Chapter 1

Digital Systems and Binary Numbers

1.1 DIGITAL SYSTEMS

Digital systems have such a prominent role in everyday life that we refer to the present technological period as the *digital age*. Digital systems are used in communication, business transactions, traffic control, space guidance, medical treatment, weather monitoring, the Internet, and many other commercial, industrial, and scientific enterprises. We have digital telephones, digital television, digital versatile discs, digital cameras, handheld devices, and, of course, digital computers. The most striking property of the digital computer is its generality. It can follow a sequence of instructions, called a program, that operates on given data. The user can specify and change the program or the data according to the specific need. Because of this flexibility, general-purpose digital computers can perform a variety of information-processing tasks that range over a wide spectrum of applications.

One characteristic of digital systems is their ability to represent and manipulate discrete elements of information. Any set that is restricted to a finite number of elements contains discrete information. Examples of discrete sets are the 10 decimal digits, the 26 letters of the alphabet, the 52 playing cards, and the 64 squares of a chessboard. Early digital computers were used for numeric computations. In this case, the discrete elements were the digits. From this application, the term *digital* computer emerged. Discrete elements of information are represented in a digital system by physical quantities called signals. Electrical signals such as voltages and currents are the most common. Electronic devices called transistors predominate in the circuitry that implements these signals. The signals in most present-day electronic digital systems use just two discrete values and are therefore said to be *binary*. A binary digit, called a *bit*, has two values: 0 and 1. Discrete elements of information are represented with groups of bits called *binary codes*. For example, the decimal digits 0 through 9 are represented in a digital system with a code of four bits (e.g., the number 7 is represented by 0111).

Through various techniques, groups of bits can be made to represent discrete symbols, which are then used to develop the system in a digital format. Thus, a digital system is a system that manipulates discrete elements of information represented internally in binary form.

Discrete quantities of information either emerge from the nature of the data being processed or may be quantized from a continuous process. On the one hand, a payroll schedule is an inherently discrete process that contains employee names, social security numbers, weekly salaries, income taxes, and so on. An employee's paycheck is processed by means of discrete data values such as letters of the alphabet (names), digits (salary), and special symbols (such as $). On the other hand, a research scientist may observe a continuous process, but record only specific quantities in tabular form. The scientist is thus quantizing continuous data, making each number in his or her table a discrete quantity. In many cases, the quantization of a process can be performed automatically by an analog-to-digital converter.

The general-purpose digital computer is the best-known example of a digital system. The major parts of a computer are a memory unit, a central processing unit, and input–output units. The memory unit stores programs as well as input, output, and intermediate data. The central processing unit performs arithmetic and other data-processing operations as specified by the program. The program and data prepared by a user are transferred into memory by means of an input device such as a keyboard. An output device, such as a printer, receives the results of the computations, and the printed results are presented to the user. A digital computer can accommodate many input and output devices. One very useful device is a communication unit that provides interaction with other users through the Internet. A digital computer is a powerful instrument that can perform not only arithmetic computations, but also logical operations. In addition, it can be programmed to make decisions based on internal and external conditions.

There are fundamental reasons that commercial products are made with digital circuits. Like a digital computer, most digital devices are programmable. By changing the program in a programmable device, the same underlying hardware can be used for many different applications. Dramatic cost reductions in digital devices have come about because of advances in digital integrated circuit technology. As the number of transistors that can be put on a piece of silicon increases to produce complex functions, the cost per unit decreases and digital devices can be bought at an increasingly reduced price. Equipment built with digital integrated circuits can perform at a speed of hundreds of millions of operations per second. Digital systems can be made to operate with extreme reliability by using error-correcting codes. An example of this strategy is the digital versatile disk (DVD), in which digital information representing video, audio, and other data is recorded without the loss of a single item. Digital information on a DVD is recorded in such a way that, by examining the code in each digital sample before it is played back, any error can be automatically identified and corrected.

A digital system is an interconnection of digital modules. To understand the operation of each digital module, it is necessary to have a basic knowledge of digital circuits and their logical function. The first seven chapters of this book present the basic tools of digital design, such as logic gate structures, combinational and sequential circuits, and programmable logic devices. Chapter 8 introduces digital design at the register transfer level (RTL). Chapters 9 and 10 deal with asynchronous sequential circuits and the various integrated digital logic families. Chapters 11 and 12 introduce commercial integrated circuits and show how they can be connected in the laboratory to perform experiments with digital circuits.

A major trend in digital design methodology is the use of a hardware description language (HDL) to describe and simulate the functionality of a digital circuit. An HDL resembles a programming language and is suitable for describing digital circuits in textual form. It is used to simulate a digital system to verify its operation before hardware is built in. It is also used in conjunction with logic synthesis tools to automate the design process. Because it is important that students become familiar with an HDL-based design methodology, HDL descriptions of digital circuits are presented throughout the book. While these examples help illustrate the features of an HDL, they also demonstrate the best practices used by industry to exploit HDLs. Ignorance of these practices will lead to cute, but worthless, HDL models that may simulate a phenomenon, but that cannot be synthesized by design tools, or to models that waste silicon area or synthesize to hardware that cannot operate correctly.

As previously stated, digital systems manipulate discrete quantities of information that are represented in binary form. Operands used for calculations may be expressed in the binary number system. Other discrete elements, including the decimal digits, are represented in binary codes. Digital circuits, also referred to as logic circuits, process data by means of binary logic elements (logic gates) using binary signals. Quantities are stored in binary (two-valued) storage elements (flip-flops). The purpose of this chapter is to introduce the various binary concepts as a frame of reference for further study in the succeeding chapters.

1.2 BINARY NUMBERS

A decimal number such as 7,392 represents a quantity equal to 7 thousands, plus 3 hundreds, plus 9 tens, plus 2 units. The thousands, hundreds, etc., are powers of 10 implied by the position of the coefficients in the number. To be more exact, 7,392 is a shorthand notation for what should be written as

$$7 \times 10^3 + 3 \times 10^2 + 9 \times 10^1 + 2 \times 10^0$$

However, the convention is to write only the coefficients and, from their position, deduce the necessary powers of 10. In general, a number with a decimal point is represented by a series of coefficients:

$$a_5a_4a_3a_2a_1a_0 \cdot a_{-1}a_{-2}a_{-3}$$

The coefficients a_j are any of the 10 digits $(0, 1, 2, \ldots, 9)$, and the subscript value j gives the place value and, hence, the power of 10 by which the coefficient must be multiplied. Thus, the preceding decimal number can be expressed as

$$10^5a_5 + 10^4a_4 + 10^3a_3 + 10^2a_2 + 10^1a_1 + 10^0a_0 + 10^{-1}a_{-1} + 10^{-2}a_{-2} + 10^{-3}a_{-3}$$

The decimal number system is said to be of *base*, or *radix*, 10 because it uses 10 digits and the coefficients are multiplied by powers of 10. The *binary* system is a different number system. The coefficients of the binary number system have only two possible values: 0 and 1. Each coefficient a_j is multiplied by 2^j, and the results are added to obtain the decimal equivalent of the number. The radix point (e.g., the decimal point when 10 is the radix) distinguishes positive powers of 10 from negative powers of 10. For example, the decimal equivalent of the binary number 11010.11 is 26.75, as shown from the multiplication of the coefficients by powers of 2:

$$1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2} = 26.75$$

In general, a number expressed in a base-r system has coefficients multiplied by powers of r:

$$a_n \cdot r^n + a_{n-1} \cdot r^{n-1} + \cdots + a_2 \cdot r^2 + a_1 \cdot r + a_0 + a_{-1} \cdot r^{-1}$$
$$+ a_{-2} \cdot r^{-2} + \cdots + a_{-m} \cdot r^{-m}$$

The coefficients a_j range in value from 0 to $r - 1$. To distinguish between numbers of different bases, we enclose the coefficients in parentheses and write a subscript equal to the base used (except sometimes for decimal numbers, where the content makes it obvious that the base is decimal). An example of a base-5 number is

$$(4021.2)_5 = 4 \times 5^3 + 0 \times 5^2 + 2 \times 5^1 + 1 \times 5^0 + 2 \times 5^{-1} = (511.4)_{10}$$

The coefficient values for base 5 can be only 0, 1, 2, 3, and 4. The octal number system is a base-8 system that has eight digits: 0, 1, 2, 3, 4, 5, 6, 7. An example of an octal number is 127.4. To determine its equivalent decimal value, we expand the number in a power series with a base of 8:

$$(127.4)_8 = 1 \times 8^2 + 2 \times 8^1 + 7 \times 8^0 + 4 \times 8^{-1} = (87.5)_{10}$$

Note that the digits 8 and 9 cannot appear in an octal number.

It is customary to borrow the needed r digits for the coefficients from the decimal system when the base of the number is less than 10. The letters of the alphabet are used to supplement the 10 decimal digits when the base of the number is greater than 10. For example, in the *hexadecimal* (base-16) number system, the first 10 digits are borrowed from the decimal system. The letters A, B, C, D, E, and F are used for the digits 10, 11, 12, 13, 14, and 15, respectively. An example of a hexadecimal number is

$$(B65F)_{16} = 11 \times 16^3 + 6 \times 16^2 + 5 \times 16^1 + 15 \times 16^0 = (46{,}687)_{10}$$

As noted before, the digits in a binary number are called *bits*. When a bit is equal to 0, it does not contribute to the sum during the conversion. Therefore, the conversion from binary to decimal can be obtained by adding only the numbers with powers of two corresponding to the bits that are equal to 1. For example,

$$(110101)_2 = 32 + 16 + 4 + 1 = (53)_{10}$$

There are four 1's in the binary number. The corresponding decimal number is the sum of the four powers of two. The first 24 numbers obtained from 2 to the power of n are listed in Table 1.1. In computer work, 2^{10} is referred to as K (kilo), 2^{20} as M (mega), 2^{30} as G (giga), and 2^{40} as T (tera). Thus, $4K = 2^{12} = 4{,}096$ and $16M = 2^{24} = 16{,}777{,}216$. Computer capacity is usually given in bytes. A *byte* is equal to eight bits and can accommodate (i.e., represent the code of) one keyboard character. A computer hard disk with four gigabytes of storage has a capacity of $4G = 2^{32}$ bytes (approximately 4 billion bytes).

Table 1.1
Powers of Two

n	2^n	n	2^n	n	2^n
0	1	8	256	16	65,536
1	2	9	512	17	131,072
2	4	10	1,024	18	262,144
3	8	11	2,048	19	524,288
4	16	12	4,096	20	1,048,576
5	32	13	8,192	21	2,097,152
6	64	14	16,384	22	4,194,304
7	128	15	32,768	23	8,388,608

Arithmetic operations with numbers in base r follow the same rules as for decimal numbers. When a base other than the familiar base 10 is used, one must be careful to use only the r-allowable digits. Examples of addition, subtraction, and multiplication of two binary numbers are as follows:

augend:	101101	minuend:	101101	multiplicand:	1011
addend:	+100111	subtrahend:	−100111	multiplier:	× 101
sum:	1010100	difference:	000110		1011
					0000
					1011
				product:	110111

The sum of two binary numbers is calculated by the same rules as in decimal, except that the digits of the sum in any significant position can be only 0 or 1. Any carry obtained in a given significant position is used by the pair of digits one significant position higher. Subtraction is slightly more complicated. The rules are still the same as in decimal, except that the borrow in a given significant position adds 2 to a minuend digit. (A borrow in the decimal system adds 10 to a minuend digit.) Multiplication is simple: The multiplier digits are always 1 or 0; therefore, the partial products are equal either to the multiplicand or to 0.

1.3 NUMBER-BASE CONVERSIONS

The conversion of a number in base r to decimal is done by expanding the number in a power series and adding all the terms as shown previously. We now present a general procedure for the reverse operation of converting a decimal number to a number in base r. If the number includes a radix point, it is necessary to separate the number into an integer part and a fraction part, since each part must be converted differently. The conversion of a decimal integer to a number in base r is done by dividing the number and all successive quotients by r and accumulating the remainders. This procedure is best illustrated by example.

EXAMPLE 1.1

Convert decimal 41 to binary. First, 41 is divided by 2 to give an integer quotient of 20 and a remainder of $\frac{1}{2}$. Then the quotient is again divided by 2 to give a new quotient and remainder. The process is continued until the integer quotient becomes 0. The *coefficients* of the desired binary number are obtained from the *remainders* as follows:

	Integer Quotient		Remainder	Coefficient
$41/2 =$	20	+	$\frac{1}{2}$	$a_0 = 1$
$20/2 =$	10	+	0	$a_1 = 0$
$10/2 =$	5	+	0	$a_2 = 0$
$5/2 =$	2	+	$\frac{1}{2}$	$a_3 = 1$
$2/2 =$	1	+	0	$a_4 = 0$
$1/2 =$	0	+	$\frac{1}{2}$	$a_5 = 1$

Therefore, the answer is $(41)_{10} = (a_5a_4a_3a_2a_1a_0)_2 = (101001)_2$.

The arithmetic process can be manipulated more conveniently as follows:

Integer	Remainder	
41		
20	1	
10	0	
5	0	
2	1	
1	0	
0	1	101001 = answer

Conversion from decimal integers to any base-*r* system is similar to this example, except that division is done by *r* instead of 2.

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EXAMPLE 1.2

Convert decimal 153 to octal. The required base *r* is 8. First, 153 is divided by 8 to give an integer quotient of 19 and a remainder of 1. Then 19 is divided by 8 to give an integer quotient of 2 and a remainder of 3. Finally, 2 is divided by 8 to give a quotient of 0 and a remainder of 2. This process can be conveniently manipulated as follows:

153	
19	1
2	3
0	$2 = (231)_8$

The conversion of a decimal *fraction* to binary is accomplished by a method similar to that used for integers. However, multiplication is used instead of division, and integers instead of remainders are accumulated. Again, the method is best explained by example.

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EXAMPLE 1.3

Convert $(0.6875)_{10}$ to binary. First, 0.6875 is multiplied by 2 to give an integer and a fraction. Then the new fraction is multiplied by 2 to give a new integer and a new fraction. The process is continued until the fraction becomes 0 or until the number of digits have sufficient accuracy. The coefficients of the binary number are obtained from the integers as follows:

	Integer		Fraction	Coefficient
$0.6875 \times 2 =$	1	+	0.3750	$a_{-1} = 1$
$0.3750 \times 2 =$	0	+	0.7500	$a_{-2} = 0$
$0.7500 \times 2 =$	1	+	0.5000	$a_{-3} = 1$
$0.5000 \times 2 =$	1	+	0.0000	$a_{-4} = 1$

Therefore, the answer is $(0.6875)_{10} = (0.a_{-1}a_{-2}a_{-3}a_{-4})_2 = (0.1011)_2$.

To convert a decimal fraction to a number expressed in base r, a similar procedure is used. However, multiplication is by r instead of 2, and the coefficients found from the integers may range in value from 0 to $r - 1$ instead of 0 and 1.

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EXAMPLE 1.4

Convert $(0.513)_{10}$ to octal.

$$0.513 \times 8 = 4.104$$

$$0.104 \times 8 = 0.832$$

$$0.832 \times 8 = 6.656$$

$$0.656 \times 8 = 5.248$$

$$0.248 \times 8 = 1.984$$

$$0.984 \times 8 = 7.872$$

The answer, to seven significant figures, is obtained from the integer part of the products:

$$(0.513)_{10} = (0.406517\ldots)_8$$

The conversion of decimal numbers with both integer and fraction parts is done by converting the integer and the fraction separately and then combining the two answers. Using the results of Examples 1.1 and 1.3, we obtain

$$(41.6875)_{10} = (101001.1011)_2$$

From Examples 1.2 and 1.4, we have

$$(153.513)_{10} = (231.406517)_8$$

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1.4 OCTAL AND HEXADECIMAL NUMBERS

The conversion from and to binary, octal, and hexadecimal plays an important role in digital computers. Since $2^3 = 8$ and $2^4 = 16$, each octal digit corresponds to three binary digits and each hexadecimal digit corresponds to four binary digits. The first 16 numbers in the decimal, binary, octal, and hexadecimal number systems are listed in Table 1.2.

The conversion from binary to octal is easily accomplished by partitioning the binary number into groups of three digits each, starting from the binary point and proceeding to the left and to the right. The corresponding octal digit is then assigned to each group. The following example illustrates the procedure:

$$\begin{array}{cccccccccc} (10 & 110 & 001 & 101 & 011 & \cdot & 111 & 100 & 000 & 110)_2 = (26153.7406)_8 \\ 2 & 6 & 1 & 5 & 3 & & 7 & 4 & 0 & 6 \end{array}$$

Table 1.2
Numbers with Different Bases

Decimal (base 10)	Binary (base 2)	Octal (base 8)	Hexadecimal (base 16)
00	0000	00	0
01	0001	01	1
02	0010	02	2
03	0011	03	3
04	0100	04	4
05	0101	05	5
06	0110	06	6
07	0111	07	7
08	1000	10	8
09	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Conversion from binary to hexadecimal is similar, except that the binary number is divided into groups of *four* digits:

$$\begin{array}{ccccccccc} (10 & 1100 & 0110 & 1011 & \cdot & 1111 & 0010)_2 & = (2C6B.F2)_{16} \\ 2 & C & 6 & B & & F & 2 & \end{array}$$

The corresponding hexadecimal (or octal) digit for each group of binary digits is easily remembered from the values listed in Table 1.2.

Conversion from octal or hexadecimal to binary is done by reversing the preceding procedure. Each octal digit is converted to its three-digit binary equivalent. Similarly, each hexadecimal digit is converted to its four-digit binary equivalent. The procedure is illustrated in the following examples:

$$\begin{array}{cccccccc} (673.124)_8 = & (110 & 111 & 011 & \cdot & 001 & 010 & 100)_2 \\ & 6 & 7 & 3 & & 1 & 2 & 4 \end{array}$$

and

$$\begin{array}{cccccc} (306.D)_{16} = & (0011 & 0000 & 0110 & \cdot & 1101)_2 \\ & 3 & 0 & 6 & & D \end{array}$$

Binary numbers are difficult to work with because they require three or four times as many digits as their decimal equivalents. For example, the binary number 111111111111 is equivalent to decimal 4095. However, digital computers use binary numbers, and it is sometimes necessary for the human operator or user to communicate directly with the machine by means of such numbers. One scheme that retains the binary system in the computer, but reduces the number of digits the human must consider, utilizes the relationship between the binary number system and the octal or hexadecimal system. By this method, the human thinks in terms of octal or hexadecimal numbers and performs the required conversion by inspection when direct communication with the machine is necessary. Thus, the binary number 111111111111 has 12 digits and is expressed in octal as 7777 (4 digits) or in hexadecimal as FFF (3 digits). During communication between people (about binary numbers in the computer), the octal or hexadecimal representation is more desirable because it can be expressed more compactly with a third or a quarter of the number of digits required for the equivalent binary number. Thus, most computer manuals use either octal or hexadecimal numbers to specify binary quantities. The choice between them is arbitrary, although hexadecimal tends to win out, since it can represent a byte with two digits.

1.5 COMPLEMENTS

Complements are used in digital computers to simplify the subtraction operation and for logical manipulation. Simplifying operations leads to simpler, less expensive circuits to implement the operations. There are two types of complements for each base-r system: the radix complement and the diminished radix complement. The first is referred to as the r's complement and the second as the $(r - 1)$'s complement. When the value of the base r is substituted in the name, the two types are referred to as the 2's complement and 1's complement for binary numbers and the 10's complement and 9's complement for decimal numbers.

Diminished Radix Complement

Given a number N in base r having n digits, the $(r - 1)$'s complement of N is defined as $(r^n - 1) - N$. For decimal numbers, $r = 10$ and $r - 1 = 9$, so the 9's complement of N is $(10^n - 1) - N$. In this case, 10^n represents a number that consists of a single 1 followed by n 0's. $10^n - 1$ is a number represented by n 9's. For example, if $n = 4$, we have $10^4 = 10{,}000$ and $10^4 - 1 = 9999$. It follows that the 9's complement of a decimal number is obtained by subtracting each digit from 9. Here are some numerical examples:

The 9's complement of 546700 is $999999 - 546700 = 453299$.

The 9's complement of 012398 is $999999 - 012398 = 987601$.

For binary numbers, $r = 2$ and $r - 1 = 1$, so the 1's complement of N is $(2^n - 1) - N$. Again, 2^n is represented by a binary number that consists of a 1 followed by n 0's. $2^n - 1$ is a binary number represented by n 1's. For example, if $n = 4$, we have $2^4 = (10000)_2$ and $2^4 - 1 = (1111)_2$. Thus, the 1's complement of a binary number is obtained by subtracting each digit from 1. However, when subtracting binary digits from 1, we can have either $1 - 0 = 1$ or $1 - 1 = 0$, which causes the bit to change from 0 to 1 or from 1 to 0, respectively. Therefore, the 1's complement of a binary number is formed by changing 1's to 0's and 0's to 1's. The following are some numerical examples:

The 1's complement of 1011000 is 0100111.

The 1's complement of 0101101 is 1010010.

The $(r - 1)$'s complement of octal or hexadecimal numbers is obtained by subtracting each digit from 7 or F (decimal 15), respectively.

Radix Complement

The r's complement of an n-digit number N in base r is defined as $r^n - N$ for $N \neq 0$ and as 0 for $N = 0$. Comparing with the $(r - 1)$'s complement, we note that the r's complement is obtained by adding 1 to the $(r - 1)$'s complement, since $r^n - N = [(r^n - 1) - N] + 1$. Thus, the 10's complement of decimal 2389 is $7610 + 1 = 7611$ and is obtained by adding 1 to the 9's-complement value. The 2's complement of binary 101100 is $010011 + 1 = 010100$ and is obtained by adding 1 to the 1's-complement value.

Since 10 is a number represented by a 1 followed by n 0's, $10^n - N$, which is the 10's complement of N, can be formed also by leaving all least significant 0's unchanged, subtracting the first nonzero least significant digit from 10, and subtracting all higher significant digits from 9. Thus,

the 10's complement of 012398 is 987602

and

the 10's complement of 246700 is 753300

The 10's complement of the first number is obtained by subtracting 8 from 10 in the least significant position and subtracting all other digits from 9. The 10's complement of the second number is obtained by leaving the two least significant 0's unchanged, subtracting 7 from 10, and subtracting the other three digits from 9.

Similarly, the 2's complement can be formed by leaving all least significant 0's and the first 1 unchanged and replacing 1's with 0's and 0's with 1's in all other higher significant digits. For example,

the 2's complement of 1101100 is 0010100

and

the 2's complement of 0110111 is 1001001

The 2's complement of the first number is obtained by leaving the two least significant 0's and the first 1 unchanged and then replacing 1's with 0's and 0's with 1's in the other four most significant digits. The 2's complement of the second number is obtained by leaving the least significant 1 unchanged and complementing all other digits.

In the previous definitions, it was assumed that the numbers did not have a radix point. If the original number N contains a radix point, the point should be removed temporarily in order to form the r's or $(r - 1)$'s complement. The radix point is then restored to the complemented number in the same relative position. It is also worth mentioning that the complement of the complement restores the number to its original value. To see this relationship, note that the r's complement of N is $r^n - N$, so that the complement of the complement is $r^n - (r^n - N) = N$ and is equal to the original number.

Subtraction with Complements

The direct method of subtraction taught in elementary schools uses the borrow concept. In this method, we borrow a 1 from a higher significant position when the minuend digit is smaller than the subtrahend digit. The method works well when people perform subtraction with paper and pencil. However, when subtraction is implemented with digital hardware, the method is less efficient than the method that uses complements.

The subtraction of two n-digit unsigned numbers $M - N$ in base r can be done as follows:

1. Add the minuend M to the r's complement of the subtrahend N. Mathematically, $M + (r^n - N) = M - N + r^n$.
2. If $M \geq N$, the sum will produce an end carry r^n, which can be discarded; what is left is the result $M - N$.
3. If $M < N$, the sum does not produce an end carry and is equal to $r^n - (N - M)$, which is the r's complement of $(N - M)$. To obtain the answer in a familiar form, take the r's complement of the sum and place a negative sign in front.

The following examples illustrate the procedure:

EXAMPLE 1.5

Using 10's complement, subtract 72532 − 3250.

$$\begin{aligned} M &= \ 72532 \\ \text{10's complement of } N &= +\ \underline{96750} \\ \text{Sum} &= \ 169282 \\ \text{Discard end carry } 10^5 &= -\underline{100000} \\ \textit{Answer} &= \ 69282 \end{aligned}$$

Note that M has five digits and N has only four digits. Both numbers must have the same number of digits, so we write N as 03250. Taking the 10's complement of N produces a 9 in the most significant position. The occurrence of the end carry signifies that $M \geq N$ and that the result is therefore positive.

EXAMPLE 1.6

Using 10's complement, subtract 3250 − 72532.

$$\begin{aligned} M &= 03250 \\ \text{10's complement of } N &= +\underline{27468} \\ \text{Sum} &= 30718 \end{aligned}$$

There is no end carry. Therefore, the answer is $-(\text{10's complement of } 30718) = -69282$.

Note that since $3250 < 72532$, the result is negative. Because we are dealing with unsigned numbers, there is really no way to get an unsigned result for this case. When subtracting with complements, we recognize the negative answer from the absence of the end carry and the complemented result. When working with paper and pencil, we can change the answer to a signed negative number in order to put it in a familiar form.

Subtraction with complements is done with binary numbers in a similar manner, using the procedure outlined previously.

EXAMPLE 1.7

Given the two binary numbers $X = 1010100$ and $Y = 1000011$, perform the subtraction **(a)** $X - Y$ and **(b)** $Y - X$ by using 2's complements.

(a)

$$\begin{array}{rr} X = & 1010100 \\ \text{2's complement of } Y = & +\ \underline{0111101} \\ \text{Sum} = & 10010001 \\ \text{Discard end carry } 2^7 = & -\underline{10000000} \\ \textit{Answer: } X - Y = & 0010001 \end{array}$$

(b)

$$\begin{array}{rr} Y = & 1000011 \\ \text{2's complement of } X = & +\ \underline{0101100} \\ \text{Sum} = & 1101111 \end{array}$$

There is no end carry. Therefore, the answer is $Y - X = -(\text{2's complement of } 1101111) = -0010001$.

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Subtraction of unsigned numbers can also be done by means of the $(r - 1)$'s complement. Remember that the $(r - 1)$'s complement is one less than the r's complement. Because of this, the result of adding the minuend to the complement of the subtrahend produces a sum that is one less than the correct difference when an end carry occurs. Removing the end carry and adding 1 to the sum is referred to as an *end-around carry*.

EXAMPLE 1.8

Repeat Example 1.7, but this time using 1's complement.

(a) $X - Y = 1010100 - 1000011$

$$\begin{array}{rr} X = & 1010100 \\ \text{1's complement of } Y = & +\ \underline{0111100} \\ \text{Sum} = & 10010000 \\ \text{End-around carry} = & +\ \underline{\qquad\quad 1} \\ \textit{Answer: } X - Y = & 0010001 \end{array}$$

(b) $Y - X = 1000011 - 1010100$

$$\begin{array}{rr} Y = & 1000011 \\ \text{1's complement of } X = & +\ \underline{0101011} \\ \text{Sum} = & 1101110 \end{array}$$

There is no end carry. Therefore, the answer is $Y - X = -(\text{1's complement of } 1101110) = -0010001$.

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Note that the negative result is obtained by taking the 1's complement of the sum, since this is the type of complement used. The procedure with end-around carry is also applicable to subtracting unsigned decimal numbers with 9's complement.

1.6 SIGNED BINARY NUMBERS

Positive integers (including zero) can be represented as unsigned numbers. However, to represent negative integers, we need a notation for negative values. In ordinary arithmetic, a negative number is indicated by a minus sign and a positive number by a plus sign. Because of hardware limitations, computers must represent everything with binary digits. It is customary to represent the sign with a bit placed in the leftmost position of the number. The convention is to make the sign bit 0 for positive and 1 for negative.

It is important to realize that both signed and unsigned binary numbers consist of a string of bits when represented in a computer. The user determines whether the number is signed or unsigned. If the binary number is signed, then the leftmost bit represents the sign and the rest of the bits represent the number. If the binary number is assumed to be unsigned, then the leftmost bit is the most significant bit of the number. For example, the string of bits 01001 can be considered as 9 (unsigned binary) or as +9 (signed binary) because the leftmost bit is 0. The string of bits 11001 represents the binary equivalent of 25 when considered as an unsigned number and the binary equivalent of −9 when considered as a signed number. This is because the 1 that is in the leftmost position designates a negative and the other four bits represent binary 9. Usually, there is no confusion in identifying the bits if the type of representation for the number is known in advance.

The representation of the signed numbers in the last example is referred to as the *signed-magnitude* convention. In this notation, the number consists of a magnitude and a symbol (+ or −) or a bit (0 or 1) indicating the sign. This is the representation of signed numbers used in ordinary arithmetic. When arithmetic operations are implemented in a computer, it is more convenient to use a different system, referred to as the *signed-complement* system, for representing negative numbers. In this system, a negative number is indicated by its complement. Whereas the signed-magnitude system negates a number by changing its sign, the signed-complement system negates a number by taking its complement. Since positive numbers always start with 0 (plus) in the leftmost position, the complement will always start with a 1, indicating a negative number. The signed-complement system can use either the 1's or the 2's complement, but the 2's complement is the most common.

As an example, consider the number 9, represented in binary with eight bits. +9 is represented with a sign bit of 0 in the leftmost position, followed by the binary equivalent of 9, which gives 00001001. Note that all eight bits must have a value; therefore, 0's are inserted following the sign bit up to the first 1. Although there is only one way to represent +9, there are three different ways to represent −9 with eight bits:

signed-magnitude representation:	10001001
signed-1's-complement representation:	11110110
signed-2's-complement representation:	11110111

In signed-magnitude, −9 is obtained from +9 by changing the sign bit in the leftmost position from 0 to 1. In signed-1's complement, −9 is obtained by complementing all the bits of +9, including the sign bit. The signed-2's-complement representation of −9 is obtained by taking the 2's complement of the positive number, including the sign bit.

Table 1.3
Signed Binary Numbers

Decimal	Signed-2's Complement	Signed-1's Complement	Signed Magnitude
+7	0111	0111	0111
+6	0110	0110	0110
+5	0101	0101	0101
+4	0100	0100	0100
+3	0011	0011	0011
+2	0010	0010	0010
+1	0001	0001	0001
+0	0000	0000	0000
−0	—	1111	1000
−1	1111	1110	1001
−2	1110	1101	1010
−3	1101	1100	1011
−4	1100	1011	1100
−5	1011	1010	1101
−6	1010	1001	1110
−7	1001	1000	1111
−8	1000	—	—

Table 1.3 lists all possible four-bit signed binary numbers in the three representations. The equivalent decimal number is also shown for reference. Note that the positive numbers in all three representations are identical and have 0 in the leftmost position. The signed-2's-complement system has only one representation for 0, which is always positive. The other two systems have either a positive 0 or a negative 0, something not encountered in ordinary arithmetic. Note that all negative numbers have a 1 in the leftmost bit position; that is the way we distinguish them from the positive numbers. With four bits, we can represent 16 binary numbers. In the signed-magnitude and the 1's-complement representations, there are eight positive numbers and eight negative numbers, including two zeros. In the 2's-complement representation, there are eight positive numbers, including one zero, and eight negative numbers.

The signed-magnitude system is used in ordinary arithmetic, but is awkward when employed in computer arithmetic because of the separate handling of the sign and the magnitude. Therefore, the signed-complement system is normally used. The 1's complement imposes some difficulties and is seldom used for arithmetic operations. It is useful as a logical operation, since the change of 1 to 0 or 0 to 1 is equivalent to a logical complement operation, as will be shown in the next chapter. The discussion of signed binary arithmetic that follows deals exclusively with the signed-2's-complement representation of negative numbers. The same procedures can be applied to the signed-1's-complement system by including the end-around carry as is done with unsigned numbers.

Arithmetic Addition

The addition of two numbers in the signed-magnitude system follows the rules of ordinary arithmetic. If the signs are the same, we add the two magnitudes and give the sum the common sign. If the signs are different, we subtract the smaller magnitude from the larger and give the difference the sign of the larger magnitude. For example, $(+25) + (-37) = -(37 - 25) = -12$ and is done by subtracting the smaller magnitude, 25, from the larger magnitude, 37, and appending the sign of 37 to the result. This is a process that requires a comparison of the signs and magnitudes and then performing either addition or subtraction. The same procedure applies to binary numbers in signed-magnitude representation. In contrast, the rule for adding numbers in the signed-complement system does not require a comparison or subtraction, but only addition. The procedure is very simple and can be stated as follows for binary numbers:

The addition of two signed binary numbers with negative numbers represented in signed-2's-complement form is obtained from the addition of the two numbers, including their sign bits. A carry out of the sign-bit position is discarded.

Numerical examples for addition follow:

$$
\begin{array}{rr}
+\ 6 & 00000110 \\
\underline{+13} & \underline{00001101} \\
+19 & 00010011
\end{array}
\qquad
\begin{array}{rr}
-\ 6 & 11111010 \\
\underline{+13} & \underline{00001101} \\
+\ 7 & 00000111
\end{array}
$$

$$
\begin{array}{rr}
+\ 6 & 00000110 \\
\underline{-13} & \underline{11110011} \\
-\ 7 & 11111001
\end{array}
\qquad
\begin{array}{rr}
-\ 6 & 11111010 \\
\underline{-13} & \underline{11110011} \\
-19 & 11101101
\end{array}
$$

Note that negative numbers must be initially in 2's-complement form and that if the sum obtained after the addition is negative, it is in 2's-complement form.

In each of the four cases, the operation performed is addition with the sign bit included. Any carry out of the sign-bit position is discarded, and negative results are automatically in 2's-complement form.

In order to obtain a correct answer, we must ensure that the result has a sufficient number of bits to accommodate the sum. If we start with two n-bit numbers and the sum occupies $n + 1$ bits, we say that an overflow occurs. When one performs the addition with paper and pencil, an overflow is not a problem, because we are not limited by the width of the page. We just add another 0 to a positive number or another 1 to a negative number in the most significant position to extend the number to $n + 1$ bits and then perform the addition. Overflow is a problem in computers because the number of bits that hold a number is finite, and a result that exceeds the finite value by 1 cannot be accommodated.

The complement form of representing negative numbers is unfamiliar to those used to the signed-magnitude system. To determine the value of a negative number in signed-2's complement, it is necessary to convert the number to a positive number to place it in a more familiar form. For example, the signed binary number 11111001 is negative because the leftmost bit is 1. Its 2's complement is 00000111, which is the binary equivalent of +7. We therefore recognize the original negative number to be equal to −7.

Arithmetic Subtraction

Subtraction of two signed binary numbers when negative numbers are in 2's-complement form is simple and can be stated as follows:

Take the 2's complement of the subtrahend (including the sign bit) and add it to the minuend (including the sign bit). A carry out of the sign-bit position is discarded.

This procedure is adopted because a subtraction operation can be changed to an addition operation if the sign of the subtrahend is changed, as is demonstrated by the following relationship:

$$(\pm A) - (+B) = (\pm A) + (-B);$$
$$(\pm A) - (-B) = (\pm A) + (+B).$$

But changing a positive number to a negative number is easily done by taking the 2's complement of the positive number. The reverse is also true, because the complement of a negative number in complement form produces the equivalent positive number. To see this, consider the subtraction $(-6) - (-13) = +7$. In binary with eight bits, this operation is written as $(11111010 - 11110011)$. The subtraction is changed to addition by taking the 2's complement of the subtrahend (-13), giving $(+13)$. In binary, this is $11111010 + 00001101 = 100000111$. Removing the end carry, we obtain the correct answer: 00000111 $(+7)$.

It is worth noting that binary numbers in the signed-complement system are added and subtracted by the same basic addition and subtraction rules as unsigned numbers. Therefore, computers need only one common hardware circuit to handle both types of arithmetic. The user or programmer must interpret the results of such addition or subtraction differently, depending on whether it is assumed that the numbers are signed or unsigned.

1.7 BINARY CODES

Digital systems use signals that have two distinct values and circuit elements that have two stable states. There is a direct analogy among binary signals, binary circuit elements, and binary digits. A binary number of n digits, for example, may be represented by n binary circuit elements, each having an output signal equivalent to 0 or 1. Digital systems represent and manipulate not only binary numbers, but also many other discrete elements of information. Any discrete element of information that is distinct among a group of quantities can be represented with a binary code (i.e., a pattern of 0's and 1's). The codes must be in binary because, in today's technology, only circuits that represent and manipulate patterns of 0's and 1's can be manufactured economically for use in computers. However, it must be realized that binary codes merely change the symbols, not the meaning of the elements of information that they represent. If we inspect the bits of a computer at random, we will find that most of the time they represent some type of coded information rather than binary numbers.

An n-bit binary code is a group of n bits that assumes up to 2^n distinct combinations of 1's and 0's, with each combination representing one element of the set that is being coded. A set of four elements can be coded with two bits, with each element assigned one of the following bit combinations: 00, 01, 10, 11. A set of eight elements requires a three-bit code and a set of

16 elements requires a four-bit code. The bit combination of an n-bit code is determined from the count in binary from 0 to $2^n - 1$. Each element must be assigned a unique binary bit combination, and no two elements can have the same value; otherwise, the code assignment will be ambiguous.

Although the *minimum* number of bits required to code 2^n distinct quantities is n, there is no *maximum* number of bits that may be used for a binary code. For example, the 10 decimal digits can be coded with 10 bits, and each decimal digit can be assigned a bit combination of nine 0's and a 1. In this particular binary code, the digit 6 is assigned the bit combination 0001000000.

BCD Code

Although the binary number system is the most natural system for a computer, most people are more accustomed to the decimal system. One way to resolve this difference is to convert decimal numbers to binary, perform all arithmetic calculations in binary, and then convert the binary results back to decimal. This method requires that we store decimal numbers in the computer so that they can be converted to binary. Since the computer can accept only binary values, we must represent the decimal digits by means of a code that contains 1's and 0's. It is also possible to perform the arithmetic operations directly on decimal numbers when they are stored in the computer in coded form.

A binary code will have some unassigned bit combinations if the number of elements in the set is not a multiple power of 2. The 10 decimal digits form such a set. A binary code that distinguishes among 10 elements must contain at least four bits, but 6 out of the 16 possible combinations remain unassigned. Different binary codes can be obtained by arranging four bits into 10 distinct combinations. The code most commonly used for the decimal digits is the straight binary assignment listed in Table 1.4. This scheme is called *binary-coded decimal* and is commonly referred to as BCD. Other decimal codes are possible and a few of them are presented later in this section.

Table 1.4
Binary-Coded Decimal (BCD)

Decimal Symbol	BCD Digit
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

Table 1.4 gives the four-bit code for one decimal digit. A number with k decimal digits will require $4k$ bits in BCD. Decimal 396 is represented in BCD with 12 bits as 0011 1001 0110, with each group of 4 bits representing one decimal digit. A decimal number in BCD is the same as its equivalent binary number only when the number is between 0 and 9. A BCD number greater than 10 looks different from its equivalent binary number, even though both contain 1's and 0's. Moreover, the binary combinations 1010 through 1111 are not used and have no meaning in BCD. Consider decimal 185 and its corresponding value in BCD and binary:

$$(185)_{10} = (0001\ 1000\ 0101)_{BCD} = (10111001)_2$$

The BCD value has 12 bits to encode the characters of the decimal value, but the equivalent binary number needs only 8 bits. It is obvious that the representation of a BCD number needs more bits than its equivalent binary value. However, there is an advantage in the use of decimal numbers, because computer input and output data are generated by people who use the decimal system.

It is important to realize that BCD numbers are decimal numbers and not binary numbers, although they use bits in their representation. The only difference between a decimal number and BCD is that decimals are written with the symbols 0, 1, 2, ..., 9 and BCD numbers use the binary code 0000, 0001, 0010, ..., 1001. The decimal value is exactly the same. Decimal 10 is represented in BCD with eight bits as 0001 0000 and decimal 15 as 0001 0101. The corresponding binary values are 1010 and 1111 and have only four bits.

BCD Addition

Consider the addition of two decimal digits in BCD, together with a possible carry from a previous less significant pair of digits. Since each digit does not exceed 9, the sum cannot be greater than $9 + 9 + 1 = 19$, with the 1 being a previous carry. Suppose we add the BCD digits as if they were binary numbers. Then the binary sum will produce a result in the range from 0 to 19. In binary, this range will be from 0000 to 10011, but in BCD, it is from 0000 to 1 1001, with the first (i.e., leftmost) 1 being a carry and the next four bits being the BCD sum. When the binary sum is equal to or less than 1001 (without a carry), the corresponding BCD digit is correct. However, when the binary sum is greater than or equal to 1010, the result is an invalid BCD digit. The addition of $6 = (0110)_2$ to the binary sum converts it to the correct digit and also produces a carry as required. This is because a carry in the most significant bit position of the binary sum and a decimal carry differ by $16 - 10 = 6$. Consider the following three BCD additions:

$$\begin{array}{rrrrrr}
4 & 0100 & 4 & 0100 & 8 & 1000 \\
\underline{+5} & \underline{+0101} & \underline{+8} & \underline{+1000} & \underline{+9} & \underline{1001} \\
9 & 1001 & 12 & 1100 & 17 & 10001 \\
 & & & \underline{+0110} & & \underline{+0110} \\
 & & & 10010 & & 10111
\end{array}$$

In each case, the two BCD digits are added as if they were two binary numbers. If the binary sum is greater than or equal to 1010, we add 0110 to obtain the correct BCD sum and a carry. In the first example, the sum is equal to 9 and is the correct BCD sum. In the second example, the binary sum produces an invalid BCD digit. The addition of 0110 produces the correct BCD sum, 0010 (i.e., the number 2), and a carry. In the third example, the binary sum produces a carry. This condition occurs when the sum is greater than or equal to 16. Although the other four bits are less than 1001, the binary sum requires a correction because of the carry. Adding 0110, we obtain the required BCD sum 0111 (i.e., the number 7) and a BCD carry.

The addition of two n-digit unsigned BCD numbers follows the same procedure. Consider the addition of $184 + 576 = 760$ in BCD:

BCD	1	1		
	0001	1000	0100	184
	+0101	0111	0110	+576
Binary sum	0111	10000	1010	
Add 6		0110	0110	
BCD sum	0111	0110	0000	760

The first, least significant pair of BCD digits produces a BCD digit sum of 0000 and a carry for the next pair of digits. The second pair of BCD digits plus a previous carry produces a digit sum of 0110 and a carry for the next pair of digits. The third pair of digits plus a carry produces a binary sum of 0111 and does not require a correction.

Decimal Arithmetic

The representation of signed decimal numbers in BCD is similar to the representation of signed numbers in binary. We can use either the familiar signed-magnitude system or the signed-complement system. The sign of a decimal number is usually represented with four bits to conform to the four-bit code of the decimal digits. It is customary to designate a plus with four 0's and a minus with the BCD equivalent of 9, which is 1001.

The signed-magnitude system is seldom used in computers. The signed-complement system can be either the 9's or the 10's complement, but the 10's complement is the one most often used. To obtain the 10's complement of a BCD number, we first take the 9's complement and then add 1 to the least significant digit. The 9's complement is calculated from the subtraction of each digit from 9.

The procedures developed for the signed-2's-complement system in the previous section also apply to the signed-10's-complement system for decimal numbers. Addition is done by summing all digits, including the sign digit, and discarding the end carry. This operation assumes that all negative numbers are in 10's-complement form. Consider the addition $(+375) + (-240) = +135$, done in the signed-complement system:

$$\begin{array}{rr} 0 & 375 \\ +9 & 760 \\ \hline 0 & 135 \end{array}$$

The 9 in the leftmost position of the second number represents a minus, and 9760 is the 10's complement of 0240. The two numbers are added and the end carry is discarded to obtain +135. Of course, the decimal numbers inside the computer, including the sign digits, must be in BCD. The addition is done with BCD digits as described previously.

The subtraction of decimal numbers, either unsigned or in the signed-10's-complement system, is the same as in the binary case: Take the 10's complement of the subtrahend and add it to the minuend. Many computers have special hardware to perform arithmetic calculations directly with decimal numbers in BCD. The user of the computer can specify programmed instructions to perform the arithmetic operation with decimal numbers directly, without having to convert them to binary.

Other Decimal Codes

Binary codes for decimal digits require a minimum of four bits per digit. Many different codes can be formulated by arranging four bits into 10 distinct combinations. BCD and three other representative codes are shown in Table 1.5. Each code uses only 10 out of a possible 16 bit combinations that can be arranged with four bits. The other six unused combinations have no meaning and should be avoided.

BCD and the 2421 code are examples of weighted codes. In a weighted code, each bit position is assigned a weighting factor in such a way that each digit can be evaluated by adding the weights of all the 1's in the coded combination. The BCD code has weights of 8, 4, 2, and 1, which correspond to the power-of-two values of each bit. The bit assignment 0110, for example, is interpreted by the weights to represent decimal 6 because $8 \times 0 + 4 \times 1 + 2 \times 1 + 1 \times 0 = 6$. The bit combination 1101, when weighted by the respective digits 2421, gives the decimal equivalent of $2 \times 1 + 4 \times 1 + 2 \times 0 + 1 \times 1 = 7$. Note that some digits can be coded in two possible ways in the 2421 code. For instance, decimal 4 can be assigned to bit combination 0100 or 1010, since both combinations add up to a total weight of 4.

Table 1.5
Four Different Binary Codes for the Decimal Digits

Decimal Digit	BCD 8421	2421	Excess-3	8, 4, −2, −1
0	0000	0000	0011	0000
1	0001	0001	0100	0111
2	0010	0010	0101	0110
3	0011	0011	0110	0101
4	0100	0100	0111	0100
5	0101	1011	1000	1011
6	0110	1100	1001	1010
7	0111	1101	1010	1001
8	1000	1110	1011	1000
9	1001	1111	1100	1111
	1010	0101	0000	0001
Unused	1011	0110	0001	0010
bit	1100	0111	0010	0011
combi-	1101	1000	1101	1100
nations	1110	1001	1110	1101
	1111	1010	1111	1110

The 2421 and the excess-3 codes are examples of self-complementing codes. Such codes have the property that the 9's complement of a decimal number is obtained directly by changing 1's to 0's and 0's to 1's (i.e., by complementing each bit in the pattern). For example, decimal 395 is represented in the excess-3 code as 0110 1100 1000. The 9's complement of 604 is represented as 1001 0011 0111, which is obtained simply by complementing each bit of the code (as with the 1's complement of binary numbers).

The excess-3 code has been used in some older computers because of its self-complementing property. Excess-3 is an unweighted code in which each coded combination is obtained from the corresponding binary value plus 3. Note that the BCD code is not self-complementing.

The 8, 4, −2, −1 code is an example of assigning both positive and negative weights to a decimal code. In this case, the bit combination 0110 is interpreted as decimal 2 and is calculated from $8 \times 0 + 4 \times 1 + (-2) \times 1 + (-1) \times 0 = 2$.

Gray Code

The output data of many physical systems are quantities that are continuous. These data must be converted into digital form before they are applied to a digital system. Continuous or analog information is converted into digital form by means of an analog-to-digital converter. It is sometimes convenient to use the Gray code shown in Table 1.6 to represent digital data that have been converted from analog data. The advantage of the Gray code over the straight binary number sequence is that only one bit in the code group changes in going from one number to the next. For example, in going from 7 to 8, the Gray code changes from 0100 to 1100. Only the first bit changes, from 0 to 1; the other three bits remain the same. By contrast, with binary numbers the change from 7 to 8 will be from 0111 to 1000, which causes all four bits to change values.

Table 1.6
Gray Code

Gray Code	Decimal Equivalent
0000	0
0001	1
0011	2
0010	3
0110	4
0111	5
0101	6
0100	7
1100	8
1101	9
1111	10
1110	11
1010	12
1011	13
1001	14
1000	15

The Gray code is used in applications in which the normal sequence of binary numbers may produce an error or ambiguity during the transition from one number to the next. If binary numbers are used, a change, for example, from 0111 to 1000 may produce an intermediate erroneous number 1001 if the value of the rightmost bit takes longer to change than do the values of the other three bits. The Gray code eliminates this problem, since only one bit changes its value during any transition between two numbers.

A typical application of the Gray code is the representation of analog data by a continuous change in the angular position of a shaft. The shaft is partitioned into segments, and each segment is assigned a number. If adjacent segments are made to correspond with the Gray-code sequence, ambiguity is eliminated between the angle of the shaft and the value encoded by the sensor.

ASCII Character Code

Many applications of digital computers require the handling not only of numbers, but also of other characters or symbols, such as the letters of the alphabet. For instance, an insurance company with thousands of policyholders will use a computer to process its files. To represent the names and other pertinent information, it is necessary to formulate a binary code for the letters of the alphabet. In addition, the same binary code must represent numerals and special characters (such as $). An alphanumeric character set is a set of elements that includes the 10 decimal digits, the 26 letters of the alphabet, and a number of special characters. Such a set contains between 36 and 64 elements if only capital letters are included, or between 64 and 128 elements if both uppercase and lowercase letters are included. In the first case, we need a binary code of six bits, and in the second, we need a binary code of seven bits.

The standard binary code for the alphanumeric characters is the American Standard Code for Information Interchange (ASCII), which uses seven bits to code 128 characters, as shown in Table 1.7. The seven bits of the code are designated by b_1 through b_7, with b_7 the most significant bit. The letter *A*, for example, is represented in ASCII as 1000001 (column 100, row 0001). The ASCII code also contains 94 graphic characters that can be printed and 34 nonprinting characters used for various control functions. The graphic characters consist of the 26 uppercase letters (A through Z), the 26 lowercase letters (a through z), the 10 numerals (0 through 9), and 32 special printable characters, such as %, *, and $.

The 34 control characters are designated in the ASCII table with abbreviated names. They are listed again below the table with their functional names. The control characters are used for routing data and arranging the printed text into a prescribed format. There are three types of control characters: format effectors, information separators, and communication-control characters. Format effectors are characters that control the layout of printing. They include the familiar word processor and typewriter controls such as backspace (BS), horizontal tabulation (HT), and carriage return (CR). Information separators are used to separate the data into divisions such as paragraphs and pages. They include characters such as record separator (RS) and file separator (FS). The communication-control characters are useful during the transmission of text between remote terminals. Examples of communication-control characters are STX (start of text) and ETX (end of text), which are used to frame a text message transmitted through telephone wires.

Table 1.7
American Standard Code for Information Interchange (ASCII)

	$b_7b_6b_5$							
$b_4b_3b_2b_1$	**000**	**001**	**010**	**011**	**100**	**101**	**110**	**111**
0000	NUL	DLE	SP	0	@	P	`	p
0001	SOH	DC1	!	1	A	Q	a	q
0010	STX	DC2	"	2	B	R	b	r
0011	ETX	DC3	#	3	C	S	c	s
0100	EOT	DC4	$	4	D	T	d	t
0101	ENQ	NAK	%	5	E	U	e	u
0110	ACK	SYN	&	6	F	V	f	v
0111	BEL	ETB	'	7	G	W	g	w
1000	BS	CAN	(	8	H	X	h	x
1001	HT	EM	)	9	I	Y	i	y
1010	LF	SUB	*	:	J	Z	j	z
1011	VT	ESC	+	;	K	[	k	{
1100	FF	FS	,	<	L	\	l	|
1101	CR	GS	-	=	M	]	m	}
1110	SO	RS	.	>	N	^	n	~
1111	SI	US	/	?	O	_	o	DEL

Control characters

NUL	Null	DLE	Data-link escape
SOH	Start of heading	DC1	Device control 1
STX	Start of text	DC2	Device control 2
ETX	End of text	DC3	Device control 3
EOT	End of transmission	DC4	Device control 4
ENQ	Enquiry	NAK	Negative acknowledge
ACK	Acknowledge	SYN	Synchronous idle
BEL	Bell	ETB	End-of-transmission block
BS	Backspace	CAN	Cancel
HT	Horizontal tab	EM	End of medium
LF	Line feed	SUB	Substitute
VT	Vertical tab	ESC	Escape
FF	Form feed	FS	File separator
CR	Carriage return	GS	Group separator
SO	Shift out	RS	Record separator
SI	Shift in	US	Unit separator
SP	Space	DEL	Delete

ASCII is a seven-bit code, but most computers manipulate an eight-bit quantity as a single unit called a *byte*. Therefore, ASCII characters most often are stored one per byte. The extra bit is sometimes used for other purposes, depending on the application. For example, some printers recognize eight-bit ASCII characters with the most significant bit set to 0. An additional

128 eight-bit characters with the most significant bit set to 1 are used for other symbols, such as the Greek alphabet or italic type font.

Error-Detecting Code

To detect errors in data communication and processing, an eighth bit is sometimes added to the ASCII character to indicate its parity. A *parity bit* is an extra bit included with a message to make the total number of 1's either even or odd. Consider the following two characters and their even and odd parity:

	With even parity	With odd parity
ASCII A = 1000001	01000001	11000001
ASCII T = 1010100	11010100	01010100

In each case, we insert an extra bit in the leftmost position of the code to produce an even number of 1's in the character for even parity or an odd number of 1's in the character for odd parity. In general, one or the other parity is adopted, with even parity being more common.

The parity bit is helpful in detecting errors during the transmission of information from one location to another. This function is handled by generating an even parity bit at the sending end for each character. The eight-bit characters that include parity bits are transmitted to their destination. The parity of each character is then checked at the receiving end. If the parity of the received character is not even, then at least one bit has changed value during the transmission. This method detects one, three, or any odd combination of errors in each character that is transmitted. An even combination of errors, however, goes undetected, and additional error detection codes may be needed to take care of that possibility.

What is done after an error is detected depends on the particular application. One possibility is to request retransmission of the message on the assumption that the error was random and will not occur again. Thus, if the receiver detects a parity error, it sends back the ASCII NAK (negative acknowledge) control character consisting of an even-parity eight bits 10010101. If no error is detected, the receiver sends back an ACK (acknowledge) control character, namely, 00000110. The sending end will respond to an NAK by transmitting the message again until the correct parity is received. If, after a number of attempts, the transmission is still in error, a message can be sent to the operator to check for malfunctions in the transmission path.

1.8 BINARY STORAGE AND REGISTERS

The binary information in a digital computer must have a physical existence in some medium for storing individual bits. A *binary cell* is a device that possesses two stable states and is capable of storing one bit (0 or 1) of information. The input to the cell receives excitation signals that set it to one of the two states. The output of the cell is a physical quantity that distinguishes between the two states. The information stored in a cell is 1 when the cell is in one stable state and 0 when the cell is in the other stable state.

Registers

A *register* is a group of binary cells. A register with n cells can store any discrete quantity of information that contains n bits. The state of a register is an n-tuple of 1's and 0's, with each bit designating the state of one cell in the register. The content of a register is a function of the interpretation given to the information stored in it. Consider, for example, a 16-bit register with the following binary content:

$$1100001111001001$$

A register with 16 cells can be in one of 2^{16} possible states. If one assumes that the content of the register represents a binary integer, then the register can store any binary number from 0 to $2^{16} - 1$. For the particular example shown, the content of the register is the binary equivalent of the decimal number 50,121. If one assumes instead that the register stores alphanumeric characters of an eight-bit code, then the content of the register is any two meaningful characters. For the ASCII code with an even parity placed in the eighth most significant bit position, the register contains the two characters C (the leftmost eight bits) and I (the rightmost eight bits). If, however, one interprets the content of the register to be four decimal digits represented by a four-bit code, then the content of the register is a four-digit decimal number. In the excess-3 code, the register holds the decimal number 9,096. The content of the register is meaningless in BCD, because the bit combination 1100 is not assigned to any decimal digit. From this example, it is clear that a register can store discrete elements of information and that the same bit configuration may be interpreted differently for different types of data.

Register Transfer

A digital system is characterized by its registers and the components that perform data processing. In digital systems, a *register transfer* operation is a basic operation that consists of a transfer of binary information from one set of registers into another set of registers. The transfer may be direct, from one register to another, or may pass through data-processing circuits to perform an operation. Figure 1.1 illustrates the transfer of information among registers and demonstrates pictorially the transfer of binary information from a keyboard into a register in the memory unit. The input unit is assumed to have a keyboard, a control circuit, and an input register. Each time a key is struck, the control circuit enters an equivalent eight-bit alphanumeric character code into the input register. We shall assume that the code used is the ASCII code with an odd-parity bit. The information from the input register is transferred into the eight least significant cells of a processor register. After every transfer, the input register is cleared to enable the control to insert a new eight-bit code when the keyboard is struck again. Each eight-bit character transferred to the processor register is preceded by a shift of the previous character to the next eight cells on its left. When a transfer of four characters is completed, the processor register is full, and its contents are transferred into a memory register. The content stored in the memory register shown in Fig. 1.1 came from the transfer of the characters "J," "O," "H," and "N" after the four appropriate keys were struck.

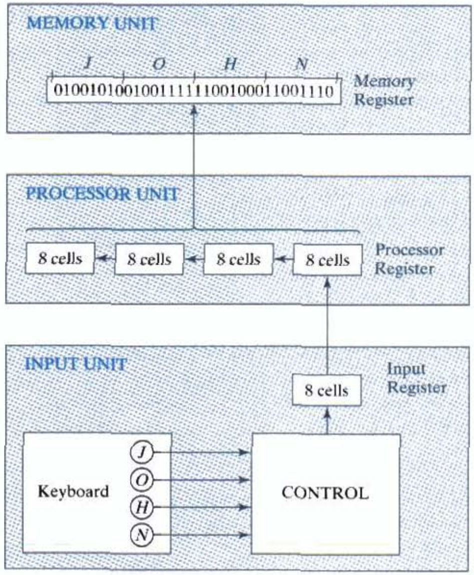


FIGURE 1.1
Transfer of information among registers

To process discrete quantities of information in binary form, a computer must be provided with devices that hold the data to be processed and with circuit elements that manipulate individual bits of information. The device most commonly used for holding data is a register. Binary variables are manipulated by means of digital logic circuits. Figure 1.2 illustrates the process of adding two 10-bit binary numbers. The memory unit, which normally consists of millions of registers, is shown with only three of its registers. The part of the processor unit shown consists of three registers—*R1*, *R2*, and *R3*—together with digital logic circuits that manipulate the bits of *R1* and *R2* and transfer into *R3* a binary number equal to their arithmetic sum. Memory registers store information and are incapable of processing the two operands. However, the information stored in memory can be transferred to processor registers, and the results obtained in processor registers can be transferred back into a memory register for storage until needed again. The diagram shows the contents of two operands transferred from two memory registers into *R1* and *R2*. The digital logic circuits produce the sum, which is transferred to register *R3*. The contents of *R3* can now be transferred back to one of the memory registers.

The last two examples demonstrated the information-flow capabilities of a digital system in a simple manner. The registers of the system are the basic elements for storing and holding the binary information. Digital logic circuits process the binary information stored in the

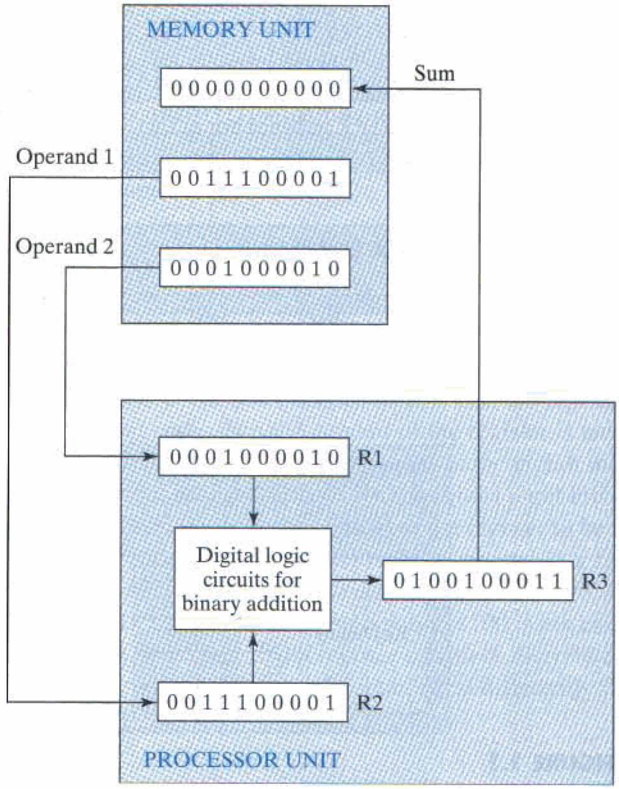


FIGURE 1.2
Example of binary information processing

registers. Digital logic circuits and registers are covered in Chapters 2 through 6. The memory unit is explained in Chapter 7. The description of register operations at the register transfer level and the design of digital systems are covered in Chapter 8.

1.9 BINARY LOGIC

Binary logic deals with variables that take on two discrete values and with operations that assume logical meaning. The two values the variables assume may be called by different names (*true* and *false*, *yes* and *no*, etc.), but for our purpose, it is convenient to think in terms of bits and assign the values 1 and 0. The binary logic introduced in this section is equivalent to an algebra called Boolean algebra. The formal presentation of Boolean algebra is covered in more detail in Chapter 2. The purpose of this section is to introduce Boolean algebra in a heuristic manner and relate it to digital logic circuits and binary signals.

Definition of Binary Logic

Binary logic consists of binary variables and a set of logical operations. The variables are designated by letters of the alphabet, such as A, B, C, x, y, z, etc., with each variable having two and only two distinct possible values: 1 and 0. There are three basic logical operations: AND, OR, and NOT.

1. AND: This operation is represented by a dot or by the absence of an operator. For example, $x \cdot y = z$ or $xy = z$ is read "x AND y is equal to z." The logical operation AND is interpreted to mean that $z = 1$ if and only if $x = 1$ and $y = 1$; otherwise $z = 0$. (Remember that x, y, and z are binary variables and can be equal either to 1 or 0, and nothing else.)
2. OR: This operation is represented by a plus sign. For example, $x + y = z$ is read "x OR y is equal to z," meaning that $z = 1$ if $x = 1$ *or* if $y = 1$ *or* if both $x = 1$ and $y = 1$. If both $x = 0$ and $y = 0$, then $z = 0$.
3. NOT: This operation is represented by a prime (sometimes by an overbar). For example, $x' = z$ (or $\overline{x} = z$) is read "not x is equal to z," meaning that z is what x is not. In other words, if $x = 1$, then $z = 0$, but if $x = 0$, then $z = 1$. The NOT operation is also referred to as the complement operation, since it changes a 1 to 0 and a 0 to 1.

Binary logic resembles binary arithmetic, and the operations AND and OR have similarities to multiplication and addition, respectively. In fact, the symbols used for AND and OR are the same as those used for multiplication and addition. However, binary logic should not be confused with binary arithmetic. One should realize that an arithmetic variable designates a number that may consist of many digits. A logic variable is always either 1 or 0. For example, in binary arithmetic, we have $1 + 1 = 10$ (read "one plus one is equal to 2"), whereas in binary logic, we have $1 + 1 = 1$ (read "one OR one is equal to one").

For each combination of the values of x and y, there is a value of z specified by the definition of the logical operation. Definitions of logical operations may be listed in a compact form called *truth tables*. A truth table is a table of all possible combinations of the variables, showing the relation between the values that the variables may take and the result of the operation. The truth tables for the operations AND and OR with variables x and y are obtained by listing all possible values that the variables may have when combined in pairs. For each combination, the result of the operation is then listed in a separate row. The truth tables for AND, OR, and NOT are given in Table 1.8. These tables clearly demonstrate the definition of the operations.

Table 1.8
Truth Tables of Logical Operations

AND			OR			NOT	
x	y	$x \cdot y$	x	y	$x + y$	x	x'
0	0	0	0	0	0	0	1
0	1	0	0	1	1	1	0
1	0	0	1	0	1		
1	1	1	1	1	1		

Logic Gates

Logic gates are electronic circuits that operate on one or more input signals to produce an output signal. Electrical signals such as voltages or currents exist as analog signals having values over a given range, say, 0 to 3 V, but in a digital system are interpreted to be either of two recognizable values, 0 or 1. Voltage-operated logic circuits respond to two separate voltage levels that represent a binary variable equal to logic 1 or logic 0. For example, a particular digital system may define logic 0 as a signal equal to 0 volts and logic 1 as a signal equal to 3 volts. In practice, each voltage level has an acceptable range, as shown in Fig. 1.3. The input terminals of digital circuits accept binary signals within the allowable range and respond at the output terminals with binary signals that fall within the specified range. The intermediate region between the allowed regions is crossed only during a state transition. Any desired information for computing or control can be operated on by passing binary signals through various combinations of logic gates, with each signal representing a particular binary variable.

The graphic symbols used to designate the three types of gates are shown in Fig. 1.4. The gates are blocks of hardware that produce the equivalent of logic-1 or logic-0 output signals

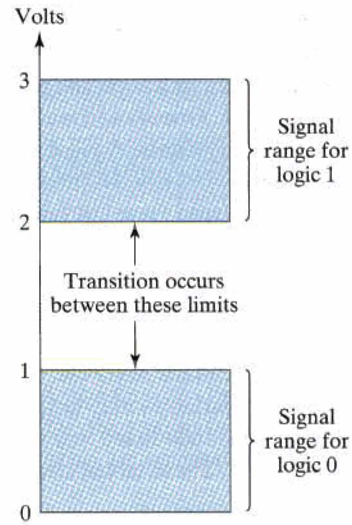


FIGURE 1.3
Example of binary signals

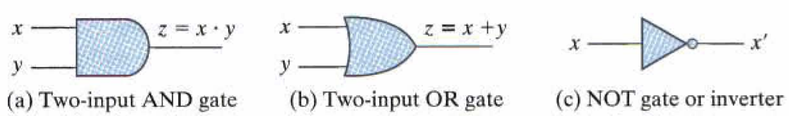


FIGURE 1.4
Symbols for digital logic circuits

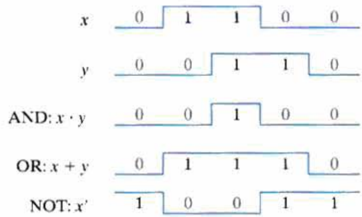
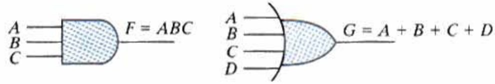


FIGURE 1.5
Input–output signals for gates



(a) Three-input AND gate (b) Four-input OR gate

FIGURE 1.6
Gates with multiple inputs

if input logic requirements are satisfied. The input signals x and y in the AND and OR gates may exist in one of four possible states: 00, 10, 11, or 01. These input signals are shown in Fig. 1.5 together with the corresponding output signal for each gate. The timing diagrams illustrate the response of each gate to the four input signal combinations. The horizontal axis of the timing diagram represents time, and the vertical axis shows the signal as it changes between the two possible voltage levels. The low level represents logic 0, the high level logic 1. The AND gate responds with a logic 1 output signal when both input signals are logic 1. The OR gate responds with a logic 1 output signal if any input signal is logic 1. The NOT gate is commonly referred to as an inverter. The reason for this name is apparent from the signal response in the timing diagram, which shows that the output signal inverts the logic sense of the input signal.

AND and OR gates may have more than two inputs. An AND gate with three inputs and an OR gate with four inputs are shown in Fig. 1.6. The three-input AND gate responds with logic 1 output if all three inputs are logic 1. The output produces logic 0 if any input is logic 0. The four-input OR gate responds with logic 1 if any input is logic 1; its output becomes logic 0 only when all inputs are logic 0.

PROBLEMS

Answers to problems marked with * appear at the end of the book.

1.1 List the octal and hexadecimal numbers from 16 to 32. Using A, B, and C for the last three digits, list the numbers from 8 to 28 in base 13.

1.2* What is the exact number of bytes in a system that contains (a) 32K bytes, (b) 64M bytes, and (c) 6.4G bytes?

1.3 Convert the following numbers with the indicated bases to decimal:
(a)* $(4310)_5$ (b)* $(198)_{12}$
(c) $(735)_8$ (d) $(525)_6$

1.4 What is the largest binary number that can be expressed with 14 bits? What are the equivalent decimal and hexadecimal numbers?

1.5* Determine the base of the numbers in each case for the following operations to be correct:
(a) $14/2 = 5$, (b) $54/4 = 13$,
(c) $24 + 17 = 40$.

1.6* The solutions to the quadratic equation $x^2 - 11x + 22 = 0$ are $x = 3$ and $x = 6$. What is the base of the numbers?

1.7* Convert the hexadecimal number 68BE to binary, and then convert it from binary to octal.

1.8 Convert the decimal number 431 to binary in two ways: (a) Convert directly to binary; (b) convert first to hexadecimal and then from hexadecimal to binary. Which method is faster?

1.9 Express the following numbers in decimal:
(a)* $(10110.0101)_2$ (b)* $(16.5)_{16}$
(c)* $(26.24)_8$ (d) $(\mathrm{FAFA})_{16}$
(e) $(1010.1010)_2$

1.10 Convert the following binary numbers to hexadecimal and to decimal: (a) 1.10010, (b) 110.010. Explain why the decimal answer in (b) is 4 times that in (a).

1.11 Perform the following division in binary: $111011 \div 101$.

1.12* Add and multiply the following numbers without converting them to decimal.
(a) Binary numbers 1011 and 101.
(b) Hexadecimal numbers 2E and 34.

1.13 Do the following conversion problems:
(a) Convert decimal 27.315 to binary.
(b) Calculate the binary equivalent of 2/3 out to eight places. Then convert from binary to decimal. How close is the result to 2/3?
(c) Convert the binary result in (b) into hexadecimal. Then convert the result to decimal. Is the answer the same?

1.14 Obtain the 1's and 2's complements of the following binary numbers:
(a) 10000000 (b) 00000000
(c) 11011010 (d) 01110110
(e) 10000101 (f) 11111111.

1.15 Find the 9's and the 10's complement of the following decimal numbers:
(a) 52,784,630 (b) 63,325,600
(c) 25,000,000 (d) 00,000,000.

1.16 (a) Find the 16's complement of B2FA.
(b) Convert B2FA to binary.
(c) Find the 2's complement of the result in (b).
(d) Convert the answer in (c) to hexadecimal and compare with the answer in (a).

1.17 Perform subtraction on the given unsigned numbers using the 10's complement of the subtrahend. Where the result should be negative, find its 10's complement and affix a minus sign. Verify your answers.

(a) 6,428 − 3,409 (b) 125 − 1,800
(c) 2,043 − 6,152 (d) 1,631 − 745

1.18 Perform subtraction on the given unsigned binary numbers using the 2's complement of the subtrahend. Where the result should be negative, find its 2's complement and affix a minus sign.

(a) 10011 − 10001 (b) 100010 − 100011
(c) 1001 − 101000 (d) 110000 − 10101

1.19* The following decimal numbers are shown in sign-magnitude form: +9,286 and +801. Convert them to signed-10's-complement form and perform the following operations (note that the sum is +10,627 and requires five digits and a sign).

(a) (+9,286) + (+801) (b) (+9,286) + (−801)
(c) (−9,286) + (+801) (d) (−9,286) + (−801)

1.20 Convert decimal +46 and +29 to binary, using the signed-2's-complement representation and enough digits to accommodate the numbers. Then perform the binary equivalent of (+29) + (−49), (−29) + (+49), and (−29) + (−49). Convert the answers back to decimal and verify that they are correct.

1.21 If the numbers $(+9{,}742)_{10}$ and $(+641)_{10}$ are in signed magnitude format, their sum is $(+10{,}383)_{10}$ and requires five digits and a sign. Convert the numbers to signed-10's-complement form and find the following sums:

(a) (+9,742) + (+641) (b) (+9,742) + (−641)
(c) (−9,742) + (+641) (d) (−9,742) + (−641)

1.22 Convert decimal 8,723 to both BCD and ASCII codes. For ASCII, an even parity bit is to be appended at the left.

1.23 Represent the unsigned decimal numbers 842 and 537 in BCD, and then show the steps necessary to form their sum.

1.24 Formulate a weighted binary code for the decimal digits, using weights

(a)* 6, 3, 1, 1

(b) 6, 4, 2, 1

1.25 Represent the decimal number 5,137 in (a) BCD, (b) excess-3 code, (c) 2421 code, and (d) a 6311 code.

1.26 Find the 9's complement of decimal 5,137 and express it in 2421 code. Show that the result is the 1's complement of the answer to (c) in Problem 1.25. This demonstrates that the 2421 code is self-complementing.

1.27 Assign a binary code in some orderly manner to the 52 playing cards. Use the minimum number of bits.

1.28 Write the expression "G. Boole" in ASCII, using an eight-bit code. Include the period and the space. Treat the leftmost bit of each character as a parity bit. Each eight-bit code should have even parity. (George Boole was a 19th century mathematician. Boolean algebra, introduced in the next chapter, bears his name.)

1.29* Decode the following ASCII code:

1000010 1101001 1101100 1101100 1000111 1100001 1110100 1100101 1110011.

1.30 The following is a string of ASCII characters whose bit patterns have been converted into hexadecimal for compactness: 73 F4 E5 76 E5 4A EF 62 73. Of the eight bits in each pair of digits, the leftmost is a parity bit. The remaining bits are the ASCII code.

(a) Convert the string to bit form and decode the ASCII.

(b) Determine the parity used: odd or even?

1.31* How many printing characters are there in ASCII? How many of them are special characters (not letters or numerals)?

1-32* What bit must be complemented to change an ASCII letter from capital to lowercase and vice versa?

1.33* The state of a 12-bit register is 100010010111. What is its content if it represents

(a) three decimal digits in BCD?

(b) three decimal digits in the excess-3 code?

(c) three decimal digits in the 84-2-1 code?

(d) a binary number?

1.34 List the ASCII code for the 10 decimal digits with an odd parity bit in the leftmost position.

1.35 By means of a timing diagram similar to Fig. 1.5, show the signals of the outputs f and g in Fig. P1.35 as functions of the three inputs a, b, and c. Use all eight possible combinations of a, b, and c.

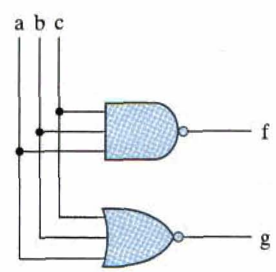


FIGURE P1.35

1.36 By means of a timing diagram similar to Fig. 1.5, show the signals of the outputs f and g in Fig. P1.36 as functions of the two inputs a and b. Use all four possible combinations of a and b.

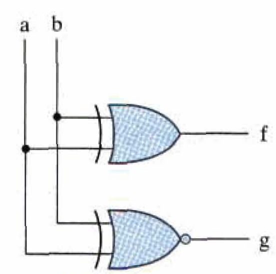


FIGURE P1.36

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