.

The ellipse is translated 4 units right and 5 units down. So, the value of h is 4 and the value of k is −5.

The equation for the original ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

The equation for the translation is $\frac{(x-4)^2}{25} + \frac{(y+5)^2}{9} = 1$.

WatchOut!

Common Misconceptions Point out to students that when the Pythagorean Theorem is used to derive the standard form of the equation of an ellipse with its center at the origin and a horizontal major axis, it is a, not c, that is used to represent the length of the hypotenuse.

Focus on Mathematical Content

Ellipses Every ellipse has two axes of symmetry. The points at which the ellipse intersects its axes of symmetry determine two segments with endpoints on the ellipse. These segments are called the *major axis* and the *minor axis*. The intersection of the axes is the center of the ellipse.

You can also determine the equation for an ellipse if you are given all four vertices.

Example 2 Write an Equation Given the Lengths of the Axes

Write an equation for the ellipse with vertices at (6, −8) and (6, 4) and co-vertices at (3, −2) and (9, −2).

The x-coordinate is the same for both vertices, so the ellipse is vertical.

The center of the ellipse is at $\left(\frac{6+6}{2}, \frac{-8+4}{2}\right)$ or (6, −2).

The length of the major axis is 4 − (−8) or 12 units, so $a = 6$.

The length of the minor axis is 9 − 3 or 6 units, so $b = 3$.

The equation for the ellipse is $\frac{(y+2)^2}{36} + \frac{(x-6)^2}{9} = 1$. $a^2 = 36, b^2 = 9$

GuidedPractice

2. Write an equation for the ellipse with vertices at (−3, 8) and (9, 8) and co-vertices at (3, 12) and (3, 4). $\frac{(x-3)^2}{36} + \frac{(y-8)^2}{16} = 1$

Many real-world phenomena can be represented by ellipses.

Real-World Example 3 Write an Equation for an Ellipse

SPACE Refer to the application at the beginning of the lesson. Mercury's greatest distance from the Sun, or *aphelion*, is about 69.2 million kilometers. Mercury's closest distance, or *perihelion*, is about 45.9 million kilometers. The diameter of the Sun is about 1,400,129.3 kilometers. Use this information to determine an equation relating Mercury's elliptical orbit around the Sun in millions of kilometers.

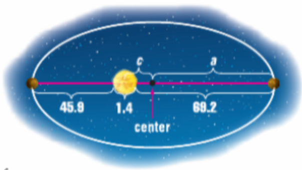
Understand We need to determine an equation representing Mercury's orbit around the Sun.

Plan Including the diameter of the Sun, the sum of the perihelion and aphelion equals the length on the major axis of the ellipse. We can use this information to determine the values of a, b, and c.

Solve Find the value of a.

The value of a is one half the length of the major axis.

$a = 0.5(69.2 + 45.9 + 1.4)$ or 58.23



Find the value of c.

The value of c is the distance from the center of the ellipse to the focus.

This distance is equal to a minus the perihelion and the radius of the Sun.

$c = 58.23 - 45.9 - 0.7$or 11.67

(continued on the next page)

Real-World Career

Aerospace Technician

Aerospace technicians work for NASA, helping engineers research and develop virtual reality and verbal communication between humans and computer systems. Although a bachelor's degree is desired, on-the-job training is available.

Source: NASA

Problem-Solving Tip

Sense-Making Draw a diagram when the problem situation involves spatial reasoning or geometric figures.

Additional Examples

2 Write an equation for the ellips with vertices at (−6, −2) and (4, −2) and co-vertices at (−1, −4) and (−1, 0).
$\frac{(x+1)^2}{25} + \frac{(y+2)^2}{4} = 1$

3 **SOUND** A listener is standing an elliptical room 150 meters wide and 320 meters long. Wh a speaker stands at one focus and whispers, the best place fo the listener to stand is at the other focus. Write an equation model this ellipse, assuming th major axis is horizontal and the center is at the origin.
$\frac{x^2}{25{,}600} + \frac{y^2}{5625} = 1$

WatchOut!

Preventing Errors Suggest that students make rough sketches of situations like the one in Example 3.

Teaching the Mathematical Practices

Sense-Making Mathematically proficient students analyze givens, constraints, relationships, and goal They can explain correspondences between equations, verbal descript tables, and graphs or draw diagran important features and relationship graph data, and search for regulari trends. Remind students that an elli has two axes of symmetry, the majo axis and the minor axis. Point out th the major axis is the longer of the two axes.

DifferentiatedInstruction OL BL

Extension Ask students to create a town on graph paper. The town should include the following components:

- An elliptical border (give equation)
- Two police headquarters located at the foci of the ellipse (give coordinates)
- Five other locations of your choice (give coordinates)
- Create a series of problems about police responding to calls at various locations. Which police headquarters should respond to the call? What other factors must be considered?

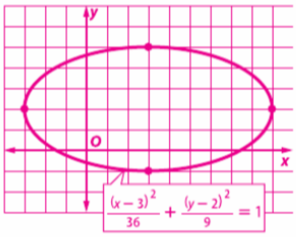
2 Graph Ellipses

Example 4 shows how to graph an equation for an ellipse that is not in standard form.

Additional Example

4 Find the coordinates of the center and foci and the lengths of the major and minor axes of an ellipse with equation $x^2 + 4y^2 - 6x - 16y - 11 = 0$. Then graph the ellipse.

center: (3, 2); foci: $(3\sqrt{3} + 3, 2)$, $(-3\sqrt{3} + 3, 2)$; major axis 12; minor axis: 6



Teaching the Mathematical Practices

Sense-Making Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?"

Find the value of b.

$$c^2 = a^2 - b^2 \quad \text{Equation relating } a, b, \text{ and } c$$
$$(11.67)^2 = (58.23)^2 - b^2 \quad c = 7.25 \text{ and } a = 36.185$$
$$136.1889 = 3390.7329 - b^2 \quad \text{Simplify.}$$
$$b^2 = 3254.544 \quad \text{Solve for } b^2.$$
$$b = 57.0486 \quad \text{Take the square root of each side.}$$

So, with the center of the orbit at the origin, the equation relating Mercury's orbit around the Sun can be modeled by $\frac{x^2}{3390.7329} + \frac{y^2}{3254.544} = 1$.

Check Use your answer to recalculate a, b, and c. Then determine the aphelion and perihelion based on your answer. Compare to the actual values.

Real-WorldLink

Earth's orbit around the Sun is nearly circular, with only about a 3% difference between perihelion and aphelion.

Source: *The Astronomer*

GuidedPractice 3. $\frac{x^2}{19.7} + \frac{y^2}{54.46} = 1$

3. **SPACE** Pluto's distance from the Sun is 4.44 billion kilometers at perihelion and about 7.38 billion kilometers at aphelion. Determine an equation relating Pluto's orbit around the Sun in billions of kilometers with the center of the horizontal ellipse at the origin.

2 Graph Ellipses When you are given an equation for an ellipse that is not in standard form, you can write it in standard form by completing the square for both x and y. Once the equation is in standard form, you can use it to graph the ellipse.

Example 4 Graph an Ellipse

Find the coordinates of the center and foci, and the lengths of the major and minor axes of an ellipse with equation $25x^2 + 9y^2 + 250x - 36y + 436 = 0$. Then graph the ellipse.

Step 1 Write in standard form. Complete the square for each variable to write this equation in standard form.

$$25x^2 + 9y^2 + 250x - 36y + 436 = 0 \quad \text{Original equation}$$
$$25x^2 + 250x + 9y^2 - 36y = -436 \quad \text{Associative Property}$$
$$25(x^2 + 10x) + 9(y^2 - 4y) = -436 \quad \text{Distributive Property}$$
$$25(x^2 + 10x + \blacksquare) + 9(y^2 - 4y + \blacksquare) = -436 + 25(\blacksquare) + 9(\blacksquare) \quad \text{Complete the squares.}$$
$$25(x^2 + 10x + 25) + 9(y^2 - 4y + 4) = -436 + 25(25) + 9(4) \quad 5^2 = 25 \text{ and } (-2)^2 = 4$$
$$25(x + 5)^2 + 9(y - 2)^2 = 225 \quad \text{Write as perfect squares.}$$
$$\frac{(x + 5)^2}{9} + \frac{(y - 2)^2}{25} = 1 \quad \text{Divide each side by 225.}$$

Step 2 Find the center.
$h = -5$ and $k = 2$, so the center of the ellipse is at $(-5, 2)$.

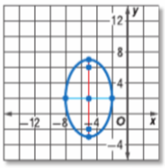
Step 3 Find the lengths of the axes and graph.
The ellipse is vertical.
$a^2 = 25$, so $a = 5$. $b^2 = 9$, so $b = 3$.
The length of the major axis is $2 \cdot 5$ or 10.
The length of the minor axis is $2 \cdot 3$ or 6.
The vertices are at $(-5, 7)$ and $(-5, -3)$.
The co-vertices are at $(-2, 2)$ and $(-8, 2)$.

DifferentiatedInstruction OL BL

Verbal/Linguistic Learners Remind students that the Moon travels around Earth in an elliptical orbit. The distance the Moon is from Earth varies from about 364,000 kilometers to about 405,000 kilometers. Ask students to research what shape the orbits are for other planets and their moons.

Step 4 Find the foci.
$c^2 = 25 - 9$ or 16, so $c = 4$.
The foci are at $(-5, 6)$ and $(-5, -2)$.

Step 5 Graph the ellipse.
Draw the ellipse that passes through the vertices and co-vertices.



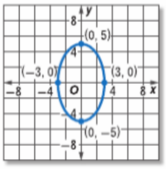
GuidedPractice

4. Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $x^2 + 4y^2 - 2x + 24y + 21 = 0$. Then graph the ellipse. **See margin.**

Check Your Understanding

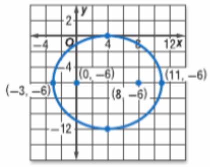
Example 1 Write an equation for each ellipse.

1.



$\frac{y^2}{25} + \frac{x^2}{9} = 1$

2.



$\frac{(x-4)^2}{49} + \frac{(y+6)^2}{36} = 1$

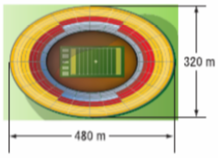
Example 2 Write an equation for an ellipse that satisfies each set of conditions.

3. vertices at $(-2, -6)$ and $(-2, 4)$, co-vertices at $(-5, -1)$ and $(1, -1)$ **3.** $\frac{(+1)^2}{25} + \frac{(+2)^2}{9} = 1$

4. vertices at $(-2, 5)$ and $(14, 5)$, co-vertices at $(6, 1)$ and $(6, 9)$ $\frac{(x-6)^2}{64} + \frac{(y-5)^2}{16} = 1$

Example 3

5. SENSE-MAKING An architectural firm sent a proposal to a city for building a coliseum, shown at the right.



a. Determine the values of a and b. $a = 240, b = 160$

b. Assuming that the center is at the origin, write an equation to represent the ellipse. **5b.** $\frac{x^2}{57{,}600} + \frac{y^2}{25{,}600} = 1$

c. Determine the coordinates of the foci. **about (179, 0) and (−179, 0)**

6. SPACE Earth's orbit is about 147.1 million kilometers at perihelion and about 152.1 million kilometers at aphelion. Determine an equation relating Earth's orbit around the Sun in millions of miles with the center of the horizontal ellipse at the origin. $\frac{x^2}{22590.09} + \frac{y^2}{22583.84} = 1$

Example 4 Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse. **7–10. See margin.**

7. $\frac{(y+1)^2}{64} + \frac{(x-5)^2}{28} = 1$

8. $\frac{(x+2)^2}{48} + \frac{(y-1)^2}{20} = 1$

9. $4x^2 + y^2 - 32x - 4y + 52 = 0$

10. $9x^2 + 25y^2 + 72x - 150y + 144 = 0$

3 Practice

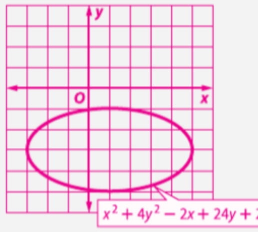
Formative Assessment

Use Exercises 1–10 to check for understanding.

Use the chart at the bottom of the n page to customize assignments for students.

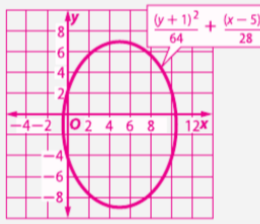
Additional Answers (Guided Pract

4. center $(1, -3)$; foci $(4.46, -3)$ ar $(-2.46, -3)$; major axis: 8; mino axis: 4

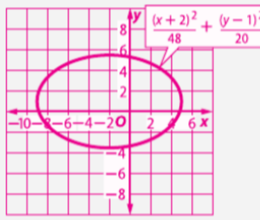


Additional Answers

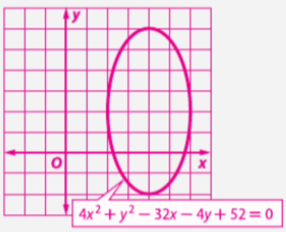
7. center $(5, -1)$; foci $(5, 5)$ and $(5, -7)$; major axis: 16; minor ax ≈ 10.58



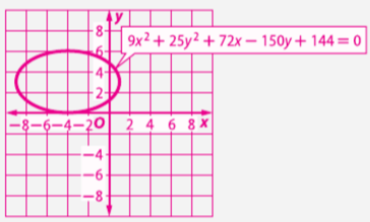
8. center $(-2, 1)$; foci $(3.29, 1)$ and $(-7.29, 1)$; major axis: ≈ 13.86; minor axis: ≈ 8.94



9. center $(4, 2)$; foci $(4, 5.46)$ and $(4, -1.46)$; major axis: 8; minor axis: 4



10. center $(-4, 3)$; foci $(0, 3)$ and $(-8, 3)$; major axis: 10; minor axis: 6



Multiple Representations

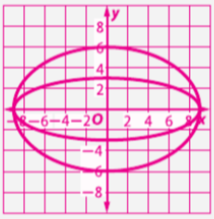
In Exercise 39, students use a graph and algebraic analysis to examine the relationship between the shape of an ellipse and the value of its eccentricity.

Teaching the Mathematical Practices

Modeling Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, analyze relationships mathematically to draw conclusions, and interpret their mathematical results in the context of a situation.

Additional Answers

39a.



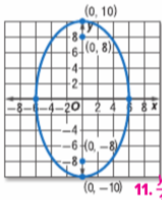
44. Sample answer: As an ellipse becomes more circular, the difference between a and b becomes smaller. This causes the value of c to become smaller since $c^2 = a^2 - b^2$. The value of $2c$ is the distance between the foci, so the foci get closer together.

45. For any point on an ellipse, the sum of the distances from that point to the foci is constant by the definition of an ellipse. So, if (2, 14) is on the ellipse, then the sum of the distances from it to the foci will be a certain value consistent with every other point on the ellipse. The distance between (−7, 2) and (2, 14) is $\sqrt{(-7-2)^2 + (2-14)^2}$ or 15. The distance between (18, 2) and (2, 14) is $\sqrt{(18-2)^2 + (2-14)^2}$ or 20. The sum of these two distances is 35.
The distance between (−7, 2) and (2, −10) is $\sqrt{(-7-2)^2 + [2-(-10)]^2}$ or 15. The distance between (18, 2) and (2, −10) is $\sqrt{(18-2)^2 + [2-(-10)]^2}$ or 20. The sum of these distances is also 35. Thus, (2, −10) also lies on the ellipse.

Practice and Problem Solving

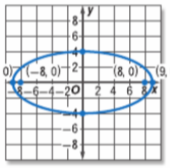
Example 1 Write an equation for each ellipse.

11.



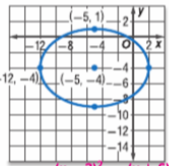
11. $\frac{y^2}{100} + \frac{x^2}{36} = 1$

12.



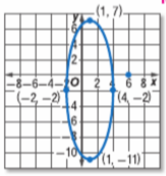
12. $\frac{x^2}{81} + \frac{y^2}{16} = 1$

13.



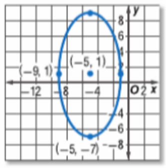
13. $\frac{(x+5)^2}{49} + \frac{(y+4)^2}{25} = 1$

14.



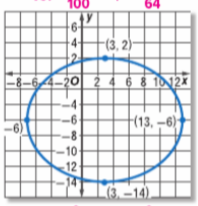
$\frac{(y+2)^2}{81} + \frac{(x-1)^2}{9} = 1$

15.



$\frac{(y-1)^2}{64} + \frac{(x+5)^2}{16} = 1$

16.



16. $\frac{(x-3)^2}{100} + \frac{(y+6)^2}{64} = 1$

Example 2 Write an equation for an ellipse that satisfies each set of conditions.

17. vertices at (−6, 4) and (12, 4), co-vertices at (3, 12) and (3, −4)

18. vertices at (−1, 11) and (−1, 1), co-vertices at (−4, 6) and (2, 6)

19. center at (−2, 6), vertex at (−2, 16), co-vertex at (1, 6)

20. center at (3, −4), vertex at (8, −4), co-vertex at (3, −2)

21. vertices at (4, 12) and (4, −4), co-vertices at (1, 4) and (7, 4)

22. vertices at (−11, 2) and (−1, 2), co-vertices at (−6, 0) and (−6, 4)

17. $\frac{(x-3)^2}{81} + \frac{(y-4)^2}{64} = 1$

18. $\frac{(y-6)^2}{25} + \frac{(x+1)^2}{9} = 1$

19. $\frac{(y-6)^2}{100} + \frac{(x+2)^2}{9} = 1$

20. $\frac{(x-3)^2}{25} + \frac{(y+4)^2}{4} = 1$

21. $\frac{(y-4)^2}{64} + \frac{(x-4)^2}{9} = 1$

22. $\frac{(x+6)^2}{25} + \frac{(y-2)^2}{4} = 1$

Example 3

23. **MODELING** The opening of a tunnel in the mountains can be modeled by semiellipses, or halves of ellipses. If the opening is 14.6 meters wide and 8.6 meters high, determine an equation to represent the opening with the center at the origin. $\frac{y^2}{73.96} + \frac{x^2}{53.29} = 1$

Example 4 Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse. **24–31. See Chapter 6 Answer Appendix.**

24. $\frac{(x-3)^2}{36} + \frac{(y-2)^2}{128} = 1$

25. $\frac{(x+6)^2}{50} + \frac{(y-3)^2}{72} = 1$

26. $\frac{x^2}{27} + \frac{(y-5)^2}{64} = 1$

27. $\frac{(x+4)^2}{16} + \frac{y^2}{75} = 1$

28. $3x^2 + y^2 - 6x - 8y - 5 = 0$

29. $3x^2 + 4y^2 - 18x + 24y + 3 = 0$

30. $7x^2 + y^2 - 56x + 6y + 93 = 0$

31. $3x^2 + 2y^2 + 12x - 20y + 14 = 0$

32. **SPACE** Like the planets, Halley's Comet travels around the Sun in an elliptical orbit. The aphelion is 5283.3 million kilometers and the perihelion is 88.3 million kilometers. Determine an equation relating the comet's orbit around the Sun in millions of kilometers with the center of the horizontal ellipse at the origin. $\frac{x^2}{2{,}786{,}629.3} + \frac{y^2}{181{,}548.8} = 1$

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	11–32, 40, 41, 44–67	11–31 odd, 47–50	12–32 even, 40, 41, 44–46, 51–67
OL Core	11–31 odd, 33–41, 44–67	11–32, 47–50	33–41, 44–46, 51–67
BL Advanced	33–64, (optional: 65–67)		

B **Write an equation for an ellipse that satisfies each set of conditions.**

33. center at (−5, −2), focus at (−5, 2), co-vertex at (−8, −2) $\frac{(y+2)^2}{25}+\frac{(x+5)^2}{9}=1$

34. center at (4, −3), focus at (9, −3), co-vertex at (4, −5) **34.** $\frac{(x-4)^2}{29}+\frac{(y+3)^2}{4}=1$

35. foci at (−2, 8) and (6, 8), co-vertex at (2, 10) $\frac{(x-2)^2}{20}+\frac{(y-8)^2}{4}=1$

36. foci at (4, 4) and (4, 14), co-vertex at (0, 9) **36.** $\frac{(y-9)^2}{41}+\frac{(x-4)^2}{16}=1$

37. GOVERNMENT The Oval Office is located in the West Wing of the White House. It is an elliptical shaped room used as the main office by the President of the United States. The long axis is 10.9 meters long and the short axis is 8.8 meters long. Write an equation to represent the outer walls of the Oval Office. Assume that the center of the room is at the origin. $\frac{x^2}{29.7025}+\frac{y^2}{19.36}=1$

38. SOUND A whispering gallery is an elliptical room in which a faint whisper at one focus cannot be heard by other people in the room, but can easily be heard by someone at the other focus. Suppose an ellipse is 121.9 meters long and 36.6 meters wide. What is the distance between the foci? **about 116.3 m**

39. MULTIPLE REPRESENTATIONS The *eccentricity* of an ellipse measures how circular the ellipse is. **39b. Sample answer: The first graph is more circular than the second graph.**

a. Graphical Graph $\frac{x^2}{81}+\frac{y^2}{36}=1$ and $\frac{x^2}{81}+\frac{y^2}{9}=1$ on the same graph. **See margin.**

b. Verbal Describe the difference between the two graphs.

c. Algebraic The eccentricity of an ellipse is $\frac{c}{a}$. Find the eccentricity for each.

d. Analytical Make a conjecture about the relationship between the value of an ellipse's eccentricity and the shape of the ellipse as compared to a circle.

39c. first graph: 0.745; second graph: 0.943

39d. Sample answer: The closer the eccentricity is to 0, the more circular the ellipse.

H.O.T. Problems Use Higher-Order Thinking Skills

C **40. ERROR ANALYSIS** Shaima and Maha are determining the equation for an ellipse with foci at (−4, −11) and (−4, 5) and co-vertices at (2, −3) and (−10, −3). Is either of them correct? Explain your reasoning.

Shaima	Maha
$\frac{(x-4)^2}{64}+\frac{(y+3)^2}{36}=1$	$\frac{(x+4)^2}{100}+\frac{(y+3)^2}{36}=1$

40. Sample answer: Neither; both are showing horizontal ellipses and answer is vertical.

41. OPEN ENDED Write an equation for an ellipse with a focus at the origin. **41. Sample answer:** $\frac{(x+4)^2}{40}+\frac{y^2}{24}=1$

42. CHALLENGE When the values of a and b are equal, an ellipse is a circle. Use this information and your knowledge of ellipses to determine the formula for the area of an ellipse in terms of a and b. $A=\pi ab$

43. CHALLENGE Determine an equation for an ellipse with foci at $(2, \sqrt{6})$ and $(2, -\sqrt{6})$ that passes through $(3, \sqrt{6})$. $\frac{y^2}{9}+\frac{(x-2)^2}{3}=1$

44. ARGUMENTS What happens to the location of the foci as an ellipse becomes more circular? Explain your reasoning. **See margin.**

45. REASONING An ellipse has foci at (−7, 2) and (18, 2). If (2, 14) is a point on the ellipse, show that (2, −10) is also a point on the ellipse. **See margin.**

46. WRITING IN MATH Explain why the domain is $\{x \mid -a \le x \le a\}$ and the range is $\{y \mid -b \le y \le b\}$ for an ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **See Chapter 6 Answer Appendix.**

Teaching the Mathematical Practices

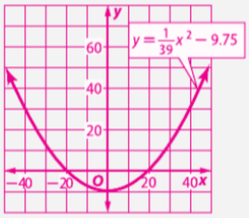
Arguments Mathematically proficie students understand and use state assumptions, definitions, and previ established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of t conjectures. And they are able to analyze situations by breaking them into cases, and can recognize and us counterexamples.

4 Assess

Yesterday's News Have students write how yesterday's lesson helped them with writing and graphing equations for ellipses in today's lesson.

Additional Answers

54c.



54d. Sample answer: The equation in meters is easier to graph because the numbers have fewer decimal places.

Standardized Test Practice

47. Multiply. **B**

$$(2 + 3i)(4 + 7i)$$

A $8 + 21i$ C $-6 + 10i$
B $-13 + 26i$ D $13 + 12i$

48. The average lifespan of American women has been tracked, and the model for the data is $y = 0.2t + 73$, where $t = 0$ corresponds to 1960. What is the meaning of the y-intercept? **H**

F In 2007, the average lifespan was 60.
G In 1960, the average lifespan was 58.
H In 1960, the average lifespan was 73.
J The lifespan is increasing 0.2 years every year.

49. GRIDDED RESPONSE If we decrease a number by 6 and then double the result, we get 5 less than the number. What is the number? **7**

50. SAT/ACT The length of a rectangular prism is one inch greater than its width. The height is three times the length. Find the volume of the prism. **C**

A $3x^3 + x^2 + 3x$
B $x^3 + x^2 + x$
C $3x^3 + 6x^2 + 3x$
D $3x^3 + 3x^2 + 3x$
E $3x^3 + 3x^2$

Spiral Review

Write an equation for the circle that satisfies each set of conditions. (Lesson 6-3)

51. center $(8, -9)$, passes through $(21, 22)$ $(x - 8)^2 + (y + 9)^2 = 1130$

52. center at $(4, 2)$, tangent to x-axis $(x - 4)^2 + (y - 2)^2 = 4$

53. center in the second quadrant; tangent to $y = -1$, $y = 9$, and the y-axis $(x + 5)^2 + (y - 4)^2 = 25$

54. ENERGY A parabolic mirror is used to collect solar energy. The mirrors reflect the rays from the Sun to the focus of the parabola. The focus of a particular mirror is 9.75 feet above the vertex, and the latus rectum is 39 feet long. (Lesson 6-2)

a. Assume that the focus is at the origin. Write an equation for the parabola formed by the mirror. $y = \frac{1}{39}x^2 - 9.75$

b. One foot is exactly 0.3048 meter. Rewrite the equation for the mirror in meters. $y = \frac{1}{11.8872}x^2 - 2.9718$

c. Graph one of the equations for the mirror. **See margin.**

d. Which equation did you choose to graph? Explain why. **See margin.**

Simplify each expression.

55. $\frac{6}{d^2 + 4d + 4} + \frac{5}{d + 2}$ $\frac{5d + 16}{(d + 2)^2}$

56. $\frac{a}{a^2 - a - 20} + \frac{2}{a + 4}$ $\frac{3a - 10}{(a - 5)(a + 4)}$

57. $\frac{x}{x + 1} + \frac{3}{x^2 - 4x - 5}$ $\frac{x^2 - 5x + 3}{(x - 5)(x + 1)}$

Solve each equation.

58. $\log_{10}(x^2 + 1) = 1$ ± 3

59. $\log_b 64 = 3$ **4**

60. $\log_b 121 = 2$ **11**

Simplify. **61.** $15a^3b^3 - 30a^4b^3 + 15a^5b^6$

61. $-5ab^2(-3a^2b + 6a^3b - 3a^4b^4)$

62. $2xy(3xy^3 - 4xy + 2y^4)$ $6x^2y^4 - 8x^2y^2 + 4xy^5$

63. $(4x^2 - 3y^2 + 5xy) - (8xy + 3y^2)$ $4x^2 - 3xy - 6y^2$

64. $(10x^2 - 3xy + 4y^2) - (3x^2 + 5xy)$ $7x^2 - 8xy + 4y^2$

Skills Review

Write an equation of the line passing through each pair of points. **65.** $y = -\frac{4}{5}x + \frac{17}{5}$ **67.** $y = -\frac{3}{5}x + \frac{16}{5}$

65. $(-2, 5)$ and $(3, 1)$

66. $(7, 1)$ and $(7, 8)$ $x = 7$

67. $(-3, 5)$ and $(2, 2)$

Differentiated Instruction OL BL

Extension Tell students that the area of an ellipse can be found using the formula $A = \pi ab$. Have students find the approximate area of the first ellipse on page 616 by counting the number of squares on the grid inside the ellipse and multiplying by 4 since each square equals 4 square units. Next, have students find the area of the same ellipse to the nearest tenth using the formula. Discuss with students how this formula is similar to the formula for the area of a circle. Sample answer: $A =$ about 160 square units; answer: $A = 159.9$ square units

CHAPTER 6

Mid-Chapter Quiz

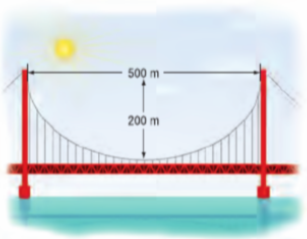
Lessons 6-1 through 6-4

Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola. (Lesson 6-2) **6–9. See margin.**

1. $y = 3x^2 - 12x + 21$
2. $x - 2y^2 = 4y + 6$
3. $y = \frac{1}{2}x^2 + 12x - 8$
4. $x = 3y^2 + 5y - 9$

$y = \frac{2}{625}x^2$

5. **BRIDGES** Write an equation of a parabola to model the shape of the suspension cable of the bridge shown. Assume that the origin is at the lowest point of the cables. (Lesson 6-2)



Identify the coordinates of the vertex and focus, the equation of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum. (Lesson 6-2) **11, 12. See Chapter 6 Answer Appendix.**

6. $y = x^2 + 6x + 5$
7. $x = -2y^2 + 4y + 1$
8. Find the center and radius of the circle with equation $(x - 1)^2 + y^2 = 9$. Then graph the circle. (Lesson 6-3) **See Chapter 6 Answer Appendix.**
9. Write an equation for a circle that has center at (3, −2) and passes through (3, 4). (Lesson 6-3) **$(x - 3)^2 + (y + 2)^2 = 36$**
10. Write an equation for a circle if the endpoints of a diameter are at (8, 31) and (32, 49). (Lesson 6-3) **$(x - 20)^2 + (y - 40)^2 = 225$**
11. **MULTIPLE CHOICE** What is the radius of the circle with equation $x^2 + 2x + y^2 + 14y + 34 = 0$? (Lesson 6-3) **B**
    - A 2
    - B 4
    - C 8
    - D 16

Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with the given equation. Then graph the ellipse. (Lesson 6-4) **17–19. See Chapter 6 Answer Appendix.**

12. $\frac{(x+4)^2}{16} + \frac{(y-2)^2}{9} = 1$
13. $\frac{(x-1)^2}{20} + \frac{(y+2)^2}{4} = 1$
14. $4y^2 + 9x^2 + 16y - 90x + 205 = 0$
15. **MULTIPLE CHOICE** Which equation represents an ellipse with endpoints of the major axis at (−4, 10) and (−4, −6) and foci at about (−4, 7.3) and (−4, −3.3)? (Lesson 6-4) **H**
    - F $\frac{(x-2)^2}{36} + \frac{(y+4)^2}{64} = 1$
    - G $\frac{(x+4)^2}{64} + \frac{(y-2)^2}{36} = 1$
    - H $\frac{(y-2)^2}{64} + \frac{(x+4)^2}{36} = 1$
    - J $\frac{(x-2)^2}{64} + \frac{(y+4)^2}{36} = 1$

Formative Assessment

Use the Mid-Chapter Quiz to assess students' progress in the first half o chapter.

For problems answered incorrectly, have students review the lessons indicated in parentheses.

FOLDABLES StudyOrganiz

Dinah Zike's Foldables®

Before students complete the Mid-Ch Quiz, encourage them to review the information for Lessons 8-1 through in their Foldables.

Additional Answers

6. $y = 3(x - 2)^2 + 9$; (2, 9); $x = 2$; opens up
7. $x = 2(y + 1)^2 + 4$; (4, −1); $y = -1$; opens to the right
8. $y = \frac{1}{2}(x + 12)^2 - 80$; (−12, −8 $x = -12$; opens up
9. $x = 3\left(y + \frac{5}{6}\right)^2 - 11\frac{1}{12}$; $\left(-\frac{133}{12}, -\frac{5}{6}\right)$; $y = -\frac{5}{6}$; opens t the right

1 Focus

VerticalAlignment

Before Lesson 6-4 Graph and analyze equations of ellipses.

Lesson 6-4 Write equations of hyperbolas. Graph hyperbolas.

After Lesson 6-4 Identify the conic section from a given equation.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- It takes about 76 years for Halley's comet to orbit the sun, and it was last seen in 1986. Approximately when will it appear again? 2062
- How is a hyperbola unlike other conic sections? A hyperbola has two branches.
- Of the conic sections studied so far, which is the only one that could be a function? a parabola with a parent function of $y = x^2$ that has a vertical axis of symmetry
- Could the graph of a hyperbola ever be a function? No.

LESSON 6-4 Hyperbolas

Then

- You graphed and analyzed equations of ellipses.

Now

1. Write equations of hyperbolas.
2. Graph hyperbolas.

Why?

- Because Halley's Comet travels around the Sun in an elliptical path, it reappears in our sky. Other comets pass through our sky only once. Many of these comets travel in paths that resemble hyperbolas.

NewVocabulary
hyperbola
transverse axis
conjugate axis
foci
vertices
co-vertices
constant difference

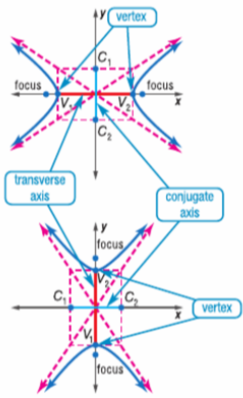
Mathematical Practices
6 Attend to precision.

1 Equations of Hyperbolas Similar to an ellipse, a **hyperbola** is the set of all points in a plane such that the absolute value of the differences of the distances from the foci is constant.

Every hyperbola has two axes of symmetry, the **transverse axis** and the **conjugate axis**. The axes are perpendicular at the center of the hyperbola.

The **foci** of a hyperbola always lie on the transverse axis. The **vertices** are the endpoints of the transverse axis. The **co-vertices** are the endpoints of the conjugate axis.

As a hyperbola recedes from the center, both halves approach asymptotes.



KeyConcept Equations of Hyperbolas Centered at the Origin

Standard Form	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Orientation	horizontal	vertical
Foci	$(\pm c, 0)$	$(0, \pm c)$
Length of Transverse Axis	$2a$ units	$2a$ units
Length of Conjugate Axis	$2b$ units	$2b$ units
Equations of Asymptotes	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$

As with ellipses, there are several important relationships among the parts of hyperbolas.

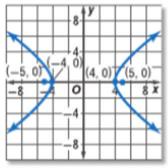
- There are two axes of symmetry.
- The values of a, b, and c are related by the equation $c^2 = a^2 + b^2$.

Math History Link

Hypatia (415–370 B.C.)
Hypatia was a mathematician, scientist, and philosopher in Alexandria, Egypt. She is considered the first woman to write on mathematical topics. Hypatia edited the book *On the Conics of Apollonius*, adding her own problems and examples to clarify the topic for her students. This book developed the ideas of hyperbolas, parabolas, and ellipses.

Example 1 Write an Equation Given Vertices and Foci

Write an equation for the hyperbola shown in the graph.



Step 1 Find the center.

The vertices are equidistant from the center.
The center is at (0, 0).

Step 2 Find the values of a, b, and c.

The value of a is the distance between a vertex and the center, or 4 units.

The value of c is the distance between a focus and the center, or 5 units.

$c^2 = a^2 + b^2$ Equation relating a, b, and c for a hyperbola

$5^2 = 4^2 + b^2$ $c = 5$ and $a = 3$

$9 = b^2$ Subtract 4^2 from each side.

Step 3 Write the equation.

The transverse axis is horizontal, so the equation is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Guided Practice

1. Write an equation for a hyperbola with vertices at (6, 0) and (−6, 0) and foci at (8, 0) and (−8, 0). $\frac{x^2}{36} - \frac{y^2}{28} = 1$

Hyperbolas can also be determined using the equations of their asymptotes.

Example 2 Write an Equation Given Asymptotes

The asymptotes for a vertical hyperbola are $y = \frac{5}{3}x$ and $y = -\frac{5}{3}x$ and the vertices are at (0, 5) and (0, −5). Write the equation for the hyperbola.

Step 1 Find the center.

The vertices are equidistant from the center.
The center of the hyperbola is at (0, 0).

Step 2 Find the values of a and b.

The hyperbola is vertical, so $a = 5$.
From the asymptotes, $b = 3$.
The value of c is not needed.

Step 3 Write the equation.

The equation for the hyperbola is $\frac{y^2}{25} - \frac{x^2}{9} = 1$.

Guided Practice 2. $\frac{x^2}{81} - \frac{y^2}{49} = 1$

2. The asymptotes for a horizontal hyperbola are $y = \frac{7}{9}x$ and $y = -\frac{7}{9}x$. The vertices are (9, 0) and (−9, 0). Write an equation for the hyperbola.

Reading Math

Standard Form In the standard form of a hyperbola, the squared terms are subtracted. For an ellipse, they are added.

1 Equations of Hyperbolas

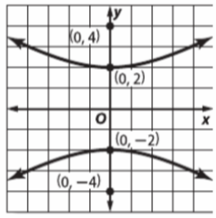
Example 1 shows how to write an equation for a hyperbola given the graph of the hyperbola. **Example 2** shows how to write an equation for hyperbola given the asymptotes an vertices.

Formative Assessment

Use the Guided Practice exercises a each example to determine student understanding of concepts.

Additional Examples

1 Write an equation for the hyperbola shown in the graph.



$\frac{y^2}{4} - \frac{x^2}{12} = 1$

2 The asymptotes for a vertical hyperbola are $y = 3x$ and $y = -3x$ and the vertices are a (0, $2\sqrt{10}$) and (0, $-2\sqrt{10}$). Wri the equation for the hyperbola

$\frac{y^2}{36} - \frac{x^2}{4} = 1$

Differentiated Instruction AL OL

If some students think that a hyperbola has the shape of two parabolas,

Then explain that this is not true, and encourage students to draw parabolas on transparent paper and place them over hyperbolas to see that the shapes of the curves are different.

Focus on Mathematical Content

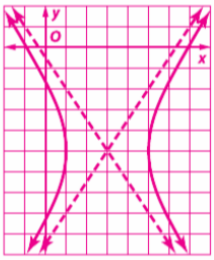
Hyperbolas Every hyperbola has *asymptotes*, or lines which the hyperbola branches approach. Every hyperbola has two axes of symmetry. The endpoints of the *transverse axis*, length $2a$, are the vertices of the hyperbola. The *conjugate axis* of length $2b$ is perpendicular to the transverse axis and passes through the midpoint of the transverse axis (the center). The distance from the center to a focus is c. For a hyperbola, $a^2 + b^2 = c^2$.

2 Graphs of Hyperbolas

Example 3 shows how to use the asymptotes to graph a hyperbola from an equation in standard form. **Example 4** shows how to graph a hyperbola that models a real-world situation.

Additional Example

3 Graph $\frac{(x-3)^2}{4} - \frac{(y+5)^2}{9} = 1$. Identify the vertices, foci, and asymptotes. vertices: $(5, -5)$, $(1, -5)$; foci: $(3 + \sqrt{13}, -5)$, $(3 - \sqrt{13}, -5)$; asymptotes: $y + 5 = \frac{3}{2}(x - 3)$, $y + 5 = -\frac{3}{2}(x - 3)$



2 Graphs of Hyperbolas

Hyperbolas can be translated in the same manner as the other conic sections.

KeyConcept Equations of Hyperbolas Centered at (h, k)

Standard Form	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Orientation	horizontal	vertical
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Co-vertices	$(h, k \pm b)$	$(h \pm b, k)$
Equations of Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{a}{b}(x - h)$

StudyTip

Calculator You can graph a hyperbola on a graphing calculator by solving for y, and then graphing the two equations on the same screen.

Example 3 Graph a Hyperbola

Graph $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{16} = 1$. Identify the vertices, foci, and asymptotes.

Step 1 Find the center. The center is at $(3, -2)$.

Step 2 Find a, b, and c. From the equation, $a^2 = 4$ and $b^2 = 16$, so $a = 2$ and $b = 4$.

$c^2 = a^2 + b^2$ — Equation relating a, b, and c for a hyperbola

$c^2 = 2^2 + 4^2$ — $a = 2$, $b = 4$

$c^2 = 20$ — Simplify.

$c = \sqrt{20}$ or about 4.47 — Take the square root of each side.

Step 3 Identify the vertices and foci. The hyperbola is horizontal and the vertices are 2 units from the center, so the vertices are at $(1, -2)$ and $(5, -2)$. The foci are about 4.47 units from the center. The foci are at $(-1.47, -2)$ and $(7.47, -2)$.

Step 4 Identify the asymptotes.

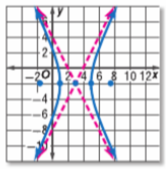
$y - k = \pm\frac{b}{a}(x - h)$ — Equation for asymptotes of a horizontal hyperbola

$y - (-2) = \pm\frac{4}{2}(x - 3)$ — $a = 2$, $b = 4$, $h = 3$, and $k = -2$

The equations for the asymptotes are $y = 2x - 8$ and $y = -2x + 4$.

Step 5 Graph the hyperbola. The hyperbola is symmetric about the transverse and conjugate axes. Use this symmetry to plot additional points for the hyperbola.

Use the asymptotes as a guide to draw the hyperbola that passes through the vertices and the other points.

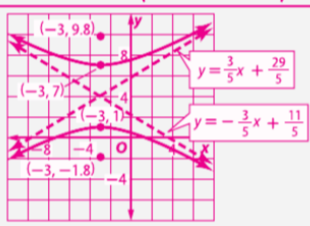


GuidedPractice 3. See margin.

3. Graph $\frac{(y-4)^2}{9} - \frac{(x+3)^2}{25} = 1$. Identify the vertices, foci, and asymptotes.

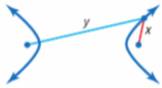
Additional Answer (Guided Practice)

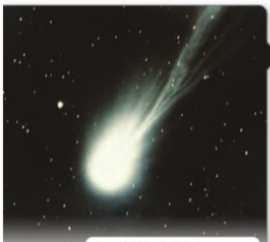
3.



In the equation for any hyperbola, the value of $2a$ represents the **constant difference**. This is the absolute value of the difference between the distances from any point on the hyperbola to the foci of the hyperbola.

Any point on the hyperbola at the right will have the same constant difference, $|y - x|$ or $2a$.





Real-WorldLink

Halley's Comet becomes visible to the unaided eye about every 76 years as it nears the Sun.

Source: NASA

Real-World Example 4 Write an Equation of a Hyperbola

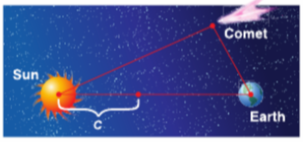
SPACE Earth and the Sun are 146 million kilometers apart. A comet follows a path that is one branch of a hyperbola. Suppose the comet is 30 million kilometers farther from the Sun than from Earth. Determine the equation of the hyperbola centered at the origin for the path of the comet.

Understand We need to determine the equation for the hyperbola.

Plan Find the center and the values of a and b. Once we have this information, we can determine the equation.

Solve The foci are Earth and the Sun, with the origin between them.

The value of c is $146 \div 2$ or 73.



The difference of the distances from the comet to each body is 30. Therefore, a is $30 \div 2$ or 15 million kilometers.

$c^2 = a^2 + b^2$ — Equation relating a, b, and c for a hyperbola

$73^2 = 15^2 + b^2$ — $a = 15$ and $c = 73$

$5104 = b^2$ — Simplify.

The equation of the hyperbola is $\frac{x^2}{225} - \frac{y^2}{5104} = 1$.

Since the comet is farther from the Sun, it is located on the branch of the hyperbola near Earth.

Check (21, 70) is a point that satisfies the equation.

The distance between this point and the Sun (−73, 0) is

$\sqrt{[21 - (-73)]^2 + (70 - 0)^2}$ or 117.2 million kilometers.

The distance between this point and Earth (73, 0) is

$\sqrt{(21 - 73)^2 + (70 - 0)^2}$ or 87.2 million kilometers.

The difference between these distances is 30. ✓

StudyTip

Exact Locations A third receiving station is necessary to determine the plane's exact location.

GuidedPractice

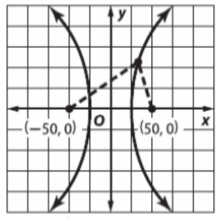
4. **SEARCH AND RESCUE** Two receiving stations that are 150 kilometers apart receive a signal from a downed airplane. They determine that the airplane is 80 kilometers farther from station A than from station B. Determine the equation of the hyperbola centered at the origin on which the plane is located. $\frac{x^2}{1600} - \frac{y^2}{4025} = 1$

WatchOut!

Preventing Errors Tell students to make rough sketches for the situations in problems such as those in Example 4.

Additional Example

4 NAVIGATION The LORAN navigational system is based o hyperbolas. Two stations send out signals at the same time. A ship notes the difference in the times at which it receives the signals. The ship is on a hyperbola with the stations at t foci. Suppose a ship determine that the difference of its distances from two stations is 50 nautical kilometers. Write a equation for a hyperbola on which the ship lies if the station are at (−50, 0) and (50, 0).



$\frac{x^2}{625} - \frac{y^2}{1875} = 1$

WatchOut!

Preventing Errors Suggest that students make lists of the values of a, b, c, h, and k. This will help avoid confusion about the signs of h and k.

Follow-ups

Students have explored ellipses, parabolas, and hyperbolas.

Ask:

- What are the similarities and differences between parabolas and ellipses? Sample answer: Similarities: Curved graphs; equations contain a variable raised to second power. Differences: Parabolas have one focus and one vertex; ellipses have two foci, 2 vertices, and 2 co-vertices.
- What are the similarities and differences between hyperbolas and the other conic sections? Sample answer: Similarities: Curved graphs; equations contain one or two variables raised to second power. Differences: Hyperbolas have two branches; other conic sections are continuous.

Teach with Tech

Web Page Have students search tl Web to find applications and real-w situations involving hyperbolas. Hav students create links on their social networking Web sites for pages wit specific examples they find interest

3 Practice

Formative Assessment

Use Exercises 1–9 to check for understanding.

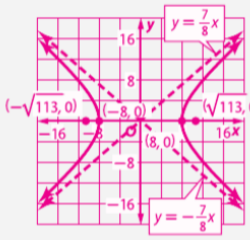
Use the chart at the bottom of this page to customize assignments for your students.

Teaching the Mathematical Practices

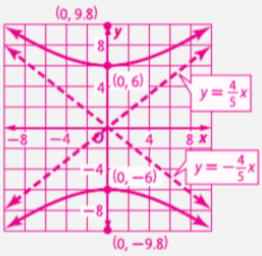
Structure Mathematically proficient students look closely to discern a pattern or structure

Additional Answers

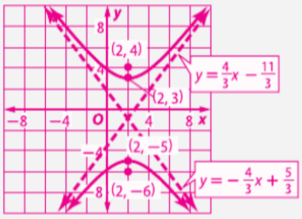
5.



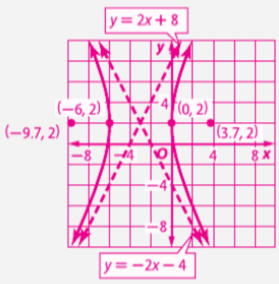
6.



7.

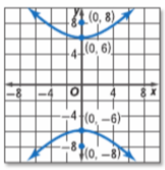


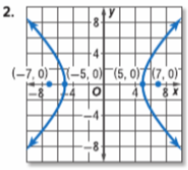
8.

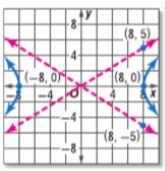


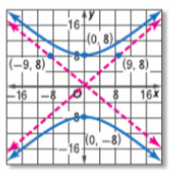
Check Your Understanding

Examples 1–2 Write an equation for each hyperbola.

1.  $\frac{y^2}{36} - \frac{x^2}{28} = 1$

2.  $\frac{x^2}{25} - \frac{y^2}{24} = 1$

3.  $\frac{x^2}{64} - \frac{y^2}{25} = 1$

4.  $\frac{y^2}{64} - \frac{x^2}{81} = 1$

Example 3 STRUCTURE Graph each hyperbola. Identify the vertices, foci, and asymptotes. **5–8. See margin.**

5. $\frac{x^2}{64} - \frac{y^2}{49} = 1$

6. $\frac{y^2}{36} - \frac{x^2}{60} = 1$

7. $9y^2 + 18y - 16x^2 + 64x - 199 = 0$

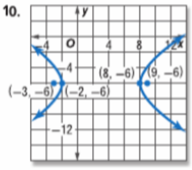
8. $4x^2 + 24x - y^2 + 4y - 4 = 0$

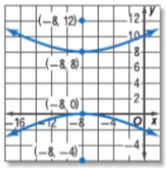
Example 4

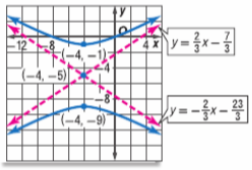
9. NAVIGATION A ship determines that the difference of its distances from two stations is 60 nautical miles. Write an equation for a hyperbola on which the ship lies if the stations are at (−80, 0) and (80, 0). $\frac{x^2}{900} - \frac{y^2}{5500} = 1$

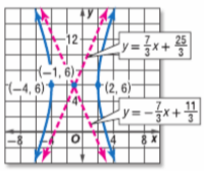
Practice and Problem Solving

Examples 1–2 Write an equation for each hyperbola.

10. 

11. 

12. 

13. 

10. $\frac{(x-3)^2}{25} - \frac{(y+6)^2}{11} = 1$

11. $\frac{(y-4)^2}{16} - \frac{(x+8)^2}{48} = 1$

12. $\frac{(y+5)^2}{16} - \frac{(x+4)^2}{36} = 1$

13. $\frac{(x+1)^2}{9} - \frac{(y-6)^2}{49} = 1$

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	10–24, 44, 46–67	11–23 odd, 50–53	10–24 even, 44, 46–49, 54–67
OL Core	11–31 odd, 32–44, 46–67	10–24, 50–53	25–44, 46–49, 54–67
BL Advanced	25–65, (optional: 66, 67)		

Example 3 **Graph each hyperbola. Identify the vertices, foci, and asymptotes.** **14–23. See Chapter 6 Answer Appendix.**

14. $\frac{x^2}{36} - \frac{y^2}{4} = 1$ **15.** $\frac{y^2}{9} - \frac{x^2}{49} = 1$

16. $\frac{y^2}{36} - \frac{x^2}{25} = 1$ **17.** $\frac{x^2}{16} - \frac{y^2}{16} = 1$

18. $\frac{(x-3)^2}{16} - \frac{(y+1)^2}{4} = 1$ **19.** $\frac{(y+5)^2}{16} - \frac{(x+2)^2}{36} = 1$

20. $9y^2 - 4x^2 - 54y + 32x - 19 = 0$ **21.** $16x^2 - 9y^2 + 128x + 36y + 76 = 0$

22. $25x^2 - 4y^2 - 100x + 48y - 144 = 0$ **23.** $81y^2 - 16x^2 - 810y + 96x + 585 = 0$

Example 4 **24.** **NAVIGATION** A ship determines that the difference of its distances from two stations is 80 nautical miles. Write an equation for a hyperbola on which the ship lies if the stations are at (−100, 0) and (100, 0). $\frac{x^2}{1600} - \frac{y^2}{8400} = 1$

B **Determine whether the following equations represent ellipses or hyperbolas.**

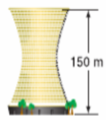
25. $4x^2 = 5y^2 + 6$ **hyperbola** **26.** $8x^2 - 2x = 8y - 3y^2$ **ellipse**

27. $-5x^2 + 4x = 6y + 3y^2$ **ellipse** **28.** $7y - 2x^2 = 6x - 2y^2$ **hyperbola**

29. $6x - 7x^2 - 5y^2 = 2y$ **ellipse** **30.** $4x + 6y + 2x^2 = -3y^2$ **ellipse**

31. **SPACE** Refer to the application at the beginning of the lesson. With the Sun as a focus and the center at the origin, a certain comet's path follows a branch of a hyperbola. If two of the coordinates of the path are (10, 0) and (30, 100) where the units are in millions of kilometers, determine the equation of the path. $\frac{x^2}{100} - \frac{y^2}{1250} = 1$

32. **COOLING** Natural draft cooling towers are shaped like hyperbolas for more efficient cooling of power plants. The hyperbola in the tower at the right can be modeled by $\frac{x^2}{16} - \frac{y^2}{225} = 1$, where the units are in meters. Find the width of the tower at the top and at its narrowest point in the middle. **8 m in the middle and 40.8 m at the top**



33. **MULTIPLE REPRESENTATIONS** Consider $xy = 16$. **a–c. See Chapter 6 Answer Appendix.**

a. **Tabular** Make a table of values for the equation for $-12 \le x \le 12$.

b. **Graphical** Graph the hyperbola represented by the equation.

c. **Logical** Determine and graph the asymptotes for the hyperbola.

d. **Analytical** What special property do you notice about the asymptotes? Hyperbolas that represent this property are called *rectangular hyperbolas*. **They are perpendicular.**

e. **Analytical** Without any calculations, what do you think will be the coordinates of the vertices for $xy = 25$? for $xy = 36$? **For $xy = 25$, the vertices will be at (5, 5) and (−5, −5), and for $xy = 36$, they will be at (−6, −6) and (6, 6).**

35. $\frac{x^2}{2,722,500} - \frac{y^2}{1,277,500} = 1$

34. **MODELING** Two receiving stations that are 250 kilometers apart receive a signal from a downed airplane. They determine that the airplane is 70 kilometers farther from station *B* than from station *A*. Determine the equation of the horizontal hyperbola centered at the origin on which the plane is located. $\frac{x^2}{1225} - \frac{y^2}{14400} = 1$

35. **WEATHER** Fatima and Ayesha live exactly 4000 feet apart. While on the phone at their homes, Fatima hears thunder out of her window and Ayesha hears it 3 seconds later out of hers. If sound travels 1100 feet per second, determine the equation for the horizontal hyperbola on which the lightning is located.

Multiple Representations

In Exercise 33, students use a table values, a graph, and logical analysi classify hyperbolas and predict thei properties.

WatchOut!

Error Analysis For Exercise 44, remind students that they must use the standard form of the equation to determine the orientation of the hyperbola.

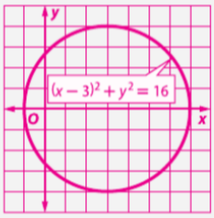
Additional Answers

44. Sample answer: Faris; The equation indicates a hyperbola that opens up and down. Fahd drew a horizontal hyperbola.

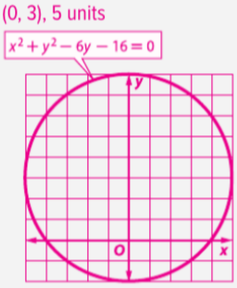
46. Sample answer: The foci move farther away from the vertices. When a is only slightly smaller than b, the value of c is only fairly larger than a. However, as b becomes much larger than a, by way of $c^2 = a^2 + b^2$, c becomes much larger than a. When this happens, the distance between the foci c becomes much greater than the distance between the vertices a. Therefore, the foci are much farther away from the vertices.

49. Sample answer: Conic sections can be used to model phenomena that can't be modeled using functions. For example, parabolas can be used to model paths of comets and ellipses can be used to model planetary orbits.

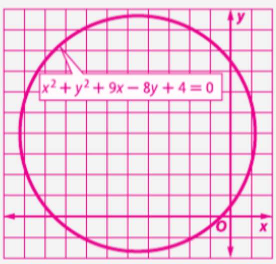
56. (3, 0), 4 units



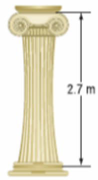
57. (0, 3), 5 units



58. $\left(-\frac{9}{2}, 4\right), \frac{\sqrt{129}}{2}$ units



C **36. ARCHITECTURE** Large pillars with cross sections in the shape of hyperbolas were popular in ancient Greece. The curves can be modeled by the equation $\frac{x^2}{0.16} - \frac{y^2}{4} = 1$, where the units are in meters. If the pillars are 2.7 meters tall, find the width of the top of each pillar and the width of each pillar at the narrowest point in the middle. Round to the nearest hundredth of a meter. **≈0.97 m, 0.8 m**



Write an equation for the hyperbola that satisfies each set of conditions.

37. vertices (−8, 0) and (8, 0), conjugate axis of length 20 units $\frac{x^2}{64} - \frac{y^2}{100} = 1$

38. vertices (0, −6) and (0, 6), conjugate axis of length 24 units **38.** $\frac{y^2}{36} - \frac{x^2}{144} = 1$

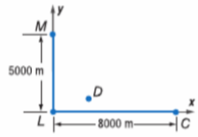
39. vertices (6, −2) and (−2, −2), foci (10, −2) and (−6, −2) $\frac{(x-2)^2}{16} - \frac{(y+2)^2}{48} = 1$

40. vertices (−3, 4) and (−3, −8), foci (−3, 9) and (−3, −13) **40.** $\frac{(y+2)^2}{36} - \frac{(x+3)^2}{85} = 1$

41. centered at the origin with a horizontal transverse axis of length 10 units and a conjugate axis of length 4 units $\frac{x^2}{25} - \frac{y^2}{4} = 1$

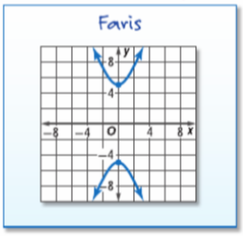
42. centered at the origin with a vertical transverse axis of length 16 units and a conjugate axis of length 12 units $\frac{y^2}{64} - \frac{x^2}{36} = 1$

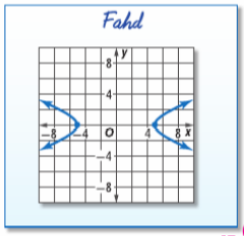
43. TRIANGULATION While looking for their lost cat in the woods, Ahmed, Mohammad, and Humaid hear a meow. Mohammad hears it 2 seconds after Ahmed and Humaid hears it 3 seconds after Ahmed. With Ahmed at the origin, determine the exact location of their cat if sound travels 1100 meters per second. **(2308, 826)**



H.O.T. Problems Use Higher-Order Thinking Skills

44. CRITIQUE Faris and Fahd are graphing $\frac{y^2}{25} - \frac{x^2}{4} = 1$. Is either of them correct? Explain your reasoning. **See margin.**





48. Sample answer: $\frac{(x-5)^2}{9} - \frac{y^2}{16} = 1$

45. CHALLENGE The origin lies on a horizontal hyperbola. The asymptotes for the hyperbola are $y = -x + 1$ and $y = x - 5$. Find the equation for the hyperbola. **45.** $\frac{(x-3)^2}{5} - \frac{(y+2)^2}{5} =$

46. REASONING What happens to the location of the foci of a hyperbola as the value of a becomes increasingly smaller than the value of b? Explain your reasoning. **See margin.**

47. REASONING Consider $\frac{y^2}{36} - \frac{x^2}{16} = 1$. Describe the change in the shape of the hyperbola and the locations of the vertices and foci if 36 is changed to 9. Explain why this happens. **See Chapter 6 Answer Appendix.**

48. OPEN ENDED Write an equation for a hyperbola with a focus at the origin.

49. WRITING IN MATH Why would you choose a conic section to model a set of data instead of a polynomial function? **See margin.**

Teaching the Mathematical Practices

Critique Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is.

Standardized Test Practice

50. You have 6 more 10 fils coins than 25 fils coins. You have a total of AED 5.15. How many 10 fils coins do you have? **C**

A 13   C 19
B 16   D 25

51. How tall is a shrub that is 15 centimeters shorter than a pole three times as tall as the shrub? **J**

F 24.5 cm
G 22.5 cm
H 21.5 cm
J 7.5 cm

52. **SHORT RESPONSE** A rectangle is 8 meters long and 6 meters wide. If each dimension is increased by the same number of meters, the area of the new rectangle formed is 32 square meters more than the area of the original rectangle. By how many meters was each dimension increased? **2**

53. **SAT/ACT** When the equation $y = 4x^2 - 5$ is graphed in the coordinate plane, the graph is which of the following? **E**

A line   D hyperbola
B circle   E parabola
C ellipse

Spiral Review

Write an equation for an ellipse that satisfies each set of conditions. (Lesson 6-4)

54. endpoints of major axis at (2, 2) and (2, −10), endpoints of minor axis at (0, −4) and (4, −4) $\frac{(y+4)2}{36}+\frac{(x-2)2}{4}=1$

55. endpoints of major axis at (0, 10) and (0, −10), foci at (0, 8) and (0, −8) $\frac{y^2}{100}+\frac{x^2}{36}=1$

Find the center and radius of the circle with the given equation. Then graph the circle. (Lesson 6-3) **56–58. See margin.**

56. $(x-3)^2 + y^2 = 16$ **57.** $x^2 + y^2 - 6y - 16 = 0$ **58.** $x^2 + y^2 + 9x - 8y + 4 = 0$

59. **BASKETBALL** Wafa plays basketball for her high school. So far this season, she has made 6 out of 10 free throws. She is determined to improve her free throw percentage. If she can make x consecutive free.

a. Graph the function. **See margin.**

b. What part of the graph is meaningful in the context of the problem? **the part in the first quadrant**

c. Describe the meaning of the y-intercept. **It represents her original free-throw percentage of 60%.**

d. What is the equation of the horizontal asymptote? Explain its meaning with respect to Wafa's shooting percentage. **$P(x) = 1$; this represents 100%, which she cannot achieve because she has already missed 4 free throws.**

Solve each equation.

60. $\left(\frac{1}{7}\right)^{y-3} = 343$ **0**   **61.** $10^{x-1} = 100^{2x-3}$ $\frac{5}{3}$ **62.** $36^{2p} = 216^{p-1}$ **−3**

Graph each inequality. 63–65. See margin.

63. $y \geq \sqrt{5x-8}$ **64.** $y \geq \sqrt{x-3} + 4$ **65.** $y < \sqrt{6x-2} + 1$

Skills Review

66. Write an equation for a parabola with vertex at the origin that passes through (2, −8) $y = -2x^2$

67. Write an equation for a parabola with vertex at (−3, −4) that opens up and has y-intercept 8. $y = \frac{4}{3}(x+3)^2 - 4$

Differentiated Instruction OL BL

Extension

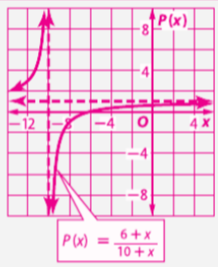
- Use a compass to draw a circle on a piece of paper. Be sure not to take up the entire page.
- Use the sharp end of your compass to poke a small hole anywhere outside of your circle. Label the point where the hole is *F*.
- Fold the paper so that point *F* lands somewhere on the circle.
- Repeat the last step using point *F* and several other points on the circle. Predict what kind of curve these creases are forming. The fixed point *F* is the focus of a hyperbola.
- Discuss with students how the shape of the curve would change if they varied the location of point *F*. As point *F* moves farther from the center of the circle, the branches of the hyperbola become wider.

4 Assess

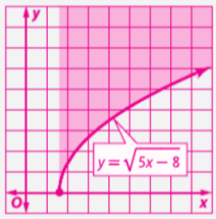
Name the Math Prepare two bags one containing an ordered pair for t center of a hyperbola on each pape the other with values for *a* and *b* on each paper. Have each student sele both an ordered pair and values for and *b*. Have students write the equations for the hyperbolas and te much about them as possible given values they chose.

Additional Answers

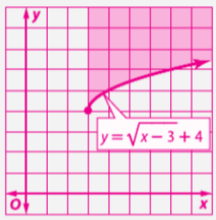
59a.



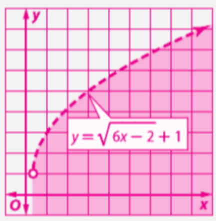
63.



64.



65.



1 Focus

VerticalAlignment

Before Lesson 6-5 Analyze different conic sections.

Lesson 6-5 Write equations of conic sections in standard form. Identify conic sections from their equations.

After Lesson 6-5 Use algebraic methods to solve systems of equations or inequalities of conic sections.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- Describe the plane that forms a hyperbola. intersects both cones
- Describe the plane that forms a parabola. parallel to the slant height of a cone
- Describe the plane that forms a circle. parallel to the base of a cone

1 Conics in Standard Form

Example 1 shows how to rewrite an equation of a conic section in standard form and determine if the equation is that of a parabola, circle, ellipse, or hyperbola.

LESSON 6-5 Identifying Conic Sections

Then

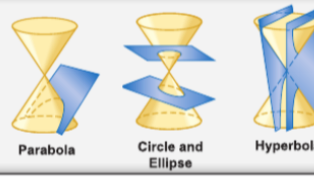
- You analyzed different conic sections.

Now

- 1 Write equations of conic sections in standard form.
- 2 Identify conic sections from their equations.

Why?

- Parabolas, circles, ellipses, and hyperbolas are called conic sections because they are the cross sections formed when a double cone is sliced by a plane.



Mathematical Practices
3 Construct viable arguments and critique the reasoning of others.
8 Look for and express regularity in repeated reasoning.

1 Conics in Standard Form The equation for any conic section can be written in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, and C are not all zero. This general form can be converted to the standard forms below by completing the square.

ConceptSummary Standard Forms of Conic Sections

Conic Section	Standard Form of Equation	
Circle	$(x-h)^2 + (y-k)^2 = r^2$	
	Horizontal Axis	Vertical Axis
Parabola	$y = a(x-h)^2 + k$	$x = a(y-k)^2 + h$
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Example 1 Rewrite an Equation of a Conic Section

Write $16x^2 - 25y^2 - 128x - 144 = 0$ in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

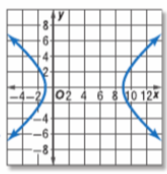
$16x^2 - 25y^2 - 128x - 144 = 0$ Original equation

$16(x^2 - 8x + ■) - 25y^2 = 144 + 16(■)$ Isolate terms.

$16(x^2 - 8x + 16) - 25y^2 = 144 + 16(16)$ Complete the square.

$16(x-4)^2 - 25y^2 = 400$ Perfect square

$\frac{(x-4)^2}{25} - \frac{y^2}{16} = 1$ Divide each side by 400.



The graph is a hyperbola with its center at (4, 0).

GuidedPractice 1. See Chapter 6 Answer Appendix.

1. Write $4x^2 + y^2 - 16x + 8y - 4 = 0$ in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

ReviewVocabulary

discriminant the expression $b^2 - 4ac$ from the Quadratic Formula

2 Identify Conic Sections

You can determine the type of conic without having to write $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ in standard form. When there is an xy-term $(B \neq 0)$, you can use the discriminant to identify the conic. $B^2 - 4AC$ is the discriminant of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

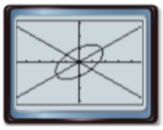
ConceptSummary Classify Conics with the Discriminant

Discriminant	Conic Section
$B^2 - 4AC < 0$; $B = 0$ and $A = C$	circle
$B^2 - 4AC < 0$; either $B \neq 0$ or $A \neq C$	ellipse
$B^2 - 4AC = 0$	parabola
$B^2 - 4AC > 0$	hyperbola

When $B = 0$, the conic will be either vertical or horizontal. When $B \neq 0$, the conic will be neither vertical nor horizontal.

Horizontal Ellipse: $B = 0$

Rotated Ellipse: $B \neq 0$



StudyTip

Identifying Conics When there is no xy-term $(B = 0)$, use A and C.
Parabola: A or $C = 0$ but not both.
Circle: $A = C$
Ellipse: A and C have the same sign but are not equal.
Hyperbola: A and C have opposite signs.

Example 2 Analyze an Equation of a Conic Section

Without writing in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

a. $y^2 + 4x^2 - 3xy + 4x - 5y - 8 = 0$

$A = 4$, $B = -3$, and $C = 1$

The discriminant is $(-3)^2 - 4(4)(1)$ or -7.

Because the discriminant is less than 0 and $B \neq 0$, the conic is an ellipse.

b. $3x^2 - 6x + 4y - 5y^2 + 2xy - 4 = 0$

$A = 3$, $B = 2$, and $C = -5$

The discriminant is $2^2 - 4(3)(-5)$ or 64.

Because the discriminant is greater than 0, the conic is a hyperbola.

c. $4y^2 - 8x + 6y - 14 = 0$

$A = 0$, $B = 0$, and $C = 4$

The discriminant is $0^2 - 4(0)(4)$ or 0.

Because the discriminant equals 0, the conic is a parabola.

GuidedPractice

2A. $8y^2 - 6x^2 + 4xy - 6x + 2y - 4 = 0$ **hyperbola**

2B. $3xy + 4x^2 - 2y + 9x - 3 = 0$ **hyperbola**

2C. $3x^2 + 16x - 12y + 2y^2 - 6 = 0$ **ellipse**

DifferentiatedInstruction AL OL

If students need a visual aid in identifying conic sections,

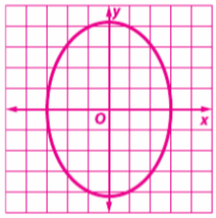
Then divide students into groups of two or three. Assign each group a different conic section and have them make a poster describing how to identify the conic section from an equation in standard form and from an equation not in standard form.

Formative Assessment

Use the Guided Practice exercises a each example to determine student understanding of concepts.

Additional Example

1 Write $y^2 = 18 - 2x^2$ in standar form. State whether the graph the equation is a *parabola*, *circ* *ellipse*, or *hyperbola*. Then gra the equation. $\frac{x^2}{9} + \frac{y^2}{18} = 1$; ellipse



2 Identify Conic Sections

Example 2 shows how to determin an equation is that of a parabola, ci ellipse, or hyperbola without writing equation in standard form.

Additional Example

2 Without writing the equation in standard form, state whether t graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

a. $3y^2 - x^2 - 9 = 0$ hyperbo

b. $2x^2 + 2y^2 + 16x - 20y =$ -32 circle

c. $y^2 - 2x - 4y + 10 = 0$ parabola

Follow-up

Students have explored the conic sections.

Ask:

- Why are parabolas, circles, ellipses, and hyperbolas called conic sections? Sample answer: They are the cross sections formed by a plane and a double napped cone.

3 Practice

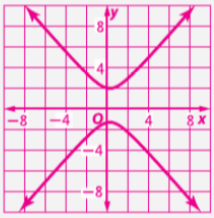
Formative Assessment

Use Exercises 1–13 to check for understanding.

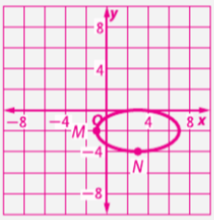
Use the chart at the bottom of this page to customize assignments for your students.

Additional Answers

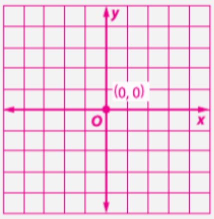
34. hyperbola



43c.



44b.



Check Your Understanding

Example 1 Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation. 1–4. See Chapter 6 Answer Appendix.

1. $x^2 + 4y^2 - 6x + 16y - 11 = 0$
2. $x^2 + y^2 + 12x - 8y + 36 = 0$
3. $9y^2 - 16x^2 - 18y - 64x - 199 = 0$
4. $6y^2 - 24y + 28 - x = 0$

Example 2 Without writing in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. 9. hyperbola

5. $4x^2 + 6y^2 - 3x - 2y = 12$ ellipse
6. $5y^2 = 2x + 6y - 8 + 3x^2$ hyperbola
7. $8x^2 + 8y^2 + 16x + 24 = 0$ circle
8. $4x^2 - 6y = 8x + 2$ parabola
9. $4x^2 - 3y^2 + 8xy - 12 = 2x + 4y$
10. $5xy - 3x^2 + 6y^2 + 12y = 18$ hyperbola
11. $8x^2 + 12xy + 16y^2 + 4y - 3x = 12$ ellipse
12. $16xy + 8x^2 + 8y^2 - 18x + 8y = 13$ parabola

13. **MODELING** A military jet performs for an air show. The path of the plane during one maneuver can be modeled by a conic section with equation $24x^2 + 1000y - 31{,}680x - 45{,}600 = 0$, where distances are represented in feet.
    a. Identify the shape of the curved path of the jet. Write the equation in standard form. parabola; $y = -0.024(x - 660)^2 + 10{,}500$
    b. If the jet begins its path upward, or ascent, at $x = 0$, what is the horizontal distance traveled by the jet from the beginning of the ascent to the end of the descent? about 402.6 ft
    c. What is the maximum height of the jet? 10,500 ft

Practice and Problem Solving

Example 1 Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation. 14–23. See Chapter 6 Answer Appendix.

14. $3x^2 - 2y^2 + 18x + 8y - 35 = 0$
15. $3x^2 + 24x + 4y^2 - 40y + 52 = 0$
16. $x^2 + y^2 = 16 + 6y$
17. $32x + 28 = y - 8x^2$
18. $7x^2 - 8y = 84x - 2y^2 - 176$
19. $x^2 + 8y = 11 + 6x - y^2$
20. $4y^2 = 24y - x - 31$
21. $112y + 64x = 488 + 7y^2 - 8x^2$
22. $28x^2 + 9y^2 - 188 = 56x - 36y$
23. $25x^2 + 384y - 64y^2 + 200x = 1776$

Example 2 Without writing in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. 29. hyperbola 31. hyperbola

24. $4x^2 - 5y = 9x - 12$ parabola
25. $4x^2 - 12x = 18y - 4y^2$ circle
26. $9x^2 + 12y = 9y^2 + 18y - 16$ hyperbola
27. $18x^2 - 16y = 12x - 4y^2 + 19$ ellipse
28. $12y^2 - 4xy + 9x^2 = 18x - 124$ ellipse
29. $5xy + 12x^2 - 16x = 5y + 3y^2 + 18$
30. $19x^2 + 14y = 6x - 19y^2 - 88$ circle
31. $8x^2 + 20xy + 18 = 4y^2 - 12 + 9x$
32. $5x - 12xy + 6x^2 = 8y^2 - 24y - 9$ hyperbola
33. $18x - 24y + 324xy = 27x^2 + 3y^2 - 5$ hyperbola

34. **LIGHT** A lamp standing near a wall throws an arc of light in the shape of a conic section. Suppose the edge of the light can be represented by the equation $3y^2 - 2y - 4x^2 + 2x - 8 = 0$. Identify the shape of the edge of the light and graph the equation. See margin.

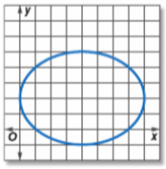
Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	14–34, 45–62	15–33 odd, 48–51	14–34 even, 45–47, 52–62
OL Core	15–33 odd, 35–43, 45–62	14–34, 48–51	35–43, 45–47, 52–62
BL Advanced	35–59, (optional: 60–62)		

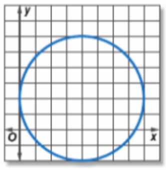
Match each graph with its corresponding equation.

a. $x^2 + y^2 - 8x - 4y = -4$ b. $9x^2 - 16y^2 - 72x + 64y = 64$ c. $9x^2 + 16y^2 = 72x + 64y - 64$

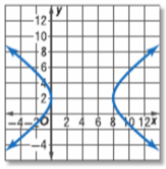
35. c



36. a



37. b



For Exercises 38–41, match each situation with an equation that could be used to represent it.

a. $47.25x^2 - 9y^2 + 18y + 33.525 = 0$

b. $25x^2 + 100y^2 - 1900x - 2200y + 45{,}700 = 0$

c. $16x^2 - 90x + y - 0.25 = 0$

d. $x^2 + y^2 - 18x - 30y - 14{,}094 = 0$

38. **COMPUTERS** the boundary of a wireless network with a range of 120 feet **d**

39. **FITNESS** the oval path of your foot on an exercise machine **b**

40. **COMMUNICATIONS** the position of a cell phone between two cell towers **a**

41. **SPORTS** the height of a ball above the ground after being kicked **c**

42. **SENSE-MAKING** The shape of the cables in a suspension bridge is approximately parabolic. If the towers for a planned bridge are 1000 meters apart and the lowest point of the suspension cables is 200 meters below the top of the towers, write the equation in standard form with the origin at the vertex. $y = 0.0008x^2$

43. **MULTIPLE REPRESENTATIONS** Consider an ellipse with center (3, −2), vertex $M(-1, -2)$, and co-vertex $N(3, -4)$. **43a.** $\frac{(x-3)^2}{16} + \frac{(y+2)^2}{4} = 1$

a. **Analytical** Determine the standard form of the equation of the ellipse.

b. **Algebraic** Convert part **a** to $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ form. $x^2 + 4y^2 - 6x + 16y + 9 = 0$

c. **Graphical** Graph the ellipse. **See margin.**

d. **Analytical** If the ellipse is rotated such that M is moved to (3, −6), determine the location of N and the angle of rotation. $N(5, -2)$; **90° counterclockwise**

H.O.T. Problems Use Higher-Order Thinking Skills

44. **CHALLENGE** When a plane passes through the vertex of a cone, a *degenerate* conic is formed.

a. Determine the type of conic represented by $4x^2 + 8y^2 = 0$. **ellipse**

b. Graph the conic. **See margin.**

c. Describe the difference between this degenerate conic and a standard conic of the same type with $A = 4$ and $B = 8$.

44c. The graph of the standard conic is an ellipse, and the graph of the degenerate conic is a single point.

45. **REASONING** Is the following statement *sometimes*, *always*, or *never* true? Explain.

When a conic is vertical and A = C, it is a circle.

45. Sample answer: Always; when a conic is vertical, $B = 0$. When this is true and $A = C$, the conic is a circle.

46. **OPEN ENDED** Write an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A = 9C$, that represents a parabola. **Sample answer:** $9x^2 + 6xy + y^2 + 2x + 2y + 8 = 0$

47. **WRITING IN MATH** Compare and contrast the graphs of the four types of conics and their corresponding equations. **See Chapter 6 Answer Appendix.**

Teaching the Mathematical Practices

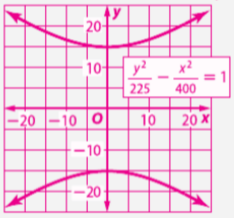
Sense-Making Mathematically proficient students start by explaini to themselves the meaning of a problem and looking for entry poin to its solution. They analyze givens constraints, relationships, and goal They check their answers to proble using a different method, and they continually ask themselves, "Does make sense?"

4 Assess

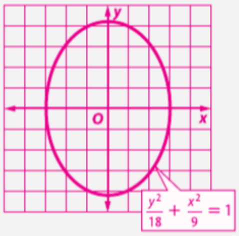
Crystal Ball Ask students to write how today's lesson will connect with solving linear-nonlinear systems.

Additional Answers

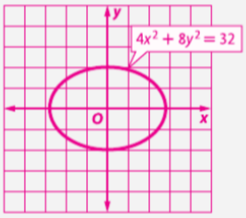
52. (0, ±15); (0, ±25); $y = \pm\frac{3}{4}x$



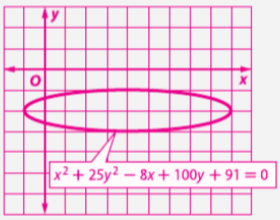
53. (0, 0); (0, ±3); $6\sqrt{2}$; 6



54. (0, 0); (±2, 0); $4\sqrt{2}$; 4



55. (4, −2); $(4 \pm 2\sqrt{6}, -2)$; 10; 2



Standardized Test Practice

48. SAT/ACT A class of 25 students took a science test. Ten students had a mean score of 80. The other students had an average score of 60. What is the average score of the whole class? **B**

A 66 D 72
B 68 E 78
C 70

49. Six times a number minus 11 is 43. What is the number? **J**

F 12
G 11
H 10
J 9

50. EXTENDED RESPONSE The amount of water remaining in a storage tank as it is drained can be represented by the equation $L = -4t^2 - 10t + 130$, where L represents the number of liters of water remaining and t represents the number of minutes since the drain was opened. How many liters of water were in the tank initially? Determine to the nearest tenth of a minute how long it will take for the tank to drain completely. **130 L; 4.6 min**

51. Ahmed has a square piece of paper with sides 4 centimeters long. He rolled up the paper to form a cylinder. What is the volume of the cylinder? **B**

A $\frac{4}{\pi}$ C 4π
B $\frac{16}{\pi}$ D 16π

Spiral Review

52. ASTRONOMY Suppose a comet's path can be modeled by a branch of the hyperbola with equation $\frac{y^2}{225} - \frac{x^2}{400} = 1$. Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola. Then graph the hyperbola. (Lesson 6-5) **See margin.**

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse. (Lesson 6-4) **53–55. See margin.**

53. $\frac{y^2}{18} + \frac{x^2}{9} = 1$

54. $4x^2 + 8y^2 = 32$

55. $x^2 + 25y^2 - 8x + 100y + 91 = 0$

Graph each function. **56–58. See Chapter 6 Answer Appendix.**

56. $f(x) = \frac{3}{x}$

57. $f(x) = \frac{-2}{x + 5}$

58. $f(x) = \frac{6}{x - 2} - 4$

59. SPACE A radioisotope is used as a power source for a satellite. The power output P (in watts) is given by $P = 50e^{-\frac{t}{250}}$, where t is the time in days.

a. Is the formula for power output an example of exponential *growth* or *decay*? Explain your reasoning. **Decay; the exponent is negative.**

b. Find the power available after 100 days. **about 33.5 watts**

c. Ten watts of power are required to operate the equipment in the satellite. How long can the satellite continue to operate? **about 402 days**

Skills Review

Solve each system of equations.

60. $6g - 8h = 50$, $6h = 22 - 4g$ **(7, −1)**

61. $3u + 5v = 6$, $2u - 4v = -7$ $\left(-\frac{1}{2}, \frac{3}{2}\right)$

62. $10m - 9n = 15$, $5m - 4n = 10$ **(6, 5)**

Differentiated Instruction OL BL

Extension Students have learned that a quadratic equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, is either a *parabola, circle, ellipse,* or a *hyperbola* when graphed. Ask students to determine the shape of the graph of the equations $x^2 + y^2 + 1 = 0$, $x^2 + y^2 = 0$ and $x^2 = 0$. empty set; single point (0, 0); the line $x = 0$

1 Focus

VerticalAlignment

Before Lesson 6-6 Solve systems of linear equations.

Lesson 6-6 Solve systems of linear and nonlinear equations algebraically and graphically. Solve systems of linear and nonlinear inequalities graphically.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- Can a person be easily located using one cellular tower? Why or why not? No; the person could be anywhere in a circular path around the tower.
- If law enforcement only knew the distance a caller was from two cellular towers, what is the maximum number of locations they would have to search to find the caller? two

LESSON 6-6 Solving Linear-Nonlinear Systems

Then

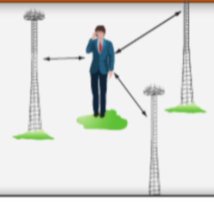
- You solved systems of linear equations.

Now

1. Solve systems of linear and nonlinear equations algebraically and graphically.
2. Solve systems of linear and nonlinear inequalities graphically.

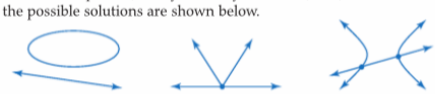
Why?

- Have you ever wondered how law enforcement agencies can track a cell phone user's location? A person using a cell phone can be located in respect to three cellular towers. The respective coordinates and distances each tower is from the caller are used to pinpoint the caller's location. This is accomplished using a system of quadratic equations.



Mathematical Practices
6 Attend to precision.

1 Systems of Equations When a system of equations consists of a linear and a nonlinear equation, the system may have zero, one, or two solutions. Some of the possible solutions are shown below.



You can solve linear-quadratic systems by using graphical or algebraic methods.

Example 1 Linear-Quadratic System

Solve the system of equations. $9x^2 + 25y^2 = 225$ (1)
$10y + 6x = 6$ (2)

Step 1 Solve the linear equation for y.

$10y + 6x = 6$ — Equation (2)
$y = -0.6x + 0.6$ — Solve for y.

Step 2 Substitute into the quadratic equation and solve for x.

$9x^2 + 25y^2 = 225$ — Quadratic equation
$9x^2 + 25(-0.6x + 0.6)^2 = 225$ — Substitute $-0.6x + 0.6$ for y.
$9x^2 + 25(0.36x^2 - 0.72x + 0.36) = 225$ — Simplify.
$9x^2 + 9x^2 - 18x + 9 = 225$ — Distribute.
$18x^2 - 18x - 216 = 0$ — Simplify.
$x^2 - x - 12 = 0$ — Divide each side by 18.
$(x - 4)(x + 3) = 0$ — Factor.
$x = 4$ or -3 — Zero Product Property

Step 3 Substitute x-values into the linear equation and solve for y.

$y = -0.6x + 0.6$ — Equation (2)
$= -0.6(4) + 0.6$ — Substitute the x-values
$= -1.8$ — Simplify.

$y = -0.6x + 0.6$
$= -0.6(-3) + 0.6$
$= 2.4$

The solutions of the system are $(4, -1.8)$ and $(-3, 2.4)$.

GuidedPractice **1A.** $\left(-11, -49\frac{1}{3}\right)$ and $(5, 4)$

1A. $3y + x^2 - 4x - 17 = 0$
$3y - 10x + 38 = 0$

1B. $3(y - 4) - 2(x - 3) = -6$
$5x^2 + 2y^2 - 53 =$

If a quadratic system contains two conic sections, the system may have anywhere from zero to four solutions. Some graphical representations are shown below.

You can use elimination to solve quadratic-quadratic systems.

Example 2 Quadratic-Quadratic System

Solve the system of equations.

$x^2 + y^2 = 45$ (1)
$y^2 - x^2 = 27$ (2)

$y^2 + x^2 = 45$	Equation (1), Commutative Property
$(+)\ y^2 - x^2 = 27$	Equation (2)
$2y^2 = 72$	Add.
$y^2 = 36$	Divide each side by 2.
$y = \pm 6$	Take the square root of each side.

Substitute 6 and −6 into one of the original equations and solve for x.

$x^2 + y^2 = 45$	Equation (1)	$x^2 + y^2 = 45$
$x^2 + 6^2 = 45$	Substitute for y.	$x^2 + (-6)^2 = 45$
$x^2 = 9$	Subtract 36 from each side.	$x^2 = 9$
$x = \pm 3$	Take the square root of each side.	$x = \pm 3$

The solutions are (−3, −6), (−3, 6), (3, −6), and (3, 6).

StudyTip
Tools If you use **ZSquare** on the **ZOOM** menu, the graph of the first equation will look like a circle.

GuidedPractice

2A. $x^2 + y^2 = 8$
$x^2 + 3y = 10$ **$(-2, 2), (2, 2), (-\sqrt{7}, 1), (\sqrt{7}, 1)$**

2B. $3x^2 + 4y^2 = 48$
$2x^2 - y^2 = -1$ **$(-2, -3), (-2, 3), (2, -3), (2, 3)$**

2 Systems of Inequalities

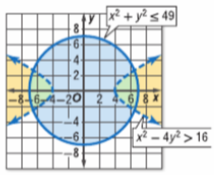
Systems of quadratic inequalities can be solved by graphing.

Example 3 Quadratic Inequalities

Solve the system of inequalities by graphing.

$x^2 + y^2 \le 49$
$x^2 - 4y^2 > 16$

The intersection of the graphs, shaded green, represents the solution of the system.



CHECK (6, 0) is in the shaded area. Use this point to check your solution.

$x^2 + y^2 \le 49$	$x^2 - 4y^2 > 16$
$6^2 + 0^2 \overset{?}{\le} 49$	$62 - 4(0)^2 \overset{?}{>} 16$
$36 \le 49$ ✓	$36 > 16$ ✓

GuidedPractice

3A. $5x^2 + 2y^2 \le 10$
$y \ge x^2 - 2x + 1$

3B. $x^2 - y^2 \le 8$
$x^2 + y^2 \ge 120$

3A, 3B. See Chapter 6 Answer Appendix.

1 Systems of Equations

Example 1 shows how to solve a linear-quadratic system using substitution. **Example 2** shows how to solve a quadratic-quadratic syste using elimination.

Formative Assessment

Use the Guided Practice exercises a each example to determine student understanding of concepts.

Additional Examples

1 Solve the system of equations.
$4x^2 - 16y^2 = 25$
$2y + x = 2$ $\left(\frac{41}{16}, -\frac{9}{32}\right)$

2 Solve the system of equations.
$x^2 + y^2 = 16$
$4x^2 + y^2 = 23$
$\left(\frac{\sqrt{21}}{3}, \frac{\sqrt{123}}{3}\right)$, $\left(-\frac{\sqrt{21}}{3}, \frac{\sqrt{123}}{3}\right)$, $\left(\frac{\sqrt{21}}{3}, -\frac{\sqrt{123}}{3}\right)$, $\left(-\frac{\sqrt{21}}{3}, -\frac{\sqrt{123}}{3}\right)$

Teaching the Mathematical Practices

Tools Mathematically proficient students consider the available too when solving a mathematical probl Encourage students to identify the types of graphs involved before se their viewing windows.

DifferentiatedInstruction OL BL

Social Learners Point out that there are combinations of graphs other than those shown on page 639 just before Example 2, that are possible for each number of solutions. As a class, challenge students to sketch as many different possibilities as they can to add to the figures shown.

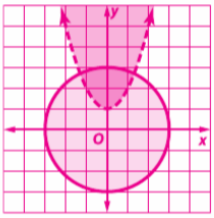
2 Systems of Inequalities

Example 3 shows how to solve a system of inequalities by graphing. **Example 4** shows how to solve a system of inequalities involving absolute value by graphing.

Additional Examples

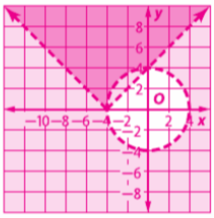
3 Solve the system of inequalities by graphing.

$y > x^2 + 1$
$x^2 + y^2 \leq 9$



4 Solve the system of inequalities by graphing.

$x^2 + y^2 > 16$
$y > |x + 4|$



3 Practice

Formative Assessment

Use Exercises 1–13 to check for understanding.

Use the chart at the bottom of the next page to customize assignments for your students.

Systems involving absolute value can also be solved by graphing.

StudyTip

Graphing Calculator Like linear inequalities, systems of quadratic and absolute value inequalities can be checked with a graphing calculator.

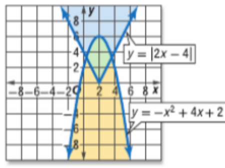
Example 4 Quadratics with Absolute Value

Solve the system of inequalities by graphing.

$y \geq |2x - 4|$
$y \leq -x^2 + 4x + 2$

Graph the boundary equations. Then shade appropriately.

The intersection of the graphs, shaded green, represents the solution to the system.



CHECK (2, 4) is in the shaded area. Use the point to check your solution.

$y \geq \|2x - 4\|$	$y \leq -x^2 + 4x + 2$
$4 \overset{?}{\geq} \|2(2) - 4\|$	$4 \overset{?}{\leq} -(2)^2 + 4(2) + 2$
$4 \geq 0$ ✓	$4 \leq 6$ ✓

GuidedPractice **4A, 4B. See Chapter 6 Answer Appendix.**

4A. $y > |-0.5x + 2|$
$\frac{x^2}{16} + \frac{y^2}{36} \leq 1$

4B. $x^2 + y^2 \leq 49$
$y \geq |x^2 + 1|$

Check Your Understanding

Examples 1–2 **Solve each system of equations.**

1. $8y = -10x$
$y^2 = 2x^2 - 7$ **(4, −5), (−4, 5)**

2. $x^2 + y^2 = 68$
$5y = -3x + 34$ **(−2, 8), (8, 2)**

3. $y = 12x - 30$
$4x^2 - 3y = 18$ **(3, 6), (6, 42)**

4. $6y^2 - 27 = 3x$
$6y - x = 13$ **(−7, 1), (−1, 2)**

5. $x^2 + y^2 = 16$
$x^2 - y^2 = 20$ **no solution**

6. $y^2 - 2x^2 = 8$
$3y^2 + x^2 = 52$

7. $x^2 + 2y = 7$
$y^2 - x^2 = 8$

8. $4y^2 - 3x^2 = 11$
$3y^2 + 2x^2 = 21$

6. **(2, 4), (−2, 4), (−2, −4), (2, −4)**
7. **(−1, 3), (1, 3), $(-\sqrt{17}, -5)$, $(\sqrt{17}, -5)$**
8. **$(-\sqrt{3}, -\sqrt{5})$, $(-\sqrt{3}, \sqrt{5})$, $(\sqrt{3}, -\sqrt{5})$, $(\sqrt{3}, \sqrt{5})$**

9. **PERSEVERANCE** Refer to the beginning of the lesson. A person using a cell phone can be located with respect to three cellular towers. In a coordinate system where one unit represents one kilometer, the location of the caller is determined to be 50 kilometers from the tower at the origin. The person is also 40 kilometers from a tower at (0, 30) and 13 kilometers from a tower at (35, 18). Where is the caller? **(40, 30)**

Examples 3–4 **Solve each system of inequalities by graphing.** **10–13. See Chapter 6 Answer Appendix.**

10. $6x^2 + 9(y - 2)^2 \leq 36$
$x^2 + (y + 3)^2 \leq 25$

11. $16x^2 + 4y^2 \leq 64$
$y \geq -x^2 + 2$

12. $4x^2 - 8y^2 \geq 32$
$y \geq |1.5x| - 8$

13. $x^2 + 8y^2 < 32$
$y < -|x - 2| + 2$

Teaching the Mathematical Practices

Perseverance Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals and make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt.

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Practice and Problem Solving

Examples 1–2 Solve each system of equations. **16–23, 25. See margin.**

14. $3x^2 - 2y^2 = -24$
$2y = -3x$ **(−4, 6), (4, −6)**

15. $5x^2 + 4y^2 = 20$
$5y = 7x + 35$ **no solution**

16. $x^2 + 3x = -4y - 2$
$y = -2x + 1$

17. $y = 2x$
$4x^2 - 2y^2 = -36$

18. $2y = x + 10$
$y^2 - 4y = 5x + 10$

19. $9y = 8x - 19$
$8x + 11 = 2y^2 + 5y$

20. $2y^2 + 5x^2 = 26$
$2x^2 - y^2 = 5$

21. $x^2 + y^2 = 16$
$x^2 - 4x + y^2 = 12$

22. $x^2 + y^2 = 8$
$5y^2 = 3x^2$

23. $y^2 - x^2 + 3y = 26$
$x^2 + 2y^2 = 34$

24. $x^2 - y^2 = 25$
$x^2 + y^2 + 7 = 0$ **no solution**

25. $x^2 - 10x + 2y^2 = 47$
$y^2 - 2x^2 = -14$

26. FIREWORKS Two fireworks are set off simultaneously but from different altitudes. The height y in feet of one is represented by $y = -16t^2 + 120t + 10$, where t is the time in seconds. The height of the other is represented by $y = -16t^2 + 60t + 310$.

a. After how many seconds are the fireworks the same height? **5 seconds**

b. What is that height? **210 ft**

39c. Sample answer: The orbit of the satellite modeled by the second equation is closer to a circle than the other orbit. The distance on the x-axis is twice as great for one satellite as the other.

Examples 3–4 **TOOLS** Solve each system of inequalities by graphing. **27–36. SEE CHAPTER 8 ANSWER APPENDIX.**

27. $x^2 + y^2 \geq 36$
$x^2 + 9(y + 6)^2 \leq 36$

28. $-x > y^2$
$4x^2 + 14y^2 \leq 56$

29. $12x^2 - 4y^2 \geq 48$
$16(x - 4)^2 + 25y^2 < 400$

30. $8y^2 - 3x^2 \leq 24$
$2y > x^2 - 8x + 14$

31. $y > x^2 - 6x + 8$
$x \geq y^2 - 6y + 8$

32. $x^2 + y^2 \geq 9$
$25x^2 + 64y^2 \leq 1600$

33. $16(x - 3)^2 + 4y^2 \leq 64$
$y \leq -|x - 2| + 2$

34. $x^2 - 4x + y^2 + 6y \leq 23$
$y > |x - 2| - 6$

35. $2y - 4 \geq |x + 4|$
$12 - 2y > x^2 + 12x + 36$

36. $18y^2 - 3x^2 \leq 54$
$y \geq |2x| - 6$

37. $x^2 + y^2 < 16$
$y \geq |x - 2| + 6$
no solution

38. $x^2 < y - 2$
$y \leq |x + 8| - 7$
no solution

39. SPACE Two satellites are placed in orbit about Earth. The equations of the two orbits are $\frac{x^2}{(300)^2} + \frac{y^2}{(900)^2} = 1$ and $\frac{x^2}{(600)^2} + \frac{y^2}{(690)^2} = 1$, where distances are in kilometers and Earth is the center of each curve.

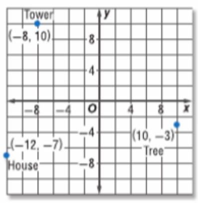
a. Solve each equation for y. **39a.** $y = \pm 900\sqrt{1 - \frac{x^2}{(300)^2}}$; $y = \pm 690\sqrt{1 - \frac{x^2}{(600)^2}}$

b. Use a graphing calculator to estimate the intersection points of the two orbits.

c. Compare the orbits of the two satellites.

39b. Sample answer: (209, 647), (−209, 647), (−209, −647), (209, −647)

40. PETS Asma's cat was missing one day. Fortunately, he was wearing an electronic monitoring device. If the cat is 10 units from the tree, 13 units from the tower, and 20 units from the house, determine the coordinates of his location. **(4, 5)**



41. BASEBALL In 1997, after Mark McGwire hit a home run, the claim was made that the ball would have traveled 538 feet if it had not landed in the stands. The path of the baseball can be modeled by $y = -0.0037x^2 + 1.77x - 1.72$ and the stands can be modeled by $y = \frac{3}{7}x - 128.6$. How far vertically and horizontally from home plate did the ball land in the stands? **440 ft from home plate, 60 ft above the playing surface**

WatchOut!

Preventing Errors Remind students that it is always a good idea to check a point inside the solution region as well as one outside that region.

Teaching the Mathematical Practices

Tools Mathematically proficient students consider the available too[ls] when solving a mathematical probl[em] and are sufficiently familiar with to[ols] appropriate for their grade or cours[e to] make sound decisions about when each of these tools might be helpfu[l].

Additional Answers

16. (3, −5), (2, −3)

17. (−3, −6), (3, 6)

18. (−2, 4), (10, 10)

19. (−1, −3), (8, 5)

20. $(-2, -\sqrt{3}), (-2, \sqrt{3}), (2, -\sqrt{3}), (2, \sqrt{3})$

21. $(1, -\sqrt{15}), (1, \sqrt{15})$

22. $(-\sqrt{5}, -\sqrt{3}), (-\sqrt{5}, \sqrt{3}), (\sqrt{5}, -\sqrt{3}), (\sqrt{5}, \sqrt{3})$

23. $(-\sqrt{2}, 4), (\sqrt{2}, 4)$

25. (5, −6), (5, 6), (−3, −2), (−3,

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	14–41, 52–74	15–41 odd, 55–58	14–40 even, 52–54, 59–74
OL Core	15–41 odd, 42–50, 52–74	14–41, 55–58	42–50, 52–54, 59–74
BL Advanced	42–70, (optional: 71–74)		

Teaching the Mathematical Practices

Arguments Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. And they are able to analyze situations by breaking them into cases, and can recognize and use counterexamples.

B **42.** **ADVERTISING** The corporate logo for an automobile manufacturer is shown at the right. Write a system of three equations to model this logo.
42. Sample answer: $x^2 + y^2 = 45$, $y = 3|x - 3| - 6$, $y = 3|x + 3| - 6$

Write a system of equations that satisfies each condition.

43. a circle and an ellipse that intersect at one point

44. a parabola and an ellipse that intersect at two points

45. a hyperbola and a circle that do not intersect

46. an ellipse and a parabola that intersect at three points

47. an ellipse and a hyperbola that intersect at four points **43–47. See Chapter 8 Answer Appendix.**

48. **FINANCIAL LITERACY** Prices are often set on an equilibrium curve, where the supply of a certain product equals its corresponding demand by consumers. An economist represents the supply of a product with $y = p^2 + 10p$ and the corresponding demand with $y = -p^2 + 40p$, where p is the price. Determine the equilibrium price. **AED 15**

49. **PAINTBALL** The shape of a paintball field is modeled by $x^2 + 4y^2 = 10{,}000$ in yards where the center is at the origin. The teams are provided with short-range walkie-talkies with a maximum range of 80 yards. Are the teams capable of hearing each other anywhere on the field? Explain your reasoning graphically.
See Chapter 8 Answer Appendix.

C **50.** **MOVING** Laila is moving to a new city and needs for the location of her new home to satisfy the following conditions. **See Chapter 8 Answer Appendix.**

- It must be less than 10 kilometers from the office where she will work.
- Because of the terrible smell of the local paper mill, it must be at least 15 kilometers away from the mill.

If the paper mill is located 9.5 kilometers east and 6 kilometers north of Laila's office, write and graph a system of inequalities to represent the area(s) were she should look for a home.

54. Sample answer: By sketching a quick graph of a quadratic system, you can determine how many real solutions there are as well as estimate the values of the solutions. This can help you when confirming the answers that you get algebraically.

H.O.T. Problems Use Higher-Order Thinking Skills

51. **CHALLENGE** Find all values of k for which the following system of equations has two solutions. **$k = a$ or $k = b$**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad x^2 + y^2 = k^2$$

52. **ARGUMENTS** When the vertex of a parabola lies on an ellipse, how many solutions can the quadratic system represented by the two graphs have? Explain your reasoning using graphs. **See Chapter 6 Answer Appendix.**

53. **OPEN ENDED** Write a system of equations, one a hyperbola and the other an ellipse, for which a solution is $(-4, 8)$. **Sample answer:** $\frac{y^2}{128} + \frac{x^2}{32} = 1$ **and** $\frac{x^2}{8} - \frac{y^2}{64} = 1$

54. **WRITING IN MATH** Explain how sketching the graph of a quadratic system can help you solve it.

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Standardized Test Practice

55. SHORT RESPONSE Solve.

$$4x - 3y = 0$$
$$x^2 + y^2 = 25 \quad \mathbf{(-3, -4), (3, 4)}$$

56. You have 16 stamps. Some are postcard stamps that cost AED 0.23, and the rest cost AED 0.41. If you spent a total of AED 5.30 on the stamps, how many postcard stamps do you have? **A**

A 7
B 8
C 9
D 10

57. Maysa received a promotion and a 7.2% raise. Her new salary is AED 53,600 a year. What was her salary before the raise? **F**

F AED 50,000
G AED 53,600
H AED 55,000
J AED 57,500

58. SAT/ACT When a number is multiplied by $\frac{2}{3}$, the result is 188. Find the number. **B**

A 292
B 282
C 272
D 262
E $125\frac{1}{3}$

Spiral Review

Match each equation with the situation that it could represent. (Lesson 6-6)

a. $9x^2 + 4y^2 - 36 = 0$
b. $0.004x^2 - x + y - 3 = 0$
c. $x^2 + y^2 - 20x + 30y - 75 = 0$

59. SPORTS the flight of a baseball **b**

60. PHOTOGRAPHY the oval opening in a picture frame **a**

61. GEOGRAPHY the set of all points 20 kilometers from a landmark **c**

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola. (Lesson 6-5) **62–64. See margin.**

62. $\frac{y^2}{16} - \frac{x^2}{25} = 1$

63. $\frac{(y-3)^2}{25} - \frac{(x-2)^2}{16} = 1$

64. $6y^2 = 2x^2 + 12$

Simplify each expression.

65. $\frac{12p^2 + 6p - 6}{4(p+1)^2} \div \frac{6p - 3}{2p + 10}$ $\mathbf{\frac{p+5}{p+1}}$

66. $\frac{x^2 + 6x + 9}{x^2 + 7x + 6} \div \frac{4x + 12}{3x + 3}$ $\mathbf{\frac{3x+9}{4x+24}}$

67. $\frac{r^2 + 2r - 8}{r^2 + 4r + 3} \div \frac{r - 2}{3r + 3}$ $\mathbf{\frac{3(r+4)}{r+3}}$

Graph each function. State the domain and range. **68–70. See Chapter 6 Answer Appendix.**

68. $f(x) = -\left(\frac{1}{5}\right)^x$

69. $y = -2.5(5)^x$

70. $f(x) = 2\left(\frac{1}{3}\right)^x$

Skills Review

Solve each equation or formula for the specified variable. **73.** $\mathbf{\frac{3V}{\pi r^2} = h}$ **74.** $\mathbf{\frac{2A}{h} - a = b}$

71. $d = rt$, for r $\mathbf{\frac{d}{t} = r}$

72. $x = \frac{-b}{2a}$, for a $\mathbf{a = \frac{-b}{2x}}$

73. $V = \frac{1}{3}\pi r^2 h$, for h

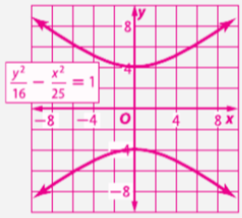
74. $A = \frac{1}{2}h(a + b)$, for b

4 Assess

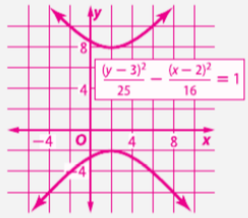
Yesterday's News Ask students to write how yesterday's lesson on co equations helped them with today's lesson.

Additional Answers

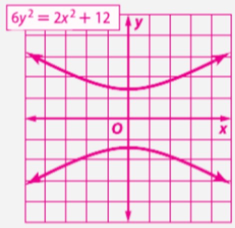
62. $(0, \pm4)$; $(0, \pm\sqrt{41})$; $y = \pm\frac{4}{5}x$



63. $(2, -2), (2, 8)$; $(2, 3 \pm \sqrt{41})$; $y - 3 = \pm\frac{5}{4}(x - 2)$



64. $(0, \pm\sqrt{2})$; $(0, \pm2\sqrt{2})$; $y = \pm\frac{\sqrt{3}}{3}x$;



Differentiated Instruction OL BL

Extension Ask students to solve this system of inequalities by graphing.

$y \geq x^2$
$x^2 + (y - 3)^2 \leq 4$
$y > -x + 2$

Uncorrected first proof - for training purposes only

1 Focus

VerticalAlignment

Before Lesson 6-7 Identify and graph conic sections.

Lesson 6-7 Find rotation of axes to write equations of rotated conic sections.
Graph rotated conic sections.

After Lesson 6-7 Write and graph the polar equation of a conic section.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- Would both elliptical gears shown turn at the same rate? Explain. No, the driver gear would turn at a constant rate while the driven gear would change speeds constantly with each revolution.

LESSON 6-7 Rotations of Conic Sections

Then

- You identified and graphed conic sections.

Now

1. Find rotation of axes to write equations of rotated conic sections.
2. Graph rotated conic sections.

Why?

- Elliptical gears are paired by rotating them about their foci. The driver gear turns at a constant speed, and the driven gear changes its speed continuously during each revolution.

1 Rotations of Conic Sections

In the previous lesson, you learned that when a conic section is vertical or horizontal with its axes parallel to the x- and y-axis, $B = 0$ in its general equation. The equation of such a conic does not contain an xy-term.

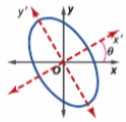
$Ax^2 + Cy^2 + Dx + Ey + F = 0$ — Axes of conic are parallel to coordinate axes.

In this lesson, you will examine conics with axes that are rotated and no longer parallel to the coordinate axes. In the general equation for such rotated conics, $B \neq 0$, so there is an xy-term.

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ — Axes of conic are rotated from coordinate axes.

If the xy-term were eliminated, the equation of the rotated conic could be written in standard form by completing the square. To eliminate this term, we rotate the coordinate axes until they are parallel to the axes of the conic.

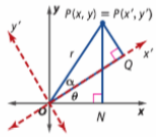
When the coordinate axes are rotated through an angle θ as shown, the origin remains fixed and new axes x' and y' are formed. The equation of the conic in the new $x'y'$-plane has the following general form.



$A(x')^2 + C(y')^2 + Dx' + Ey' + F = 0$ — Equation in $x'y'$-plane

Trigonometry can be used to develop formulas relating a point $P(x, y)$ in the xy-plane and $P(x', y')$ in the $x'y'$ plane.

Consider the figure at the right. Notice that in right triangle PNO, $OP = r$, $ON = x$, $PN = y$, and $m\angle NOP = \alpha + \theta$. Using $\triangle PNO$, you can establish the following relationships.



$x = r\cos(\alpha + \theta)$ — Cosine ratio
$\quad = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$ — Cosine Sum Identity

$y = r\sin(\alpha + \theta)$ — Sine ratio
$\quad = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$ — Sine Sum Identity

Using right triangle POQ, in which $OP = r$, $OQ = x'$, $PQ = y'$, and $m\angle QOP = \alpha$, you can establish the relationships $x' = r\cos\alpha$ and $y' = r\sin\alpha$. Substituting these values into the previous equations, you obtain the following.

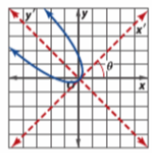
$x = x'\cos\theta - y'\sin\theta$ $\qquad$ $y = y'\cos\theta + x'\sin\theta$

KeyConcept Rotation of Axes of Conics

An equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ in the xy-plane can be rewritten as $A(x')^2 + C(y')^2 + Dx' + Ey' + F = 0$ in the rotated $x'y'$-plane.

The equation in the $x'y'$-plane can be found using the following equations, where θ is the angle of rotation.

$x = x'\cos\theta - y'\sin\theta$ $\qquad$ $y = x'\sin\theta + y'\cos\theta$



Additional Example

2 Using a suitable angle of rotation for the conic with equation $x^2 - 4xy - 2y^2 - 6 = 0$, write the equation in standard form.

$\frac{(y')^2}{3} - \frac{(x')^2}{2} = 1$

Tips for New Teachers

Trigonometric Identities The identities shown in Example 2 can be utilized to find $\sin\theta$ and $\cos\theta$ when the angle of rotation is not given in the problem.

Teach with Tech

Journal Have students write an entry in a journal explaining how to rotate the coordinate axes of a conic through an angle θ from the xy-plane to the $x'y'$-plane and from the $x'y'$-plane to the xy-plane. Be sure that students include the equations for x, y, x', and y' in their explanations.

Example 2 Write an Equation in Standard Form

Using a suitable angle of rotation for the conic with equation $8x^2 + 12xy + 3y^2 = 4$, write the equation in standard form.

The conic is a hyperbola because $B^2 - 4AC > 0$. Find θ.

$\cot 2\theta = \frac{A-C}{B}$ — Rotation of the axes

$= \frac{5}{12}$ — $A = 8$, $B = 12$, and $C = 3$

The figure illustrates a triangle for which $\cot 2\theta = \frac{5}{12}$. From this, $\sin 2\theta = \frac{12}{13}$ and $\cos 2\theta = \frac{5}{13}$.

Use the half-angle identities to determine $\sin\theta$ and $\cos\theta$.

$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$ — Half-Angle Identities — $\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$

$= \sqrt{\frac{1-\frac{5}{13}}{2}}$ — $\cos 2\theta = \frac{5}{13}$ — $= \sqrt{\frac{1+\frac{5}{13}}{2}}$

$= \frac{2\sqrt{13}}{13}$ — Simplify. — $= \frac{3\sqrt{13}}{13}$

StudyTip

$x'y'$ **Term** When you correctly substitute values of x' and y' in for x and y, the coefficient of the $x'y'$ term will become zero. If the coefficient of this term is not zero, then an error has occurred.

Next, find the equations for x and y.

$x = x'\cos\theta - y'\sin\theta$ — Rotation equations for x and y — $y = x'\sin\theta + y'\cos\theta$

$= \frac{3\sqrt{13}}{13}x' - \frac{2\sqrt{13}}{13}y'$ — $\sin\theta = \frac{2\sqrt{13}}{13}$ and $\cos\theta = \frac{3\sqrt{13}}{13}$ — $= \frac{2\sqrt{13}}{13}x' + \frac{3\sqrt{13}}{13}y'$

$= \frac{3\sqrt{13}x' - 2\sqrt{13}y'}{13}$ — Simplify. — $= \frac{2\sqrt{13}x' + 3\sqrt{13}y'}{13}$

Substitute these values into the original equation.

$$8x^2 + 12xy + 3y^2 = 4$$

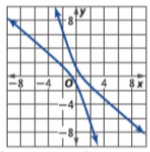
$$8\left(\frac{3\sqrt{13}x' - 2\sqrt{13}y'}{13}\right)^2 + 12\,\frac{3\sqrt{13}x' - 2\sqrt{13}y'}{13}\cdot\frac{2\sqrt{13}x' + 3\sqrt{13}y'}{13} + 3\left(\frac{2\sqrt{13}x' + 3\sqrt{13}y'}{13}\right)^2 = 4$$

$$\frac{72(x')^2 - 96x'y' + 32(y')^2}{13} + \frac{72(x')^2 + 60x'y' - 72(y')^2}{13} + \frac{12(x')^2 + 36x'y' + 27(y')^2}{13} = 4$$

$$\frac{156(x')^2 - 13(y')^2}{13} = 4$$

$$3(x')^2 - \frac{(y')^2}{4} = 1$$

The standard form of the equation in the $x'y'$-plane is $\frac{(x')^2}{\frac{1}{3}} - \frac{(y')^2}{4} = 1$. The graph of this hyperbola is shown.



GuidedPractice

Using a suitable angle of rotation for the conic with each given equation, write the equation in standard form.

2A. $2x^2 - 12xy + 18y^2 - 4y = 2$ **2A.** $\left(y' - \frac{3\sqrt{10}}{100}\right)^2 = \frac{\sqrt{10}}{50}\left(x' + \frac{109\sqrt{10}}{200}\right)$

2B. $20x^2 + 20xy + 5y^2 - 12x - 36y - 200 = 0$ $\left(x' - \frac{6\sqrt{5}}{25}\right)^2 = \frac{12\sqrt{5}}{25}\left(y' + \frac{259\sqrt{5}}{75}\right)$

380 | Lesson 6-7 | Rotations of Conic Sections

Two other formulas relating x' and y' to x and y can be used to find an equation in the xy-plane for a rotated conic.

KeyConcept Rotation of Axes of Conics

When an equation of a conic section is rewritten in the $x'y'$-plane by rotating the coordinate axes through θ, the equation in the xy-plane can be found using

$$x' = x\cos\theta + y\sin\theta, \text{ and } y' = y\cos\theta - x\sin\theta.$$

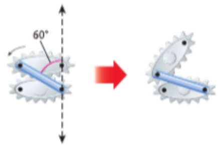
Real-WorldLink

In a system of gears where both gears spin, such as a bicycle, the speed of the gears in relation to each other is related to their size. If the diameter of one of the gears is $\frac{1}{2}$ of the diameter of the second gear, the first gear will rotate twice as fast as the second gear.

Source: *How Stuff Works*

Example 3 Write an Equation in the xy-Plane

PHYSICS Elliptical gears can be used to generate variable output speeds. After a 60° rotation, the equation for the rotated gear in the $x'y'$-plane is $\frac{(x')^2}{36} + \frac{(y')^2}{18} = 1$. Write an equation for the ellipse formed by the rotated gear in the xy-plane.



Use the rotation formulas for x' and y' to find the equation of the rotated conic in the xy-plane.

$x' = x\cos\theta + y\sin\theta$	Rotation equations for x' and y'	$y' = y\cos\theta - x\sin\theta$
$= x\cos 60° + y\sin 60°$	$\theta = 60°$	$= y\cos 60° - x\sin 60°$
$= \frac{1}{2}x + \frac{\sqrt{3}}{2}y$	$\sin 60° = \frac{1}{2}$ and $\cos 60° = \frac{\sqrt{3}}{2}$	$= \frac{1}{2}y - \frac{\sqrt{3}}{2}x$

Substitute these values into the original equation.

$$\frac{(x')^2}{36} + \frac{(y')^2}{18} = 1 \quad \text{Original equation}$$

$$(x')^2 + 2(y')^2 = 36 \quad \text{Multiply each side by 36.}$$

$$\left(\frac{x+\sqrt{3}y}{2}\right)^2 + 2\left(\frac{y-\sqrt{3}x}{2}\right)^2 = 36 \quad \text{Substitute.}$$

$$\frac{x^2 + 2\sqrt{3}xy + 3y^2}{4} + \frac{2y^2 - 4\sqrt{3}xy + 6x^2}{4} = 36 \quad \text{Simplify.}$$

$$\frac{7x^2 - 2\sqrt{3}xy + 5y^2}{4} = 36 \quad \text{Combine like terms.}$$

$$7x^2 - 2\sqrt{3}xy + 5y^2 = 144 \quad \text{Multiply each side by 4.}$$

$$7x^2 - 2\sqrt{3}xy + 5y^2 - 144 = 0 \quad \text{Subtract 144 from each side.}$$

The equation of the rotated ellipse in the xy-plane is $7x^2 - 2\sqrt{3}xy + 5y^2 - 144 = 0$.

GuidedPractice

3. If the equation for the gear after a 30° rotation in the $x'y'$-plane is $(x')^2 + 4(y')^2 - 40 = 0$, find the equation for the gear in the xy-plane. $\frac{7}{4}x^2 - \frac{3\sqrt{3}}{2}xy + \frac{13}{4}y^2 - 40 = 0$

2 Graph Rotated Conics

When the equations of rotated conics are given for the $x'y'$-plane, they can be graphed by finding points on the graph of the conic and then converting these points to the xy-plane.

Additional Example

3 **PHYSICS** If the equation for an elliptical gear after a 45° rotatio in the $x'y'$-plane is $\frac{(x')^2}{16} + \frac{(y')^2}{8} = 1$, find the equation for the gear in the xy-plane. $3x^2 - 2xy + 3y^2 -$ $32 = 0$

Focus on Mathematical Conte

The xy-term The presence of a nonzero xy-term indicates rotation the graph of the conic section in the plane. For example, in the case of a ellipse, the major axis is no longer parallel to the x-axis or y-axis. The major axis is rotated by an amount depends on A, B, and C, which is ze when $B = 0$.

2 Graph Rotated Conics

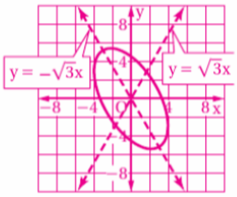
Example 4 shows how to graph a c using rotations. **Example 5** shows h to graph a conic with an xy-term usi the Quadratic Formula and a graphi calculator.

DifferentiatedInstruction AL OL BL

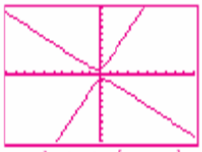
Interpersonal Learners Divide the class into groups of three or four, mixing abilities. After working through Examples 1–3 with the class, have groups work together to complete the Guided Practice Exercises for each example. Have them compare their results with another group and discuss any discrepancies. Ask each group to share its results with the class for each problem. Discuss with the whole class any questions, difficulties, or discrepancies that were found.

Additional Examples

4 Graph $\frac{(x')^2}{36} + \frac{(y')^2}{9} = 1$ if it has been rotated 60° from its position in the *xy*-plane.



5 Use a graphing calculator to graph the conic given by $8x^2 + 5xy - 4y^2 = -2$.



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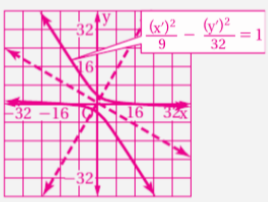
Tips for New Teachers

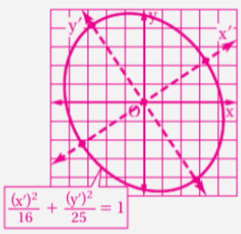
Graphing Calculators When using a graphing calculator to graph a conic, first solve the equation for *y* using the Quadratic Formula. Enter the equation with a positive square root as the first equation and then the negative square root as the second equation. Graph both equations on the same screen to see the whole graph.

Additional Answers (Guided Practice)

4A.



4B.



Example 4 Graph a Conic Using Rotations

Graph $(x' - 2)^2 = 4(y' - 3)$ if it has been rotated 30° from its position in the *xy*-plane.

The equation represents a parabola, and it is in standard form. Use the vertex (2, 3) and axis of symmetry $x' = 2$ in the $x'y'$-plane to determine the vertex and axis of symmetry for the parabola in the *xy*-plane.

Find the equations for *x* and *y* for $\theta = 30°$.

$x = x'\cos\theta - y'\sin\theta$	Rotation equations for *x* and *y*	$y = x'\sin\theta + y'\cos\theta$
$= \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$	$\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$	$= \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$

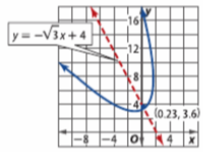
Use the equations to convert the $x'y'$-coordinates of the vertex into *xy*-coordinates.

$x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$	Conversion equation	$y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$
$= \frac{\sqrt{3}}{2}(2) - \frac{1}{2}(3)$	$x' = 2$ and $y' = 3$	$= \frac{1}{2}(2) + \frac{\sqrt{3}}{2}(3)$
$= \sqrt{3} - \frac{3}{2}$ or about 0.23	Multiply.	$= 1 + \frac{3\sqrt{3}}{2}$ or about 3.60

Find the equation for the axis of symmetry.

$x' = x\cos\theta + y\sin\theta$	Conversion equation
$2 = \frac{\sqrt{3}}{2}x + \frac{1}{2}y$	$\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$
$y = -\sqrt{3}x + 4$	Solve for *y*.

The new vertex and axis of symmetry can be used to sketch the graph of the parabola in the *xy*-plane.



StudyTip

Graphing Convert other points on the conic from $x'y'$-coordinates to *xy*-coordinates. Then make a table of these values to complete the sketch of the conic.

GuidedPractice **Graph each equation at the indicated angle.** **4A–B. See margin.**

4A. $\frac{(x')^2}{9} - \frac{(y')^2}{32} = 1$; 60°

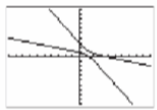
4B. $\frac{(x')^2}{16} + \frac{(y')^2}{25} = 1$; 30°

One method of graphing conic sections with an *xy*-term is to solve the equation for *y* and graph with a calculator. Write the equation in quadratic form and then use the Quadratic Formula.

Example 5 Graph a Conic in Standard Form

Use a graphing calculator to graph the conic given by $4y^2 + 8xy - 60y + 2x^2 - 40x + 155 = 0$.

$4y^2 + 8xy - 60y + 2x^2 - 40x + 155 = 0$	Original equation
$4y^2 + (8x - 60)y + (2x^2 - 40x + 155) = 0$	Quadratic form
$y = \frac{-(8x - 60) \pm \sqrt{(8x - 60)^2 - 4(4)(2x^2 - 40x + 155)}}{2(4)}$	$a = 4$, $b = 8x - 60$, and $c = 2x^2 - 40x + 155$
$= \frac{-8x + 60 \pm \sqrt{32x^2 - 320x + 1120}}{8}$	Multiply and combine like terms.
$= \frac{-8x + 60 \pm 4\sqrt{2x^2 - 20x + 70}}{8}$	Factor out $\sqrt{16}$.
$= \frac{-2x + 15 \pm \sqrt{2x^2 - 20x + 70}}{2}$	Divide each term by 4.



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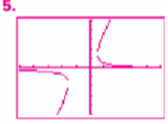
Graphing both of these equations on the same screen yields the hyperbola shown.

5.



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StudyTip

Arranging Terms Arrange the terms in descending powers of *y* in order to convert the equation to quadratic form.

GuidedPractice

5. Use a graphing calculator to graph the conic given by $4x^2 - 6xy + 2y^2 - 60x - 20y + 275 = 0$.

Additional Answers

1. $(x')^2 + 2\sqrt{3}x'y' - (y')^2 + 18 = 0$; hyperbola
2. $(x')^2 - (y')^2 + 16 = 0$; hyperbola
3. $(y')^2 - 8x' = 0$; parabola
4. $(x')^2 + (y')^2 - 4 = 0$; circle
5. $(x')^2 + 2\sqrt{3}x'y' + 3(y')^2 + 16\sqrt{3}x' - 16y' = 0$; parabola
6. $21(x')^2 + 10\sqrt{3}x'y' + 31(y')^2 - 144 = 0$; ellipse
7. $2(x')^2 + 2(y')^2 - 5\sqrt{2}x + 5\sqrt{2}y - 6 = 0$; ellipse
8. $33(x')^2 - 130x'y' + 33(y')^2 - 1568 = 0$; hyperbola
9. $25(x')^2 - 4(y')^2 + 64 = 0$; hyperbola
10. $23(x')^2 - 2\sqrt{3}x'y' + 21(y')^2 - 120 = 0$; ellipse

Exercises

Write each equation in the $x'y'$-plane for the given value of θ. Then identify the conic. (Example 1)

1. $x^2 - y^2 = 9$, $\theta = \frac{\pi}{3}$ **1–10. See margin.**
2. $xy = -8$, $\theta = 45°$
3. $x^2 - 8y = 0$, $\theta = \frac{\pi}{2}$
4. $2x^2 + 2y^2 = 8$, $\theta = \frac{\pi}{6}$
5. $y^2 + 8x = 0$, $\theta = 30°$
6. $4x^2 + 9y^2 = 36$, $\theta = 30°$
7. $x^2 - 5x + y^2 = 3$, $\theta = 45°$
8. $49x^2 - 16y^2 = 784$, $\theta = \frac{\pi}{4}$
9. $4x^2 - 25y^2 = 64$, $\theta = 90°$
10. $6x^2 + 5y^2 = 30$, $\theta = 30°$

Using a suitable angle of rotation for the conic with each given equation, write the equation in standard form. (Example 1) **15–18. See margin.**

11. $xy = -4$ $\frac{(y')^2}{8} - \frac{(x')^2}{8} = 1$
12. $x^2 - xy + y^2 = 2$ **12.** $\frac{(x')^2}{4} + \frac{(y')^2}{\frac{4}{3}} = 1$
13. $145x^2 + 120xy + 180y^2 = 900$ $\frac{(x')^2}{4} + \frac{(y')^2}{9} = 1$
14. $16x^2 - 24xy + 9y^2 - 5x - 90y + 25 = 0$ $(y' - 1)^2 = 3x'$
15. $2x^2 - 72xy + 23y^2 + 100x - 50y = 0$
16. $x^2 - 3y^2 - 8x + 30y = 60$
17. $8x^2 + 12xy + 3y^2 + 4 = 6$
18. $73x^2 + 72xy + 52y^2 + 25x + 50y - 75 = 0$

Write an equation for each conic in the xy-plane for the given equation in $x'y'$ form and the given value of θ. (Example 3)

19. $(x')^2 + 3(y')^2 = 8$, $\theta = \frac{\pi}{4}$ $x^2 - xy + y^2 - 4 = 0$
20. $\frac{(x')^2}{25} - \frac{(y')^2}{225} = 1$, $\theta = \frac{\pi}{4}$ $4x^2 + 10xy + 4y^2 - 225 = 0$
21. $\frac{(x')^2}{9} - \frac{(y')^2}{36} = 1$, $\theta = \frac{\pi}{3}$ $x^2 + 10\sqrt{3}xy + 11y^2 - 144 = 0$
22. $(x')^2 = 8y'$, $\theta = 45°$ $x^2 + 2xy + y^2 + 8\sqrt{2}x - 8\sqrt{2}y = 0$
23. $\frac{(x')^2}{7} + \frac{(y')^2}{28} = 1$, $\theta = \frac{\pi}{6}$ $13x^2 + 6\sqrt{3}xy + 7y^2 - 112 = 0$
24. $4x' = (y')^2$, $\theta = 30°$ $x^2 - 2\sqrt{3}xy + 3y^2 - 8\sqrt{3}x - 8y = 0$
25. $\frac{(x')^2}{64} - \frac{(y')^2}{16} = 1$, $\theta = 45°$ $3x^2 - 10xy + 3y^2 + 128 = 0$
26. $(x')^2 = 5y'$, $\theta = \frac{\pi}{3}$ $x^2 + 2\sqrt{3}xy + 3y^2 + 10\sqrt{3}x - 10y = 0$
27. $\frac{(x')^2}{4} - \frac{(y')^2}{9} = 1$, $\theta = 30°$ $23x^2 + 26\sqrt{3}xy - 3y^2 - 144 = 0$
28. $\frac{(x')^2}{3} + \frac{(y')^2}{4} = 1$, $\theta = 60°$ $13x^2 + 2\sqrt{3}xy + 15y^2 - 48 = 0$

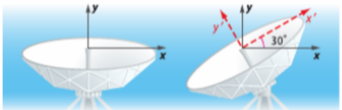
29. **ASTRONOMY** Suppose $144(x')^2 + 64(y')^2 = 576$ models the shape in the $x'y'$-plane of a reflecting mirror in a telescope. (Example 4) **a–b. See margin.**
    a. If the mirror has been rotated 30°, determine the equation of the mirror in the xy-plane.
    b. Graph the equation.

30–35. See Chapter 6 Answer Appendix.

Graph each equation at the indicated angle.

30. $\frac{(x')^2}{4} + \frac{(y')^2}{9} = 1$; 60°
31. $\frac{(x')^2}{25} - \frac{(y')^2}{36} = 1$; 45°
32. $(x')^2 + 6x' - y' = -9$; 30°
33. $8(x')^2 + 6(y')^2 = 24$; 30°
34. $\frac{(x')^2}{4} - \frac{(y')^2}{16} = 1$; 45°
35. $y' = 3(x')^2 - 2x' + 5$; 60°

36. **COMMUNICATION** A satellite dish tracks a satellite directly overhead. Suppose $y = \frac{1}{6}x^2$ models the shape of the dish when it is oriented in this position. Later in the day, the dish is observed to have rotated approximately 30°. (Example 4) **a–b. See margin.**
    a. Write an equation that models the new orientation of the dish.
    b. Use a graphing calculator to graph both equations on the same screen. Sketch this graph on your paper.



37–46. See Chapter 6 Answer Appendix.

GRAPHING CALCULATOR Graph the conic given by each equation. (Example 5)

37. $x^2 - 2xy + y^2 - 5x - 5y = 0$
38. $2x^2 + 9xy + 14y^2 = 5$
39. $8x^2 + 5xy - 4y^2 = -2$
40. $2x^2 + 4\sqrt{3}xy + 6y^2 + 3x = y$
41. $2x^2 + 4xy + 2y^2 + 2\sqrt{2}x - 2\sqrt{2}y = -12$
42. $9x^2 + 4xy + 6y^2 = 20$
43. $x^2 + 10\sqrt{3}xy + 11y^2 - 64 = 0$
44. $x^2 + y^2 - 4 = 0$
45. $x^2 - 2\sqrt{3}xy - y^2 + 18 = 0$
46. $2x^2 + 9xy + 14y^2 - 5 = 0$

B

The graph of each equation is a degenerate case. Describe the graph.

47. $y^2 - 16x^2 = 0$ intersecting lines $y = 4x$ and $y = -4x$
48. $(x + 4)^2 - (x - 1)^2 = y + 8$ line $y = 10x + 7$
49. $(x + 3)^2 + y^2 + 6y + 9 - 6(x + y) = 18$ point at (0, 0)

3 Practice

Formative Assessment

Use Exercises 1–46 to check for understanding.

Then use the table below to custom[ize] your assignments for students.

WatchOut!

Common Error For Exercises 19–28, make sure students use the trigonometric equations $x' = x\cos\theta + y\sin\theta$ and $y' = y\cos\theta - x\sin\theta$ to get values to substitute in for x' and y'.

Additional Answers

15. $\frac{(y' - 1)^2}{\frac{1}{2}} - \frac{(x' - 1)^2}{1} = 1$

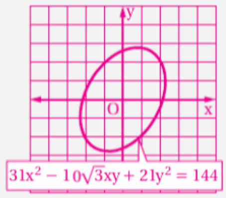
16. $\frac{(x - 4)^2}{1} - \frac{(y - 5)^2}{\frac{1}{3}} = 1$

17. $\frac{(x')^2}{\frac{1}{6}} - \frac{(y')^2}{2} = 1$

18. $\frac{\left(x' + \frac{1}{4}\right)^2}{\frac{7}{8}} + \frac{\left(y' + \frac{1}{2}\right)^2}{\frac{7}{2}} = 1$

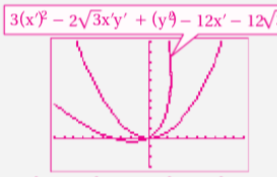
29a. $31x^2 - 10\sqrt{3}xy + 21y^2 = 14$[4]

29b.



36a. $3(x')^2 - 2\sqrt{3}x'y' + (y')^2 -$ … $- 12\sqrt{3}y' = 0$

36b.



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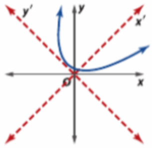
Differentiated Homework Options OL BL AL

Level	Assignment	Two-Day Option	
AL Approaching Level	1–46, 60, 62–82	1–45 odd, 79–82	2–46 even, 60, 62–78
OL On Level	1–53 odd, 54, 55–59 odd, 60, 62–82	1–46, 79–82	47–60, 62–78
BL Beyond Level	47–82		

Match the graph of each conic with its equation.

50. 
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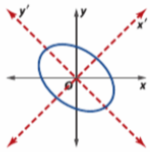
51. 
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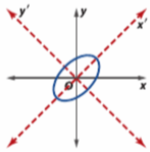
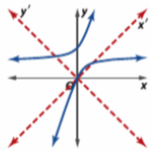
52. 
a

53. 
c

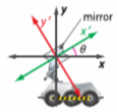
a. $x^2 - xy + y^2 = 2$

b. $145x^2 + 120xy + 180y^2 - 900 = 0$

c. $2x^2 - 72xy + 23y^2 + 100x - 50y = 0$

d. $16x^2 - 24xy + 9y^2 - 5x - 90y + 25 = 0$

54. **ROBOTICS** A hyperbolic mirror used in robotic systems is attached to the robot so that it is facing to the right. After it is rotated, the shape of its new position is represented by $51.75x^2 + 184.5\sqrt{3}xy - 132.75y^2 = 32{,}400$.



a. Solve the equation for y. **a–b. See margin.**

b. Use a graphing calculator to graph the equation.

c. Determine the angle θ through which the mirror has been rotated. Round to the nearest degree. **30°**

55. **INVARIANTS** When a rotation transforms an equation from the xy-plane to the $x'y'$-plane, the new equation is equivalent to the original equation. Some values are invariant under the rotation, meaning their values do not change when the axes are rotated. Use reasoning to explain how $A + C = A' + C'$ is a rotation invariant.
See Chapter 6 Answer Appendix.

GRAPHING CALCULATOR Graph each pair of equations and find any points of intersection. If the graphs have no points of intersection, write *no solution*. **56–58. See margin.**

56. $x^2 + 2xy + y^2 - 8x - y = 0$
$8x^2 + 3xy - 5y^2 = 15$

57. $9x^2 + 4xy + 5y^2 - 40 = 0$
$x^2 - xy - 2y^2 - x - y + 2 = 0$

58. $x^2 + \sqrt{3}xy - 3 = 0$
$16x^2 - 20xy + 9y^2 = 40$

59. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate angles of rotation that produce the original graphs.

a. **TABULAR** For each equation in the table, identify the conic and find the minimum angle of rotation needed to transform the equation so that the rotated graph coincides with its original graph.

Equation	Conic	Minimum Angle of Rotation
$x^2 - 5x + 3 - y = 0$	par.	360°
$6x^2 + 10y^2 = 15$	ell.	180°
$2xy = 9$	hyp.	180°

See margin.

b. **VERBAL** Describe the relationship between the lines of symmetry of the conics and the minimum angles of rotation needed to produce the original graphs.

c. **ANALYTICAL** A noncircular ellipse is rotated 50° about the origin. It is then rotated again so that the original graph is produced. What is the second angle of rotation?

H.O.T. Problems Use Higher-Order Thinking Skills

60. **ERROR ANALYSIS** Mahmoud and Ahmed are describing the graph of $x^2 + 4xy + 6y^2 + 3x - 4y = 75$. Mahmoud says that it is an ellipse. Ahmed thinks it is a parabola. Is either of them correct? Explain your reasoning.
Leon; $B^2 - 4AC = 4^2 - 4(1)(6) < 0$ and $A \neq C$

61. **CHALLENGE** Show that a circle with the equation $x^2 + y^2 = r^2$ remains unchanged under any rotation θ.
See Chapter 6 Answer Appendix.

62. **REASONING** *True* or *false*: Every angle of rotation θ can be described as an acute angle. Explain.
True; see Chapter 6 Answer Appendix for explanation.

63. **PROOF** Prove $x' = x\cos\theta + y\sin\theta$ and $y' = y\cos\theta - x\sin\theta$. (*Hint:* Solve the system $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ by multiplying one equation by $\sin\theta$ and the other by $\cos\theta$.)
See Chapter 6 Answer Appendix.

64. **REASONING** The angle of rotation θ can also be defined as $\tan 2\theta = \frac{B}{A - C}$, when $A \neq C$, or $\theta = \frac{\pi}{4}$, when $A = C$. Why does defining the angle of rotation in terms of cotangent not require an extra condition with an additional value for θ? **See Chapter 6 Answer Appendix.**

65. **WRITING IN MATH** The discriminant can be used to classify a conic $A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$ in the $x'y'$-plane. Explain why the values of A' and C' determine the type of conic. Describe the parameters necessary for an ellipse, a circle, a parabola, and a hyperbola.
See Chapter 6 Answer Appendix.

66. **REASONING** *True* or *false*: Whenever the discriminant of an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is equal to zero, the graph of the equation is a parabola. Explain.
False; see Chapter 6 Answer Appendix for explanation.

Additional Answers

54a. $y = \dfrac{184.5\sqrt{3}x \pm \sqrt{129{,}600x^2 - 17{,}204{,}400}}{265.5}$

54b. 

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Spiral Review

Graph the hyperbola given by each equation. **67–69. See Chapter 6 Answer Appendix.**

67. $\frac{x^2}{9} - \frac{y^2}{64} = 1$

68. $\frac{y^2}{25} - \frac{x^2}{49} = 1$

69. $\frac{(x-3)^2}{64} - \frac{(y-7)^2}{25} = 1$

Determine the eccentricity of the ellipse given by each equation.

70. $\frac{(x+17)^2}{39} + \frac{(y+7)^2}{30} = 1$ **0.480**

71. $\frac{(x-6)^2}{12} + \frac{(y+4)^2}{15} = 1$ **0.447**

72. $\frac{(x-10)^2}{29} + \frac{(y+2)^2}{24} = 1$ **0.415**

73. **INVESTING** Mansour has a total of AED 5000 in his savings account and in a certificate of deposit. His savings account earns 3.5% interest annually. The certificate of deposit pays 5% interest annually if the money is invested for one year. Mansour calculates that his interest earnings for the year will be AED 227.50.

   a. Write a system of equations for the amount of money in each investment. **a. $s + d = 5000$, $0.035s + 0.05d = 227.50$**

   b. Use Cramer's Rule to determine how much money is in Mansour's savings account and in the certificate of deposit. **savings account: AED 1500; certificate of deposit: AED 3500**

74. **OPTICS** The amount of light that a source provides to a surface is called the *illuminance*. The illuminance E in foot candles on a surface that is R feet from a source of light with intensity I candelas is $E = \frac{I \cos \theta}{R^2}$, where θ is the measure of the angle between the direction of the light and a line perpendicular to the surface being illuminated. Verify that $E = \frac{I \cot \theta}{R^2 \csc \theta}$ is an equivalent formula.

$$\frac{I \cot \theta}{R^2 \csc \theta} = \frac{I \frac{\cos \theta}{\sin \theta}}{R^2 \frac{1}{\sin \theta}} = \frac{I \frac{\cos \theta}{\sin \theta}}{R^2 \frac{1}{\sin \theta}} \cdot \frac{\sin \theta}{\sin \theta} = \frac{I \cos \theta}{R^2} ✓$$

Solve each equation.

75. $\log_4 8n + \log_4 (n-1) = 2$ **2**

76. $\log_9 9p + \log_9 (p+8) = 2$ **1**

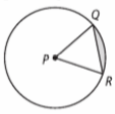
Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$.

77. $f(x) = x^4 - x^3 - 16x^2 + 4x + 48$; $(x-4)$, $(x-2)$ **yes, yes; $(x-4)(x-2)(x+2)(x+3)$**

78. $f(x) = 2x^4 + 9x^3 - 23x^2 - 81x + 45$; $(x+5)$, $(x+3)$ **yes, yes; $(x+5)(x+3)(x-3)(2x-1)$**

Skills Review for Standardized Tests

79. **SAT/ACT** P is the center of the circle and $PQ = QR$. If $\triangle PQR$ has an area of $9\sqrt{3}$ square units, what is the area of the shaded region in square units? **D**



A $24\pi - 9\sqrt{3}$

B $9\pi - 9\sqrt{3}$

C $18\pi - 9\sqrt{3}$

D $6\pi - 9\sqrt{3}$

E $12\pi - 9\sqrt{3}$

80. **REVIEW** Which is NOT the equation of a parabola? **H**

F $y = 2x^2 + 4x - 9$

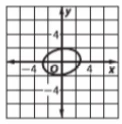
G $3x + 2y^2 + y + 1 = 0$

H $x^2 + 2y^2 + 8y = 8$

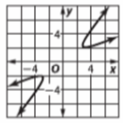
J $x = \frac{1}{2}(y-1)^2 + 5$

81. Which is the graph of the conic given by the equation $4x^2 - 2xy + 8y^2 - 7 = 0$? **A**

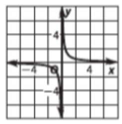
A



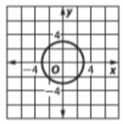
C



B



D



82. **REVIEW** How many solutions does the system $\frac{x^2}{5^2} - \frac{y^2}{3^2} = 1$ and $(x-3)^2 + y^2 = 9$ have? **H**

F 0

G 1

H 2

J 4

WatchOut!

Error Analysis In Exercise 60, students should use what they know about the discriminant to determine that Mahmoud is corre

4 Assess

Yesterday's News Have each stud write how yesterday's lesson on hyperbolas helped with today's concepts about rotating conics.

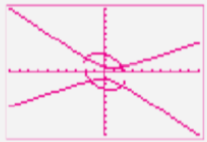
Additional Answers

56.

[−10, 10] scl: 1 by [−10, 10] scl:

(1.9, 2.3)

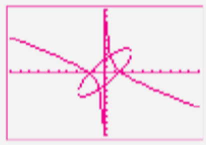
57.



[−10, 10] scl: 1 by [−10, 10] scl:

(−1.9, 2.2), (−1.5, −1.5), (1.9, −2.2), (1.9, 0.8)

58.



[−10, 10] scl: 1 by [−10, 10] scl:

(−1.6, −0.1), (0.6, 2.7), (−0.6, −2.7), (1.6, 0.1)

59b. Sample answer: A parabola h 1 line of symmetry and the minimum angle of rotation is complete circle. An ellipse an hyperbola have 2 lines of symmetry and the minimum a of rotation is a half circle.

Differentiated Instruction BL

Extension Have students write a system of two equations of conic sections with exactly two solutions. They should name the type of conic represented by each equation. Then they should graph the system and find the solutions of the system.

1 Focus

VerticalAlignment

Before Lesson 6-8 Model motion using quadratic functions.

Lesson 6-8 Graph parametric equations.
Solve problems related to the motion of projectiles.

After Lesson 6-8 Model situations with parametric equations.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What shape could be used to model the trajectory of a basketball shot toward the basket? parabola
- What do *trajectory* and *range* mean with regard to a basketball? Trajectory is the path along which the basketball is traveling. Range is the horizontal distance the basketball travels.

LESSON 6-8

Parametric Equations

Then

- You modeled motion using quadratic functions.

Now

1. Graph parametric equations.
2. Solve problems related to the motion of projectiles.

Why?

- You have used quadratic functions to model the paths of projectiles such as a tennis ball. Parametric equations can also be used to model and evaluate the trajectory and range of projectiles.

NewVocabulary
parametric equation
parameter
orientation
parametric curve

1 Graph Parametric Equations

So far in this text, you have represented the graph of a curve in the xy-plane using a single equation involving two variables, x and y. In this lesson you represent some of these same graphs using two equations by introducing a third variable.

Consider the graphs below, each of which models different aspects of what happens when a certain object is thrown into the air. Figure 6.5.1 shows the vertical distance the object travels as a function of time, while Figure 6.5.2 shows the object's horizontal distance as a function of time. Figure 6.5.3 shows the object's vertical distance as a function of its horizontal distance.

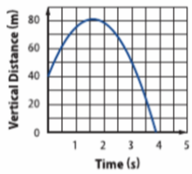


Figure 6.5.1

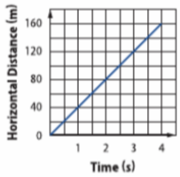


Figure 6.5.2

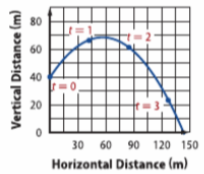


Figure 6.5.3

Each of these graphs and their equations tells part of what is happening in this situation, but not the whole story. To express the position of the object, both horizontally and vertically, as a function of time we can use **parametric equations**. The equations below both represent the graph shown in Figure 6.5.3.

Rectangular Equation

$y = -\frac{2}{225}x^2 + x + 40$

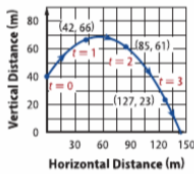
Parametric Equations

$x = 30\sqrt{2}t$ Horizontal component

$y = -16t^2 + 30\sqrt{2}t + 40$ Vertical component

From the parametric equations, we can now determine where the object was at a given time by evaluating the horizontal and vertical components for t. For example, when $t = 0$, the object was at (0, 40). The variable t is called a **parameter**.

The graph shown is plotted over the time interval $0 \le t \le 4$. Plotting points in the order of increasing values of t traces the curve in a specific direction called the **orientation** of the curve. This orientation is indicated by arrows on the curve as shown.



KeyConcept Parametric Equations

If f and g are continuous functions of t on the interval I, then the set of ordered pairs $(f(t), g(t))$ represent a **parametric curve**. The equations

$$x = f(t) \text{ and } y = g(t)$$

are parametric equations for this curve, t is the parameter, and I is the parameter interval.

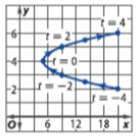
Example 1 Sketch Curves with Parametric Equations

Sketch the curve given by each pair of parametric equations over the given interval.

a. $x = t^2 + 5$ and $y = \frac{t}{2} + 4$; $-4 \le t \le 4$

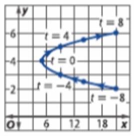
Make a table of values for $-4 \le t \le 4$. Then, plot the (x, y) coordinates for each t-value and connect the points to form a smooth curve. The arrows in the graph indicate the orientation of the curve as t moves from -4 to 4.

t	x	y	t	x	y
−4	21	2	1	6	4.5
−3	14	2.5	2	9	5
−2	9	3	3	14	5.5
−1	6	3.5	4	21	6
0	5	4			



b. $x = \frac{t^2}{4} + 5$ and $y = \frac{t}{4} + 4$; $-8 \le t \le 8$

t	x	y	t	x	y
−8	21	2	2	6	4.5
−6	14	2.5	4	9	5
−4	9	3	6	14	5.5
−2	6	3.5	8	21	6
0	5	4			



GuidedPractice **1A–B. See margin.**

1A. $x = 3t$ and $y = \sqrt{t} + 6$; $0 \le t \le 8$

1B. $x = t^2$ and $y = 2t + 3$; $-10 \le t \le 10$

> **StudyTip**
> **Plane Curves** Parametric equations can be used to represent curves that are not functions, as shown in Example 1.

Notice that the two different sets of parametric equations in Example 1 trace out the same curve. The graphs differ in their *speeds* or how rapidly each curve is traced out. If t represents time in seconds, then the curve in part **b** is traced in 16 seconds, while the curve in part **a** is traced out in 8 seconds.

Another way to determine the curve represented by a set of parametric equations is to write the set of equations in rectangular form. This can be done using substitution to eliminate the parameter.

Example 2 Write in Rectangular Form

Write $x = -3t$ and $y = t^2 + 2$ in rectangular form.

To eliminate the parameter t, solve $x = -3t$ for t. This yields $t = -\frac{1}{3}x$. Then substitute this value for t in the equation for y.

$y = t^2 + 2$ — Equation for y

$= \left(-\frac{1}{3}x\right)^2 + 2$ — Substitute $-\frac{1}{3}x$ for t.

$= \frac{1}{9}x^2 + 2$ — Simplify.

This set of parametric equations yields the parabola $y = \frac{1}{9}x^2 + 2$.

GuidedPractice

2. Write $x = t^2 - 5$ and $y = 4t$ in rectangular form. $y = 4\sqrt{x + 5}$

> **StudyTip**
> **Eliminating a Parameter** When you are eliminating a parameter to convert to rectangular form, you can solve either of the parametric equations first.

In Example 2, notice that a parameter interval for t was not specified. When not specified, the implied parameter interval is all values for t which produce real number values for x and y.

1 Graph Parametric Equation

Example 1 shows how to sketch the graph of parametric equations over given interval. **Examples 2–4** demonstrate how to write parametr equations with no restrictions, with restrictions, and with θ as a parame in rectangular form. **Example 5** sho how to write parametric equations given the equation in rectangular fo and one parameter.

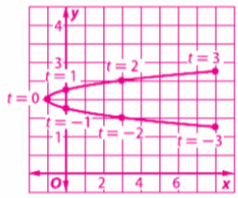
Formative Assessment

Use the Guided Practice exercises a each example to determine student understanding of concepts.

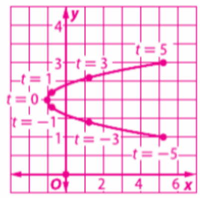
Additional Examples

1 Sketch the curve given by each pair of parametric equations ov the given interval.

a. $x = t^2 - 1$ and $y = \frac{t}{4} + 2$; $-3 \le t \le 3$



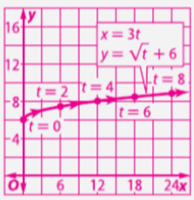
b. $x = \frac{t^2}{4} - 1$ and $y = \frac{t}{5} + 2$; $-5 \le t \le 5$



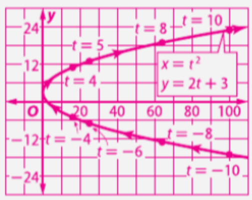
2 Write $x = t^2 + 2$ and $y = 2t$ in rectangular form. $y = 2\sqrt{x -$

Additional Answers (Guided Practice)

1A.

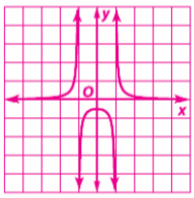


1B.



Additional Examples

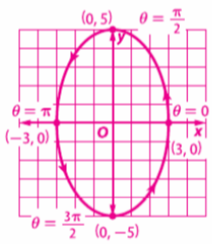
3 Write $x = \sqrt{t+1}$ and $y = \frac{1}{2t}$ in rectangular form. Then graph the equation. State any restrictions on the domain. $y = \frac{1}{2x^2 - 2}$



$x \neq -1$ and $x \neq 1$

4 Write $x = 3\cos\theta$ and $y = 5\sin\theta$ in rectangular form. Then graph the equation.

$\frac{y^2}{25} + \frac{x^2}{9} = 1$



Focus on Mathematical Content

Parametric Equations Parametric equations are a set of equations in which the coordinates are each expressed in terms of a parameter such as time or angle measure. When writing parametric equations in rectangular form without the parameter, always check that extraneous portions of the conic section have not been introduced.

Sometimes the domain must be restricted after converting from parametric to rectangular form.

Example 3 Rectangular Form with Domain Restrictions

Write $x = \frac{1}{\sqrt{t}}$ and $y = \frac{t+1}{t}$ in rectangular form. Then graph the equation. State any restrictions on the domain.

To eliminate t, square each side of $x = \frac{1}{\sqrt{t}}$. This yields $x^2 = \frac{1}{t}$, so $t = \frac{1}{x^2}$. Substitute this value for t in parametric equation for y.

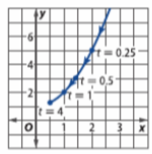
$y = \frac{t+1}{t}$ — Parametric equation for y

$= \frac{\frac{1}{x^2} + 1}{\frac{1}{x^2}}$ — Substitute $\frac{1}{x^2}$ for t.

$= \frac{\frac{x^2+1}{x^2}}{\frac{1}{x^2}}$ — Simplify the numerator.

$= x^2 + 1$ — Simplify.

While the rectangular equation is $y = x^2 + 1$, the curve is only defined for $t > 0$. From the parametric equation $x = \frac{1}{\sqrt{t}}$, the only possible values for x are values greater than zero. As shown in the graph, the domain of the rectangular equation needs to be restricted to $x > 0$.



Guided Practice

3. Write $x = \sqrt{t+4}$ and $y = \frac{1}{t}$ in rectangular form. Graph the equation. State any restrictions on the domain. **See margin.**

The parameter in a parametric equation can also be an angle, θ.

Technology Tip

Parameters When graphing parametric equations on a calculator, θ and t are interchangeable.

Example 4 Rectangular Form with θ as Parameter

Write $x = 2\cos\theta$ and $y = 4\sin\theta$ in rectangular form. Then graph the equation.

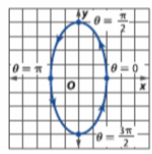
To eliminate the angular parameter θ, first solve the equations for $\cos\theta$ and $\sin\theta$ to obtain $\cos\theta = \frac{x}{2}$ and $\sin\theta = \frac{y}{4}$. Then use the Pythagorean Identity to eliminate the parameter θ.

$\cos^2\theta + \sin^2\theta = 1$ — Pythagorean Identity

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$ — $\cos\theta = \frac{x}{2}$ and $\sin\theta = \frac{y}{4}$

$\frac{x^2}{4} + \frac{y^2}{16} = 1$ — Simplify.

You should recognize this equation as that of an ellipse centered at the origin with vertices at (0, 4) and (0, −4) and covertices at (2, 0) and (−2, 0) as shown. As θ varies from 0 to 2π, the ellipse is traced out counterclockwise.

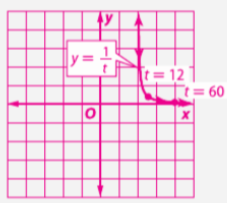


Guided Practice **See margin for graph.**

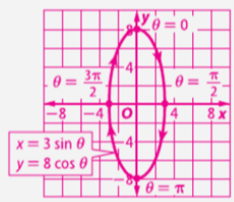
4. Write $x = 3\sin\theta$ and $y = 8\cos\theta$ in rectangular form. Then sketch the graph. $\frac{x^2}{9} + \frac{y^2}{64} = 1$

Additional Answers (Guided Practice)

3. $y = \frac{1}{x^2 - 4}$, $x \geq 0$, $x \neq 2$



4.



390 | Lesson 6-8 | Parametric Equations

As you saw in Example 1, parametric representations of rectangular graphs are not unique. By varying the definition for the parameter, you can obtain parametric equations that produce graphs that vary only in speed and/or orientation.

StudyTip

Parametric Form The easiest method of converting an equation from rectangular to parametric form is to use $x = t$. When this is done, the other parametric equation is the original equation with t replacing x.

Example 5 Write Parametric Equations from Graphs

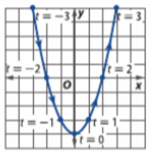
Use each parameter to write the parametric equations that can represent $y = x^2 - 4$. Then graph the equation, indicating the speed and orientation.

a. $t = x$

$y = x^2 - 4$ Original equation

$= t^2 - 4$ Substitute for x in original equation.

The parametric equations are $x = t$ and $y = t^2 - 4$. The associated speed and orientation are indicated on the graph.



b. $t = 4x + 1$

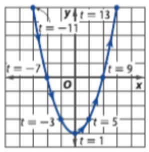
$x = \frac{t-1}{4}$ Solve for x.

$y = \left(\frac{t-1}{4}\right)^2 - 4$ Substitute for x in original equation.

$= \frac{t^2}{16} - \frac{t}{8} - \frac{63}{16}$ Simplify.

$x = \frac{t-1}{4}$ and $y = \frac{t^2}{16} - \frac{t}{8} - \frac{63}{16}$ are the parametric equations.

Notice that the speed is much slower than part **a**.



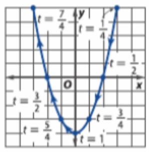
c. $t = 1 - \frac{x}{4}$

$4 - 4t = x$ Solve for x.

$y = (4 - 4t)^2 - 4$ Substitute for x in original equation.

$= 16t^2 - 32t + 12$ Simplify.

The parametric equations are $x = 4 - 4t$ and $y = 16t^2 - 32t + 12$. Notice that the speed is much faster than part **a**. The orientation is also reversed, as indicated by the arrows.



5A–C. See margin for graphs.

5A. $x = t - 1$ and $y = \sqrt{7 - t}$

5B. $x = \frac{t}{3}$ and $y = \sqrt{6 - \frac{t}{3}}$

5C. $x = \frac{4 - t}{2}$ and $y = \sqrt{4 + \frac{t}{2}}$

GuidedPractice

Use each parameter to determine the parametric equations that can represent $x = 6 - y^2$. Then graph the equation, indicating the speed and orientation.

5A. $t = x + 1$ **5B.** $t = 3x$ **5C.** $t = 4 - 2x$

2 Projectile Motion

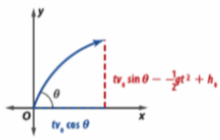
Parametric equations are often used to simulate projectile motion. The path of a projectile launched at an angle other than 90° with the horizontal can be modeled by the following parametric equations.

KeyConcept Projectile Motion

For an object launched at an angle θ with the horizontal at an initial velocity v_0, where g is the gravitational constant, t is time, and h_0 is the initial height:

Horizontal Distance $x = tv_0 \cos \theta$

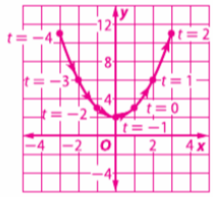
Vertical Position $y = tv_0 \sin \theta - \frac{1}{2}gt^2 + h_0$



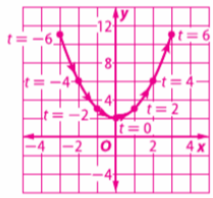
Additional Example

5 Use each parameter to write th[e] parametric equations that can represent $y = x^2 + 2$. Then graph the equation, indicating the speed and orientation.

a. $t = x - 1$ $y = t^2 + 2t + 3$

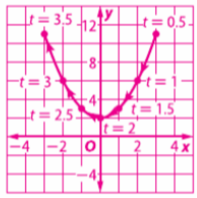


b. $t = 2x$ $y = \frac{t^2}{4} + 2$



c. $t = 2 - \frac{x}{2}$

$y = 4t^2 - 16t + 18$

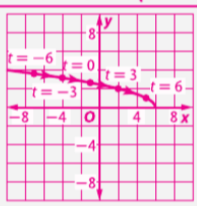


Tips for New Teachers

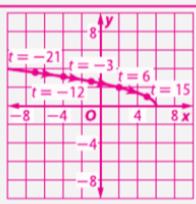
Graphs of Parametric Equations [...] Example 5, emphasize that all three graphs have the same x-values and y-values, which define the shape of [the] curve. It is the graph of $y = x^2 - 4$[.] [The] only aspect that changes is the parameter, t.

Additional Answers (Guided Practice)

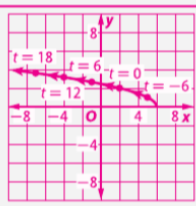
5A.



5B.



5C.



2 Projectile Motion

Example 6 shows how parametric equations can be used to model projectile motion.

Additional Example

6 **SOCCER** Omar holds his team's record for the longest goal punted at 46.47 meteres. Suppose that he kicked the ball at an initial height of 2 meters and with an initial velocity of 16 meteres per second at an angle of 72°. How far will the ball travel horizontally? about 46 yd

Teach with Tech

Journal In a journal, students should write an entry about parametric equations. Make sure they include information about both rectangular form and parametric form.

Follow-up

Students have explored writing parametric equations.

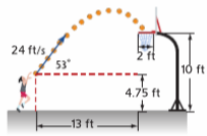
Ask:

- How do parametric equations help you to see "the whole picture?" Sample answer: Parametric equations offer a way to describe both the horizontal and vertical position of an object as a function of time. This is helpful because it allows you to determine where the object is at any given time.

Real-World Example 6 Projectile Motion

BASKETBALL **Khadija is practicing free throws for an upcoming basketball game. She releases the ball with an initial velocity of 24 feet per second at an angle of 53° with the horizontal. The horizontal distance from the free throw line to the front rim of the basket is 13 feet. The vertical distance from the floor to the rim is 10 feet. The front of the rim is 2 feet from the backboard. She releases the shot 4.75 feet from the ground. Does Khadija make the basket?**

Make a diagram of the situation.



StudyTip

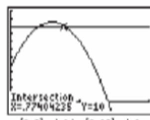
Gravity At the surface of Earth, the acceleration due to gravity is 9.8 meters per second squared or 32 feet per second squared. When solving problems, be sure to use the appropriate value for gravity based on the units of the velocity and position.

To determine whether she makes the shot, you need the horizontal distance that the ball has traveled when the height of the ball is 10 feet. First, write a parametric equation for the vertical position of the ball.

$y = tv_0 \sin \theta - \frac{1}{2}gt^2 + h_0$ — Parametric equation for vertical position

$= t(24) \sin 53 - \frac{1}{2}(32)t^2 + 4.75$ — $v_0 = 24$, $\theta = 53°$, $g = 32$, and $h_0 = 4.75$

Graph the equation for the vertical position and the line $y = 10$. The curve will intersect the line in two places. The second intersection represents the ball as it is moving down toward the basket. Use 5: intersect on the CALC menu to find the second point of intersection with $y = 10$. The value is about 0.77 second.



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Determine the horizontal position of the ball at 0.77 second.

$x = tv_0 \cos \theta$ — Parametric equation for horizontal position

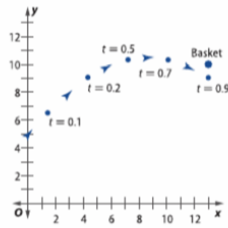
$= 0.77(24) \cos 53$ — $v_0 = 24$, $\theta = 53°$, and $t \approx 0.77$

≈ 11.1 — Use a calculator.

Because the horizontal position is less than 13 feet when the ball reaches 10 feet for the second time, the shot is short of the basket. Khadija does not make the free throw.

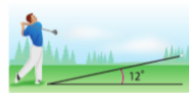
CHECK You can confirm the results of your calculation by graphing the parametric equations and determining the path of the ball in relation to the basket.

t	x	y	t	x	y
0	0	4.75	0.5	7.22	10.33
0.1	1.44	6.51	0.6	8.67	10.49
0.2	2.89	7.94	0.7	10.11	10.32
0.3	4.33	9.06	0.8	11.55	9.84
0.4	5.78	9.86	0.9	13.00	9.04



GuidedPractice

6. **GOLF** Saeed drives a golf ball with an initial velocity of 56 meters per second at an angle of 12° down a flat driving range. How far away will the golf ball land? 130 m



Real-WorldLink

In April 2007, Morgan Pressel became the youngest woman ever to win a major LPGA championship.

Source: *LPGA*

DifferentiatedInstruction OL BL

Visual/Spatial Learners Have the class work in groups of three or four. Each group creates a digital video recording of students throwing a softball or a baseball. Have students time the ball while it is in the air. Then, playing the video in slow motion, trace the path of the ball. Students should determine the maximum height, range, initial angle of release, and initial velocity for the throw. Have groups use this data to write word problems and exchange them with other groups in the class. Have each group show its video to the class.

Exercises

Sketch the curve given by each pair of parametric equations over the given interval. (Example 1) **1–8. See Chapter 6 Answer Appendix.**

1. $x = t^2 + 3$ and $y = \frac{t}{4} - 5$; $-5 \le t \le 5$
2. $x = \frac{t^2}{2}$ and $y = -4t$; $-4 \le t \le 4$
3. $x = -\frac{5t}{2} + 4$ and $y = t^2 - 8$; $-6 \le t \le 6$
4. $x = 3t + 6$ and $y = \sqrt{t} + 1$; $0 \le t \le 9$
5. $x = 2t - 1$ and $y = -\frac{t^2}{2} + 7$; $-4 \le t \le 4$
6. $x = -2t^2$ and $y = \frac{t}{3} - 6$; $-6 \le t \le 6$
7. $x = \frac{t}{2}$ and $y = -\sqrt{t} + 5$; $0 \le t \le 8$
8. $x = t^2 - 4$ and $y = 3t - 8$; $-5 \le t \le 5$

9–16. See Chapter 6 Answer Appendix for graphs.
Write each pair of parametric equations in rectangular form. Then graph the equation and state any restrictions on the domain. (Examples 2 and 3)

9. $x = 2t - 5$, $y = t^2 + 4$ $y = 0.25x^2 + 2.5x + 10.25$
10. $x = 3t + 9$, $y = t^2 - 7$ $y = \frac{x^2}{9} - 2x + 2$
11. $x = t^2 - 2$, $y = 5t$ $y = \pm 5\sqrt{x + 2}$
12. $x = t^2 + 1$, $y = -4t + 3$ $y = \pm 4\sqrt{x - 1} + 3$
13. $x = -t - 4$, $y = 3t^2$ $y = 3x^2 + 24x + 48$
14. $x = 5t - 1$, $y = 2t^2 + 8$ $y = \frac{2x^2}{25} + \frac{4x}{25} + \frac{202}{25}$
15. $x = 4t^2$, $y = \frac{6t}{5} + 9$ $y = \pm\frac{3\sqrt{x}}{5} + 9$
16. $x = \frac{t}{3} + 2$, $y = \frac{t^2}{6} - 7$ $y = \frac{3x^2}{2} - 6x - 1$

17. **MOVIE STUNTS** During the filming of a movie, a stunt double leaps off the side of a building. The pulley system connected to the stunt double allows for a vertical fall modeled by $y = -16t^2 + 15t + 100$, and a horizontal movement modeled by $x = 4t$, where x and y are measured in feet and t is measured in seconds. Write and graph an equation in rectangular form to model the stunt double's fall for $0 \le t \le 3$. (Example 3) **See margin.**

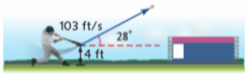
Write each pair of parametric equations in rectangular form. Then graph the equation. (Example 4) **18–25. See Chapter 6 Answer Appendix.**

18. $x = 3\cos\theta$ and $y = 5\sin\theta$
19. $x = 7\sin\theta$ and $y = 2\cos\theta$
20. $x = 6\cos\theta$ and $y = 4\sin\theta$
21. $x = 3\cos\theta$ and $y = 3\sin\theta$
22. $x = 8\sin\theta$ and $y = \cos\theta$
23. $x = 5\cos\theta$ and $y = 6\sin\theta$
24. $x = 10\sin\theta$ and $y = 9\cos\theta$
25. $x = \sin\theta$ and $y = 7\cos\theta$

Use each parameter to write the parametric equations that can represent each equation. Then graph the equations, indicating the speed and orientation. (Example 5) **26–31. See Chapter 6 Answer Appendix.**

26. $t = 3x - 2$; $y = x^2 + 9$
27. $t = 8x$; $y^2 = 9 - x^2$
28. $t = 2 - \frac{x}{3}$; $y = \frac{x^2}{12}$
29. $t = \frac{x}{5} + 4$; $y = 10 - x^2$
30. $t = 4x + 7$; $y = \frac{x^2 - 1}{2}$
31. $t = \frac{1 - x}{2}$; $y = \frac{3 - x^2}{4}$

32. **BASEBALL** A baseball player hits the ball at a 28° angle with an initial speed of 103 feet per second. The bat is 4 feet from the ground at the time of impact. Assuming that the ball is not caught, determine the distance traveled by the ball. (Example 6) **282 ft**



33. **PLAY BALL** Obaid attempts a 43-yard goal. He kicks the ball at a 41° angle with an initial speed of 70 feet per second. The goal is 15 feet high. Is the kick long enough to make the goal? (Example 6) **yes**

B **34.** $4x^2 - 32x + 67, x \ge 4$
Write each pair of parametric equations in rectangular form. Then state the restriction on the domain.

34. $x = \sqrt{t} + 4$
    $y = 4t + 3$
35. $x = \log t$
    $y = t + 3$ $y = 10^x + 3$
36. $x = \sqrt{t - 7}$
    $y = -3t - 8$ $y = -3x^2 - 29, x \ge 0$
37. $x = \log(t - 4)$
    $y = t$ $y = 10^x + 4$
38. $x = \frac{1}{\sqrt{t + 3}}$
    $y = t$ $y = \frac{1}{x^2} - 3, x > 0$
39. $x = \frac{1}{\log(t + 2)}$
    $y = 2t - 4$ $y = 2 \cdot 10^{\frac{1}{x}} - 8, x \ne 0$

40. **TENNIS** Mazen hits a tennis ball 55 centimeters above the ground at an angle of 15° with the horizontal. The ball has an initial speed of 18 meters per second.
    a. Use a graphing calculator to graph the path of the tennis ball using parametric equations. **See margin.**
    b. How long does the ball stay in the air before hitting the ground? **about 1.06 s**
    c. If Mazen is 10 meters from the net and the net is 1.5 meters above the ground, will the tennis ball clear the net? If so, by how many meters? If not, by how many meters is the ball short? **See margin.**

Write a set of parametric equations for the line or line segment with the given characteristics. **41–44. See margin.**

41. line with a slope of 3 that passes through (4, 7)
42. line with a slope of −0.5 that passes through (3, −2)
43. line segment with endpoints (−2, −6) and (2, 10)
44. line segment with endpoints (7, 13) and (13, 11)

3 Practice

Formative Assessment

Use Exercises 1–33 to check for understanding.

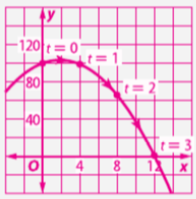
Then use the table below to custom[ize] your assignments for students.

WatchOut!

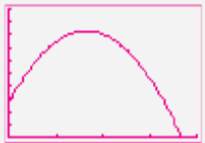
Common Error For Exercises 1–[8] make sure students use the x- an[d] y-coordinates to define the shape of their sketch. The locations of the parameter t should then be written on the curve.

Additional Answers

17. $y = -x^2 + \frac{15}{4}x + 100$



40a.



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t[0, 20]; tstep: 0.1

40c. Yes; the tennis ball reaches th[e] net in about 0.575 seconds. A[t] this time, the ball is at a heigh[t of] about 1.6 m, so it clears the n[et]

41. Sample answer: $x = t + 4$ and $y = 3t + 7$

42. Sample answer: $x = t + 3$ and $y = -0.5t - 2$

43. Sample answer: $x = t - 2$ an[d] $y = 4t - 6, 0 \le t \le 4$

44. Sample answer: $x = t + 7$ an[d] $y = -\frac{1}{3}t + 13, 0 \le t \le 6$

Differentiated Homework Options OL BL AL

Level	Assignment	Two-Day Option	
AL Approaching Level	1–33, 54, 55, 57–79	1–33 odd, 77–79	2–32 even, 54, 55, 57–76
OL On Level	1–39 odd, 40, 41–47 odd, 49–52, 54, 55, 57–79	1–33, 77–79	34–52, 54, 55, 57–76
BL Beyond Level	34–79		

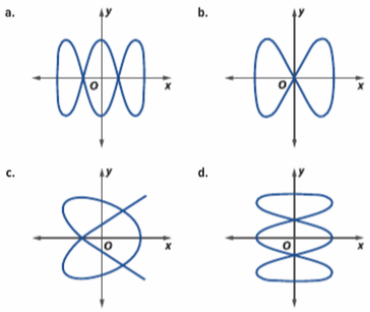
Match each set of parametric equations with its graph.

45. $x = \cos 2t$, $y = \sin 4t$ **b**

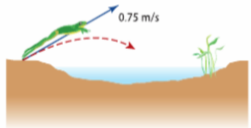
46. $x = \cos 3t$, $y = \sin t$ **d**

47. $x = \cos t$, $y = \sin 3t$ **a**

48. $x = \cos 4t$, $y = \sin 3t$ **c**



49. **BIOLOGY** A frog jumps off the bank of a creek with an initial velocity of 0.75 meter per second at an angle of 45° with the horizontal. The surface of the creek is 0.3 meter below the edge of the bank. Let g equal 9.8 meters per second squared.



a. Write the parametric equations to describe the position of the frog at time t. Assume that the surface of the water is located at the line $y = 0$. **See margin.**

b. If the creek is 0.5 meter wide, will the frog reach the other bank, which is level with the surface of the creek? If not, how far from the other bank will it hit the water? **no; 0.34 m**

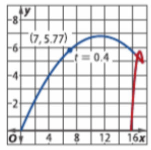
c. If the frog was able to jump on a lily pad resting on the surface of the creek 0.4 meter away and stayed in the air for 0.38 second, what was the initial speed of the frog? **about 1.49 m/s**

50. **RACE** Hala and Hidaya are competing in a 100-meter dash. When the starter gun fires, Hala runs 8.0 meters per second after a 0.1 second delay from the point (0, 2) and Hidaya runs 8.1 meters per second after a 0.3 second delay from the point (0, 5).

a. Using the y-axis as the starting line and assuming that the women run parallel to the x-axis, write parametric equations to describe each runner's position after t seconds. **See margin.**

b. Who wins the race? If the women ran 200 meters instead of 100 meters, who would win? Explain your answer. **See margin.**

51. **FOOTBALL** The graph below models the path of a football kicked by one player and then headed back by another player. The path of the initial kick is shown in blue, and the path of the headed ball is shown in red.



b. about 1 second

a. If the ball is initially kicked at an angle of 50°, find the initial speed of the ball. **about 27.2 ft/s**

b. At what time does the ball reach the second player if the second player is standing about 17.5 feet away?

c. If the second player heads the ball at an angle of 75°, an initial speed of 8 feet per second, and at a height of 4.75 feet, approximately how long does the ball stay in the air from the time it is first kicked until it lands? **about 1.84 seconds**

52. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate a *cycloid*, the curve created by the path of a point on a circle with a radius of 1 unit as it is rolled along the x-axis. **a–c. See margin.**

a. **GRAPHICAL** Use a graphing calculator to graph the parametric equations $x = t - \sin t$ and $y = 1 - \cos t$, where t is measured in radians.

b. **ANALYTICAL** What is the distance between x-intercepts? Describe what the x-intercepts and the distance between them represent.

c. **ANALYTICAL** What is the maximum value of y? Describe what this value represents and how it would change for circles of differing radii.

H.O.T. Problems Use Higher-Order Thinking Skills

53. **CHALLENGE** Consider a line ℓ with parametric equations $x = 2 + 3t$ and $y = -t + 5$. Write a set of parametric equations for the line m perpendicular to ℓ containing the point (4, 10). **Sample answer: $x = 4 + t$, $y = 10 + 3t$**

54. **WRITING IN MATH** Explain why there are infinitely many sets of parametric equations to describe one line in the xy-plane. **See margin.**

55. **REASONING** Determine whether parametric equations for projectile motion can apply to objects thrown at an angle of 90°. Explain your reasoning. **See Chapter 6 Answer Appendix.**

56. **CHALLENGE** A line in three-dimensional space contains the points $P(2, 3, -8)$ and $Q(-1, 5, -4)$. Find two sets of parametric equations for the line. **See Chapter 6 Answer Appendix.**

57. **WRITING IN MATH** Explain the advantage of using parametric equations versus rectangular equations when analyzing the horizontal/vertical components of a graph. **See Chapter 6 Answer Appendix.**

Additional Answers

49a. $x = t \cdot 0.75 \cos 45°$, $y = t \cdot 0.75 \sin 45° - 4.9t^2 + 0.3$

50a. Hala: $x = 8(t - 0.1)$, $y = 2$; Hidaya: $x = 8.1(t - 0.3)$, $y = 5$

50b. Hala wins the 100-meter race finishing in 12.6 seconds, while Hidaya finishes in 12.65 seconds. Hidaya would win the 200 m race finishing in 25 seconds, while Hala would finish in 25.1 seconds.

Spiral Review

Graph each equation at the indicated angle. **58–59. See Chapter 6 Answer Appendix.**

58. $\frac{(x')^2}{9} - \frac{(y')^2}{4} = 1$ at a 60° rotation from the *xy*-axis

59. $(x')^2 - (y')^2 = 1$ at a 45° rotation from the *xy*-axis

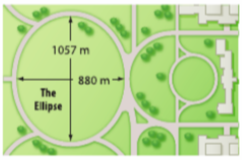
60–61. See margin.

Write an equation for the hyperbola with the given characteristics.

60. vertices (5, 4), (5, −8); conjugate axis length of 4

61. transverse axis length of 4; foci (3, 5), (3, −1)

62. **WHITE HOUSE** There is an open area known as The Ellipse. Write an equation to model The Ellipse. Assume that the origin is at its center. $\frac{x^2}{193{,}600} + \frac{y^2}{279{,}312.25} = 1$



Simplify each expression.

63. $\frac{\sin x}{\csc x - 1} + \frac{\sin x}{\csc x + 1}$ **$2\tan^2 x$**

64. $\frac{1}{1 - \cos x} + \frac{1}{1 + \cos x}$ **$2\csc^2 x$**

Use the properties of logarithms to rewrite each logarithm below in the form $a \ln 2 + b \ln 3$, where a and b are constants. Then approximate the value of each logarithm given that $\ln 2 \approx 0.69$ and $\ln 3 \approx 1.10$.

65. ln 54 **3 ln 3 + ln 2; 3.99**

66. ln 24 **3 ln 2 + ln 3; 3.17**

67. $\ln \frac{8}{3}$ **3 ln 2 − ln 3; 0.97**

68. $\ln \frac{9}{16}$ **2 ln 3 − 4 ln 2; −0.56**

For each function, determine any asymptotes and intercepts. Then graph the function and state its domain. **69–72. See Chapter 6 Answer Appendix.**

69. $h(x) = \frac{x}{x + 6}$

70. $h(x) = \frac{x^2 + 6x + 8}{x^2 - 7x - 8}$

71. $f(x) = \frac{x^2 + 8x}{x + 5}$

72. $f(x) = \frac{x^2 + 4x + 3}{x^3 + x^2 - 6x}$

Solve each equation.

73. $\sqrt{3z - 5} - 3 = 1$ **7**

74. $\sqrt{5n - 1} = 0$ **$\frac{1}{5}$**

75. $\sqrt{2c + 3} - 7 = 0$ **23**

76. $\sqrt{4a + 8} + 8 = 5$ **no solution**

Skills Review for Standardized Tests

77. **SAT/ACT** With the exception of the shaded squares, every square in the figure contains the sum of the number in the square directly above it and the number in the square directly to its left. For example, the number 4 in the unshaded square is the sum of the 2 in the square above it and the 2 in the square directly to its left. What is the value of *x*? **E**

0	1	2	3	4	5
1	2	4			
2					
3			x		
4					
5					

A 7 B 8 C 15 D 23 E 30

78. Saleh and Sultan are performing a physics experiment in which they will launch a model rocket. The rocket is supposed to release a parachute 91.5 meters in the air, 7 seconds after liftoff. They are firing the rocket at a 78° angle from the horizontal. To protect other students from the falling rockets, the teacher needs to place warning signs 45.7 meters from where the parachute is released. How far should the signs be from the point where the rockets are launched? **G**

F 111.6 meters
G 116.2 meters
H 121.6 meters
J 126.2 meters

79. **FREE RESPONSE** An object moves along a curve according to $y = \frac{10\sqrt{3t} \pm \sqrt{496 - 2304t}}{62}$, $x = \sqrt{t}$.

a. Convert the parametric equations to rectangular form. **$21x^2 + 31y^2 - 10\sqrt{3}xy = 4$**

b. Identify the conic section represented by the curve. **ellipse**

c. Write an equation for the curve in the $x'y'$-plane, assuming it was rotated 30°. **$4(x')^2 + 9(y')^2 = 1$**

d. Determine the eccentricity of the conic. **≈ 0.745**

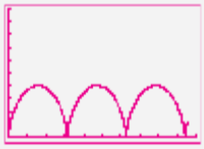
e. Identify the location of the foci in the $x'y'$-plane, if they exist. **$\left(\frac{\sqrt{5}}{6}, 0\right), \left(-\frac{\sqrt{5}}{6}, 0\right)$**

4 Assess

Ticket Out the Door Have each student write the steps involved in writing the rectangular form of the equation when given the two parametric equations. Solve one equation for *t*, then substitute that v into the other equation and simplify

Additional Answers

52a.



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52b. 2π; Sample answer: The *x*-intercepts represent the instances when the point on t circle touches the *x*-axis as it rolled. Since the entire circumference of the circle wi touch the *x*-axis as it is rolled distance between the *x*-interc will be equal to the circumfer of the circle, which is equal to 2π.

52c. 2; Sample answer: This value represents the maximum heig the point reaches as the circle rolled along the *x*-axis. It is eq to the diameter of the circle. A circle with a radius of *r* will produce a maximum *y*-value of 2*r*.

54. Sample answer: Parametric equations are written using a p on the line and a parallel vecto infinite number of equations ca written using an infinite numbe points on any line.

60. $\frac{(y + 2)^2}{36} - \frac{(x - 5)^2}{4} = 1$

61. $\frac{(y - 2)^2}{4} - \frac{(x - 3)^2}{5} = 1$

Differentiated Instruction BL

Extension Graphing calculators can be used to graph parametric equations by changing the Mode from Func to Par. A cycloid is a function that can be modeled using the parametric equations $x = t - \sin t$ and $y = 1 - \cos t$. Have students do Internet research to find out the characteristics of a cycloid. Have each student draw a sketch of a cycloid. Then using a graphing calculator in parametric mode, he or she can plot the graph of a cycloid and compare it to his or her sketch.

Study Guide and Review

Study Guide

KeyConcepts

Midpoint and Distance Formulas (Lesson 6-1)

- $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Parabolas (Lesson 6-2)

- Standard Form: $y = a(x - h)^2 + k$
  $x = a(y - k)^2 + h$

Circles (Lesson 6-3)

- The equation of a circle with center (h, k) and radius r can be written in the form $(x - h)^2 + (y - k)^2 = r^2$.

Ellipses (Lesson 6-4)

- Standard Form: horizontal $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
  vertical $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$

Hyperbolas (Lesson 6-5)

- Standard Form: horizontal $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
  vertical $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Solving Linear-Nonlinear Systems (Lesson 6-7)

- Systems of quadratic equations can be solved using substitution and elimination.
- A system of quadratic equations can have zero, one, two, three, or four solutions.

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.

KeyVocabulary

center (of a circle)	foci (of an ellipse)
center (of an ellipse)	focus
circle	hyperbola
conjugate axis	latus rectum
constant difference	major axis
constant sum	minor axis
co-vertices (of a hyperbola)	parabola
co-vertices (of an ellipse)	radius
directrix	transverse axis
ellipse	vertices (of a hyperbola)
foci (of a hyperbola)	vertices (of an ellipse)

VocabularyCheck

State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.

1. The set of all points in a plane that are equidistant from a given point in the plane, called the focus, forms a circle. **false, center**
2. A(n) ellipse is the set of all points in a plane such that the sum of the distances from the two fixed points is constant. **true**
3. The endpoints of the major axis of an ellipse are the foci of the ellipse. **false, vertices**
4. The radius is the distance from the center of a circle to any point on the circle. **true**
5. The line segment with endpoints on a parabola, through the focus of the parabola, and perpendicular to the axis of symmetry is called the latus rectum. **true**
6. Every hyperbola has two axes of symmetry, the transverse axis and the major axis. **false, conjugate axis**
7. A directrix is the set of all points in a plane that are equidistant from a given point in the plane, called the center. **false, circle**
8. A hyperbola is the set of all points in a plane such that the absolute value of the sum of the distances from any point on the hyperbola to two given points is constant. **false, difference**
9. A parabola can be defined as the set of all points in a plane that are the same distance from the focus and a given line called the directrix. **true**
10. The major axis is the longer of the two axes of symmetry of an ellipse. **true**
11. The equation for a graph can be written using the variables x and y, or using _______ equations, generallu using t or the angle θ.
12. The graph of $f(t) = (\sin t, \cos t)$ is a _______ with a shape that is circle traced clockwise.

Formative Assessment

KeyVocabulary The page referen after each word denote where that was first introduced. If students hav difficulty answering questions 1–10, remind them that they can use thes page references to refresh their memories about the vocabulary.

FOLDABLES StudyOrganize

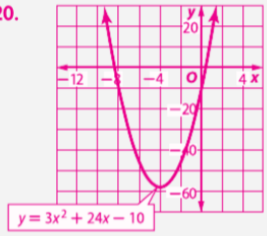
Dinah Zike's Foldables®
Have students look through the cha to make sure they have included examples in their Foldables. Sugge that students keep their Foldables handy while completing the Study Guide and Review pages. Point out their Foldables can serve as a quick review tool when studying for the chapter test.

Lesson-by-Lesson Review

Intervention If the given examples are not sufficient to review the topics covered by the questions, remind students that the lesson references tell them where to review that topic in their textbook.

Additional Answers

20.



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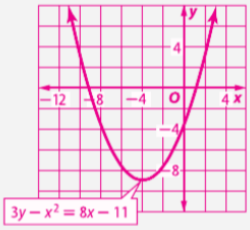

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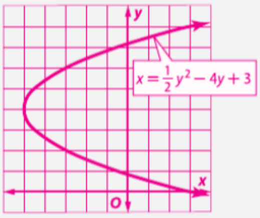


CHAPTER 6 Study Guide and Review *Continued*

Lesson-by-Lesson Review

6-2 Parabolas

Graph each equation. **20–23. See margin.**

13. $y = 3x^2 + 24x - 10$

14. $x = \frac{1}{2}y^2 - 4y + 3$

15. $3y - x^2 = 8x - 11$

16. $x = y^2 - 14y + 25$

24–27. See margin.

Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

17. $y = -\frac{1}{2}x^2$

18. $x - 6y = y^2 + 4$

19. $y = 4x^2 - 16x + 9$

20. $x = y^2 + 14y + 20$

21. **SPORTS** When a football is kicked, the path it travels is shaped like a parabola. Suppose a football is kicked from ground level, reaches a maximum height of 50 feet, and lands 200 feet away. Assuming the football was kicked at the origin, write an equation of the parabola that models the flight of the football. $y = -\frac{1}{200}(x - 100)^2 + 50$

Example 1

Write $3y - x^2 = 4x + 7$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

Write the equation in the form $y = a(x - h)^2 + k$ by completing the square.

$3y = x^2 + 4x + 7$	Isolate the terms with x.
$3y = (x^2 + 4x + ■) + 7 - ■$	Complete the square.
$3y = (x^2 + 4x + 4) + 7 - 4$	$\left(\frac{4}{2}\right)^2 = 4$
$3y = (x + 2)^2 + 3$	$(x^2 + 4x + 4) = (x + 2)$
$y = \frac{1}{3}(x + 2)^2 + 1$	Divide each side by 3.

Vertex: $(-2, 1)$; axis of symmetry: $x = -2$; direction of opening: upward since $a > 0$.

6-3 Circles

Write an equation for the circle that satisfies each set of conditions. **29. $(x + 1)^2 + (y - 6)^2 = 9$**

22. center $(-1, 6)$, radius 3 units

23. endpoints of a diameter $(2, 5)$ and $(0, 0)$

24. endpoints of a diameter $(4, -2)$ and $(-2, -6)$

Find the center and radius of each circle. Then graph the circle. **32–35. See margin.**

25. $(x + 5)^2 + y^2 = 9$

26. $(x - 3)^2 + (y + 1)^2 = 25$

27. $(x + 2)^2 + (y - 8)^2 = 1$

28. $x^2 + 4x + y^2 - 2y - 11 = 0$

29. **SOUND** A loudspeaker in a school is located at the point (65, 40). The speaker can be heard in a circle with a radius of 30.5 meters. Write an equation to represent the possible boundary of the loudspeaker sound. $(x - 65)^2 + (y - 40)^2 = 100^2$

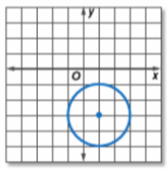
Example 2

Find the center and radius of the circle with equation $x^2 - 2x + y^2 + 6y + 6 = 0$. Then graph the circle.

Complete the squares.

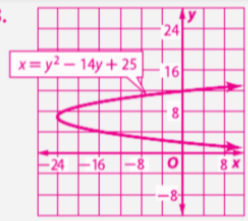
$$x^2 - 2x + y^2 + 6y + 6 = 0$$
$$(x^2 - 2x + ■) + (y^2 + 6y + ■) = -6 + ■ + ■$$
$$(x^2 - 2x + 1) + (y^2 + 6y + 9) = -6 + 1 + 9$$
$$(x - 1)^2 + (y + 3)^2 = 4$$

The center of the circle is at $(1, -3)$ and the radius is 2.



30. $(x - 1)^2 + (y - 2.5)^2 = \frac{29}{4}$ **31.** $(x - 1)^2 + (y + 4)^2 = 13$

23.



24. $y = -\frac{1}{2}x^2$; vertex: (0, 0); axis of symme
$x = 0$; opens: upward

25. $y = 4(x - 2)^2 - 7$; vertex: $(2, -7)$; axis
symmetry: $x = 2$; opens: upward

26. $x = (y + 3)^2 - 5$; vertex: $(-5, -3)$; axis
symmetry: $y = -3$; opens to the right

27. $x = (y + 7)^2 - 29$; vertex $= (-29, -7)$
axis of symmetry $y = -7$; opens to the rig

6-4 Ellipses

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

30. $\frac{x^2}{9} + \frac{y^2}{36} = 1$ **31.** $\frac{y^2}{10} + \frac{x^2}{5} = 1$

32. $\frac{x^2}{36} + \frac{(y-4)^2}{4} = 1$ **33.** $27x^2 + 9y^2 = 81$

34. $\frac{(x+1)^2}{25} + \frac{(y-2)^2}{16} = 1$ **37–44. See Chapter 6 Answer Appendix.**

35. $9x^2 + 4y^2 + 54x - 8y + 49 = 0$

36. $9x^2 + 25y^2 - 18x + 50y - 191 = 0$

37. $7x^2 + 3y^2 - 28x - 12y = -19$

38. **LANDSCAPING** Saeed's family has a garden in their front yard that is shaped like an ellipse. The major axis is 16 meters and the minor axis is 10 meters. Write an equation to model the garden. Assume the origin is at the center of the garden and the major axis is horizontal.
$\frac{x^2}{64} + \frac{y^2}{25} = 1$

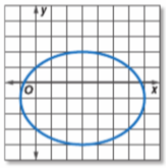
Example 3

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with equation $9x^2 + 16y^2 - 54x + 32y - 47 = 0$. Then graph the ellipse.

First, convert to standard form.

$$9x^2 + 16y^2 - 54x + 32y - 47 = 0$$
$$9(x^2 - 6x + ■) + 16(y^2 + 2y + ■) = 47 + 9(■) + 16(■)$$
$$9(x^2 - 6x + 9) + 16(y^2 + 2y + 1) = 47 + 9(9) + 16(1)$$
$$9(x-3)^2 + 16(y+1)^2 = 144$$
$$\frac{(x-3)^2}{16} + \frac{(y+1)^2}{9} = 1$$

The center of the ellipse is $(3, -1)$. The ellipse is horizontal. $a^2 = 16$, so $a = 4$. $b^2 = 9$, so $b = 3$. The length of the major axis is $2 \cdot 4$ or 8. The length of the minor axis is $2 \cdot 3$ or 6. To find the foci: $c^2 = 16 - 9$ or 7, so $c = \sqrt{7}$. The foci are $(3 + \sqrt{7}, -1)$ and $(3 - \sqrt{7}, -1)$.



6-5 Hyperbolas

Graph each hyperbola. Identify the vertices, foci, and asymptotes. **46–50. See margin.**

39. $\frac{y^2}{9} - \frac{x^2}{4} = 1$

40. $\frac{(x-3)^2}{1} - \frac{(y+2)^2}{4} = 1$

41. $\frac{(y+1)^2}{16} - \frac{(x-4)^2}{9} = 1$

42. $4x^2 - 9y^2 = 36$

43. $9y^2 - x^2 - 4x + 18y + 4 = 0$

44. **MIRRORS** A hyperbolic mirror is shaped like one branch of a hyperbola. It reflects light rays directed at one focus toward the other focus. Suppose a hyperbolic mirror is modeled by the upper branch of the hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$. A light source is located at $(-10, 0)$. Where should the light hit the mirror so that the light will be reflected to $(0, -5)$? **See margin.**

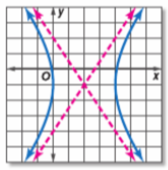
Example 4

Graph $9x^2 - 4y^2 - 36x - 8y - 4 = 0$. Identify the vertices, foci, and asymptotes.

Complete the square.

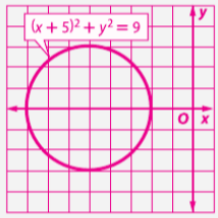
$$9x^2 - 4y^2 - 36x - 8y - 4 = 0$$
$$9(x^2 - 4x + ■) - 4(y^2 + 2y + ■) = 4 + 9(■) - 4(■)$$
$$9(x^2 - 4x + 4) - 4(y^2 + 2y + 1) = 4 + 9(4) - 4(1)$$
$$9(x-2)^2 - 4(y+1)^2 = 36$$
$$\frac{(x-2)^2}{4} - \frac{(y+1)^2}{9} = 1$$

The center is at $(2, -1)$. The vertices are at $(0, -1)$ and $(4, -1)$. The foci are at $(2 + \sqrt{13}, -1)$ and $(2 - \sqrt{13}, -1)$. The equations of the asymptotes are $y + 1 = \pm\frac{3}{2}(x - 2)$

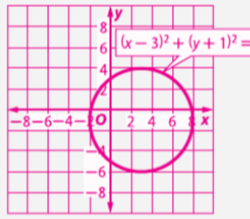


Additional Answers

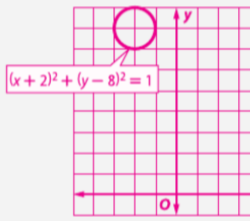
32. $(-5, 0)$; $r = 3$



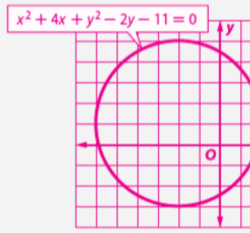
33. $(3, -1)$; $r = 5$



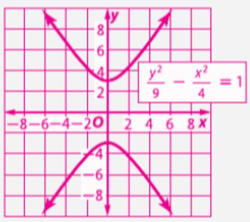
34. $(-2, 8)$; $r = 1$



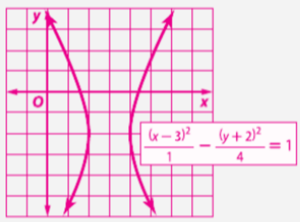
35. $(-2, 1)$; $r = 4$



46. $(0, \pm 3)$; $(0, \pm\sqrt{13})$; $y = \pm\frac{3}{2}x$

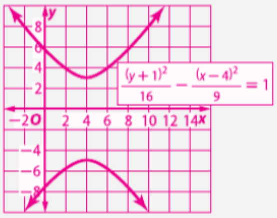


47. $(2, -2)$, $(4, -2)$; $(3 \pm \sqrt{5}, -2)$; $y + 2 = \pm 2(x - 3)$

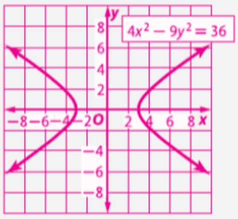


Additional Answers

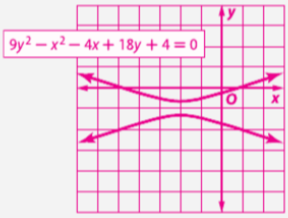
48. (4, −5), (4, 3), (4, −6), (4, 4); $y + 1 = \pm\frac{4}{3}(x - 4)$



49. (±3, 0); $(\pm\sqrt{13}, 0)$; $y = \pm\frac{2}{3}x$



50. $\left(-2, -\frac{2}{3}\right), \left(-2, -\frac{4}{3}\right)$; $\left(-2, -1 \pm \frac{\sqrt{10}}{3}\right)$; $y + 1 = \pm\frac{1}{3}(x + 2)$



51. $\left(\frac{40 - 24\sqrt{5}}{5}, \frac{45 - 12\sqrt{5}}{5}\right)$

CHAPTER 6 Study Guide and Review *Continued*

6-6 Identifying Conic Sections

Write each equation in standard form. State whether the graph of the equation is a *parabola, circle, ellipse,* or *hyperbola*. Then graph.

43. $3x^2 + 12x - y + 8 = 0$ **52–55. See Chapter 6 Answer Appendix.**

44. $9x^2 + 16y^2 = 144$

45. $x^2 + y^2 - 8x - 2y + 8 = 0$

46. $-9x^2 + y^2 + 36x - 45 = 0$

Without writing the equation in standard form, state whether the graph of the equation is a *parabola, circle, ellipse,* or *hyperbola*.

47. $7x^2 + 9y^2 = 63$ **ellipse**

48. $5y^2 + 2y + 4x - 13x^2 = 81$ **hyperbola**

49. $x^2 - 8x + 16 = 6y$ **parabola**

50. $x^2 + 4x + y^2 - 285 = 0$ **circle**

51. **LIGHT** Suppose the edge of a shadow can be represented by the equation $16x^2 + 25y^2 - 32x - 100y - 284 = 0$.

a. What is the shape of the shadow? **ellipse**

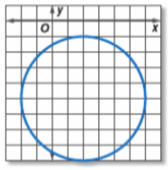
b. Graph the equation. **See margin.**

Example 5

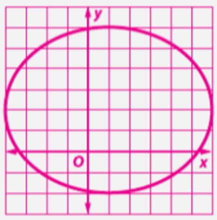
Write $3x^2 + 3y^2 - 12x + 30y + 39 = 0$ in standard form. State whether the graph of the equation is a *parabola, circle, ellipse,* or *hyperbola*. Then graph the equation.

$$3x^2 + 3y^2 - 12x + 30y + 39 = 0$$
$$3(x^2 - 4x + ■) + 3(y^2 + 10y + ■) = -39 + 3(■) + 3(■)$$
$$3(x^2 - 4x + 4) + 3(y^2 + 10y + 25) = -39 + 3(4) + 3(25)$$
$$3(x - 2)^2 + 3(y + 5)^2 = 48$$
$$(x - 2)^2 + (y + 5)^2 = 16$$

In this equation $A = 3$ and $C = 3$. Since A and C are both positive and $A = C$, the graph is a circle. The center is at (2, −5), and the radius is 4.



60b.



6-7 Solving Linear-Nonlinear Systems

Solve each system of equations.

52. $x^2 + y^2 = 8$ **(2, −2), (−2, 2)**
$x + y = 0$

51. $x - 2y = 2$ **(−2, −2)**
$y^2 - x^2 = 2x + 4$

53. $y + x^2 = 4x$ **(4, 0)**
$y + 4x = 16$

54. $3x^2 - y^2 = 11$
$x^2 + 4y^2 = 8$

54. $5x^2 + y^2 = 30$
$9x^2 - y^2 = -16$
(1, ±5), (−1, ±5)

56. $\frac{x^2}{30} + \frac{y^2}{6} = 1$
$x = y$
$(\sqrt{5}, \sqrt{5}), (-\sqrt{5}, -\sqrt{5})$

55. PHYSICAL SCIENCE Two balls are launched into the air at the same time. The heights they are launched from are different. The height y in feet of one is represented by $y = -16t^2 + 80t + 25$ where t is the time in seconds. The height of the other ball is represented by $y = -16t^2 + 30t + 100$. **64. (−2, −1), (−2, 1), (2, −1), (2, 1)**

a. After how many seconds are the balls at the same height? **1.5 seconds**

b. What is this height? **109 ft**

56. ARCHITECTURE An architect is building the front entrance of a building in the shape of a parabola with the equation $y = -\frac{1}{10}(x - 10)^2 + 20$. While the entrance is being built, the construction team puts in two support beams with equations $y = -x + 10$ and $y = x - 10$. Where do the support beams meet the parabola? **(0, 10) and (20, 10)**

Solve each system of inequalities by graphing. **69–74. See margin.**

57. $x^2 + y^2 < 64$
$x^2 + 16(y - 3)^2 < 16$

58. $x^2 + y^2 < 49$
$16x^2 - 9y^2 \geq 144$

59. $x + y < 4$
$9x^2 - 4y^2 \geq 36$

60. $x^2 + y^2 < 25$
$4x^2 - 9y^2 < 36$

61. $x^2 + y^2 < 36$
$4x^2 + 9y^2 > 36$

62. $y^2 < x$
$x^2 - 4y^2 < 16$

Example 6

Solve the system of equations.
$x^2 + y^2 = 100$
$3x - y = 10$

Use substitution to solve the system.
First, rewrite $3x - y = 10$ as $y = 3x - 10$.

$$x^2 + y^2 = 100$$
$$x^2 + (3x - 10)^2 = 100$$
$$x^2 + 9x^2 - 60x + 100 = 100$$
$$10x^2 - 60x + 100 = 100$$
$$10x^2 - 60x = 0$$
$$10x(x - 6) = 0$$

$10x = 0$ or $x - 6 = 0$
$x = 0$ $\quad x = 6$

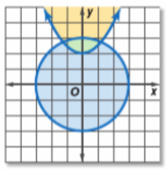
Now solve for y.

$y = 3x - 10 = 3(0) - 10 = -10$
$y = 3x - 10 = 3(6) - 10 = 8$

The solutions of the system are (0, −10) and (6, 8)

Example 7

Solve the system of inequalities by graphing.
$x^2 + y^2 \leq 9$
$2y \geq x^2 + 4$



The solution is the green shaded region.

Additional Answers

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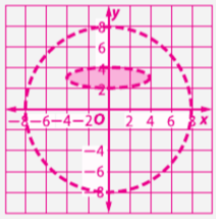

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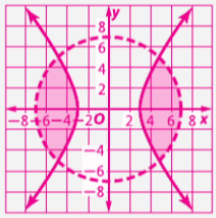

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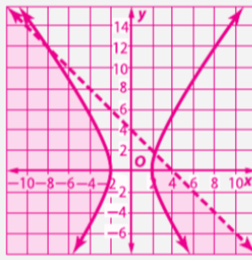

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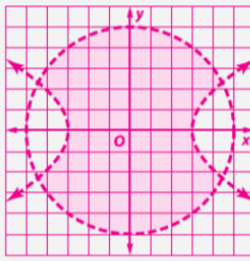

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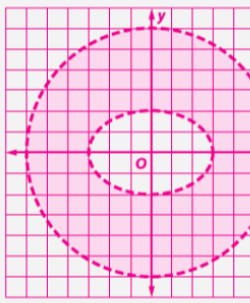
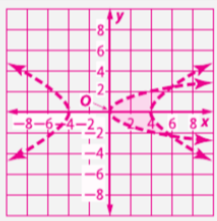
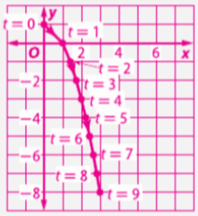


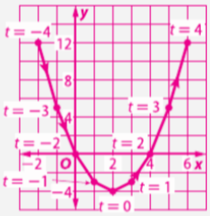
74.


Additional Answers

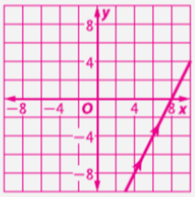
50.



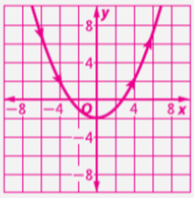
51.



52. $y = 2x - 16$



53. $y = \frac{x^2}{4} - 2$



CHAPTER 6 Study Guide and Review *Continued*

6-4 Rotations of Conic Sections

Use a graphing calculator to graph the conic given by each equation. **41–45. See Chapter 6 Answer Appendix.**

63. $x^2 - 4xy + y^2 - 2y - 2x = 0$

64. $x^2 - 3xy + y^2 - 3y - 6x + 5 = 0$

65. $2x^2 + 2y^2 - 8xy + 4 = 0$

66. $3x^2 + 9xy + y^2 = 0$

67. $4x^2 - 2xy + 8y^2 - 7 = 0$

46–49. See Chapter 6 Answer Appendix.

Write each equation in the $x'y'$-plane for the given value of θ. Then identify the conic.

68. $x^2 + y^2 = 4$; $\theta = \frac{\pi}{4}$

69. $x^2 - 2x + y = 5$; $\theta = \frac{\pi}{3}$

70. $x^2 - 4y^2 = 4$; $\theta = \frac{\pi}{2}$

71. $9x^2 + 4y^2 = 36$; $\theta = 90°$

Example 8

Use a graphing calculator to graph $x^2 + 2xy + y^2 + 4x - 2y = 0$.

$x^2 + 2xy + y^2 + 4x - 2y = 0$ — Original equation

$1y^2 + (2x - 2)y + (x^2 + 4x) = 0$ — Quadratic form

Use the Quadratic Formula.

$$y = \frac{-(2x-2) \pm \sqrt{(2x-2)^2 - 4(1)(x^2+4x)}}{2(1)}$$

$$= \frac{-2x + 2 \pm \sqrt{4x^2 - 8x + 4 - 4x^2 - 16x}}{2}$$

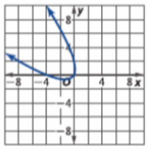
$$= \frac{-2x + 2 \pm 2\sqrt{1 - 6x}}{2}$$

$$= -x + 1 \pm \sqrt{1 - 6x}$$

Graph as

$y_1 = -x + 1 + \sqrt{1 - 6x}$ and

$y_2 = -x + 1 - \sqrt{1 - 6x}$.



6-5 Parametric Equations

Sketch the curve given by each pair of parametric equations over the given interval. **50–51. See margin.**

72. $x = \sqrt{t}$, $y = 1 - t$; $0 \le t \le 9$

73. $x = t + 2$, $y = t^2 - 4$; $-4 \le t \le 4$

Write each pair of parametric equations in rectangular form. Then graph the equation. **52–55. See margin.**

74. $x = t + 5$ and $y = 2t - 6$

75. $x = 2t$ and $y = t^2 - 2$

76. $x = t^2 + 3$ and $y = t^2 - 4$

77. $x = t^2 - 1$ and $y = 2t + 1$

Example 9

Write $x = 5 \cos t$ and $y = 9 \sin t$ in rectangular form. Then graph the equation.

$x = 5 \cos t$ $\quad$ $y = 9 \sin t$

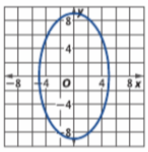
$\cos t = \frac{x}{5}$ $\quad$ $\sin t = \frac{y}{9}$ — Solve for sin *t* and cos *t*

$\sin^2 t + \cos^2 t = 1$

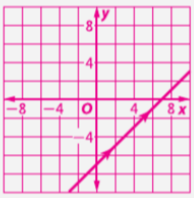
$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{9}\right)^2 = 1$

$\frac{x^2}{25} + \frac{y^2}{81} = 1$

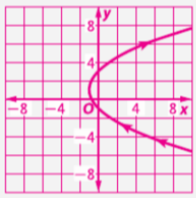
The parametric equations represent the graph of an ellipse.



54. $y = x - 7$



55. $(y - 1)^2 = 4(x + 1)$



CHAPTER 6 Practice Test

Find the midpoint of the line segment with endpoints at the given coordinates.

1. $(8, 3), (-4, 9)$ **(2, 6)**

2. $\left(\frac{3}{4}, 0\right), \left(\frac{1}{2}, -1\right)$ $\left(\frac{5}{8}, -\frac{1}{2}\right)$

3. $(-10, 0), (-2, 6)$ **(−6, 3)**

Find the distance between each pair of points with the given coordinates.

4. $(-5, 8), (4, 3)$ $\sqrt{106}$

5. $\left(\frac{1}{3}, \frac{2}{3}\right), \left(-\frac{5}{6}, -\frac{11}{6}\right)$ $\frac{\sqrt{274}}{6}$

6. $(4, -5), (4, 9)$ **14**

State whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation. **7–16. See Chapter 6 Answer Appendix.**

7. $y^2 = 64 - x^2$

8. $4x^2 + y^2 = 16$

9. $4x^2 - 9y^2 + 8x + 36y = 68$

10. $\frac{1}{2}x^2 - 3 = y$

11. $y = -2x^2 - 5$

12. $16x^2 + 25y^2 = 400$

13. $x^2 + 6x + y^2 = 16$

14. $\frac{y^2}{4} - \frac{x^2}{16} = 1$

15. $(x + 2)^2 = 3(y - 1)$

16. $4x^2 + 16y^2 + 32x + 63 = 0$

17. MULTIPLE CHOICE Which equation represents a hyperbola that has vertices at $(-3, -3)$ and $(5, -3)$ and a conjugate axis of length 6 units? **B**

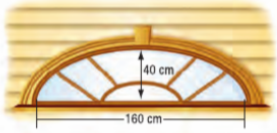
A $\frac{(y-1)^2}{16} - \frac{(x+3)^2}{9} = 1$

B $\frac{(x-1)^2}{16} - \frac{(y+3)^2}{9} = 1$

C $\frac{(y+1)^2}{16} - \frac{(x-3)^2}{9} = 1$

D $\frac{(x+1)^2}{16} - \frac{(y-3)^2}{9} = 1$

18. CARPENTRY Ayoub built a small window frame shaped like the top half of an ellipse. The window is 40 centimeters tall at its highest point and 160 centimeters wide at the bottom. What is the height of the window 20 centimeters from the center of the base?



18. $10\sqrt{15}$ **or about 38.73 centimeters**

Solve each system of equations.

19. $x^2 + y^2 = 100$, $y = -x - 2$ **(−8, 6), (6, −8)**

20. $x^2 + 2y^2 = 11$, $x + y = 2$ **(3, −1),** $\left(-\frac{1}{3}, \frac{7}{3}\right)$

21. $x^2 + y^2 = 34$, $y^2 - x^2 = 9$ $\left(\pm\frac{5\sqrt{2}}{2}, \pm\frac{\sqrt{86}}{2}\right)$

22, 23. See Chapter 6 Answer Appendix.

Solve each system of inequalities.

22. $x^2 + y^2 \leq 9$, $y > -x^2 + 2$

23. $\frac{(x-2)^2}{4} - \frac{(y-4)^2}{9} \geq 1$, $x - 4y < 8$

24. MULTIPLE CHOICE Which is NOT the equation of a parabola? **J**

F $y = 3x^2 + 5x - 3$

G $2y + 3x^2 + x - 9 = 0$

H $x = 3(y + 1)^2$

J $x^2 + 2y^2 + 6x = 10$

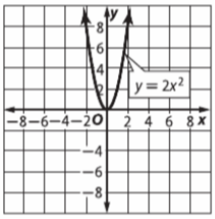
25. FORESTRY A forest ranger at an outpost in the Sam Houston National Forest and another ranger at the primary station both heard an explosion. The outpost and the primary station are 6 kilometers apart. **a, b. See Chapter 6 Answer Appendix.**

a. If one ranger heard the explosion 6 seconds before the other, write an equation that describes all the possible locations of the explosion. Place the two ranger stations on the x-axis with the midpoint between the stations at the origin. The transverse axis is horizontal. (*Hint*: The speed of sound is about 0.35 kilometer per second.)

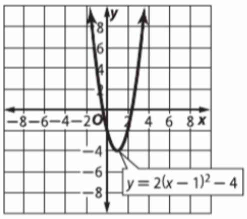
b. Draw a sketch of the possible locations of the explosion. Include the ranger stations in the drawing.

Lesson 6-1 (Guided Practice)

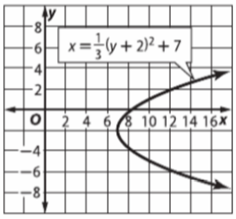
2A.



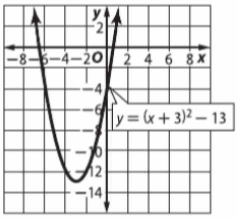
2B.



3A. $x = \frac{1}{3}(y + 2)^2 + 7$

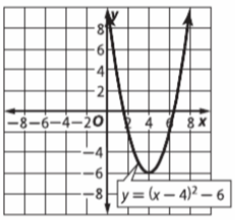


3B. $y = (x + 3)^2 - 13$



Lesson 6-1

5.



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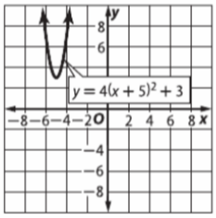

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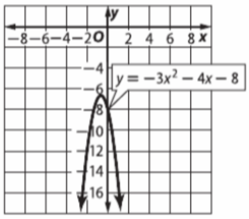
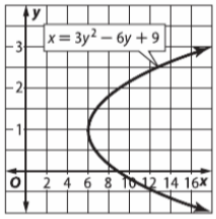
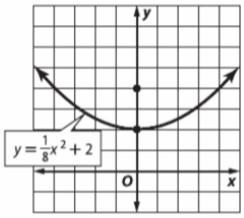


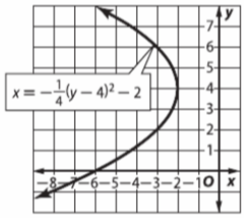
8.



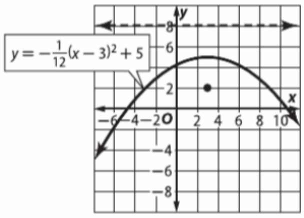
9. $y = \frac{1}{8}x^2 + 2$



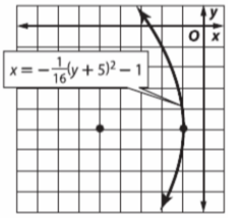
10. $x = -\frac{1}{4}(y - 4)^2 - 2$



11. $y = -\frac{1}{12}(x - 3)^2 + 5$



12. $x = -\frac{1}{16}(y + 5)^2 - 1$



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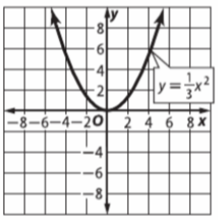

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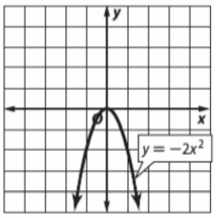

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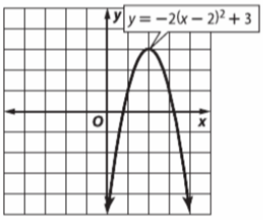

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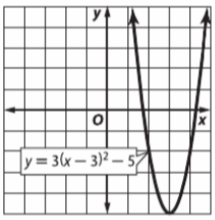

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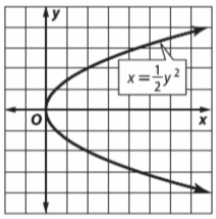

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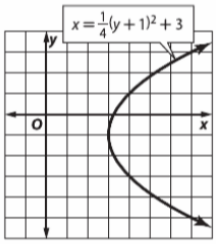
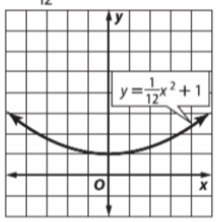
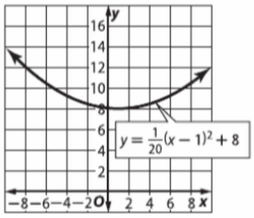


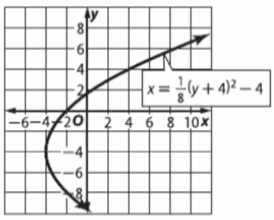
26. $y = \frac{1}{12}x^2 + 1$



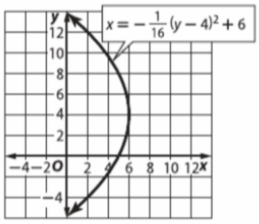
27. $y = \frac{1}{20}(x - 1)^2 + 8$



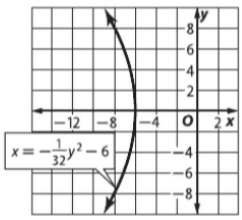
28. $x = \frac{1}{8}(y + 4)^2 - 4$



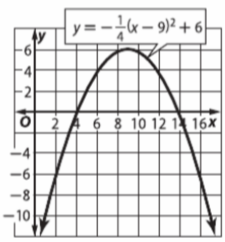
29. $x = -\frac{1}{16}(y - 4)^2 + 6$



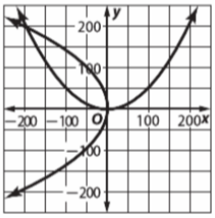
30. $x = -\frac{1}{32}y^2 - 6$



31. $y = -\frac{1}{4}(x - 9)^2 + 6$

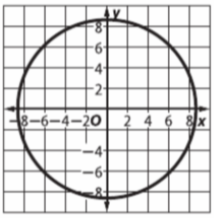


33a.

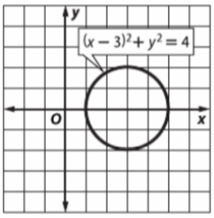


Lesson 6-2

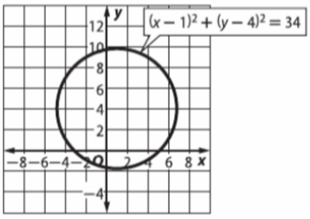
31. center: (0,0); radius: $5\sqrt{3}$



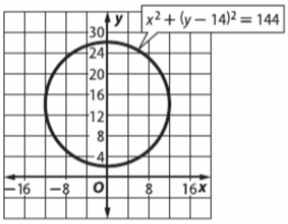
32. center: (3, 0); radius: 2



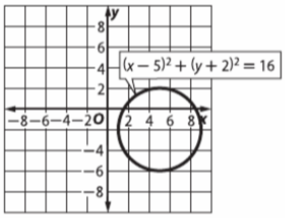
33. center: (1, 4); radius: $\sqrt{34}$



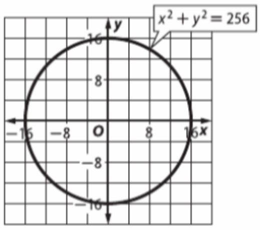
34. center: (0, 14); radius: 12



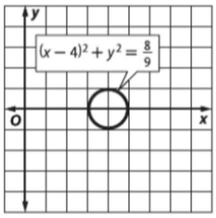
35. center: (5, −2); radius: 4



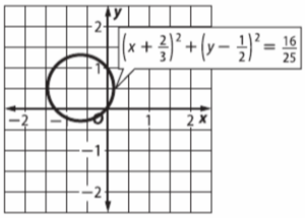
36. center: (0, 0); radius: 16



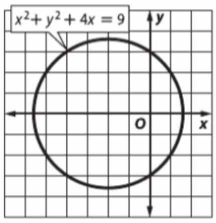
37. center: (4, 0); radius: $\frac{\sqrt{8}}{3}$



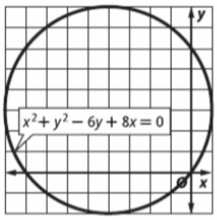
38. center: $\left(-\frac{2}{3}, \frac{1}{2}\right)$; radius: $\frac{4}{5}$



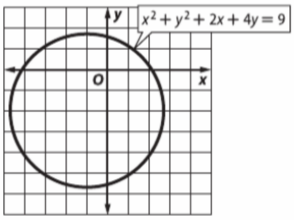
39. center: (−2, 0); radius: $\sqrt{13}$



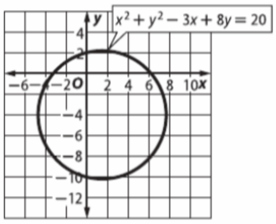
40. center: $(-4, 3)$; radius: 5



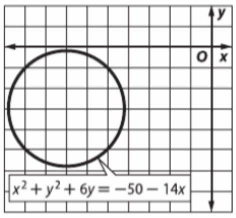
41. center: $(-1, -2)$; radius: $\sqrt{14}$



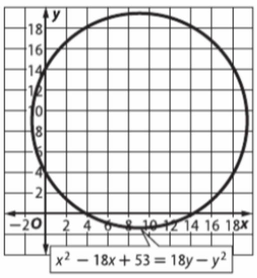
42. center: $\left(\frac{3}{2}, -4\right)$; radius: $\frac{3\sqrt{17}}{2}$



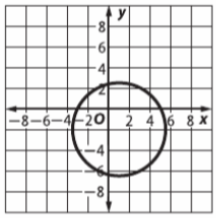
43. center: $(-7, -3)$; radius: $2\sqrt{2}$ units



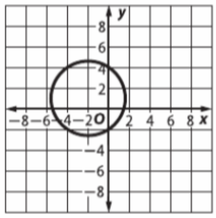
44. center: $(9, 9)$; radius: $\sqrt{109}$ units



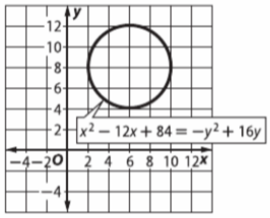
45. center: $(1, -2)$; radius: $\sqrt{21}$



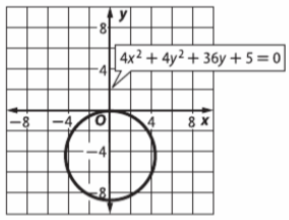
46. center: $(-2, 1)$; radius: $\sqrt{13}$



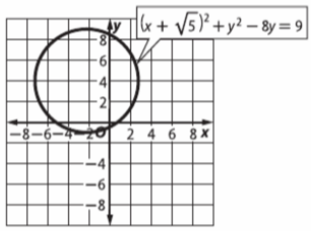
58. center: $(6, 8)$, radius: 4



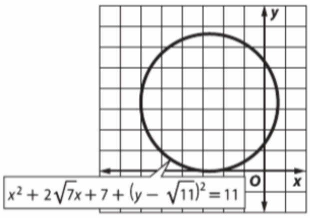
59. center: $\left(0, -\frac{9}{2}\right)$; radius: $\sqrt{19}$



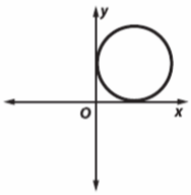
60. center: $\left(-\sqrt{5}, 4\right)$; radius: 5



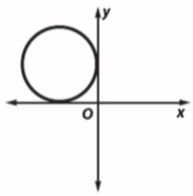
61. center: $(-\sqrt{7}, \sqrt{11})$; radius: $\sqrt{11}$



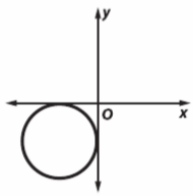
67. Quadrant I
$a > 0, b > 0, a = b, r > 0$



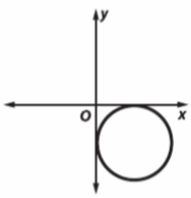
Quadrant II
$a < 0, b > 0, a = -b, r > 0$



Quadrant III
$a < 0, b < 0, a = b, r > 0$

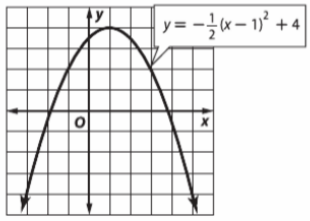


Quadrant IV
$a > 0, b < 0, a = -b, r > 0$

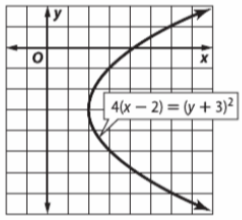


Sample answer: The circle is rotated 90° about the origin from one quadrant to the next.

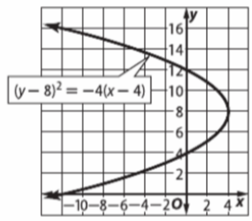
72.



73.

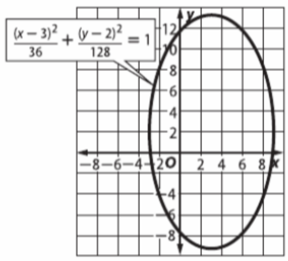


74.

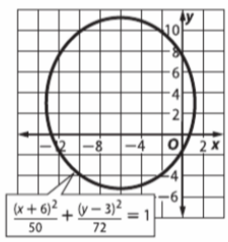


Lesson 6-3

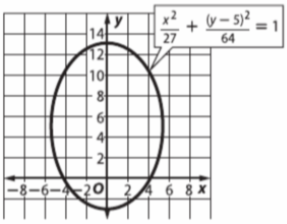
24. center (3, 2); foci (3, 11.59) and (3, −7.59); major axis: ≈22.63; minor axis: 12



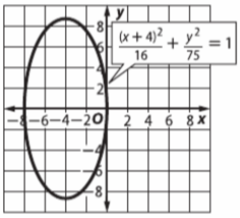
25. center (−6, 3); foci (−6, 7.69) and (−6, −1.69); major axis: ≈16.97; minor axis: ≈14.14



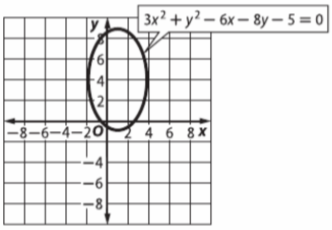
26. center (0, 5); foci (0, 11.08) and (0, −1.08); major axis: 16; minor axis: ≈10.39



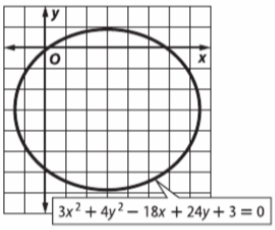
27. center (−4, 0); foci (−4, 7.68) and (−4, −7.68); major axis: ≈17.32; minor axis: 8



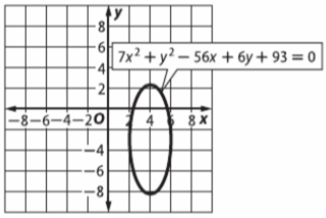
28. center (1, 4); foci (1, 8) and (1, 0); major axis: ≈9.80; minor axis: ≈5.66



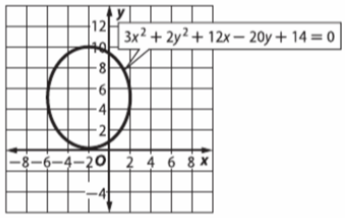
29. center (3, −3); foci (5.24, −3) and (0.76, −3); major axis: ≈8.94; minor axis: ≈7.75



30. center (4, −3); foci (4, 1.90) and (4, −7.90); major axis: ≈10.58; minor axis: 4



31. center (−2, 5); foci (−2, 7.83) and (−2, 2.17); major axis: ≈9.80 minor axis: 8



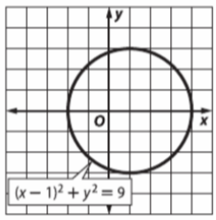
46. Sample answer: The domain is $\{x \mid -a \le x \le a\}$ because, if $|x| > a$, then $\frac{x^2}{a^2}$ would be greater than 1. This will force $\frac{y^2}{b^2}$ to be negative since $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ must equal 1. In order for $\frac{y^2}{b^2}$ to be negative, either y^2 or b^2 must be negative, which cannot happe with real numbers. For any values of $\{x \mid -a \le x \le a\}$, $\frac{x^2}{a^2}$ will b between 0 and 1. The value of $\frac{y^2}{b^2}$ will also be between 0 and 1 for $\{y \mid -b \le y \le b\}$ and there are infinite combinations of $\frac{x^2}{a^2}$ and $\frac{y^2}{b^2}$ for which $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Thus, the domain must be $\{x \mid -a \le x \le a\}$. The same method proves that the range must be $\{y \mid -b \le y \le b\}$.

Mid-Chapter Quiz

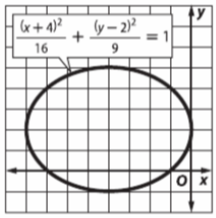
11. $(-3, -4)$; $\left(-3, -\frac{15}{4}\right)$; $x = -3$; $y = -\frac{17}{4}$; opens upward; 1 unit

12. $(3, 1)$; $\left(\frac{23}{8}, 1\right)$; $y = 1$; $x = \frac{25}{8}$; opens to the left; $\frac{1}{2}$ unit

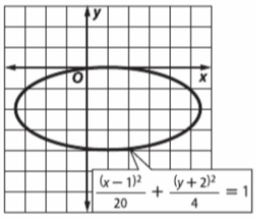
13. (1, 0); 3 units



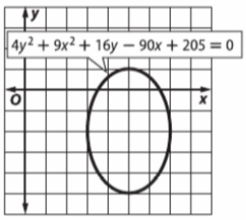
17. $(-4, 2)$; $(-4 \pm \sqrt{7}, 2)$; 8; 6



18. $(1, -2)$; $(5, -2)$, $(-3, -2)$; $4\sqrt{5}$; 4

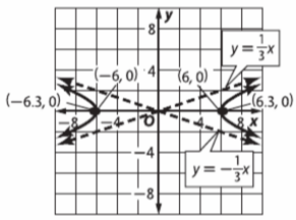


19. $(5, -2)$; $(5, -2 \pm \sqrt{5})$; 6; 4

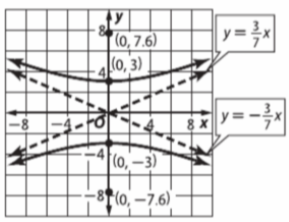


Lesson 6-4

14.



15.



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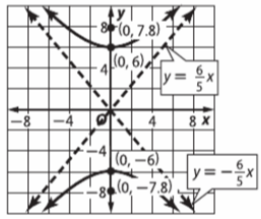

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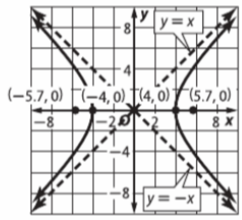
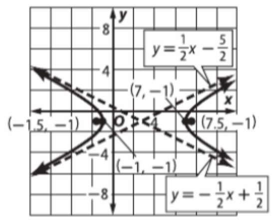
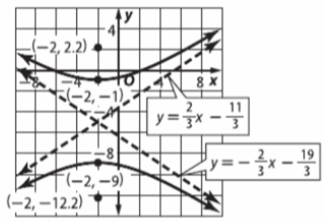


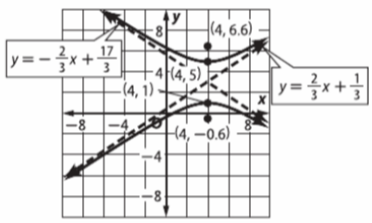
18.



19.



20.



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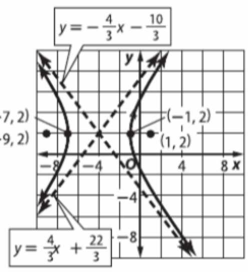
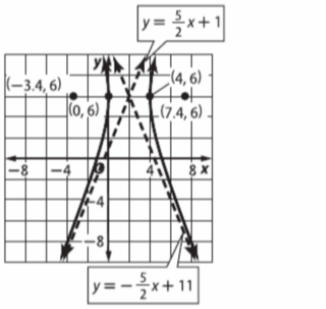
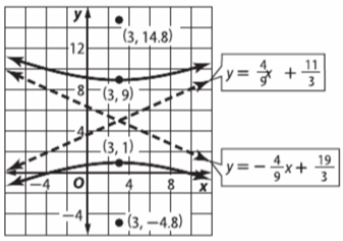


22.

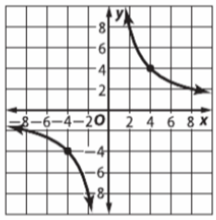


23.

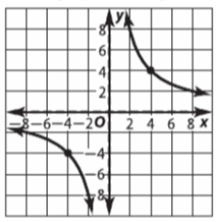


33a.

x	y
−12	−1.33
−10	−1.6
−8	−2
−6	−2.67
−4	−4
−2	−8
0	undef
2	8
4	4
6	2.67
8	2
10	1.6
12	1.33



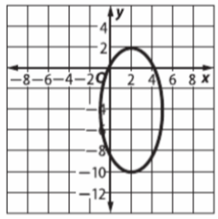
33c. The asymptotes are $y = 0$ and $x = 0$.



47. Sample answer: When 36 changes to 9, the vertical hyperbola widens (splits out from the y-axis faster). This is due to a smaller value of y being needed to produce the same value to x. The vertices are moved closer together due to the value of a decreasing from 6 to 3. The foci moved father from the vertices because the difference between c and a increased.

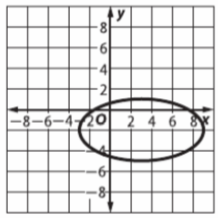
Lesson 6-5 (Guided Practice)

1. $\frac{(x-2)^2}{9} + \frac{(y+4)^2}{36} = 1$; ellipse

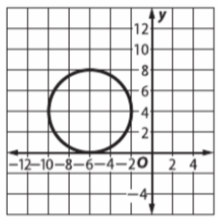


Lesson 6-5

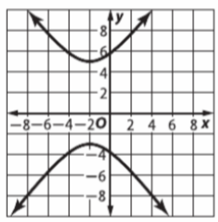
1. $\frac{(x-3)^2}{36} + \frac{(y+2)^2}{9} = 1$; ellipse



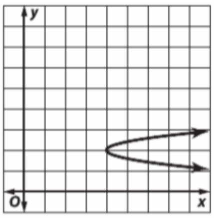
2. $(x+6)^2 + (y-4)^2 = 16$; circle



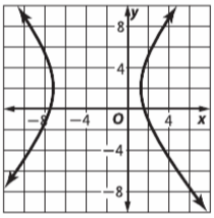
3. $\frac{(y-1)^2}{16} - \frac{(x+2)^2}{9}$; hyperbola



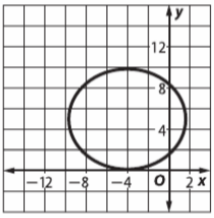
4. $x = 6(y - 2)^2 + 4$; parabola



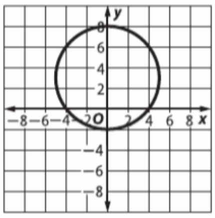
14. $\frac{(x + 3)^2}{18} - \frac{(y - 2)^2}{27} = 1$; hyperbola



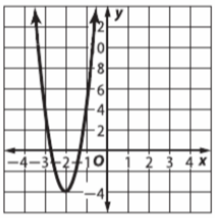
15. $\frac{(x + 4)^2}{32} + \frac{(y - 5)^2}{24} = 1$; ellipse



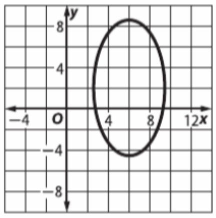
16. $x^2 + (y - 3)^2 = 25$; circle



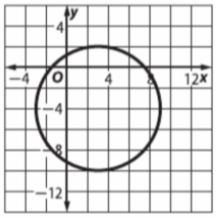
17. $y = 8(x + 2)^2 - 4$; parabola



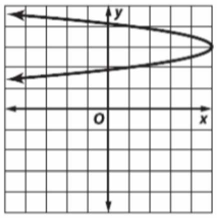
18. $\frac{(y - 2)^2}{42} + \frac{(x - 6)^2}{12} = 1$; ellipse



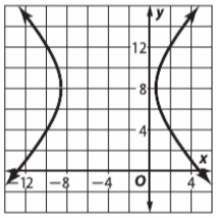
19. $(x - 3)^2 + (y + 4)^2 = 36$; circle



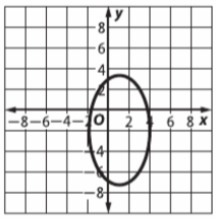
20. $x = -4(y - 3)^2 + 5$; parabola



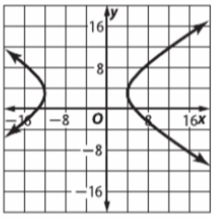
21. $\frac{(x + 4)^2}{21} - \frac{(y - 8)^2}{24} = 1$; hyperbola



22. $\frac{(y + 2)^2}{28} + \frac{(x - 1)^2}{9} = 1$; ellipse

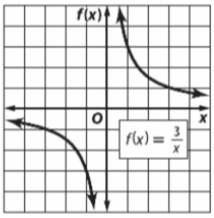


23. $\frac{(x+4)^2}{64} - \frac{(y-3)^2}{25} = 1$; hyperbola

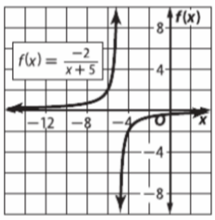


47. Sample answer: An ellipse is a flattened circle. Both circles and ellipses are enclosed regions while hyperbolas and parabolas are not. A parabola has one branch, which is a smooth curve that never ends, and a hyperbola has two such branches that are reflections of each other. In standard form and when there is no *xy*-term: an equation for a parabola consists of only one squared term, an equation for a circle has values for *A* and *C* that are equal, an equation for an ellipse has values for *A* and *C* that are the same sign but not equal, and an equation for a hyperbola has values of *A* and *C* that have opposite signs.

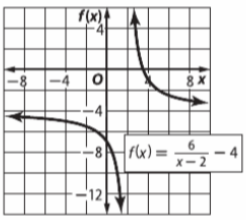
56.



57.

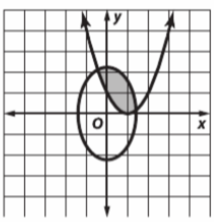


58.

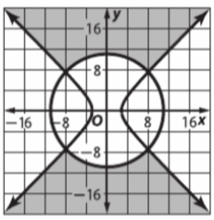


Lesson 6-7 (Guided Practice)

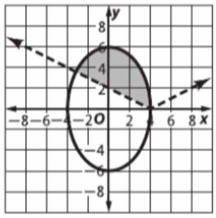
3A.



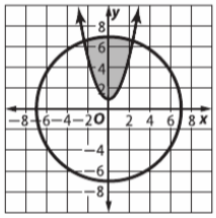
3B.



4A.

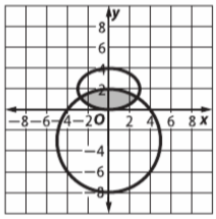


4B.



Lesson 6-7

10.



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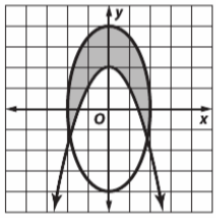
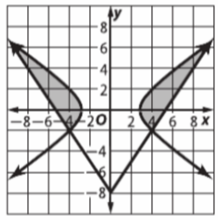


12.



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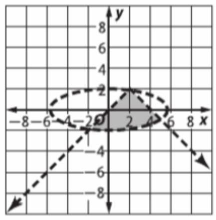

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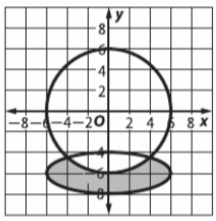

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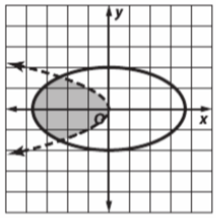

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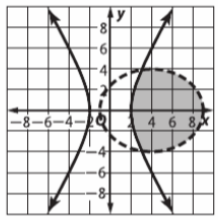

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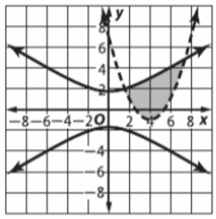

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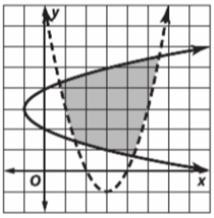

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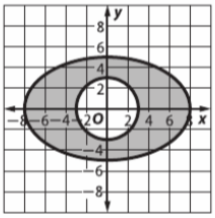

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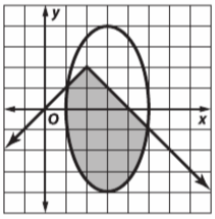

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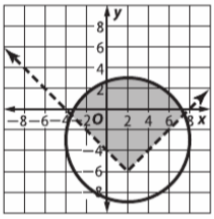

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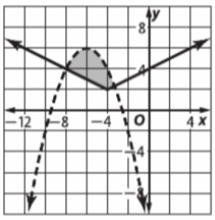
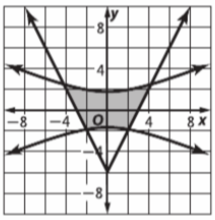


36.



43. Sample answer: $\frac{x^2}{16} + \frac{y^2}{36} = 1$ and $(x + 10)^2 + y^2 = 36$

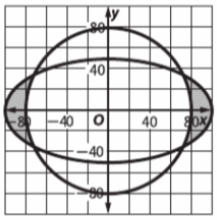
44. Sample answer: $y = x^2$ and $\frac{x^2}{16} + \frac{y^2}{36} = 1$

45. Sample answer: $x^2 + y^2 = 1$ and $\frac{x^2}{16} - \frac{y^2}{36} = 1$

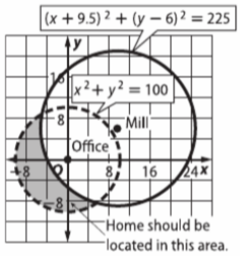
46. Sample answer: $y = x^2$ and $\frac{x^2}{64} + \frac{(y-4)^2}{16} = 1$

47. Sample answer: $\frac{x^2}{64} + \frac{y^2}{100} = 1$ and $x^2 - y^2 = 1$

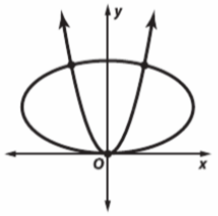
49. Sample answer: No; if one player is in one of the shaded areas and the other player is in the other shaded area, they will not be able to hear each other.



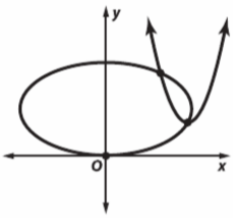
50. If we assume that Laila's office is located at the origin, then a system of equations describing this situation is $x^2 + y^2 < 100$ and $(x + 9.5)^2 + (y + 6)^2 \geq 225$.



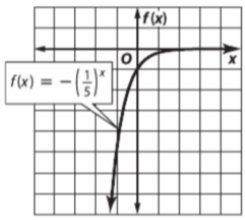
52. 1, 2 or 3; the parabola could be tangent to the ellipse and have one solution. The parabola could intersect the ellipse at three points like this.



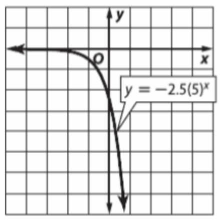
Or it could have two solutions like this.



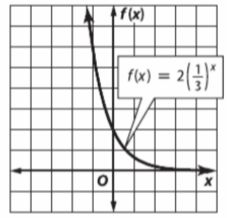
68. D = {all real numbers}, R = $\{f(x) \mid f(x) < 0\}$



69. D = {all real numbers}, R = $\{y \mid y < 0\}$

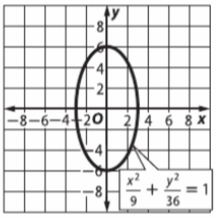


70. D = {all real numbers}, R = {$f(x)$ | $f(x) > 0$}

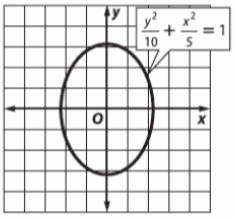


Study Guide and Review

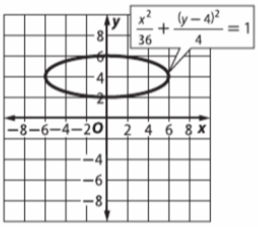
37. (0, 0); (0 , $\pm 3\sqrt{3}$); 12; 6



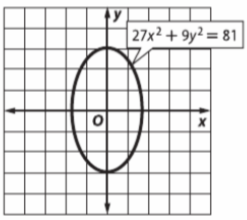
38. (0, 0); (0, $\pm\sqrt{5}$); $2\sqrt{10}$; $2\sqrt{5}$



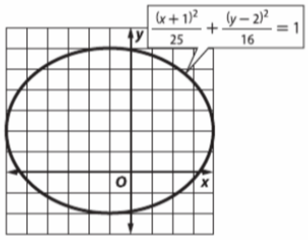
39. (0, 4); ($\pm 4\sqrt{2}$, 4); 12; 4



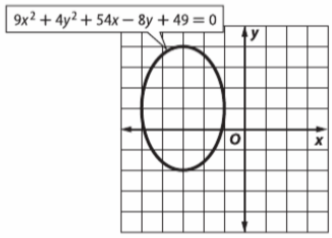
40. (0, 0); (0, $\pm\sqrt{6}$) ; 6; $2\sqrt{3}$



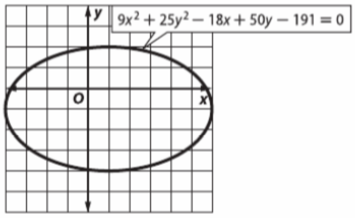
41. (−1, 2); (−4, 2), (2, 2); 10; 8



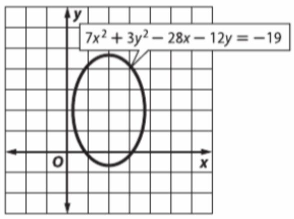
42. (−3, 1); (−3, 1 $\pm\sqrt{5}$); 6; 4



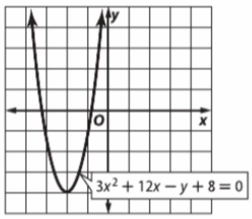
43. (1, −1); (−3, −1), (5, −1); 10; 6



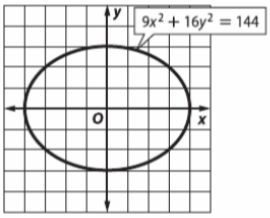
44. (2, 2); (2, 4), (2, 0); $2\sqrt{7}$; $2\sqrt{3}$



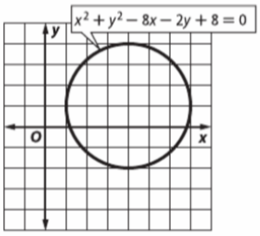
52. $y = 3(x + 2)^2 - 4$; parabola



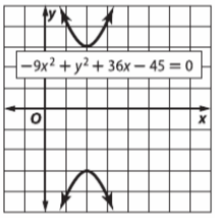
53. $\frac{x^2}{16} + \frac{y^2}{9} = 1$; ellipse



54. $(x - 4)^2 + (y - 1)^2 = 9$; circle

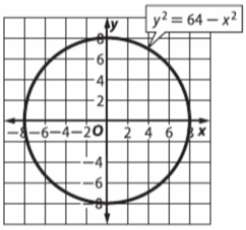


55. $\frac{y^2}{9} - \frac{(x - 2)^2}{1} = 1$; hyperbola

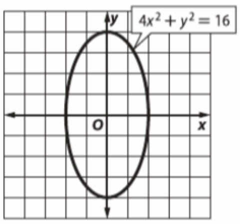


Practice Test

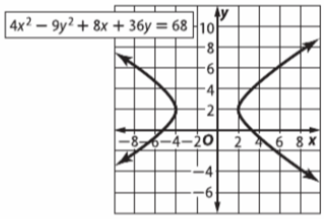
7. circle



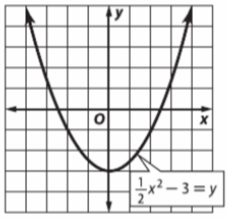
8. ellipse



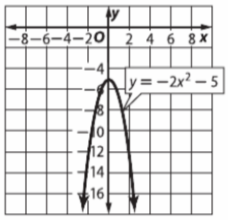
9. hyperbola



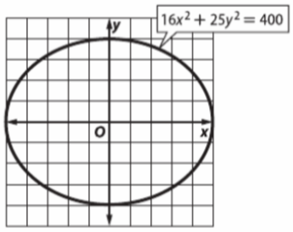
10. parabola



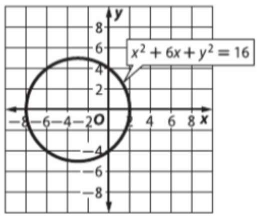
11. parabola



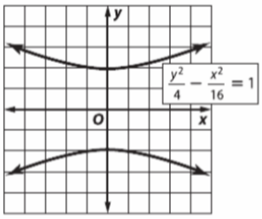
12. ellipse



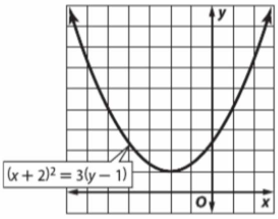
13. circle



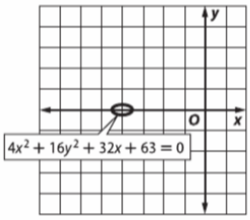
14. hyperbola



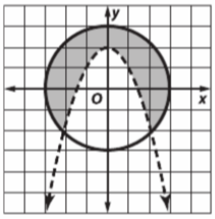
15. parabola



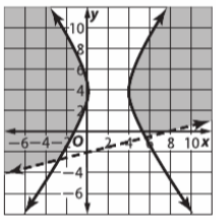
16. ellipse



22.

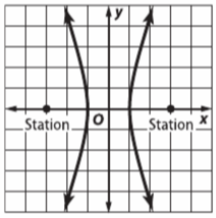


23.



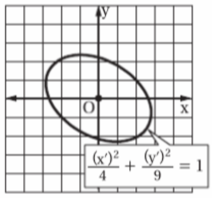
25a. $\frac{x^2}{1.1025} - \frac{y^2}{7.8975} = 1$

25b.



Lesson 6-7

30.



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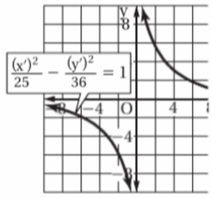
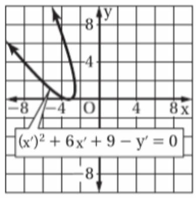


32.



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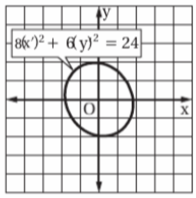
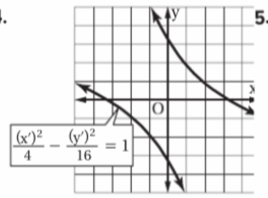
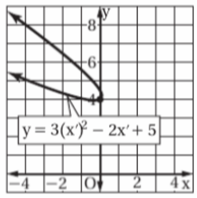


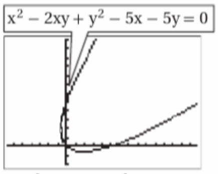
34.



5.



37.



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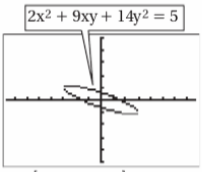
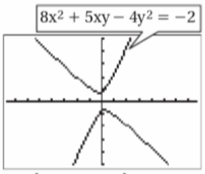
38.



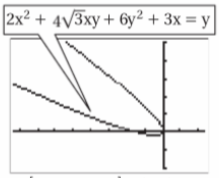
[−7.58, 7.58] scl: 1 by [−5, 5] scl: 1

39.



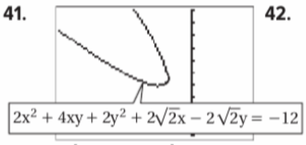
[−7.58, 7.58] scl: 1 by [−5, 5] scl: 1

40.



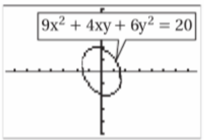
[−8.31, 2.31] scl: 1 by [−2, 5] scl: 1

41.



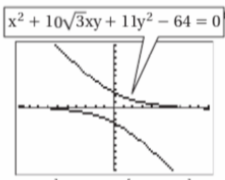
[−10.58, 4.58] scl: 1 by [−2, 8] scl: 1

42.



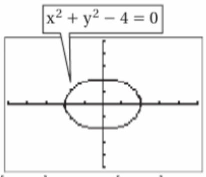
[−7.85, 7.85] scl: 1 by [−5, 5] scl: 1

43.



[−10, 10] scl: 1 by [−10, 10] scl: 1

44.



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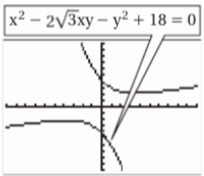
45.



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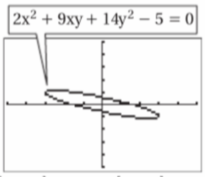
46.



[−5, 5] scl: 1 by [−5, 5] scl: 1

55. $Ax^2 = A(x'\cos\theta - y'\sin\theta)^2$

$= A(x')^2\cos^2\theta - 2Ax'y'\sin\theta\cos\theta + A(y')^2\sin^2\theta$

$Bxy = B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$

$= B(x')^2\sin\theta\cos\theta - Bx'y'\sin^2\theta + Bx'y'\cos^2\theta -$
$B(y')^2\sin\theta\cos\theta$

$= B(x')^2\sin\theta\cos\theta + Bx'y'(\cos^2\theta - \sin^2\theta) -$
$B(y')^2\sin\theta\cos\theta$

$Cy^2 = C(x'\sin\theta + y'\cos\theta)^2$

$= C(x')^2\sin^2\theta + 2Cx'y'\sin\theta\cos\theta + C(y')^2\cos^2\theta$

Combine like terms to confirm that $A + C = A' + C'$. A is the coefficient of the $(x')^2$-term and C is the coefficient of the $(y')^2$-term.

$A(x')^2\cos^2\theta + B(x')^2\sin\theta\cos\theta + C(x')^2\sin^2\theta =$
$[A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta](x')^2$

$A(y')^2\sin^2\theta - B(y')^2\sin\theta\cos\theta + C(y')^2\cos^2\theta =$
$[A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta](y')^2$

$A' + C' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta +$
$[A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta]$
$= A(\cos^2\theta + \sin^2\theta) + B(\sin\theta\cos\theta -$
$\sin\theta\cos\theta) + C(\sin^2\theta + \cos^2\theta)$
$= A(1) + B(0) + C(1)$
$= A + C$

61. Let $x = x'\cos\theta + y'\sin\theta$ and $y = -x'\sin\theta + y'\cos\theta$.

$r^2 = x^2 + y^2$
$= (x'\cos\theta + y'\sin\theta)^2 + (-x'\sin\theta + y'\cos\theta)^2$
$= (x')^2\cos^2\theta + 2x'y'\cos\theta\sin\theta + (y')^2\sin^2\theta +$
$(x')\sin^2\theta - 2x'y'\cos\theta\sin\theta + (y')^2\cos^2\theta$
$= [(x')^2 + (y')^2]\cos^2\theta + [(x')^2 + (y')^2]\sin^2\theta$
$= [(x')^2 + (y')^2](\cos^2\theta + \sin^2\theta)$
$= [(x')^2 + (y')^2](1)$
$= (x')^2 + (y')^2$

62. Sample answer: The axes have rotational symmetry of 90°. Any rotation of the axes greater than 90° will rotate the axes onto itself for every 90°. The final rotational angle θ will be the remaining angle measure after multiples of 90° have been subtracted out of the original angle.

63.
$\cos\theta\,(x = x'\cos\theta - y'\sin\theta)$
$\sin\theta\,(y = x'\sin\theta + y'\cos\theta)$

$x\cos\theta = x'\cos^2\theta - y'\sin\theta\cos\theta$
$+\, y\sin\theta = x'\sin^2\theta + y'\sin\theta\cos\theta$
$x\cos\theta + y\sin\theta = x'\cos^2\theta + x'\sin^2\theta$
$x\cos\theta + y\sin\theta = x'(\cos^2\theta + \sin^2\theta)$
$x\cos\theta + y\sin\theta = x'$

$\sin\theta\,(x = x'\cos\theta - y'\sin\theta)$
$\cos\theta\,(y = x'\sin\theta + y'\cos\theta)$

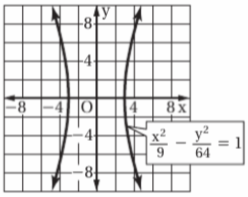
$x\sin\theta = x'\cos\theta\sin\theta - y'\sin^2\theta$
$-\, y\cos\theta = x'\cos\theta\sin\theta + y'\cos^2\theta$
$x\sin\theta - y\cos\theta = -y'\sin^2\theta - y'\cos^2\theta$
$x\sin\theta - y\cos\theta = -y'(\sin^2\theta + \cos^2\theta)$
$y\cos\theta - x\sin\theta = y'$

64. Sample answer: When $A = C$, the expression for θ written in terms of tangent has 0 in the denominator and is undefined. However, there is still a rotation of $\frac{\pi}{4}$. Thus, there is an extra condition. The angle of rotation θ written in terms of cotangent is $\cot 2\theta = \frac{A - C}{B}$. The only way this expression is undefined is if $B = 0$. If $B = 0$, then there is no reason to find a value for θ because there is no rotation necessary.

65. Sample answer: The discriminant is defined as $B^2 - 4AC$, or in this instance, $(B')^2 - 4A'C'$. Since a conic that is rotated has no B' term, the discriminant reduces to $-4A'C'$. Thus, only the A' and C' terms determine the type of conic. Therefore, $-4A'C' < 0$ would be an ellipse or a circle, $-4A'C' = 0$ would be a parabola, and $-4A'C' > 0$ would be a hyperbola. For a circle or an ellipse, A' and C' need to share the same sign. For a parabola, either A' or C' has to be equal to 0. For a hyperbola, A' and C' need to have opposite signs.

66. False; Sample answer: The equation may result in a pair of intersecting lines, a single line, a set of parallel lines or a single point.

67.



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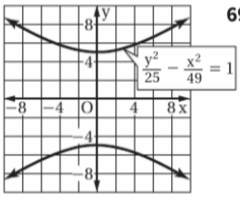

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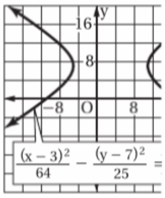
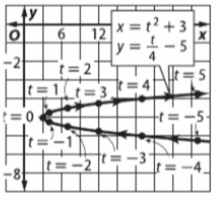


Lesson 6-5

1.



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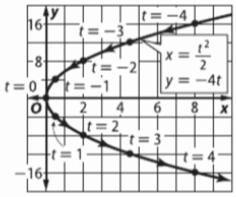

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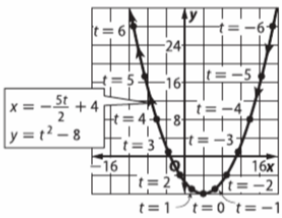

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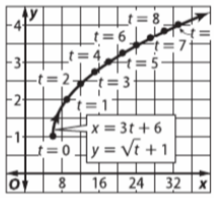

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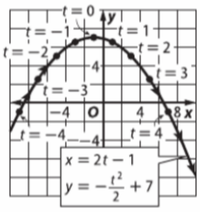

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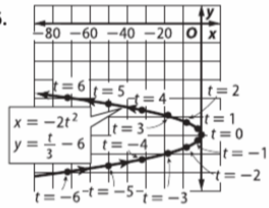

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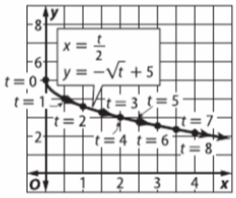

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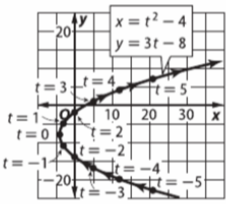

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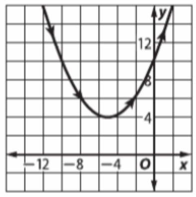

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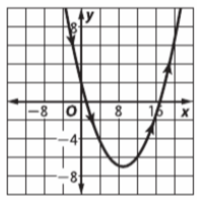

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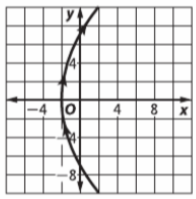

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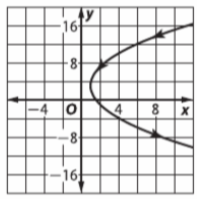

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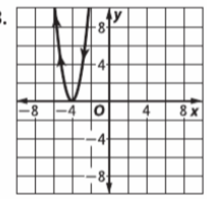

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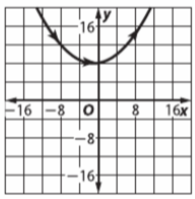
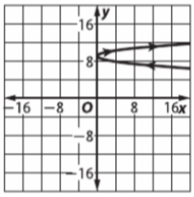
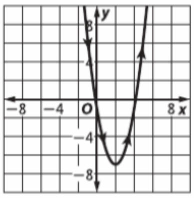
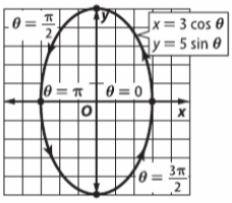


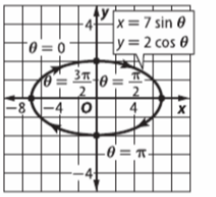
15. 

16. 

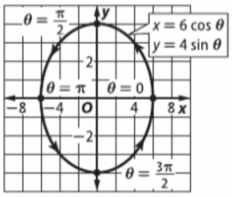
18. $\frac{x^2}{9} + \frac{y^2}{25} = 1$



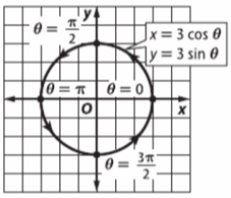
19. $\frac{x^2}{49} + \frac{y^2}{4} = 1$



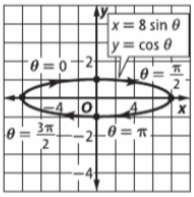
20. $\frac{x^2}{36} + \frac{y^2}{16} = 1$



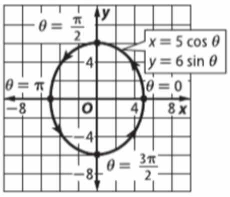
21. $\frac{x^2}{9} + \frac{y^2}{9} = 1$



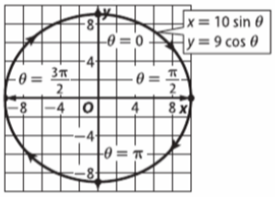
22. $\frac{x^2}{64} + y^2 = 1$



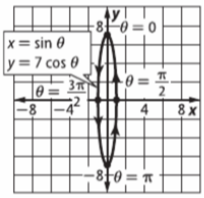
23. $\frac{x^2}{25} + \frac{y^2}{36} = 1$



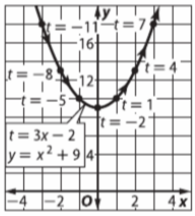
24. $\frac{x^2}{100} + \frac{y^2}{81} = 1$



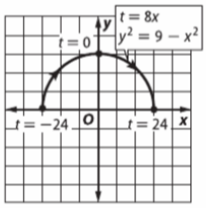
25. $x^2 + \frac{y^2}{49} = 1$



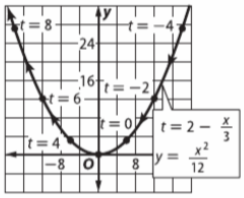
26. $x = \frac{t+2}{3}$ and $y = \frac{t^2}{9} + \frac{4t}{9} + \frac{85}{9}$



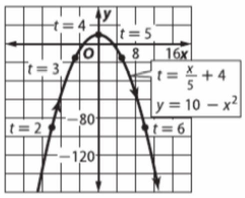
27. $x = \frac{t}{8}$ and $y = \sqrt{9 - \frac{t^2}{64}}$



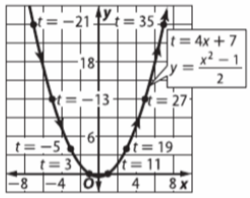
28. $x = 6 - 3t$ and $y = \frac{3}{4}t^2 - 3t + 3$



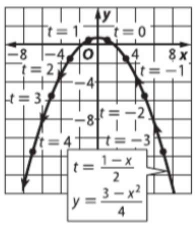
29. $x = 5t - 20$ and $y = -25t^2 + 200t - 390$



30. $x = \frac{t-7}{4}$ and $y = \frac{t^2}{32} - \frac{7t}{16} + \frac{33}{32}$



31. $x = 1 - 2t$ and $y = -t^2 + t + \frac{1}{2}$

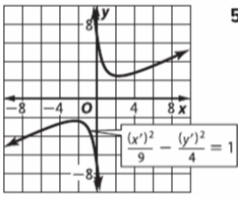


55. Sample answer: The horizontal distance is modeled by the cosine function, which is 0 at 90°. This would imply that the projectile has no horizontal movement. The corresponding parametric equation would be $x = 0$.

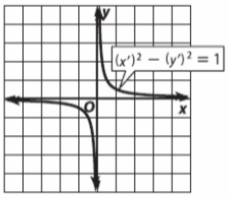
56. Sample answer: $x = 2 - 3t, y = 3 + 2t, z = -8 + 4t$; $x = -1 - 3t, y = 5 + 2t, z = -4 + 4t$

57. Sample answer: Parametric equations show both horizontal and vertical positions of an object over time, while rectangular equations can only show one or the other.

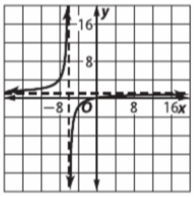
58.



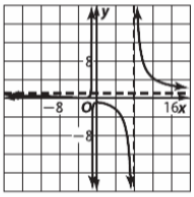
59.



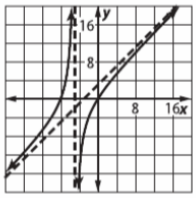
69. horizontal asymptote at $y = 1$; vertical asymptote at $x = -6$; x-intercept: 0; y-intercept: 0; $D = \{x \mid x \neq -6, x \in \mathbb{R}\}$



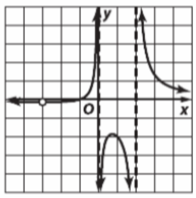
70. horizontal asymptote at $y = 1$; vertical asymptotes at $x = 8, x = -1$; x-intercepts: $-4, -2$; y-intercept: -1; $D = \{x \mid x \neq 8 \text{ or } -1, x \in \mathbb{R}\}$



71. vertical asymptotes at $x = -5$; oblique asymptote at $y = x + 3$; x-intercepts 0 and -8; y-intercept 0; $D = (-\infty, -5) \cup (-5, \infty)$

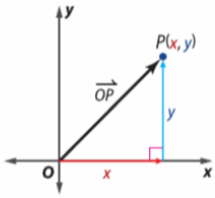


72. vertical asymptotes at $x = 0$ and $x = 2$; horizontal asymptote at $y = 0$; removable discontinuity at $x = -3$; x-intercept -1; no y-intercept; $D = (-\infty, -3) \cup (-3, 0) \cup (0, 2) \cup (2, \infty)$

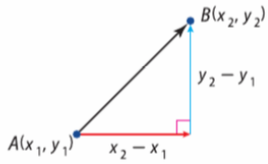


7-2 Vectors in the Coordinate Plane

The component form of a vector, the algebraic vector $\langle x, y\rangle$, is another way of denoting the geometric vector $\overrightarrow{OP}$ when in standard position. The angle θ formed by the positive x-axis and a vector, called the direction angle, specifies the direction of the vector.



To find the component form of vector $\overrightarrow{AB}$ when not in standard position, use the coordinates of its terminal and initial points: $\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1\rangle$. The magnitude of $\overrightarrow{AB}$ is $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



Addition, subtraction, and scalar multiplication of algebraic vectors are similar to those same operations with matrices. The result of these operations yields a vector.

The vector sum $a\mathbf{i} + b\mathbf{j}$ is a linear combination of the vectors **i** and **j**.

7-3 Dot Products and Vector Projections

The dot product of $a = \langle a_1, a_2\rangle$ and $b = \langle b_1, b_2\rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$. The dot product of two vectors yields a scalar.

If the dot product is 0, then the two vectors are perpendicular and are said to be orthogonal.

The projection of vector **u** onto vector **v** is a vector parallel to vector **v**.

Vector projections can be used to find a force and to calculate the work done by a force.

7-4 Vectors in Three-Dimensional Space

A three-dimensional coordinate system consists of the following:

- x-, y-, and z-axes, and
- eight regions called octants.

A point in space is represented by an ordered triple of rea numbers, (x, y, z).

Given two points in space $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

- the distance between the points is given by $\overline{AB} = \sqrt{x_2 - x_1^2 + y_2 - y_1^2 + z_2 - z_1^2}$ and
- the midpoint M of $\overline{AB}$ is $M\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2}\right)$

The component form of vector $\overrightarrow{AB}$ in standard position wi terminal point at (x_1, y_1, z_1) is $\langle x_1, y_1, z_1\rangle$.

The component form of vector $\overrightarrow{AB}$ not in standard positio with a terminal point at $B(x_2, y_2, z_2)$ and initial point at $A(x_1, y_1, z_1)$, is $\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$.

7-5 Dot and Cross Products of Vectors in Space

The dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ space is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

If the dot product is 0, then the two vectors are perpendicular and are orthogonal vectors.

The cross product of two vectors **a** and **b** in space is $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$, where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

The magnitude of the cross product of two vectors in spa represents the area of the parallelogram that has two adjacent sides formed by the two vectors.

Three vectors that lie in different planes but have the same initial point determine the adjacent edges of a parallelepiped. The magnitude of the triple scalar produc these vectors represents the volume of the parallelepipe